mum vapour tension of the day occurs about the time when the breeze from the lake falls to a calm, and before the land breeze springs up.

A cursory examination of the curves suffices to show that there is a close connection between their critical phases and the corresponding phases of atmospheric pressure and temperature; and the idea is suggested that in this great contribution of Plantamour's to the meteorology of Geneva, we are put in possession of data of the utmost importance as regards the relations of the vapour of the atmosphere and its movements to changes of atmospheric pressure in a way such as could be done by the observations of few observatories.

5. On Knots. By Professor Tait.

(*Abstract.*)

At the last meeting of the Society I stated that I had just procured a remarkable essay by Listing, part of which bears on the subject of knots, and that I had found in it an example of a change of form not producible by the modes of deformation I had employed.

It had for some time struck me as very singular that, thoug I could easily prove that (when nugatory intersections are removed) a knot in which the crossings are alternately over and under is not farther reducible, I could not prove all its possible deformations to be producible by inversions or projections of the kinds specified in my paper; but, as soon as I recognised the existence of amphicheiral forms, I saw that it was probable that they would furnish a key to my difficulty. I immediately set to work to classify the simpler of such forms; and while I was thus engaged I got the *Göttinger Studien* for 1847, in which is Listing's paper, with the title *Vorstudien zur Topologie*.

By this title Listing means *qualitative* as distinguished from *quantitative* space-relations. He commences with a study of inversion (*Umkehrung*) and perversion (*Verkehrung*),—the first being the effect of a rotation through two right angles about any axis, the second the result of reflexion in a plane mirror.

He next treats of screwing of all kinds, including twisting and plaiting.

He then applies the notion of lines winding or screwing round one another to the projection of a knot on a surface, and shows that we can thus obtain a knowledge of the relative situation of the various coils. At each crossing portions of the two branches may be regarded as small parts of lines twisted round one another. Of the four angles so formed, two vertical or opposite ones are bounded towards the *right*, the other two towards the *left*, by that line which passes *over* the other. We thus distinguish these pairs of vertical angles into right- and left-handed. [Listing uses right-handed for what we should call left-handed in screwing, but the difference is of no importance, so far as his results are concerned.]

Next he shows that perversion, but not inversion, changes right-handed into left-handed angles.

He then gives the complete knot with three intersections, and shows that when it is in a *reduced* (as distinguished by him from a *reducible*) form, all the angles in each separate mesh have a common character; but that, when it is reducible, some of the meshes have angles of both kinds. He distinguishes between the right- and left-handed forms of the reduced knot, and shows that they are not convertible into one another; also that (including external space, or the *Amplexum*) there are three meshes with two corners each, and two with three corners; one class being right-handed, the other left. And he states that the Amplex may be made to change its character from right to left by being changed from a three-cornered to a two-cornered mesh, or *vice versa*.

He points out that a loop (*i.e.*, a mesh with only one corner) does not appear in the reduced form, and then writes, as the type-symbol of the reduced right-handed knot of three intersections, the expression

$$\left.\begin{matrix} 3r^2 \\ 2l^3 \end{matrix}\right\},$$

denoting three right-handed meshes with two corners each (*Oesen*), and two left-handed meshes with three corners each (*Maschen*). [The perverted, or left-handed, form is of course represented by the same symbols, with the interchange of r and l.]

The sum of the numerical coefficients in the symbol is the number of meshes (the Amplex included), and is greater by two than the number of crossings. The sum of the products of the coefficients into the corresponding exponents gives, in each of the two parts of the symbol, double the number of crossings. These symbols contain the topologic character of any particular knotting.

Listing next points out that the reduced three-crossing knot may be obtained by three half-turns about one another of two originally parallel cords whose ends are afterwards joined into one ring, and that the character depends upon the direction of the torsion.

He proceeds to give a symmetrical knot, with seven crossings, in two different reducible, and one reduced, forms. The reduction of the first gives the three crossing knot, that of the second the four forms of the essentially non-clear coil of five intersections. (Figs. 6–9 of my first paper.)

Their common symbol he writes as

$$\left.\begin{array}{l} 2r^4 + r^2 \\ 2l^3 + 2l^2 \end{array}\right\},$$

and he points out that, in this case, the Amplex belongs to each in succession of these four kinds of meshes.

He then states that the symbol

$$\left.\begin{array}{l} 2r^4 + 3r^2 \\ 2l^5 + 2l^2 \end{array}\right\}$$

gives five different reduced forms, each with seven crossings; while the symbol

$$\left.\begin{array}{l} r^5 + 3r^3 \\ l^4 + 2l^3 + 2l^2 \end{array}\right\}$$

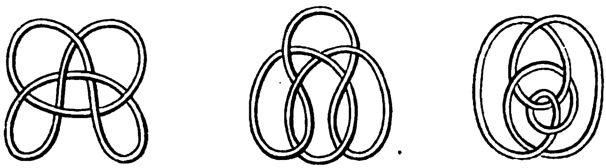
has six distinct forms.

But he adds the following extremely important remark:—"In certain cases one symbol is equivalent to another, so that the reduced forms of the one can be transformed into those of the other." He states that this is the case with the last written symbol and the following:—

$$\left.\begin{array}{l} 2r^4 + 2r^3 \\ l^4 + 2l^3 + 2l^2 \end{array}\right\}$$

which has five reduced forms.

Thus there are, in all, *eleven* reduced forms of these kinds, all equivalent to one another, and all having seven crossings. The following are his figures: the first and second belonging to the former of the two equivalent symbols, the third to the latter—



He concludes this part of his Essay by saying that "these examples (confined as they are to single, closed lines), and the remarks made upon them, serve to show that the fundamental conception of twisting of lines is capable of being applied to the most complex space relations."

It may be added that these very elegant and important results are given as statements merely, without any hint of how they were arrived at, or how they may be extended. In fact brevity has been sedulously studied, for all that is given about knots forms a comparatively small part of the whole of Listing's extremely valuable, but too brief, Essay.

The rest treats, rather more fully, the whole subjects of inversion and perversion, screws of various kinds, plaiting and twisting, (with their applications to vegetable spirals, &c.), the numbers of lines joining given sets of points, the extensions of the meaning of the word *Area*, &c., &c.

The above abstract, which contains almost all of Listing's remarks on knots, shows that he has long ago anticipated a very great deal of what I have lately sent to the Society. For myself, however, I may say that I have had to learn only two things (about *knots*) from Listing, viz. (1), a special case (which will be examined immediately), in which two forms are equivalent, though not transformable into one another by the methods given in my first paper; and (2), to value more highly than I had hitherto done the method of classifying forms by the numbers of each kind of mesh, and the right-handed or left-handed character of each.

My first paper, as sent to the Society, was essentially confined (as indeed its title indicates) to the results deducible from a special elementary theorem,—one of two which occurred to me long ago when designing Vortex-Atoms of various forms, and which I gave to Section A at the late meeting of the British Association. The second of these theorems (as will be seen by reference to my British Association paper, *Messenger of Mathematics*, Jan. 1877), was virtually the same as Listing's division of the meshes of a reduced knot into right- and left-handed,—only I called them black and white, but as I did not see how to connect this theorem directly with the measure of beknottedness, I did not formally introduce it into my papers read to the Society. It is, of course, virtually included in the statements regarding coins thrown into the corners of cells,—for, taking the case of the silver and copper coins, the pair of left-handed vertical angles are those in which, or in one of them, there is silver—the right-handed, copper.

Nothing can be clearer than Listing's statements on several parts of the subject: it is greatly to be desired that he had made many more. Still, with a cordial recognition of the great value of all that is to be found in Listing's paper, I adhere to what I said in my last communication, to the effect that the full character of a knot cannot be learned except from its "scheme," or something equivalent to it. That the type-symbol (when such a representation is possible) is ultimately equivalent to the scheme may possibly be true,* especially when we consider that it virtually contains *two independent* descriptions of a knot (*i.e.* in terms of its right-handed and its left-handed meshes separately); that for purposes of classification it is superior is, I think, obvious, but I think it equally obvious that for the purpose of drawing the knot it is inferior. And the scheme for a reducible knot is quite as simple as that for a reduced one, while it is not easy to see what would be called the symbol of a reducible knot. Nor can I represent by

* (*Added Feb.* 7.)—I have just found symbols for which this is not the case. The following single instance is sufficient, for the present, to show that the type-symbol is not always equivalent to the scheme. The symbol

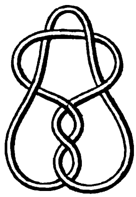
$$\left.\begin{matrix} r^4+2r^3+2r^2 \\ l^5+l^4+l^3+l^2 \end{matrix}\right\}$$

may represent either a continuous curve with 7 intersections, or a complex system consisting of a circle intersected at six points by a skewed figure of 8. I shall discuss the subject fully in a paper "On Links," which I have in preparation.

Listing's notation the double trefoil knot which has appeared in each of my papers; for, although irreducible (at least so far as I am aware), it contains several meshes which have angles of essentially different characters. Listing's avowed object was to simplify notation as far as possible. My impression is that, in one respect at least, he has carried simplification a little too far; for it cost me some little time and trouble to draw, from his type-symbol, the one knot which he speaks of, but leaves undrawn, viz., as above,—

$$\left.\begin{matrix} 2r^4 + 3r^2 \\ 2l^5 + 2l^3 \end{matrix}\right\} .$$

Here is one of its forms: transformation will give the four others.



In fact the type-symbol, even in this specially simple and symmetrical case, where it is much condensed, contains just as many separate typographical characters as the scheme; and I think there can be no doubt whatever that it is almost incomparably more easy to draw the figure from the scheme than from the symbol. Given the scheme, the symbol can be formed from it in a moment; while the finding of the scheme from the symbol is very troublesome. But in such a matter experience is the only guide, and I have had almost no practice in trying to draw from the symbol. Listing's type-symbol leads directly, however, to an inquiry not even suggested by the scheme; (for the latter, as I have given it, is essentially confined to a single closed curve),—viz., the forms of more than one closed curve, intersecting one another or not, which jointly divide an unlimited plane into given numbers of meshes with given numbers of sides.

The first idea of this was suggested to me when I endeavoured to draw the curve with six intersections, whose type symbol is

$$\left.\begin{matrix} 3r^4 \\ 2l^3+3l^2 \end{matrix}\right\}.$$

This symbol obviously satisfies the three numerical conditions; but, on trying to draw the corresponding figure, I found that it always came out as a species of endless chain of three separate links. One of its forms, from which the others can be found by transformation, is three circles, every two of which intersect.

Two figures of 8, linked together at each end, give the symbol

$$\left.\begin{matrix} 4r^3 \\ 2l^4+2l^2 \end{matrix}\right\}.$$

And by shifting the twist from one to the other, as explained in the latter part of this paper, the symbol may be changed to

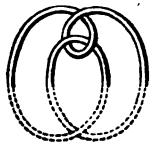
$$\left.\begin{matrix} r^4+2r^3+r^2 \\ 2l^4+2l^2 \end{matrix}\right\}.$$

I have not as yet studied the theory of type-symbols, as it differs so much from my own method; but it is obviously desirable to find the criterion by which to distinguish from one another the type-symbols necessarily denoting one closed curve, and those necessarily denoting two or more intersecting curves. It is probable that there are symbols which may represent either kind of figure. The inquiry will no doubt be found very simple, if only approached from the proper side.

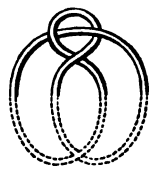
I now pass to the sole point of Listing's paper which (so far as knots are concerned) was thoroughly new to me, though not unexpected; and I shall lead up gradually to the special case which he gives, using for the purpose the properties of the amphicheiral knots mentioned in my last paper.

To apply the amphicheiral property to the production of new forms, we may begin by studying under what conditions the internal arrangements of a knot can be altered while *four* points of its contour, and *two* of the parts of the cord or wire joining pairs of these, are fixed. [The reason for the number *four* is, that when two only are employed to mark off separate parts of a knot, it is either (*a*) virtually unconstrained, or (*b*) it is divisible (and is actually divided) into two separate knots.] If, under these circumstances, changes can be made on one side of the fixtures, they will be practicable also *in whatever way these points be connected by the rest of the string*, provided always that it be not led through the amphicheiral part. Hence, if there be an amphicheiral part of any knot, it may often be transformed *in situ;* the rest of the knot being unaltered, but the amphicheiral part being made to present, as it were, another hand to the rest.

To begin with a very simple case, let us take the simplest amphicheiral form—the complete knot with four crossings—as below.



1.



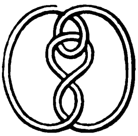
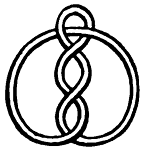
2.

When the first of these is inverted, O being within either of the *interior* three-sided meshes, the second is produced: if O be in one of the *boundary* three-side meshes the perversion of the second is produced. But we may easily convert 1 into 2 by fixing the lower crossing (*i.e.*, by fixing a point near it in each of the four lines diverging from it, so that the dotted part remains fixed), and making the upper loop rotate so as to banish the upper crossing. Thus the upper parts of these two figures are equivalent.

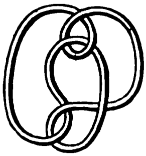
And we can now suppose the (dotted) portions between the fixed points to be cut open and reknotted in any way we choose,—subject, of course, always to the rule of alternate over and under, else the knot would in general be reducible. Of course, unless we wish to study *linking*, the ends must be joined so as to preserve continuity throughout the string.

The simplest modes of joining (without additional intersections) give us at once two different aspects of the same "trefoil" knot: —with one crossing additional we have the original figures

(1 and 2) above: with two additional we have the following,—figured in my first paper.

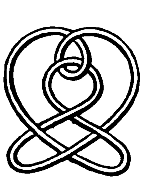
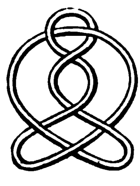
 

With three we have two apparently different forms,

 and 

These are, however, only transformations of the six-crossing amphicheiral form (figs. 3 and 4 of my last paper), and are directly transformable into one another. They are, of course, unchanged if the lower part be reversed, for the upper parts are symmetrical.

But when we add four new crossings, as below, instead of the single crossing removed, we get the two equivalent figures

which are not transformable into one another by the processes of my first paper. In fact the *schemes* will be found to be incompatible, begin each where we choose. In Listing's notation their type-symbols would stand respectively, thus—

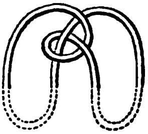
$$\left.\begin{matrix}2r^4+2r^3\\2l^4+3l^2\end{matrix}\right\} \text{ and } \left\{\begin{matrix}r^5+r^4+r^3+r^2\\2l^4+3l^2\end{matrix}\right.$$

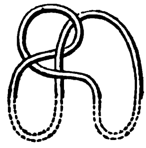
The schemes are

AEBFCBDAEGFCGD | A and ADBFCADGEBFEGC | A.

But even this simplest amphicheiral form has other applications.

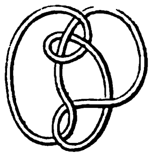
Thus, it will easily be seen that the figures below are mere *distortions* of fig. 2 above; and that, the dotted parts being fixed, they can be changed on an actual cord into one another, and even reversed (as from left to right), thus giving four distinguishable forms, of which I figure only two,—

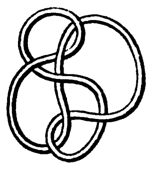




Each of these may be changed without undoing the fixtures to its reverse (from left to right) by the first simple process just described, the loop, in fact, being transferred from one branch of the (undotted) cord to the other.

Of course the number of ways in which the dotted part can be varied is infinite. I therefore give here only that which reproduces the forms already quoted from Listing as equivalent.





A glance shows that their type-symbols are

$$\left.\begin{matrix} r^6+3r^3 \\ l^4+2l^3+2l^2 \end{matrix}\right\} \quad \text{and} \quad \left\{\begin{matrix} 2r^4+2r^3 \\ l^4+2l^3+2l^2 \end{matrix}\right.$$

which are those already quoted from Listing.

In fact, looking at Listing's figures above, we see that in each there is a part of the curve, containing four crossings, exactly the same as one or other of the two (partly dotted) distortions of the 4-crossing knot above.

The paper contains many other instances of these applications of amphicheiral forms.

In conclusion, it appears that the problem of finding all the absolutely distinct forms of knots, with a given number of intersections, is a much more difficult one than I at first thought; and it is so because the number of really distinct species of each order is very much *less* than I was prepared to find it. The question now belongs more to quantitative than to qualitative relations. It resembles, in fact, the species of problem originally suggested by Crum Brown, and resolved by Sylvester and Cayley, of determining the number of conceivable Hydrocarbons under given conditions of limitation. And here I am glad to leave it, for at this stage it is entirely out of my usual sphere of work, and it has already occupied too much of my time.

[*April* 11*th*.—Prof. Listing, to whom I sent in "proof" the above abstract of part of his paper, has kindly written some remarks upon it, from which I extract the following very interesting passages, which increase our regret that such a master has published so little on a subject which he has evidently made his own:—

"In dem . . . proof . . . sollte die Aufmerksamkeit vorzugsweise auf den allerdings sehr kurzen Theil gelenkt werden, welcher sich auf die Knoten oder Curvenverschlingungen bezieht. Anderenfalls würde ich gewünscht haben, dass das—obwohl sehr elementäre—Capitel über die 'Position' etwas näher besprochen worden wäre, zumal darin das Motiv zu finden, warum ich unsere gewöhnliche Schraube laeotrop nenne statt dexiotrop, wie Sie p. 307 rügen. Ich habe längst die Benennungen rechts und links bei Schraubenwindungen, sowie die lateinischen oder griechischen Ausdrücke, welche stets zu den widerwärtigsten Verwechselungen Anlass geben, durch die Namen *Delta*- und *Lambda*-Windung ersetzt, welche, wie p. 42 meiner Schrift erwähnt ist, mnemonish und intuitiv die Vorstellungen fixiren. Der Botaniker—und einige haben in der That diesen Modus befolgt—würde also dem Humulus lupulus Deltawindung, dem Convolvulus Lambdawindung zuschreiben, so wie unsere künstliche Schrauben der Technik meistens Lambda-Schrauben und nur in seltneren Fällen Delta-Schrauben sind. Die Unterscheidung zwischen Inversion und Perversion im Anschluss an sogenannte positive und negative Positionen ist nur ein gelegentliches specielles Ergebniss."

"Die Verschlingungen cyclischer Curven im Raume d. h. die Knoten betreffend, so sollten, wie in den 'Vorstudien' es nicht anders erwartet werden durfte, die Geometer auf die Bedeutsamkeit dieses überaus schwierigen Theils der Topologie durch Aufzeigung einiger der einfachsten Anfänge aufmerksam gemacht werden, z. B. durch Andeutungen über mögliche Bezeichnungs-methoden, Symbole und dgl. durch welche wir, wie es die Wissenschaft in allen analogen Fällen zu erstreben hat, von der Intuition zu den Begriffen fortzuschreiten vermögen. Die von mir empfohlenen Symbole sollen nichts weiter sein als ein derartiger Fingerzeig, und wenn ich auch ganz mit Ihnen darin übereinstimme dass das Schema einen Nodalcomplex leichter construirbar macht als das Symbol, so bleibt doch das Schema viel weiter von dem Begriffe entfernt als das Symbol, auch abgesehen von den auf beiden Seiten noch übrig bleibenden Vieldeutigkeiten. Wie fragmentär das, was ich vor 30 Jahren darüber angedeutet habe, gewesen, kann ich Ihnen unter vielen an dem einen Punkte erläutern, dass ich mich dort nur auf die Verknotung einer einzigen cyklischen oder einer cyklodischen Curve beschränkte, und nur auf solche Verschlingungen unter der Benennung 'reducirt' einging, deren sämmtliche Parcellen (incl. Amplexum) monotyp sind, wiewohl Ihnen selbst bekannt ist, dass es Knoten giebt mit Maschen, deren Ecken promiscue δ und λ heissen, ohne auf einfachere Formen, d. h. eine geringere Zahl von crossings reducirbar zu sein. . . . Es versteht sich von selbst, das die in den Vorstudien angeführten Symbole nur auf monotype reducirte Complexe anwendbar sind, und durch Verallgemeinerung andere Gestalten annehmen, worauf ich indessen hier weiter einzugehen unterlasse."

I have thought it better to give Listing's own comments on my remarks than to seek permission from the Council to alter these remarks in the text.]