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On Pontrjagin classes and homotopy types of manifolds.

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1. Introduction.

In the present paper we shall obtain manifolds of the same homotopy type with different Pontrjagin classes, belonging therefore to different classes in the sense of diffeomorphism (i. e. differentiable homeomorphism).

This reveals the Pontrjagin classes as no homotopy invariants and the problem of "topological invariance of Pontrjagin classes," except for mod 2 and mod 3, as not provable by means of homotopy invariants, such as (co)homology groups, homotopy groups, Steenrod operators etc.

In section 2 of this paper we define some sphere bundles over spheres and determine the homotopy types of them by method of A. Dold. In section 4, Pontrjagin classes of these bundles are calculated from the Chern classes of associated bundles by the obstruction theory prepared in section 3. Section 5 is devoted to the description of the cohomology groups of total spaces of bundles. The principal tool here is the Gysin exact sequence.

In section 6, C^{∞}-manifolds are defined from the bundles and their Pontrjagin classes are computed. Our final results are exposed in section 7.

We use in this paper the results on the homotopy groups of spheres and classical groups which can be found, for example, in Steenrod [11], Borel et Serre [1], Serre [10].

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2. Fibre bundles over the q-sphere.

Let $\mathfrak{B}=\{B, p, S^q, S^r, SO(r+1)\}$ be fibre bundles over q-sphere S^q with total space B, r-sphere S^r as fibre and the rotation group $SO(r+1)$ as structural group.

Let S^{q-1} be a great $(q-1)$-sphere on S^q and let E_1, E_2 be the closed hemispheres of S^q determined by S^{q-1}. Let $V_i\,(i=1,2)$ be an open q-cell on S^q containing E_i and bounded by an $(q-1)$-sphere parallel to S^{q-1}, and x_0 be a reference point on S^{q-1}.

Bundle $\mathfrak{B}$ is strictly equivalent to a bundle with coordinate neighbourhoods $V_i\,(i=1,2)$ and coordinate functions $\phi_i: V_i\times S^r\to\mathfrak{B}_i\,(i=1,2)$, where $\mathfrak{B}_i$ is the portion of $\mathfrak{B}$ over V_i, and with coordinate transformations g_{12} such that $g_{12}(x_0)=e$.

The map $T=g_{12}|S^{q-1}$ which maps S^{q-1} into $SO(r+1)$ is characteristic map of $\mathfrak{B}$. The equivalence classes of bundles are in 1-1 correspondence with homotopy classes of maps T.

For $q=4, 8$, we shall describe the homotopy classes of maps T explicitly.

As is well-known, we have

$$\pi_3(SO(3))\approx Z,\ \pi_3(SO(4))\approx Z+Z,\ \pi_3(SO(r))\approx Z \qquad (r\geqq 5),$$
$$\pi_7(SO(7))\approx Z,\ \pi_7(SO(8))\approx Z+Z,\ \pi_7(SO(r))\approx Z \qquad (r\geqq 9).$$

(Z means as usual the additive group of integers, Z_n the group Z mod n.)

Let $i_r: SO(r)\to SO(r+1)$ be natural injection. Then the generators

$$\{\rho_3\},\ \{\rho_4, \sigma_4\},\ \{\sigma_r\}\ (r>5),\ \{\bar\rho_7\},\ \{\bar\rho_8, \bar\sigma_8\},\ \{\bar\sigma_r\} \qquad (r\geqq 9)$$

of $\pi_3(SO(3))$, $\pi_3(SO(4))$, $\pi_3(SO(r))$, $\pi_7(SO(7))$, $\pi_7(SO(8))$, $\pi_7(SO(r))$ respectively are given as follows;

$$\rho_3(u)v=uvu^{-1},\ \rho_4=(i_3)_*\rho_3,\ \sigma_4(u)v=uv,$$

where u, v denote quaternions as usual. And

$$\sigma_r=(i_{r-1})_*\circ(i_{r-2})_*\circ\cdots\circ(i_4)_*\sigma_4 \qquad (r\geqq 5).$$
$$t\bar\rho_7(x)y=xyx^{-1},\ \bar\rho_8=(i_7)_*\bar\rho_7,\ \bar\sigma_8(x)y=xy, \qquad (2.1)$$

where x, y denote Cayley numbers as usual and t is a odd integer.[1] Furthermore

$$\bar\sigma_9=(i_8)_*(\bar\sigma_8-[t/2]\bar\rho_8),^{2)}\ \bar\sigma_r=(i_{r-1})_*\circ(i_{r-2})_*\circ\cdots\circ(i_9)_*\bar\sigma_9 \qquad (r\geqq 10).$$

Between these generators hold the following relations.

$$(i_4)_*\rho_4=2\sigma_5\,,\ (i_8)_*\bar{\rho}_8=2\bar{\sigma}_9\,,\ (i_8)_*\bar{\sigma}_8=t\bar{\sigma}_9\,. \tag{2.2}$$

Now we define the bundles $\mathfrak{B}^{(q,r)}$ by:

$$\mathfrak{B}^{(4,3)}_{m,n}=\{B^{(4,3)}_{m,n}, p, S^4, S^3, SO(4)\}\,,$$
$$\mathfrak{B}^{(4,r)}_{n}=\{B^{(4,r)}_{n}, p, S^4, S^r, SO(r+1)\} \qquad (r\geqq 4)\,,$$
$$\mathfrak{B}^{(8,7)}_{m,n}=\{B^{(8,7)}_{m,n}, p, S^8, S^7, SO(8)\}\,,$$
$$\mathfrak{B}^{(8,r)}_{n}=\{B^{(8,r)}_{n}, p, S^8, S^r, SO(r+1)\} \qquad (r\geqq 8)\,,$$

where $\mathfrak{B}^{(4,3)}_{m,n}$, $\mathfrak{B}^{(4,r)}_{n}$, $\mathfrak{B}^{(8,7)}_{m,n}$, $\mathfrak{B}^{(8,r)}_{n}$ have characteristic maps $m\rho_4+n\sigma_4$, $n\sigma_r$, $m\bar{\rho}_8+n\bar{\sigma}_8$, $n\bar{\sigma}_r$ respectively.

Let us consider the commutative diagram;

$$\begin{array}{ccc} \pi_{q-1}(SO(r)) & \xrightarrow{(i_r)_*} & \pi_{q-1}(SO(r+1)) \\ \downarrow (\bar{j}_0)_* & (\bar{i})_* & \downarrow (\bar{j})_* \\ \pi_{q-1}(G^r) & \longrightarrow & \pi_{q-1}(\mathfrak{G}^r) \end{array} \tag{2.3}$$

where $\mathfrak{G}^r$ is the space of all continuous mappings from S^r into S^r with the compact open topology, G^r is the subspace of $\mathfrak{G}^r$ consisting of mappings which leave invariant one fixed point of S^r and $\bar{j}_0$, $\bar{j}$, $\bar{i}$ are natural injections.

We owe Dold [2] the following theorem on the homotopy equivalence between sphere bundles over spheres.

THEOREM 2.1. *Let T_i be the characteristic map of $\mathfrak{B}_i=\{B_i, p, S^q, S^r, SO(r+1)\}$ $(i=1,2)$. Let $(\bar{j})_*$ be the map of $\pi_{q-1}(SO(r+1))$ into $\pi_{q-1}(\mathfrak{G}^r)$ as given by* (1.3). *If $(\bar{j})_*T_1=(\bar{j})_*T_2$, then T_1 and T_2 define homotopically equivalent bundles.*

As is well known, we have $\iota:\pi_{q-1}(G^r)\approx\pi_{r+q-1}(S^r)$ and $J=\iota\circ(\bar{j}_0)_*$: $\pi_{q-1}(SO(r))\to\pi_{r+q-1}(S^r)$ is the so-called J homomorphism (G. W. Whitehead [14]). And

$$\pi_6(S^3)\approx Z_{12},\ \pi_7(S^4)\approx Z+Z_{12},\qquad \pi_{3+r}(S^r)\approx Z_{24}\qquad (r\geqq 5)\,;$$
$$\pi_{14}(S^7)\approx Z_{120},\ \pi_{15}(S^8)\approx Z+Z_{120},\ \pi_{7+r}(S^r)\approx Z_{240}\qquad (r\geqq 9)\,.$$

Therefore, in the following commutative diagram,

$$\begin{array}{ccc} \pi_3(SO(3)) & \xrightarrow{(i_3)_*} & \pi_3(SO(4)) \\ \downarrow (\bar{j}_0)_* & (\bar{i})_* & \downarrow (\bar{j})_* \\ Z_{12}\approx\pi_3(G^3) & \longrightarrow & \pi_3(\mathfrak{G}^3) \end{array}$$

we have

$$(\bar{j})_*(12\rho_4)=(\bar{j})_*\circ(i_3)_*(12\rho_3)=(\bar{i})_*\circ(\bar{j}_0)_*(12\rho_3)=0\ .$$

Similarly we have following commutative diagram,

$$\begin{array}{ccccc}
\pi_3(SO(3)) & \xrightarrow{(i_3)_*} & \pi_3(SO(4)) & \xrightarrow{(i_4)_*} & \pi_3(SO(5)) \\
 & & \downarrow{\scriptstyle (\bar{j}_0)_*} & & \downarrow{\scriptstyle (\bar{j})_*} \\
J\downarrow & & \pi_3(G^4) & \xrightarrow{(\bar{i})_*} & \pi_3(\mathfrak{G}^4) \\
 & & \updownarrow{\scriptstyle \iota} & & \\
Z_{12}\approx\pi_6(S^3) & \xrightarrow{E} & \pi_7(S^4)\approx Z+Z_{12} & &
\end{array}$$

where E is the suspension homomorphism. Therefore

$$\begin{aligned}(\bar{j})_*(24\sigma_5)&=(\bar{j})_*\circ(i_4)_*(12\rho_4)=(\bar{i})_*\circ(\bar{j}_0)_*(12\rho_4)\\ &=(\bar{i})_*\circ(\bar{j}_0)_*\circ(i_3)_*(12\rho_3)=(\bar{i})_*\circ\iota^{-1}\circ E\circ J(12\rho_3)=0\ .\end{aligned}$$

For $r\geqq 5$, we have the following commutative diagram,

$$\begin{array}{ccc}
\pi_3(SO(r)) & \xrightarrow{(i_r)_*} & \pi_3(SO(r+1)) \\
\downarrow{\scriptstyle (\bar{j}_0)_*} & & \downarrow{\scriptstyle (\bar{j})_*} \\
Z_{24}\approx\pi_3(G^r) & \xrightarrow{(\bar{i})_*} & \pi_3(\mathfrak{G}^r)
\end{array}$$

and

$$(\bar{j})_*(24\sigma_{r+1})=(\bar{j})_*\circ(i_r)_*(24\sigma_r)=(\bar{i})_*\circ(\bar{j}_0)_*(24\sigma_r)=0\ .$$

By Theorem 2.1 and the above results, we obtain easily the following Theorem using weak equivalence of bundles.

THEOREM 2.2. (i) *If* $m\equiv m'$ (*mod* 12), $n=n'$ *; or* $m=-m'$, $n=-n'$, *then* $B^{(4,3)}_{m,n}$, $B^{(4,3)}_{m',n'}$ *have the same homotopy type.*

(ii) *Let* $r\geqq 4$. *If* $n\equiv\pm n'$ (*mod* 24), *then* $B^{(4,r)}_{n}$, $B^{(4,r)}_{n'}$ *have the same homotopy type.*

Similarly we obtain for $\mathfrak{B}^{(8,7)}_{m,n}$ and $\mathfrak{B}^{(8,r)}_{n}$.

THEOREM 2.3. (i) *If* $m\equiv m'$ (*mod* 120), $n=n'$*; or* $m=-m'$, $n=-n'$*; then* $B^{(8,7)}_{m,n}$, $B^{(8,7)}_{m',n'}$ *have the same homotopy type.*

(ii) *Let* $r\geqq 8$, *If* $n\equiv\pm n'$ (*mod* 240), *then* $B^{(8,r)}_{n}$, $B^{(8,r)}_{n'}$ *have the same homotopy type.*

REMARK 2.4. James and J. H. C. Whitehead [7] have given a necessary and sufficient condition for sphere bundles over spheres with cross sections to have the same homotopy type. We see by their result that the conditions of Theorem 2.2 are also necessary in case of $B^{(4,3)}_{m,0}$, $B^{(4,q)}_{n}$.

3. Obstructions of bundles over q-sphere.

Let $\mathfrak{B}'=\{B', p', S^q, Y, SO(r+1)\}$ be a bundle (weakly) associated to $\mathfrak{B}$, and we now assume that $\pi_i(Y)=0$ $(0\leqq i<q-1)$, $\pi_{q-1}(Y)\approx Z$. We denote with ε a generator of $\pi_{q-1}(Y)$.

By our assumptions obstruction cocycle $\bar{c}(\mathfrak{B}')\in H^q(S^q)$ is defined (Steenrod [11] § 32). Let us compute $\bar{c}(\mathfrak{B}')$ from the characteristic map T of $\mathfrak{B}$.

Let E_1, E_2 be hemispheres of S^q as in section 2. They constitute a cell subdivision of S^q. Orient E_1, E_2 concordantly with S^q and orient S^{q-1} so as to be positively incident with E_1.

We construct a cross section of $\mathfrak{B}'$ over E_2 by $\phi_2(E_2\times y_0)$, where y_0 is a fixed point of fibre Y. This cross section is defined over S^{q-1} and obstruction to extend it over E_1 is given by $\phi_1^{-1}\phi_2(S^{q-1}\times y_0)=T(x)(y_0)$, $x\in S^{q-1}$. Define now $[T(x)]$ by

$$T(x)(y_0)\in[T(x)]\varepsilon\,. \tag{3.1}$$

Then we have $\bar{c}(\mathfrak{B}')(E_1)=[T(x)]$ and $\bar{c}(\mathfrak{B}')(E_2)=0$. Hence we obtain the following theorem:

THEOREM 3.1. *Under above hypothesis, the primary obstruction cocycle* $\bar{c}(\mathfrak{B}')$ *of* $\mathfrak{B}'$ *is given by* $\pm[T(x)]\alpha$, *where* α *is a generator of* $H^q(S^q)$.

4. Pontrjagin classes of fibre bundles.

Pontragin classes of a fibre bundle $\mathfrak{B}$ with structural group $SO(r)$ are defined as follows (Hirzebruch [5] § 4).
Let $j_r: SO(r)\to SU(r)$ be natural injection of the rotation group into the special unitary group. We can regard bundle $\mathfrak{B}$ provided with structural group $SU(r)$.

We construct associate bundles $\mathfrak{B}'_i$ $(i=1, 2,\cdots, r)$ with fibre $SU(r)/SU(i-1)$. Then Pontrjagin classes p_i $(i=1, 2,\cdots, [r/2])$ of $\mathfrak{B}$ are defined by

$$p_i(\mathfrak{B})=(-1)^i\,\bar{c}(\mathfrak{B}'_{2i})\,.$$

Let us calculate p_1 of $\mathfrak{B}^{(4,r)}$ $(r\geqq 3)$ and p_2 of $\mathfrak{B}^{(8,r)}$ $(r\geqq 7)$. We denote $\mathfrak{B}'^{(4,r)}_2$, $\mathfrak{B}'^{(8,r)}_4$ simply by $\mathfrak{B}'^{(4,r)}$, $\mathfrak{B}'^{(8,r)}$.

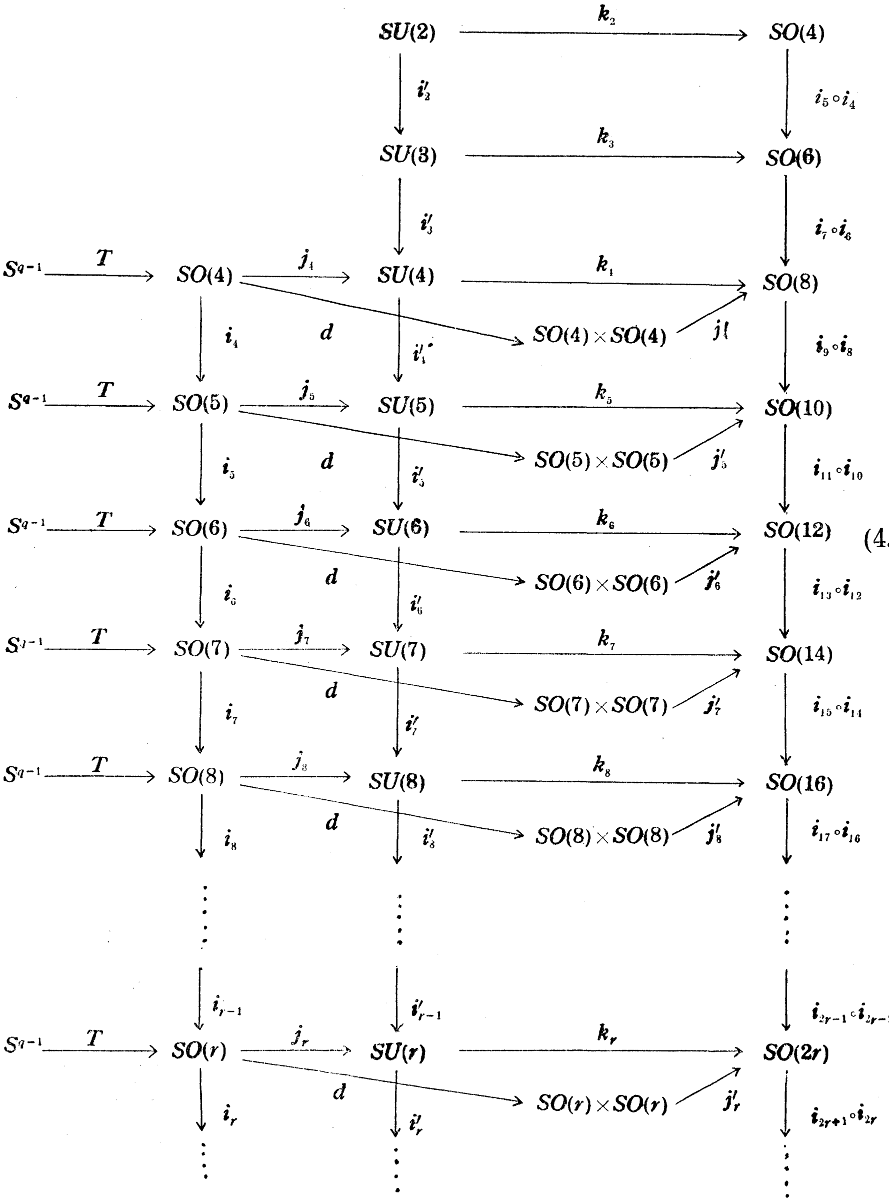
(I) *The first Pontrjagin class of* $\mathfrak{B}^{(4,r)}$ $(r\geqq 3)$.

Since natural projection $p: SU(r)\to SU(r)/SU(1)$ induces the isomorphism of homotopy groups

$$p_* : \pi_3(SU(r)) \approx \pi_3(SU(r)/SU(1)) \qquad (r \geqq 3)\,,$$

we may consider $\pi_3(SU(r))$, instead of $\pi_3(SU(r)/SU(1))$.

Let us consider the following (not commutative) diagram, where i'_r, k_r, j'_r are natural injections and $d: SO(r) \to SO(r) \times SO(r)$ is defined by $a \to (a, a)$.



(4.1)

We have $\pi_3(SU(r)) \approx Z$ $(r \geqq 2)$. Let μ_r be a generator of $\pi_3(SU(r))$ such that

$$(i'_r)_* \mu_r = \mu_{r+1}, \quad (k_2)_* \mu_2 = \sigma_4 .$$

Then we have

$$(k_4)_* \circ (j_4)_* (m\rho_4 + n\sigma_4) = (j'_4)_* \circ (d)_* (m\rho_4 + n\sigma_4)$$
$$= 2(i_7)_* \circ (i_6)_* \circ (i_5)_* \circ (i_4)_* (m\rho_4 + n\sigma_4) .$$

Since $(i_r)_*$ is isomorphism onto for $r \geqq 5$ and

$$(i_4)_* \sigma_4 = \sigma_5, \; (i_4)_* \rho_4 = 2\sigma_5$$

we have

$$(k_4)_* \circ (j_4)_* (m\rho_4 + n\sigma_4) = 2(2m+n)\sigma_8 .$$

On the other hand

$$\mu_4 = (i'_3)_* \circ (i'_2)_* \mu_2, \quad \sigma_8 = (i_7)_* \circ (i_6)_* \circ (i_5)_* \circ (i_4)_* \circ (k_2)_* \mu_2,$$
$$(k_4)_* \circ (i'_3)_* \circ (i'_2)_* = (i_7)_* \circ (i_6)_* \circ (i_5)_* \circ (i_4)_* \circ (k_2)_* .$$

Therefore

$$\mu_4 = (k_4)_*^{-1} \sigma_8$$

and

$$(j_4)_* (m\rho_4 + n\sigma_4) = 2(2m+n)\mu_4 .$$

Hence, if α_4 is a generator of $H^4(S^4)$, we obtain from Theorem 3.1

$$p_1(\mathfrak{B}^{(4,3)}_{m,n}) = -\bar{c}(\mathfrak{B}'^{(4,3)}_{m,n}) = \pm 2(2m+n)\alpha_4 .$$

For $r \geqq 5$, in the same way

$$(k_r)_* \circ (j_r)_* (n\sigma_r) = (j'_r)_* \circ (d)_* (n\sigma_r) = 2n\sigma_{2r} = (k_r)_* (2n\mu_r)$$

therefore from Theorem 3.1

$$p_1(\mathfrak{B}^{(4,r)}_n) = -\bar{c}(\mathfrak{B}'^{(4,r)}_n) = \pm 2n\alpha_4 .$$

Thus we have

THEOREM 4.1. (i) *The first Pontrjagin class* p_1 *of* $\mathfrak{B}^{(4,3)}_{m,n}$ *is* $\pm 2(2m+n)\alpha_4$.

(ii) *For* $r \geqq 4$, *the first Pontrjagin class* p_1 *of* $\mathfrak{B}^{(4,r)}_n$ *is* $\pm 2n\alpha_4$, *where* α_4 *is a generator of* $H^4(S^4)$.

REMARK 4.2. In the case of $\mathfrak{B}^{(4,3)}_{m,n}$, Milnor obtained the same result using Pontrjagin class of quaternion projective plane calculated by Hirzebruch (Milnor [8]).

(II) *The second Pontrjagin class of* $\mathfrak{B}^{(8,r)}$ $(r \geqq 7)$.

For $r \geqq 4$, we have $\pi_7(SU(r)) \approx Z$. Let $\bar{\mu}_r$ be a generator of $\pi_7(SU(r))$ such that

$$(i'_r)_* \bar{\mu}_r = \bar{\mu}_{r+1}, \quad (k_5)_* \bar{\mu}_5 = 2\bar{\sigma}_{10}.^{3)}$$

LEMMA 4.3. *Let $r \geqq 7$ and $p: SU(r) \to SU(r)/SU(3)$ be natural projection, then*

$$p_* \bar{\mu}_r = \pm 6\lambda,$$

where λ is a generator of $\pi_7(SU(r)/SU(3)) \approx Z$.

PROOF. We consider exact sequence of homotopy groups of the principal bundle $(SU(r), p, SU(r)/SU(3), SU(3))$.

$$\cdots \to \pi_7(SU(3)) \to \pi_7(SU(r)) \to \pi_7(SU(r)/SU(3)) \to \pi_6(SU(3)) \to \pi_6(SU(r)) \to \cdots$$

Now we have

$$\pi_7(SU(3)) = 0, \ \pi_7(SU(r)) \approx Z, \ \pi_7(SU(r)/SU(3)) \approx Z,$$
$$\pi_6(SU(3)) \approx Z_6, \ \pi_6(SU(r)) = 0$$

whence our lemma is easily proved.

Now from the diagram (4.1) follows

$$(k_8)_* \circ (j_8)_* (m\bar{\rho}_8 + n\bar{\sigma}_8) = (j'_8)_* \circ (d)_* (m\bar{\rho}_8 + n\bar{\sigma}_8)$$
$$= 2(i_{15})_* \circ (i_{14})_* \circ \cdots \circ (i_8)_* (m\bar{\rho}_8 + n\bar{\sigma}_8).$$

Since $(i_r)_*$ is isomorphism onto for $r \geqq 9$, and

$$(i_8)_* (\bar{\rho}_8) = 2\bar{\sigma}_9, \ (i_8)_* \bar{\sigma}_8 = t\bar{\sigma}_9,$$

we have

$$(k_8)_* \circ (j_8)_* (m\bar{\rho}_8 + n\bar{\sigma}_8) = 2(2m + tn)\bar{\sigma}_{16}. \tag{4.2}$$

On the other hand, we have

$$\bar{\mu}_8 = (i'_7)_* \circ \cdots \circ (i'_4)_* \bar{\mu}_4, \ 2\bar{\sigma}_{16} = (i_{15})_* \circ (i_{14})_* \circ \cdots \circ (i_8)_* \circ (k_5)_* \bar{\mu}_5$$
$$(k_8)_* \circ (i'_7)_* \circ \cdots \circ (i'_5)_* = (i_{15})_* \circ (i_{14})_* \circ \cdots \circ (i_{10})_* \circ (k_5)_*$$

therefore

$$(j_8)_* (m\bar{\rho}_8 + n\bar{\sigma}_8) = (2m + tn)\bar{\mu}_8.$$

For $r \geqq q$, we obtain in the same way

$$(k_r)_* \circ (j_r)_* (n\bar{\sigma}_r) = (j'_r)_* \circ (d)_* (n\bar{\sigma}_r) = 2n\bar{\sigma}_{2r} = (k_r)_* (n\bar{\mu}_r).$$

Hence from the Lemma 4.3 and Theorem 3.1 follows

$$p_2(\mathfrak{B}^{(8,7)}_{m,n})=\bar{c}(\mathfrak{B}'^{(8,7)}_{m,n})=\pm 6(2m+tn)\alpha_8$$

$$p_2(\mathfrak{B}^{(8,r)}_{n})=\bar{c}(\mathfrak{B}'^{(8,r)}_{n})=\pm 6n\alpha_8 \qquad (r\geqq 8),$$

where α_8 denotes a generator of $H^8(S^8)$.

So we obtain the following theorem:

THEOREM 4.4. (i) *The second Pontrjagin class p_2 of $\mathfrak{B}^{(8,7)}_{m,n}$ is $\pm 6(2m+tn)\alpha_8$, where t is odd integer determined by* (2.1) *and α_8 is a generator of $H^8(S^8)$.*

(ii) *For $r\geqq 8$, the second Pontrjagin class p_2 of $\mathfrak{B}^{(8,r)}_{n}$ is $\pm 6n\alpha_8$.*

REMARK 4.5. Pontrjagin classes defined here are different from classical ones which are defined by Grassmann manifolds. But since the difference between them is 2-torsion, both definitions coincide in our cases (Wu [16]).

REMARK 4.6. By Theorem 4.4 (i) and the fact that the homogeneous part Ω^{15} of the 15th degree of Thom algebra Ω is a finite group (Thom [13]), we obtain the manifolds which are homeomorphic, but not diffeomorphic with 15-sphere S^{15}, making use of the invariant λ constructed in the same way as Milnor [8].

5. Cohomology groups of total spaces.

We shall first prove the following lemmas.

LEMMA 5.1. *The primary obstruction of $\mathfrak{B}^{(4,3)}_{m,n}$ is $\pm n\alpha_4$.*

PROOF. Take quaternion unit 1 for y_0 of section 3. Then we have $T(x)(1)=n\sigma_4(1)$ by definitions of ρ_4, σ_4, and this defines n multiple of a generator of $\pi_3(S^3)$. So lemma is proved by Theorem 3.1.

LEMMA 5.2. *The primary obstruction of $\mathfrak{B}^{(8,7)}_{m,n}$ is $\pm n\alpha_8$.*

PROOF. This is proved similarly as in the preceeding lemma 5.1. We may only replace ρ_4, σ_4 by $\bar{\rho}_8$, $\bar{\sigma}_8$, quaternion unit 1 by Cayley unit 1.

REMARK 5.3. Obviously the primary obstruction is trivial in $\mathfrak{B}^{(4,r)}_{n}$ $(r\geqq 4)$, $\mathfrak{B}^{(8,r)}_{n}$ $(r\geqq 8)$.

We now consider the Gysin exact sequence (Serre [9] Prop. 6)

$$\cdots\rightarrow H^i(S^q,Z)\rightarrow H^i(B^{(q,r)},Z)\rightarrow H^{i-r}(S^q,Z)\xrightarrow{h} H^{i+1}(S^q,Z)\rightarrow\cdots$$

where $h(x)=x\cdot\bar{c}(\mathfrak{B}^{(q,r)})$.

This enables us to compute the cohomology groups of total spaces from Lemma 4.1, 4.2. We obtain:

THEOREM 5.4. (i) *Non-trivial cohomology groups of $B^{(4,r)}$ are*

$$H^4(B^{(4,3)}_{m,n}) \approx Z_n$$
$$H^4(B^{(4,4)}_n) \approx Z+Z$$
$$H^4(B^{(4,r)}_n) \approx H^r(B^{(4,r)}_n) \approx Z \qquad (r \geqq 5).$$

(ii) *Non-trivial cohomology groups of* $B^{(8,r)}$ *are*

$$H^8(B^{(8,7)}_{m,n}) \approx Z_n$$
$$H^8(B^{(8,8)}_n) \approx Z+Z$$
$$H^8(B^{(8,r)}_n) \approx H^r(B^{(8,r)}_n) \approx Z \qquad (r \geqq 9).$$

6. Pontrjagin classes of manifolds.

In the following sections, all manifolds and differentiable structures considered are always C^∞-differentiable.

S^q and $SO(r+1)$ have natural differentiable structures and $\rho_4, \sigma_4, \sigma_r, t\bar{\rho}_8, \bar{\sigma}_8, t\bar{\sigma}_r$ defined in section 2 are differentiable mappings.

Therefore we can define differentiable coordinate transformations in $V_1 \cap V_2$ from the characteristic map T (Steenrod [11] § 18). Then open covering $V_i \times S^r$ $(i=1,2)$ with natural differentiable structure determines a differentiable structure on $B^{(q,r)}$.

Let $M^{(q,r)}_{m,n}$, $M^{(q,r)}_n$ be manifolds thus obtained from $B^{(q,r)}_{m,n}$, $B^{(q,r)}_n$. Now, Pontrjagin classes of a manifold M mean, as usual, Pontrjagin classes of its tangential bundle $\mathfrak{T}(M)$.

Let β_4, β_8 be the generators of $H^4(M^{(4,r)})$, $H^8(M^{(8,r)})$ respectively given by $\beta_4 = p^*(\alpha_4)$, $\beta_8 = p^*(\alpha_8)$, where p is the projection.

THEOREM 6.1. (i) *The first Pontrjagin class* p_1 *of* $M^{(4,3)}_{m,n}$ *is* $\pm 4m\beta_4$ *(mod n).*

(ii) *For* $r \geqq 4$, *the first Pontrjagin class* p_2 *of* $M^{(4,r)}_n$ *is* $\pm 2n\beta_4$.

THEOREM 6.2. (i) *The second Pontrjagin class* p_2 *of* $M^{(8,7)}_{tm,n}$ *is* $\pm 12tm\beta_8$ *(mod n).*

(ii) *For* $r \geqq 8$, *the second Pontrjagin class* p_2 *of* $M^{(8,r)}_{tn}$ *is* $\pm 6tn\beta_8$.

PROOF. Let σ^{r+1} be $(r+1)$-cell of closed interior of S^r. We associate to bundle $\mathfrak{B}^{(q,r)}$, the bundle $\overline{\mathfrak{B}}^{(q,r)}$ with fibre σ^{r+1}:

$$\overline{\mathfrak{B}}^{(q,r)} = \{\overline{B}^{(q,r)}, \bar{p}, S^q, \sigma^{r+1}, SO(r+1)\}.$$

Natural differentiable structure is defined on $\overline{B}^{(q,r)}$ as $B^{(q,r)}$ above and obtain $(q+r+1)$-dimensional manifold $V^{(q,r)}$ with boundary. Obviously

$M^{(q,r)}$ is boundary manifold of $V^{(q,r)}$.

$V^{(q,r)}$ is of the same homotopy type as S^q and $\bar{p}^*\alpha_q$ is a generator of $H^q(V^{(q,r)})$, where α_q is a generator of $H^q(S^q)$.

Let $\mathfrak{T}(V^{(q,r)})$ be the tangential bundle of $V^{(q,r)}$. Then $\mathfrak{T}(V^{(q,r)})$ is the Whitney sum of the bundle of vectors tangent to the fibre $\mathfrak{T}_1(V^{(q,r)})$ and the bundle of vectors normal to the fibre $\mathfrak{T}_2(V^{(q,r)})$. $\mathfrak{T}_1(V^{(q,r)})$ is the induced bundle induced by $\mathfrak{B}^{(q,r)}$ and $\bar{p}$, and $\mathfrak{T}_2(V^{(q,r)})$ is the one induced by tangent bundle of S^q and $\bar{p}$. Since S^4, S^8 are boundary manifolds, we have $p_1(S^4)=0$, $p_2(S^8)=0$. Therefore, for $q=4,8$

$$p_i(\mathfrak{T}(V^{(q,r)}))=p_i(\mathfrak{T}_1(V^{(q,r)}))=\bar{p}^*p_i(\mathfrak{B}^{(q,r)}) \qquad (i=q/4) \tag{6.1}$$

Now let $i: M^{(q,r)}\rightarrow V^{(q,r)}$ be injection. Over $M^{(q,r)}$, $i^*\mathfrak{T}(V^{(q,r)})$ is the Whitney sum of $\mathfrak{T}(M^{(q,r)})$ and the 1-vector bundle $\mathfrak{N}(M^{(q,r)})$ normal to $\mathfrak{T}(M^{(q,r)})$ in $V^{(q,r)}$. Obviously $\mathfrak{N}(M^{(q,r)})$ is trivial, therefore

$$p_i(\mathfrak{T}(M^{(q,r)}))=i^*p_i(\mathfrak{T}(V^{(q,r)})) \qquad (i=q/4) \tag{6.2}$$

Since i is a bundle map

$$i^*\bar{p}^*\alpha_q=p^*\alpha_q=\beta_q\,. \tag{6.3}$$

We obtain from (6.1), (6.2), (6.3)

$$p_i(M^{(q,r)})=p^*p_i(\mathfrak{B}^{(q,r)}) \qquad q=4,8,\ i=q/4$$

Hence our theorems follow from Theorem 4.1, Theorem 4.4, Theorem 5.4.

7. Pontrjagin classes and homotopy types of manifolds.

From Theorem 2.3, Theorem 2.4, Theorem 6.1, Theorem 6.2, we obtain

THEOREM 7.1. (i) $M^{(4,3)}_{m+12i,n}$, $M^{(4,3)}_{-m+12i',-n}$ $(i, i'=0, \pm1, \pm2,\cdots)$ *are of the same homotopy type and their first Pontrjagin classes* p_1 *are*

$$p_1(M^{(4,3)}_{m+12i,n})=\pm(4m+48i)\beta_4 \quad \mod n\,.$$

(ii) *For* $r\geqq 4$, $M^{(4,r)}_{\pm n+24i}$ $(i=0, \pm1, \pm2,\cdots)$ *are of the same homotopy type and their first Pontrjagin classes* p_1 *are*

$$p_1(M^{(4,r)}_{n+24i})=\pm(2n+48i)\beta_4\,.$$

THEOREM 7.2. (i) $M^{(8,7)}_{tm+\{120,t\}i,n}$, $M^{(8,7)}_{-tm+\{120,t\}i',-n}$ $(i, i'=0, \pm1, \pm2,\cdots)$ *are of the same homotopy type and their second Pontrjagin classes* p_2 *are*

$$p_2(M^{(8,7)}_{tm+\{120,t\}i,n})=\pm(12tm+12\{120, t\})\beta_8 \quad \mod n\,.$$

(ii) *For* $r \geqq 8$, $M^{(8,r)}_{\pm tn+\{240,t\}i}$ $(i=0, \pm 1, \pm 2, \cdots)$ *are of the same homotopy type and their second Pontrjagin classes* p_2 *are*

$$p_2(M^{(8,r)}_{tn+\{240,t\}i}) = \pm(6tn+6\{240, t\}i)\beta_8 .$$

Where t is odd integer determined by (2.1) *and* $\{\ ,\ \}$ *means l. c. m.*

In particular, $M^{(q,r)}_{0,0}$, $M^{(q,r)}_0$ are $S^q \times S^r$, and so we have

COROLLARY 7.3. *For* $r \geqq 3$, *there are infinitely many* $(4+r)$-*dimensional manifolds of the same homotopy type as* $S^4 \times S^r$, *whose first Pontrjagin classes* p_1 *are divisible by* 48, *and between which there exists no diffeomorphism* (*i. e. differentiable homeomorphism*).

COROLLARY 7.4. *For* $r \geqq 7$, *there are infinitely many* $(8+r)$-*dimensional manifolds of the same homotopy type as* $S^8 \times S^r$, *whose second Pontrjagin classes* p_2 *are divisible by* $12\{120, t\}$, *and between which there exists no diffeomorphism.*

REMARK 7.5. Thom proved topological (homotopy) invariance of p_1 mod 2 (Thom [12]). Wu and Hirzebruch proved topological (homotopy) invariance of p_1 mod 3 (Wu [15], [16], Hirzebruch [3]).

The results obtained above would disprove the topological invariance of Pontrjagin classes, if the conjecture of Hurewicz "Two closed manifolds of the same homotopy type is homeomorphic" (Hurewicz [6]) is true, or if, what would be easier to prove, one of $M_{\pm 12m,0}$ $(m=1, 2, \cdots)$ is homeomorphic to $S^4 \times S^3$ (Hirzebruch [4] Problem 1).

Conversely if topological invariance of p_1 for mod p, p being a prime other than 2, 3, or topological invariance of p_2 for mod p, p being a prime other than 2, 3, 5 and factors of t, could be proved, then the conjecture of Hurewicz would be denied.

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Notes

1) The author does not know whether $t=1$ or not, and also whether $\bar{\rho}_7$ has C^∞-differentiable representative or not.
2) Notice that the kernel of $(i_8)_*$ is generated by $2\bar{\sigma}_8 - t\bar{\rho}_8$.
3) The coefficient of $\bar{\sigma}_{10}$ must be ± 1 or ± 2 by (4.2). On the other hand, we can show that it is even, by using relations (2.2) and the exact sequence of homotopy groups of the principal bundle $(SU(4), S^7, SU(3))$.

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