61.

ON A REMARKABLE MODIFICATION OF STURM'S THEOREM.

[*Philosophical Magazine,* v. (1853), pp. 446—456.]

LET me be allowed to use the term *improper* continued fraction to denote a fraction differing from an ordinary continued fraction, in the sole circumstance of the numerators being all negative units instead of positive units, as thus:

$$\cfrac{1}{q_1 - \cfrac{1}{q_2 - \cfrac{1}{q_3 - \&c.}}}$$

The successive convergents of such a fraction as that written above will be

$$\frac{1}{q_1},\ \frac{q_2}{q_2q_1 - 1},\ \frac{q_3q_2 - 1}{q_3q_2q_1 - q_3 - q_1},\ \&c.$$

If we call these respectively

$$\frac{N_1}{D_1},\ \frac{N_2}{D_2},\ \frac{N_3}{D_3},\ \&c.$$

we have the general scale of formation

$$N_\iota = q_\iota N_{\iota-1} - N_{\iota-2},$$

$$D_\iota = q_\iota D_{\iota-1} - D_{\iota-2}.$$

Moreover, we shall have universally

$$N_\iota D_{\iota-1} - N_{\iota-1} D_\iota \text{ equal to } +1,$$

instead of alternating between $+1$ and -1, as is the case in continued fractions of the ordinary kind.

Again, let me be allowed to use the term *signaletic* series to denote a series of disconnected terms, designed to exhibit a certain succession of algebraical signs $+$ and $-$, and to speak of two series being *signaletically* equivalent when the *number* of continuations of signs and of variations of signs between the several terms and those that are immediately contiguous to them is the same for the two series; a condition which evidently may be satisfied without the *order* of such changes and continuations being identical. I am now able to enunciate the following remarkable *theorem of signaletic equivalence* between two distinct series of terms, each generated from the same improper continued fraction. But first I must beg to introduce yet another new term in addition to those already employed, namely *reverse convergents*, to denote the convergents generated from a given continued fraction by reading the quotients in a reverse order, or if we like so to say, the convergents corresponding to the given continued fraction reversed.

The two forms

$$\cfrac{1}{q_1 - \cfrac{1}{q_2 - \cfrac{1}{q_3 \atop \vdots \; - \cfrac{1}{q_n}}}} \qquad \text{and} \qquad \cfrac{1}{q_n - \cfrac{1}{q_{n-1} - \cfrac{1}{q_{n-2} \atop \vdots \; - \cfrac{1}{q_1}}}},$$

are obviously reciprocal; and if the two last convergents of either one of them be respectively

$$\frac{N_{n-1}}{D_{n-1}}, \; \frac{N_n}{D_n},$$

$\frac{D_{n-1}}{D_n}$ will serve to generate the other. For the clearer and more simple enunciation of the theorem about to be given, it will be better to take as our first convergent $\frac{0}{1}$, so that 1 will be treated as the denominator of the first convergent in every case; and calling D_0 such denominator, we shall always understand that $D_0 = 1$. Let now $D_0, D_1, D_2 \dots D_n$ be the $(n+1)$ denominators of any improper continued fraction of n quotients, and $\mathfrak{D}_0, \mathfrak{D}_1, \mathfrak{D}_2 \dots \mathfrak{D}_n$ the corresponding denominator series for the same fraction reversed; then, I say, that these *two series* are *signaletically equivalent*.

I do not here propose to demonstrate this proposition, to which I was led unconsciously by researches connected with the theory of elimination, which afford a complete and general but somewhat indirect and circuitous proof. Doubtless some simple and direct proof cannot fail ere long to be discovered*. For the present I shall content myself with showing *à posteriori* the truth of the theorem for a particular case. Let $n = 3$. The two series which are to be proved to be signaletically equivalent may be written

$$1, \quad A, \quad BA - 1, \quad CBA - C - A,$$
$$1, \quad C, \quad BC - 1, \quad ABC - A - C.$$

* See Postscript [p. 616 below].

Call these respectively S and (S). In S we may substitute in the third term, in place of $BA-1$, CA without affecting the signaletic value of the series; for if the second and fourth terms have different signs, the third term may be taken anything whatever, since the sequence of the second, third, and fourth terms will give one continuation and one change, whatever the middle one may be. Suppose, then, that the second and fourth terms have the same sign, and let

$$CBA - C - A = m^2 A,$$

therefore $$C(BA-1) = (m^2+1)A,$$

therefore $$(BA-1)AC = (m^2+1)A^2.$$

Hence $BA-1$ and AC will have the same sign; hence S is signaletically equivalent to S', where S' denotes the series

$$1, \quad A, \quad CA, \quad CBA - C - A.$$

Now, again, if CA is negative, we may put instead of A anything whatever, and therefore, if we please, C, without affecting signaletically the value of S'. But if CA is positive, A and C will have the same sign, and therefore on this supposition also C may be substituted for A. Hence always S' is signaletically equivalent to S'', where S'' denotes

$$1, \quad C, \quad CA, \quad CBA - C - A.$$

Again, if C and $CBA-C-A$ have different signs, the value of the intermediate term is immaterial; but if C and $CBA-C-A$ have the same sign, let

$$CBA - C - A = m^2 C;$$

then $$A(CB-1) = (1+m^2)C,$$

and $$A^2(CB-1) = (1+m^2)AC;$$

and consequently $CB-1$ and AC have the same sign. In every case, therefore, S'' is signaletically equivalent to

$$1, \quad C, \quad CB-1, \quad ACB - A - C;$$

that is S is signaletically equivalent to S', and therefore to S'', and therefore to (S), as was to be proved.

The application of the foregoing theory to Sturm's process for finding the number of real roots of an equation is apparent; for a very little consideration will serve to show, that if we expand $\frac{f'x}{fx}$, fx being of the nth degree in x, algebraically under the form of a continued fraction

$$\cfrac{1}{Q_1 - \cfrac{1}{Q_2 - \cfrac{1}{Q_3 \atop \vdots \; - \cfrac{1}{Q_n}}}},$$

where Q_1, Q_2, $Q_3 \dots Q_n$ may be supposed linear functions of x (although, in fact, this restriction, as will be hereafter noticed, is unnecessary), the denominators of the reverse convergents

$$\frac{0}{1}, \frac{1}{Q_n}, \frac{Q_{n-1}}{Q_n Q_{n-1} - 1} \dots \frac{Q_{n-1} Q_{n-2} \dots Q_1 - \&c.}{Q_n Q_{n-1} \dots Q_1 - \&c.},$$

will be signaletically equivalent with the Sturmian series of functions for determining the number of real roots of fx within given limits; in fact,

$$1, \quad Q_n, \quad Q_n Q_{n-1} - 1, \dots, Q_n Q_{n-1} \dots Q_1 - \&c.$$

will be the Sturmian functions themselves, divided out by the negative of the last or constant residue which arises in the application of the process of continued division, according to Sturm's rule; and as we have shown that the series of the denominators to the convergents of any continued fraction, and the series of the denominators to the convergents of the same fraction reversed, are signaletically equivalent, we have this surprisingly new, interesting, and suggestive mode of stating Sturm's theorem, namely, the denominators to the convergents of the continued fraction which represents $\frac{f'x}{fx}$ constitute a Rhizoristic series for fx, that is a signaletic series which serves to determine the number of roots of fx comprised within any prescribed limits. Moreover, in applying this theorem it is by no means necessary that, in the continued fraction which represents $\frac{f'x}{fx}$, all or any of the quotients should be taken linear functions of x. A very little consideration of the principles upon which the demonstration of Sturm's theorem is founded will serve to show that the convergent denominators to any continued fraction whatever which represents $\frac{f'x}{fx}$, whether the quotients be linear or non-linear, integral or fractional, or mixed functions of x, and whatever the number of quotients, which, it may be observed, cannot be less than, but may be made to any extent greater than the exponent of the degree of fx, will equally well furnish a Rhizoristic series for fixing the position of the roots, provided only that the last *divisor* in the process of expanding $\frac{f'x}{fx}$ under the form of an improper continued fraction be a constant quantity or any function of x incapable of changing its sign.

Let us, however, for the present confine our attention to the ordinary Sturmian form, where all the quotients are linear functions of x. Let these quotients be respectively

$$a_1x + b_1, \quad a_2x + b_2, \quad a_3x + b_3 \dots a_nx + b_n.$$

In order to determine the total number of real and imaginary roots of fx, we must count the loss of continuations of sign in the Rhizoristic series in passing from $x = +\infty$ to $x = -\infty$. When x is infinitely great, it is clear that, whether positive or negative, the parts $b_1, b_2 \dots b_n$ may be neglected, and only the highest powers of x need be attended to in writing down the signaletic series corresponding to these two values of x. Accordingly for $x = \pm\infty$ the signaletic series becomes

$$1, \quad a_1x, \quad a_1a_2x^2, \quad \dots, \quad a_1a_2 \dots a_nx^n,$$

and consequently the number of pairs of imaginary roots of fx is the number of changes of sign in the series

$$1, \ a_1, \ a_1a_2, \ \dots, \ a_1a_2 \dots a_n,$$

that is, is the number of negative quantities in the series

$$a_1, \ a_2, \ a_3, \ \dots, \ a_n.$$

Hence we have the curious and hitherto strangely overlooked theorem, that in applying Sturm's process of successive division to fx and $f'x$, the number of negative coefficients of x in the successive quotients gives the number of pairs of imaginary roots of fx; as a corollary, we learn the somewhat curious fact that never more than half of these coefficients can be negative; and in general it would appear that the better practical method of applying Sturm's theorem would be not to deal with the Residues, which have hitherto been the sole things considered, but rather with the linear quotients which have been treated as merely incidental to the formation of the Residues.

To find the value of the Rhizoristic series corresponding to a given value of x, the better method would accordingly seem to be to commence with finding the arithmetical values of the n quotients

$$a_1x + b_1, \ a_2x + b_2 \dots a_nx + b_n.$$

We thus obtain n numbers $\mu_1, \mu_2 \dots \mu_n$, and have only to form a progression according to the well-known law

$$1, \ N_1, \ N_2 \dots N_n,$$

where $N_1 = \mu$, and in general $N_\iota = \mu_\iota N_{\iota-1} - N_{\iota-2}$.

The number of arithmetical operations required by this method (after the division part of the process which is common to the two methods has been performed) will be* $2n$ multiplications and $2n$ additions or subtractions; whereas if we deal with the residues directly, the number of multiplications will be

$$n + (n-1) + \dots + 1,$$

that is
$$\frac{n(n+1)}{2},$$

(besides having to raise x to the nth power), and the same number of additions. The practical advantage, however, of this method over the old

[* *footnote*, p. 622 below.]

method is not quite so great as it may at first sight appear, in consequence of the quantities operated with on applying it being larger numbers than those which have to be used in the old method.

If we were to employ, instead of the direct series,

$$1,\ N_1,\ N_2N_1-1,\ \&c.,$$

the signaletically equivalent reverse series

$$1,\ N_n,\ N_{n-1}N_n-1,\ \&c.,$$

the arithmetical difficulty would be much increased in consequence of the quotients becoming rapidly more complex as the division proceeds. It were much to be desired that some person practically conversant with the application of Sturm's method, such as that excellent and experienced mathematician, my esteemed friend Professor J. R. Young, would perpend and give his opinion upon the relative practical advantages of the two methods of substitution; the one that where the residues are employed, the other that where the quotients.

I am bound to state, that but for a valuable hint furnished to me by my friend, that most profound mathematician, M. Hermite, who discovered a theorem virtually involving the transformation of Sturm's theorem here presented, but founded upon entirely different and less general considerations, and in the origin of which hint, as arising out of my own previous speculations upon which I was in correspondence with M. Hermite, I may perhaps myself claim a share, this theory would probably not have come to light. It is of course not confined to Sturm's theorem, which deals only with the special case of two functions, whereof one is the first derivative of the other.

There is a larger theory, to which M. Sturm's is a corollary, which contemplates the relations of the roots of any two functions whatever. This is what I term the *theory of interpositions*, upon which I do not propose here to enter, but which will be fully developed in a memoir nearly completed, and which I shortly propose to present* to the Royal Society, wherein will be found combined and flowing into one current various streams of thought bearing upon this subject which had previously existed disunited, and appearing to follow each a separate course.

Remark.

I am not aware that anyone has observed what the effect would be of omitting to change the signs of the successive residues in the application of Sturm's method, that is, of employing a proper in lieu of an improper continued fraction to express $\frac{f'x}{fx}$.

[* pp. 429—586 above.]

Although easily made out, it is well worthy of being remarked. Suppose

$$\frac{\phi}{f} = \frac{1}{Q_1 - \frac{1}{Q_2 - \frac{1}{Q_3 \vdots - \frac{1}{Q_n}}}},$$

and in general (P being any letter) use $\bar{P}$ to denote $-P$. Now we may write

$$\begin{aligned} f &= Q_1\phi - \rho_1, \\ \phi &= Q_2\rho_1 - \rho_2, \\ \rho_1 &= Q_3\rho_2 - \rho_3, \\ \rho_2 &= Q_4\rho_3 - \rho_4, \\ \rho_3 &= Q_5\rho_4 - \rho_5, \\ \rho_4 &= Q_6\rho_5 - \rho_6, \\ \&c. &= \&c. \end{aligned}$$

This gives

$$\begin{aligned} f &= Q_1\phi + \bar{\rho}_1, \\ \phi &= \bar{Q}_2\bar{\rho}_1 + \bar{\rho}_2, \\ \bar{\rho}_1 &= Q_3\bar{\rho}_2 + \rho_3, \\ \bar{\rho}_2 &= \bar{Q}_4\rho_3 + \rho_4, \\ \rho_3 &= Q_5\rho_4 + \bar{\rho}_5, \\ \&c. &= \&c. \end{aligned}$$

The law evidently being that the quotients change their sign alternately, that is in the 2nd, 4th, 6th, &c. places, and remain unaltered in the 1st, 3rd, 5th, &c. places; whereas the residues or excesses change their signs in the 1st and 2nd, 5th and 6th, 9th and 10th, &c., and remain unaltered in the 3rd and 4th, 7th and 8th, 11th and 12th, &c. places. The effect is, that if, in applying Sturm's method, we omit to change the signs of the remainders, and take as our signaletic series

$$fx,\ f'x,\ R_1,\ R_2,\ R_3 \ldots R_{n-1},$$

R_1, R_2, R_3, &c. being the successive unaltered residues, the signaletic index corresponding to any value of x instead of being the number of continuations in the above series, will become the number of continuations in going from a term in an odd place to a term in an even place *plus* the number of variations in going from a term in an odd place to a term in an even place.

If we adopt the quotient method, the rule will be simply to change the sign of the alternate quotients (beginning with the second) in forming the signaletic series.

As an artist delights in recalling the particular time and atmospheric effects under which he has composed a favourite sketch, so I hope to be excused putting upon record that it was in listening to one of the magnificent choruses in the 'Israel in Egypt' that, unsought and unsolicited, like a ray of light, silently stole into my mind the idea (simple, but previously unperceived) of the equivalence of the Sturmian residues to the denominator series formed by the reverse convergents. The idea was just what was wanting,—the key-note to the due and perfect evolution of the theory.

Postscript.

Immediately after leaving the foregoing matter in the hands of the printer, a most simple and complete proof has occurred to me of the theorem left undemonstrated in the text [p. 610].

Suppose that we have any series of terms $u_1, u_2, u_3 \dots u_n$, where

$$u_1 = A_1, \quad u_2 = A_1A_2 - 1, \quad u_3 = A_1A_2A_3 - A_1 - A_3, \text{ \&c.}$$

and in general

$$u_\iota = A_\iota u_{\iota-1} - u_{\iota-2},$$

then $u_1, u_2, u_3 \dots u_n$ will be the successive principal coaxal determinants of a symmetrical matrix. Thus suppose $n = 5$; if we write down the matrix

$$\begin{matrix} A_1, & 1, & 0, & 0, & 0, \\ 1, & A_2, & 1, & 0, & 0, \\ 0, & 1, & A_3, & 1, & 0, \\ 0, & 0, & 1, & A_4, & 1, \\ 0, & 0, & 0, & 1, & A_5, \end{matrix}$$

(the mode of formation of which is self-apparent), these successive coaxal determinants will be

$$1 \quad \begin{vmatrix} A_1 \end{vmatrix} \quad \begin{vmatrix} A_1, & 1 \\ 1, & A_2 \end{vmatrix} \quad \begin{vmatrix} A_1, & 1, & 0 \\ 1, & A_2, & 1 \\ 0, & 1, & A_3 \end{vmatrix} \quad \begin{vmatrix} A_1, & 1, & 0, & 0 \\ 1, & A_2, & 1, & 0 \\ 0, & 1, & A_3, & 1 \\ 0, & 0, & 1, & A_4 \end{vmatrix} \quad \begin{vmatrix} A_1, & 1, & 0, & 0, & 0 \\ 1, & A_2, & 1, & 0, & 0 \\ 0, & 1, & A_3, & 1, & 0 \\ 0, & 0, & 1, & A_4, & 1 \\ 0, & 0, & 0, & 1, & A_5 \end{vmatrix}$$

that is

$$1, \; A_1, \; A_1A_2 - 1, \; A_1A_2A_3 - A_1 - A_3, \; A_1A_2A_3A_4 - A_1A_2 - A_1A_4 - A_3A_4 + 1,$$
$$A_1A_2A_3A_4A_5 - A_1A_2A_5 - A_1A_4A_5 - A_3A_4A_5 - A_1A_2A_3 + A_5 + A_3 + A_1.$$

It is proper to introduce the *unit* because it is, in fact, the value of a determinant of zero places, as I have observed elsewhere. Now I have demonstrated directly in this very *Magazine* (August 1852)*, under cover of the *umbral* notation, that the *signaletic* value of a regularly ascending series of principal coaxal determinants formed from any symmetrical matrix is unaffected by any such transposition whatever of the lines and columns of the matrix as does not destroy the symmetry about the principal axis. Hence, then, beginning from the lower extremity of the axis A_5, and reading off the ascending series of coaxal minors from that point, we obtain the reverse series,

$$1,\ A_5,\ A_5A_4-1,\ A_5A_4A_3-A_5-A_3,\ A_5A_4A_3A_2-A_5A_4-A_5A_2-A_3A_2+1,$$
$$A_5A_4A_3A_2A_1-A_5A_4A_1-A_5A_2A_3-A_3A_2A_1-A_5A_4A_3+A_1+A_3+A_5.$$

Hence we see that the denominators to the convergents of

$$\cfrac{1}{A_1-\cfrac{1}{A_2-\cfrac{1}{A_3-\cfrac{1}{A_4-\cfrac{1}{A_5}}}}},$$

beginning with 1, form a series signaletically equivalent to that similarly formed from the fraction

$$\cfrac{1}{A_5-\cfrac{1}{A_4-\cfrac{1}{A_3-\cfrac{1}{A_2-\cfrac{1}{A_1}}}}};$$

and the reasoning is of course general, and establishes the theorem in question.

It seems only proper and natural that I should not leave unstated here the signaletic properties of the series of *numerators* to the convergents to $\frac{f'x}{fx}$ expanded under the form of a continued fraction.

Let the number of changes of sign in the denominator series for any given value a of x be called $D(a)$, and for the numerator series $N(a)$. Then $N(a)-N(b)$ may be equal to, or at most can only differ by a positive or negative unit from $D(a)-D(b)$. The relation between these differences depends on the nature of the interval between the greater of the two limits a and b, and the root of $f(x)$ next less than that limit, and of the interval between the less of the two limits a and b, and the root of fx next greater than such limit. If a root of $f'x$ is contained in each such interval,

$$N(a)-N(b)=D(a)-D(b)+1;$$

if a root of $f'x$ is contained within one interval, but no root within the other,

$$N(a)-N(b)=D(a)-D(b);$$

if no root of $f'x$ is contained within either interval,

$$N(a)-N(b)=D(a)-D(b)-1.$$

[* p. 380 above.]

I may conclude with noticing that the determinantive form of exhibiting the successive convergents to an improper continued fraction affords an instantaneous demonstration of the equation which connects any two consecutive such convergents as

$$\frac{N_{\iota-1}}{D_{\iota-1}} \text{ and } \frac{N_\iota}{D_\iota},$$

namely

$$N_\iota D_{\iota-1} - N_{\iota-1} D_\iota = 1.$$

For if we construct the matrix, which for greater simplicity I limit to five lines and columns,

$$\begin{vmatrix} A, & 1, & 0, & 0, & 0 \\ 1, & B, & 1, & 0, & 0 \\ 0, & 1, & C, & 1, & 0 \\ 0, & 0, & 1, & D, & 1 \\ 0, & 0, & 0, & 1, & E \end{vmatrix}, \qquad \text{(M)}$$

and represent umbrally as

$$\begin{pmatrix} a_1, & a_2, & a_3, & a_4, & a_5 \\ b_1, & b_2, & b_3, & b_4, & b_5 \end{pmatrix},$$

and if, by way of example, we take the fourth and fifth convergents, these will be in the umbral notation represented by

$$\frac{\begin{pmatrix} a_2, & a_3, & a_4 \\ b_2, & b_3, & b_4 \end{pmatrix}}{\begin{pmatrix} a_1, & a_2, & a_3, & a_4 \\ b_1, & b_2, & b_3, & b_4 \end{pmatrix}} \text{ and } \frac{\begin{pmatrix} a_2, & a_3, & a_4, & a_5 \\ b_2, & b_3, & b_4, & b_5 \end{pmatrix}}{\begin{pmatrix} a_1, & a_2, & a_3, & a_4, & a_5 \\ b_1, & b_2, & b_3, & b_4, & b_5 \end{pmatrix}},$$

respectively. Hence

$$N_5 D_4 - N_4 D_5 = \begin{pmatrix} a_2, & a_3, & a_4, & a_5 \\ b_2, & b_3, & b_4, & b_5 \end{pmatrix} \times \begin{pmatrix} a_2, & a_3, & a_4, & a_1 \\ b_2, & b_3, & b_4, & b_1 \end{pmatrix}$$

$$- \begin{pmatrix} a_2, & a_3, & a_4 \\ b_2, & b_3, & b_4 \end{pmatrix} \times \begin{pmatrix} a_2, & a_3, & a_4, & a_5, & a_1 \\ b_2, & b_3, & b_4, & b_5, & b_1 \end{pmatrix},$$

which [p. 252 above]

$$= \begin{pmatrix} a_2, & a_3, & a_4, & a_5 \\ b_2, & b_3, & b_4, & b_5 \end{pmatrix} \times \begin{pmatrix} a_2, & a_3, & a_4, & a_1 \\ b_2, & b_3, & b_4, & b_1 \end{pmatrix} - \begin{pmatrix} a_2, & a_3, & a_4, & a_5 \\ b_2, & b_3, & b_4, & b_5 \end{pmatrix} \begin{pmatrix} a_2, & a_3, & a_4, & a_1 \\ b_2, & b_3, & b_4, & b_1 \end{pmatrix}$$

$$+ \begin{pmatrix} a_2, & a_3, & a_4, & a_5 \\ b_2, & b_3, & b_4, & b_1 \end{pmatrix} \times \begin{pmatrix} a_2, & a_3, & a_4, & a_1 \\ b_2, & b_3, & b_4, & b_5 \end{pmatrix}$$

$$= \begin{pmatrix} a_2, & a_3, & a_4, & a_5 \\ b_1, & b_2, & b_3, & b_4 \end{pmatrix} \times \begin{pmatrix} a_1, & a_2, & a_3, & a_4 \\ b_2, & b_3, & b_4, & b_5 \end{pmatrix},$$

that is

$$\begin{vmatrix} 1, & B, & 1, & 0 \\ 0, & 1, & C, & 1 \\ 0, & 0, & 1, & D \\ 0, & 0, & 0, & 1 \end{vmatrix} \times \begin{vmatrix} 1, & 0, & 0, & 0 \\ B, & 1, & 0, & 0 \\ 1, & C, & 1, & 0 \\ 0, & 1, & D, & 1 \end{vmatrix} = 1 \times 1 = 1,$$

as was to be proved. And the demonstration is evidently general in its nature. We may treat a proper continued fraction in precisely the same manner, substituting throughout $\sqrt{(-1)}$ in place of 1 in the generating matrix, and we shall thus, by the same process as has been applied to improper continued fractions, obtain

$$N_{\iota+1} D_\iota - N_\iota D_{\iota+1} = \{\sqrt{(-1)}\}^\iota \times \{\sqrt{(-1)}\}^\iota$$
$$= (-1)^\iota.$$

I believe that the introduction of the method of determinants into the algorithm of continued fractions cannot fail to have an important bearing upon the future treatment and development of the theory of Numbers*.

* If in the above matrix (M) we write throughout $\sqrt{(-1)}$ in place of 1, we have a representation of the numerators and denominators of the convergents to a proper continued fraction, and such representation gives an immediate and visible proof of the simple and elegant rule (not stated in the ordinary treatises on the subject, nor so well known as it deserves to be) for forming any such numerators or denominators by means of the principal terms in each; the rule, I mean, according to which the ιth denominator may be formed from $q_1q_2q_3q_4 \ldots q_\iota$ ($q_1, q_2 \ldots q_\iota$ being the successive quotients), and the ιth numerator from $q_2q_3 \ldots q_\iota$, by leaving out from the above products respectively any pair or any number of pairs of consecutive quotients as $q_\rho q_{\rho+1}$. For instance, from $q_1q_2q_3q_4q_5$, by leaving out q_1q_2, q_2q_3, q_3q_4 and q_4q_5, we obtain

$$q_3q_4q_5 + q_1q_4q_5 + q_1q_2q_5 + q_1q_2q_3;$$

and by leaving out $q_1q_2 \times q_3q_4$, $q_1q_2 \times q_4q_5$, $q_2q_3 \times q_4q_5$, we obtain $q_5 + q_3 + q_1$; so that the total denominator becomes

$$q_1q_2q_3q_4q_5 + q_3q_4q_5 + q_1q_4q_5 + q_1q_2q_5 + q_1q_2q_3 + q_1 + q_3 + q_5;$$

and in like manner the numerator of the same convergent is

$$q_2q_3q_4q_5 \left\{1 + \frac{1}{q_2q_3} + \frac{1}{q_3q_4} + \frac{1}{q_4q_5} + \frac{1}{q_2q_3q_4q_5}\right\},$$

that is
$$q_2q_3q_4q_5 + q_4q_5 + q_2q_5 + q_2q_3 + 1.$$

The most cursory inspection of the form of the generating matrix will show at once the reason of this rule. It may furthermore be observed, that every progression of terms constructed in conformity with the equation

$$u_n = a_n u_{n-1} - b_n u_{n-2} + c_n u_{n-3} \pm \&c.,$$

may be represented as an ascending series of principal coaxal determinants to a common matrix. Thus if each term in such progression is to be made a linear function of the three preceding terms, it will be representable by means of the matrix

$$\begin{matrix} A, & B, & C'', & 0, & 0 \\ 1, & A', & B'', & C''', & 0 \\ 0, & 1, & A'', & B''', & C'''' \\ 0, & 0, & 1, & A''', & B'''' \\ 0, & 0, & 0, & 1, & A'''' \end{matrix}$$

indefinitely continued, which gives the terms

$$1,\ A,\ AA' - B,\ AA'A'' - BA'' - AB'' + C'',\ \&c.$$

62.

NOTE ON A REMARKABLE MODIFICATION OF STURM'S THEOREM, AND ON A NEW RULE FOR FINDING SUPERIOR AND INFERIOR LIMITS TO THE ROOTS OF AN EQUATION.

[*Philosophical Magazine,* VI. (1853), pp. 14—20.]

IN my paper [p. 609 above] on this subject in the preceding Number of the *Magazine,* I showed how by means of the quotients a_1x+b_1, $a_2x+b_2 \ldots a_nx+b_n$, obtained by throwing $\dfrac{f'x}{fx}$ under the form of a continued fraction, the process for finding the signaletic index for any given value of x in the series for determining the number of real roots of fx within given limits was reduced to performing two sets of n multiplications and as many additions or subtractions. But by means of a very simple observation, I can now show that the second and more laborious set of multiplications may be dispensed with and replaced by the simple operation of finding reciprocals, which can be done by mere inspection by means of Barlow's or similar tables, which are familiar to all computers. If we call the quotients

$$a_1x + b_1,\ a_2x + b_2 \ldots a_nx + b_n,$$

we must, as explained in the preceding article, find the n numerical values $\mu_1, \mu_2 \ldots \mu_n$ which these quotients assume for any assigned value of x. This being done, the signaletic index corresponding to such value of x, that is the number of continuations of sign in the signaletic series

$$1,\ \mu_1,\ \mu_1\mu_2 - 1,\ \mu_3\mu_2\mu_1 - \mu_2 - \mu_1,\ \&\text{c.},$$

is evidently the number of positive terms in the series

$$\frac{1}{\mu_1},\ \frac{1}{\mu_2 - \dfrac{1}{\mu_1}},\ \frac{1}{\mu_3 - \dfrac{1}{\mu_2 - \dfrac{1}{\mu_1}}} \cdots \frac{1}{\mu_n - \dfrac{1}{\mu_{n-1} \; \vdots - \dfrac{1}{\mu_1}}}.$$

These terms may be found with the utmost facility in succession from one another; for if M_ι be one of them, the next will be $(\mu_{\iota+1} - M_\iota)^{-1}$. Thus, then, the necessity for the more operose set of multiplications is done away with, and the actual labour of computation reduced much more than 50 per cent. below that required by the method indicated in the preceding article on the subject. I need hardly add, that the old method of Sturm would admit of a similar abbreviation; but in using it we should be subjected to the great practical disadvantage of having to begin with the more heavy and complicated quotients μ_n, μ_{n-1}, &c. instead of μ_1, μ_2, &c., which would very greatly enhance the labour of computation. I will conclude by a remark of some interest under an algebraical point of view.

It has been stated that the denominators of the successive convergents to

$$\frac{1}{q_n} - \frac{1}{q_{n-1}} - \frac{1}{q_{n-2}} \text{ \&c.} \;\vdots\; - \frac{1}{q_1}$$

are equivalent (to a constant factor *près*) with the Sturmian functions, and the reader may be curious to know something of the nature of the signaletically equivalent series formed by the denominators of the convergents to the direct fraction

$$\frac{1}{q_1} - \frac{1}{q_2} - \frac{1}{q_3} \text{ \&c.} \;\vdots\; - \frac{1}{q_n}.$$

These denominators are (abstracting from a constant factor not affecting the signs) the Sturmian residues resulting from performing the process of common measure between $f'x$ and f_1x; f_1x being related in a remarkable manner in point of form to $f'x$. Call the roots of fx $a_1, a_2 \ldots a_n$; we know that $f'x$ is

$$\Sigma \{(x-a_2)(x-a_3)\ldots(x-a_n)\},$$

and I am able to state that f_1x is (to a constant factor *près*) equal to

$$\Sigma [\zeta(a_2, a_3 \ldots a_n) \{(x-a_2)(x-a_3)\ldots(x-a_n)\}],$$

$\zeta(a_2, a_3 \ldots a_n)$ denoting the product of the squares of the differences between the $(n-1)$ quantities $a_2, a_3 \ldots a_n$. Accordingly it will be seen that whenever x is indefinitely near, whether on the side of excess or defect, to a real root of fx, $f'x$ and f_1x will have the same sign; which serves to show, upon an independent and specific algebraical ground, why the two series of residues corresponding to $\frac{f'x}{fx}$ and $\frac{f_1x}{fx}$ are (as by a deduction from a general principle they have been previously shown to be) *rhizoristically* equivalent.

Observation.

In comparing the relative merits of the old and new methods of substitution for the purposes of Sturm's theorem, the effect of the introduction of positive multipliers into the dividends in order to keep all the numerical quantities integral ought not to be disregarded. If we call the quotients corresponding to this modification of the dividends Q_1, Q_2, Q_3, Q_4, &c., and the factors thus introduced m_1, m_2, m_3, m_4, &c., the true quotients will be

$$\frac{Q_1}{m_1},\ \frac{m_1}{m_2}Q_2,\ \frac{m_2}{m_1 m_3}Q_3,\ \frac{m_1 m_3}{m_2 m_4}Q_4,\ \&\text{c.};$$

and it will be found that we may employ as our rhizoristic index either the number of continuations of sign in the series

$$1,\ Q_1,\ Q_2Q_1 - m_2,\ Q_3(Q_2Q_1 - m_2) - m_3Q_1,\ \&\text{c.}$$

the law of formation of the successive terms u_0, u_1, u_2, &c. being

$$u_{\iota+1} = Q_{\iota+1}u_\iota - m_{\iota+1}u_{\iota-1},$$

or the number of positive signs in the series

$$Q_1,\ Q_2 - \frac{m_2}{Q_1},\ Q_3 - \frac{m_3}{Q_2 - \dfrac{m_2}{Q_1}},\ \&\text{c.}$$

the law of formation of the successive terms v_1, v_2, v_3, &c. being

$$v_\iota = Q_\iota - \frac{m_\iota}{v_{\iota-1}}.$$

There may therefore, in fact, be in each case $(n-1)$ more multiplications than have been taken account of in the text above.

If integer numbers be used *throughout* (so that accordingly the u series is that made use of), the total number of multiplications will in general be $n + 2(n-1)$* or $3n - 2$; the old method, as previously stated, would require $\frac{1}{2}n(n+1)$ multiplications; for if we call any one of the Sturmian functions

$$A_0x^\iota + A_1x^{\iota-1} + A_2x^{\iota-2} + \ldots + A_\iota,$$

we shall, using the most abbreviated method of computation, have to calculate successively

$$xA_0 + A_1,\ x(xA_0 + A_1) + A_2,\ \&\text{c.},$$

* If all the extraneous factors are units, the number of multiplications (like that of the additions) would be $2n-1$, and not $2n$, as inadvertently stated in the preceding number of the *Magazine*.

giving rise to ι operations (but, it must be admitted, with the practical advantage of the use of a constant multiplier); and as ι may take all values from n to 1, the total number of such operations will be $\frac{1}{2}n(n+1)$. When $n = 4$,

$$\tfrac{1}{2}n(n+1) = 3n - 2.$$

Consequently (if it be thought necessary to adhere to integers throughout), for values of n not exceeding 4, the old method would be probably the more expeditious.

ADDENDUM.

On a method of finding Superior and Inferior Limits to the real Roots of any Algebraical Equation.

The theory above considered has incidentally led me to the discovery of a new and very remarkable method for finding superior and inferior limits to the real roots of any algebraical equation. Suppose in general that

$$\frac{N}{D} = \frac{1}{q_1 +}\,\frac{1}{q_2 +}\,\frac{1}{q_3 +}\cdots + \frac{1}{q_n};$$

then it is easily seen that

$$D = M_1 M_2 M_3 \ldots M_n,$$

where

$$M_1 = q_1,\ M_2 = q_2 + \frac{1}{q_1},\ M_3 = q_3 + \frac{1}{M_2} \cdots M_n = q_n + \frac{1}{M_{n-1}}.$$

In general let any numerical quantity within brackets be used to denote its *positive* numerical value; so that, for instance, whether $q = \pm 3$, (q) will equally denote $+3$.

And now suppose that neither q_1 nor q_n, the first or last of the quotients, lies between $+1$ and -1, and that no one of the intermediate quotients $q_2, q_3 \ldots q_{n-1}$ lies between $+2$ and -2; so that, in other words,

$$(q_1) > 1,\ (q_2) > 2,\ (q_3) > 2 \ldots (q_{n-1}) > 2,\ (q_n) > 1;$$

then, I say, that $M_1, M_2, M_3 \ldots M_n$ will have the same signs as $q_1, q_2, q_3 \ldots q_n$ respectively; for

$$M_1 = q_1,$$

therefore
$$(M_1) > 1;$$

but
$$M_2 = q_2 + \frac{1}{M_1},$$

therefore $$(M_2) = (q_2) \pm \left(\frac{1}{M_1}\right) > 2 \pm 1\,;$$

therefore

$$M_2 \text{ has the same sign as } q_2, \text{ and } \textit{also } (M_2) > 1\,;$$

therefore in like manner,

$$(M_3) \text{ has the same sign as } q_3, \text{ and } \textit{also } (M_3) > 1\,;$$

therefore in like manner,

$$(M_4) \text{ has the same sign as } q_4, \text{ and } \textit{also } (M_4) > 1\,;$$

and so on until we come to M_{n-1}, and we shall find

$$M_{n-1} \text{ of the same sign as } q_{n-1}, \text{ and } \textit{also } (M_{n-1}) > 1.$$

Finally,

$$M_n = q_n \pm \frac{1}{M_{n-1}},$$

where $(q_n) > 1$ and $\left(\frac{1}{M_{n-1}}\right) < 1$, therefore

$$M_n \text{ has the same sign as } q_n\,;$$

but we cannot say (nor is there any occasion to say) that $(M_n) > 1$; therefore

$$D = M_1 M_2 M_3 \ldots M_n \text{ has the same sign as } q_1 q_2 q_3 \ldots q_n.$$

Now let fx be any given function of x of the nth degree, and ϕx any *assumed* function whatever of x of the $(n-1)$th degree, and let

$$\frac{\phi x}{fx} = \frac{1}{q_1 +}\,\frac{1}{q_2 +}\,\frac{1}{q_3 +} \ldots \frac{1}{q_n},$$

where $q_1, q_2, q_3 \ldots q_n$ are now supposed to be linear functions of x, which, except for *special relations* between f and ϕ, will always exist, and can be found by the ordinary process of successive division.

Write down the n pairs of equations,

$$u_1 = q_1 + 1 = 0,\quad u_2 = q_2 + 2 = 0,\quad u_3 = q_3 + 2 = 0 \ldots u_n = q_n + 1 = 0,$$

$$u'_1 = q_1 - 1 = 0,\quad u'_2 = q_2 - 2 = 0,\quad u'_3 = q_3 - 2 = 0 \ldots u'_n = q_n - 1 = 0.$$

If the greatest of the values of x determined from these $2n$ equations be called L, and the least of these values be called Λ, it may easily be made out that between $+\infty$ and L, each of the quantities $q_1, q_2, q_3 \ldots q_n$ will remain unaltered in sign; and between $-\infty$ and Λ also the same invariability of sign obtains; and, moreover, between $+\infty$ and L, and between Λ and $-\infty$, $(q_1), (q_2) \ldots (q_{n-1}), (q_n)$ will be respectively greater than 1, 2 ... 2, 1. Consequently, by virtue of the preceding theorem, between $+\infty$ and L, and between Λ and $-\infty$, D will always retain the same sign as $q_1 q_2 q_3 \ldots q_n$, and therefore no root of fx will be contained within either such interval. And hence fx, which is manifestly identical with D (the denominator of the continued fraction last above written), affected with a certain constant factor, will retain an invariable sign within each such interval respectively. Hence, then, the following rule.

Calling $q_1, q_2, q_3 \ldots q_n$ respectively

$$a_1x - b_1,\ a_2x - b_2,\ a_3x - b_3 \ldots a_nx - b_n,$$

if we form the $2n$ quantities

$$\frac{b_1 \pm 1}{a_1},\ \frac{b_2 \pm 2}{a_2},\ \frac{b_3 \pm 2}{a_3} \ldots \frac{b_{n-1} \pm 2}{a_{n-1}},\ \frac{b_n \pm 1}{a_n},$$

the greatest of these will be a superior limit, and the least of them an inferior limit to the roots of fx.

The values of these fractions will depend upon the form of the assumed subsidiary function ϕ. Hence, then, arises a most curious question for future discussion—*to wit,* to discover whether in any case the subsidiary function can be so assumed as that the superior limit can be brought to coincide with the greatest, or the inferior limit with the least real root, supposing that there are any real roots. I believe that it will be found that this is always impossible to be done. Then, again, if all the roots are imaginary, can *inconsistent limits* (evincing this imaginariness) be obtained by giving different forms to the subsidiary function, which would be the case if we could find that the superior limit brought out by one form were less than the inferior limit brought out by another, or the inferior limit brought out by one form greater than the superior brought out by another? If, as I suspect, this also can never be done, then the general question remains to determine for all cases the form to be given to the subsidiary function, which will make the interval between either limit and its nearest root, or between the two limits themselves, a minimum. Thus, it appears to me, a fine field of research is thrown open to those who are interested in the theory of maxima minimorum, and minima maximorum, and one likely to lead to unexpected and important discoveries [cf. p. 533 above, and the Author's footnote, p. 495].

It may be asked how is the above rule to be applied if any of the leading coefficients in ϕx, or of the successive residues of fx and ϕx vanish; in which case, instead of the coefficients being linear, some of them will be, as in fact all might be, polynomial functions of x. The rule, it may be proved, will still subsist.

Equating the first and last quotients each of them to $+1$ and to -1, and the intermediate ones to $+2$ and to -2, the greatest root of all the equations so formed continues to be a superior, and the least root an inferior limit to the roots of fx. Nor is it ever necessary, even in these special cases, actually to *solve* any of these equations; for evidently it will be sufficient to find a superior limit and an inferior limit to each of them, and adopt the greatest of the superior and the least of the inferior limits as the superior and inferior limits to the roots of the given equation. Thus, then, we should have to repeat upon the quotients increased and diminished by 1 or 2 (as the case may be), the same process as is supposed to be originally applied to fx, and thus by a continued process of *trituration* (since every new function so to be operated upon is of a lower degree than the original function) we must finally descend to linear equations exclusively.

It is interesting thus to see that there are no failing cases in the application of the rule, and that a solution of equations of a higher degree than the first is never necessary. But as a matter of fact, the chances are infinitely improbable (if ϕx is chosen at random), of any of the quotients after the first ceasing to be linear; and the first is of course linear, provided that the degree of ϕx is taken only one unit below that of fx.

In working with Sturm's theorem, a system of quotients is supplied ready to hand; and these quotients, by virtue of the rule given above, may be used to assign a superior and inferior limit in the first instance, before setting about to determine the distribution of the roots between these limits by aid either of these same quotients or of the residues. For the change of sign of the residues required by the Sturmian process will only affect the signs, and not the forms of the quotients; but in the application of the above rule for finding the limits, the sign of any quotient is evidently immaterial.

63.

ON THE NEW RULE FOR FINDING SUPERIOR AND INFERIOR LIMITS TO THE REAL ROOTS OF ANY ALGEBRAICAL EQUATION.

[*Philosophical Magazine*, VI. (1853), pp. 138—140.]

THE lemma accessory to the demonstration of the rule for finding limits to the roots of an equation, given in the addendum [p. 623 above] to my paper in the *Magazine* for this month, admits of two successive and large steps of generalization, in which the scope of the principal theorem will participate in an equal degree.

1. Whatever the signs may be of $q_1, q_2, q_3 \dots q_r$, the denominator of the continued fraction

$$\frac{1}{q_1+}\frac{1}{q_2+}\frac{1}{q_3}\dots\frac{1}{q_r}$$

will have the same sign as $q_1 q_2 q_3 \dots q_r$, provided that

$$[q_1] > \mu_1,\ [q_2] > \mu_2 + \frac{1}{\mu_1},\ [q_3] > \mu_3 + \frac{1}{\mu_2} \dots$$

$$\dots [q_{r-1}] > \mu_{r-1} + \frac{1}{\mu_{r-2}},\ [q_r] > \frac{1}{\mu_{r-1}},$$

where $\mu_1,\ \mu_2 \dots \mu_{r-1}$ signify any *positive* quantities whatsoever; in the particular case where $\mu_1 = \mu_2 = \mu_3 = \dots = \mu_{r-1} = 1$, we fall back upon the lemma as originally stated.

2. But the lemma admits of another modification, which will in general impose far less stringent limits upon the arithmetical values of the series of q's.

Let all the possible sequences of q's be taken which present only variations of sign; for example if the entire series be q_1, q_2, q_3, q_4, and the corresponding algebraical signs are $+ - - +$, we shall have the two sequences q_1, q_2; q_3, q_4. If the entire series be $q_1, q_2, q_3 \dots q_{15}$, and the signs be

$$- - - + - + + + + - + + + + -,$$

then the sequences to be taken will be

$$q_3, q_4, q_5, q_6;\ q_9, q_{10}, q_{11};\ q_{14}, q_{15},$$

and so in general.

Suppose, now, that $q_{\rho+1}, q_{\rho+2} \ldots q_{\rho+i}$ are the terms of any one such sequence. Then, provided that

$$[q_{\rho+1}] > \mu_1,\ [q_{\rho+2}] > \mu_2 + \frac{1}{\mu_1} \ldots q_{\rho+i-1} > \mu_{i-1} + \frac{1}{\mu_{i-2}},$$

and
$$q_{\rho+i} > \frac{1}{\mu_{i-1}},$$

(it being understood that the values of $\mu_1, \mu_2 \ldots \mu_{i-1}$ are perfectly arbitrary, except being subject to the condition of being all positive, and that there are as many distinct and independent systems of such values as there are sequences of variations of sign), it will continue to be true (and capable of being demonstrated to be so by precisely the same reasoning as was applied to the demonstration of the lemma in its original form) that the denominator of $\frac{1}{q_1 +}\frac{1}{q_2 +} \ldots \frac{1}{q_r}$ will have the same sign as the product $q_1 q_2 q_3 \ldots q_r$. It will be observed that, as regards the *residual* quotients not comprised in any sequence, their values are absolutely unaffected by any condition whatever. As a direct consequence from this lemma, we derive the following greatly improved *Theorem* for the discovery of the limits.

Let, as before, $fx = 0$ be any given algebraical equation; ϕx any assumed arbitrary function of x of an inferior degree to that of fx; and let

$$\frac{\phi x}{fx} = \frac{1}{X_1 +}\frac{1}{X_2 +}\frac{1}{X_3 +} \ldots \frac{1}{X_r};$$

let the leading coefficients of $X_1, X_2, X_3 \ldots X_r$ be $q_1, q_2, q_3 \ldots q_r$, and let this latter series be divided into sequences of variations and residual terms not comprised in any such sequence, as explained above. Let the X's corresponding to the residual terms be called

$$P_1,\ P_2 \ldots P_\omega,$$

and let the successive sets of X's corresponding to the sequences be called respectively

$$V_1,\ \ V_2\ \ldots V_\rho,$$
$$V_1',\ \ V_2'\ \ldots V'_{\rho'},$$
$$V_1'',\ \ V_2''\ \ldots V''_{\rho''},$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$
$$(V_1), (V_2) \ldots (V_{(\rho)}).$$

And let

$$X = P_1 P_2 \dots P_\omega$$
$$\times (V_1^2 - c_1^2)(V_2^2 - c_2^2) \dots (V_\rho^2 - c_\rho^2)$$
$$\times (V_1'^2 - c_1'^2)(V_2'^2 - c_2'^2) \dots (V'_{\rho'}{}^2 - c'_{\rho'}{}^2)$$
$$\&\text{c.} \qquad\qquad \&\text{c.}$$
$$\times \{(V_1)^2 - (c_1)^2\}\{(V_2)^2 - (c_2)^2\} \dots \{(V_{(\rho)})^2 - (c_{(\rho)})^2\},$$

where, in general, any system of values

$$c_1, c_2, c_3 \dots c_{\rho-1}, c_\rho,$$

represents

$$\mu_1, \ \mu_2 + \frac{1}{\mu_1} \dots \mu_{\rho-1} + \frac{1}{\mu_{\rho-2}}, \ \frac{1}{\mu_{\rho-1}}.$$

Then the largest root of $X = 0$ is a superior limit, and the smallest root of $X = 0$ is an inferior limit to the real roots of $fx = 0$; and if $X = 0$ has no real roots, neither will $fx = 0$ have any. For the complete demonstration and some further developments of this theorem see the forthcoming number of Terquem's *Nouvelles Annales* for the present month*.

[* p. 423 and p. 424 above.]

64.

NOTE ON THE NEW RULE OF LIMITS.

[*Philosophical Magazine*, VI. (1853), pp. 210—213.]

It may appear like harping too long on the same string to add any further remarks on the rule relating to so simple and elementary a matter as that of assigning limits to the roots of a given algebraical equation; but it will be remembered that some of the greatest masters of analysis, including the honoured names of Newton and Cauchy, have not disdained to treat, and to give to the world their comparatively imperfect results on this very subject. I hope, therefore, to stand excused of any undue egotism in adding some observations which may tend to present, under a clearer aspect and more finished form, the new and beautifully flexible rule laid before the readers of this *Magazine* in the two preceding Numbers.

Firstly, I observe that any succession of signs may be considered as made up of, and decomposable into, sequences of changes exclusively, if we agree to consider, where necessary, a single isolated sign + or − as a sequence of zero changes. Thus, for instance, + − − + + + + − + + + − + − − may be treated as made up of the variation sequences

$$+-,\ -+,\ +,\ +,\ +-+,\ +,\ +-+-,\ -\text{*}.$$

Secondly, I observe that if $X_1, X_2 \dots X_i$ be all linear functions of x, and the signs of the coefficients of x in these functions constitute a single unbroken series of variations, the denominator of the continued fraction

$$\frac{1}{X_1+}\,\frac{1}{X_2+}\,\frac{1}{X_3+}\cdots\frac{1}{X_i}$$

(reduced to the form of an ordinary algebraical fraction) will have all its roots real.

* The rule is, that the given series of signs is to be separated into distinct sequences of variations, so that the final term of one sequence and the initial term of the next shall form a continuation, that is we must have variation sequences connected together by continuations at their joinings.

Thirdly, suppose, for greater simplicity, that ϕx is of one degree in x lower than fx, and that by the ordinary process of common measure we obtain

$$\frac{\phi x}{fx} = \frac{1}{X_1 +} \frac{1}{X_2 +} \frac{1}{X_3 +} \cdots \frac{1}{X_n},$$

where $X_1, X_2, X_3 \ldots X_n$ are all of them linear functions of x.

Let $X_1, X_2 \ldots X_n$ be divided into distinct and unblending sequences,

$$X_1 X_2 \ldots X_i, \quad X_{i+1} X_{i+2} \ldots X_{i'}, \quad X_{i'+1} \ldots X_{i''}, \quad \ldots, \quad X_{(i)+1} X_{(i)+2} \ldots X_n;$$

so that in each sequence the signs of the coefficients of x present a single unbroken series of variations, which by virtue of observation (1), may be considered to be always capable of being done, and let

$$\frac{\phi_1 x}{f_1 x} = \frac{1}{X_1 +} \frac{1}{X_2 +} \frac{1}{X_3 +} \cdots \frac{1}{X_i},$$

$$\frac{\phi_2 x}{f_2 x} = \frac{1}{X_{i+1} +} \frac{1}{X_{i+2}} \cdots\cdots \frac{1}{X_{i'}},$$

....

....

$$\frac{(\phi) x}{(f) x} = \frac{1}{X_{(i)+1} +} \cdots\cdots\cdots\cdots \frac{1}{X_n};$$

then, according to observation (2), the equations

$$f_1 x = 0, \quad f_2 x = 0 \ldots (f)\, x = 0,$$

have each of them all their roots real; and the observation now to be made is, that the highest of the highest roots and the lowest of the lowest roots of these equations furnish respectively a superior and inferior limit to the roots of $fx = 0$*.

* This theorem may be more concisely stated as follows:—"If U with any subscript be understood to mean a linear function of x in which the sign of the coefficient of x is constant, then the finite roots of the equation

$$\frac{1}{U_1 -} \frac{1}{U_2 -} \frac{1}{U_3 -} \cdots \frac{1}{U_i +} \frac{1}{U_{i+1} -} \frac{1}{U_{i+2} -} \cdots \frac{1}{U_{i'} +} \cdots \frac{1}{U_{(i)+1} -} \frac{1}{U_{(i)+2} -} \cdots \frac{1}{U_n} = \infty$$

lie between the greatest and least finite roots of the equations

$$\frac{1}{U_1 -} \frac{1}{U_2 -} \cdots \frac{1}{U_i} = \infty,$$

$$\frac{1}{U_{i+1} -} \frac{1}{U_{i+2} -} \cdots \frac{1}{U_{i'}} = \infty,$$

....

....

$$\frac{1}{U_{(i)} -} \frac{1}{U_{(i)+1} -} \cdots \frac{1}{U_n} = \infty."$$

The theorem under this form suggests a much more general one relating to para-symmetrical determinants, that is determinants partly normal and partly *gauche*, which will be given hereafter; one example among the many confirming the importance of the view first stated in this *Magazine* by the author of this paper, whereby continued fractions are incorporated with the doctrine of determinants.

N.B. The single root of any one or more of these which may be of the first degree in x is to be treated, in applying the preceding observation, as being at the same time the highest and the lowest root of such equation or equations.

Fourthly and lastly, the problem of assigning limits to the roots of $fx = 0$ reduces itself to that of finding limits to

$$f_1x = 0, \quad f_2x = 0 \ldots (f)\,x = 0;$$

for the greatest and least of these collectively will evidently, *à fortiori*, by virtue of the preceding observation, be limits to the roots of $fx = 0$. Of any such of these as are linear, the root or roots themselves may be treated as known; leaving these out of consideration, the functional part of any other of them, such as f_1x, is the denominator of a continued fraction of the form

$$\frac{1}{(a_1x+b_1)+}\;\frac{1}{(a_2x+b_2)+}\;\frac{1}{(a_3x+b_3)+}\cdots\frac{1}{(a_ix+b_i)},$$

in which $a_1, a_2, a_3 \ldots a_i$ present a single sequence of variations of sign, and the limits to the roots of $f_1x = 0$ may be found as follows.

Form the two systems of equations (in which $\mu_1, \mu_2 \ldots \mu_{i-1}$ are numerical quantities having all the same algebraical sign, but are otherwise arbitrary and independent),

$$\begin{array}{ll|ll}
a_1x+b_1 = & \mu_1 & a_1x+b_1 = & -\mu_1 \\
a_2x+b_2 = & -\mu_2 - \dfrac{1}{\mu_1} & a_2x+b_2 = & \mu_2 + \dfrac{1}{\mu_1} \\
a_3x+b_3 = & \mu_3 + \dfrac{1}{\mu_2} & a_3x+b_3 = & -\mu_3 - \dfrac{1}{\mu_2} \\
\cdots\cdots & & \cdots\cdots & \\
a_{i-1}x+b_{i-1} = & (-)^{i-2}\mu_{i-1} + \dfrac{(-)^{i-2}}{\mu_{i-2}} & a_{i-1}x+b_{i-1} = & (-)^{i-1}\mu_{i-1} + \dfrac{(-)^{i-1}}{\mu_{i-2}} \\
a_ix+b_i = & \dfrac{(-)^{i-1}}{\mu_{i-1}} & a_ix+b_i = & \dfrac{(-)^{i}}{\mu_{i-1}}
\end{array}$$

then (supposing μ_1 to have the same sign as a_1) the highest of the values of x obtained from the first system, and the lowest of the values of x found from the second system of these equations, will be a superior and inferior limit respectively to the roots of $f_1x = 0$; and so for all the rest of the equations

$$f_2(x) = 0, \; f_3(x) = 0 \ldots (f)\,x = 0,$$

excluding those of the first degree.

It will be seen that the theorems contained in the observations (3) and (4) combined (which presuppose the statements made in observations (1) and (2)), contain between them the theorem given in the last Number of the *Magazine* [p. 627 above], but rendered in one or two particulars more simple and precise, and, as it were, reduced to its lowest terms. In the whole course of my experience I never remember a theory which has undergone so many successive transformations in my mind as this very simple one, since the day when I first unexpectedly discovered the germ of it in results obtained for quite a different purpose. In fact, it never entered into my thoughts that in so beaten a track, and in so hackneyed a subject as that of finding numerical limits to the roots of an equation, there was left anything to be discovered; and my sole merit, if any, in bringing the new rule to light, consists in having been able to detect the presence and appreciate the value of a truth which fortune or providence had put into my hands.