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ON THE VANISHING OF THE THIRD SPIN COBORDISM GROUP Ω_3^{Spin}

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An elementary proof of the theorem given in the title is presented. Bibliography: 6 *titles.*

Here, we prove the following theorem.

Theorem 1.1. *A spin* 3*-manifold M is the spin boundary of a spin* 4*-manifold; in short,* $\Omega_3^{\mathrm{Spin}} = 0$.

The proof given below can be found in various sources (cf. [3, 2]); in our presentation, we try to be as elementary as possible. Preceding the proof, we collect the basics about spin structures and handlebodies, and list their fundamental properties. (For a more detailed treatment of these notions, see [3, 1].) The purpose of this survey is to complete the proofs discussed in [5, 6].

§1. Preliminaries on spin structures and handlebodies

Definition 1.2. A manifold X has a spin structure if its stable tangent bundle $TX \oplus \epsilon^k$ admits a trivialization over the 1-skeleton of X which extends over the 2-skeleton.[1] A *spin structure* is a homotopy class of such trivializations. (Here, ϵ denotes a trivialized bundle. It can also be shown that the definition does not depend on k if $k \geq 1$.)

An oriented manifold X admits a spin structure if and only if $w_2(X) = 0$. An oriented 3-manifold always admits a spin structure, since its tangent bundle is trivial. Since for a 4-manifold X we have $w_2(X) \cup \alpha \equiv \alpha \cup \alpha$ (mod 2), the existence of a spin structure on X implies that the cup square of every $\alpha \in H^2(X;\mathbb{Z})$ is even. Moreover, if X is simply connected and $\alpha \cup \alpha$ is even for all $\alpha \in H^2(X;\mathbb{Z})$, then $w_2(X) = 0$; this direction does not hold in the presence of 2-torsion in $H_1(X;\mathbb{Z})$. Spin structures on a manifold (of arbitrary dimension) with vanishing $w_2(X)$ can be parametrized by $H^1(X;\mathbb{Z}_2)$.

If X is a spin manifold and $Y \subset X$ is a codimension-0 submanifold, then Y inherits a natural spin structure from X. If $N \subset X$ has trivialized normal bundle (e.g., $N = \partial X$), then N inherits a spin structure from X. Now assume that X_1 and X_2 are two spin manifolds with boundary and $N_i \subset \partial X_i$ are codimension-0 submanifolds of the boundaries. If $f\colon N_1 \to N_2$ is an orientation-reversing diffeomorphism mapping the induced spin structures into each other, then the manifold $X_1 \cup_{N_2 = f(N_1)} X_2$ has a canonical spin structure induced from those of X_1 and X_2.

Using the above properties, we define the spin cobordism group Ω_n^{Spin} of closed spin n-manifolds. Finally, we note that there is an orientation-preserving self-diffeomorphism $f\colon S^1 \times D^2 \to S^1 \times D^2$ of the solid torus interchanging its two spin structures.

We quickly review the part of handlebody theory that is relevant in our subsequent discussions. (See [1] for further details.)

Definition 1.3. For $0 \leq k \leq n$, an *n-dimensional k-handle h* is a copy of $D^k \times D^{n-k}$ attached to the boundary of an n-manifold X along $\partial D^k \times D^{n-k}$ by an embedding $\varphi\colon \partial D^k \times D^{n-k} \to \partial X$.

$D^k \times 0$ is the *core*, $0 \times D^{n-k}$ the *cocore*, φ the *attaching map*, $\partial D^k \times D^{n-k}$ (or its image $\varphi(\partial D^k \times D^{n-k})$) the *attaching region*, and $\partial D^k \times 0$ (or its image) the *attaching sphere* of the handle.

$X \cup h$ is specified by an embedding $\varphi_0\colon S^{k-1} \to \partial X$ (i.e., a knot in ∂X) with trivial normal bundle together with a *framing* f of $\varphi_0(S^{k-1})$ (i.e., an identification of the normal bundle $\nu\varphi_0(S^{k-1})$ with $S^{k-1} \times \mathbb{R}^{n-k}$).

If ∂X is connected and we restrict ourselves to the study of oriented manifolds, then 1-handles are attached uniquely (up to isotopy). In order to glue a 2-handle, we specify a (1-dimensional) framed knot in ∂X. Homotopically different framings can be parametrized by the group $\pi_1(SO(n-2))$, which is isomorphic to $\mathbb{Z}$ for $n = 4$. If $X = D^4$, then this framing can be identified with the linking number of $\varphi(\partial D^2 \times \{0\})$ with $\varphi(\partial D^2 \times \{p\})$ for some $p \in D^2$ close to the origin 0. The extendability of the spin structure of D^4 to a 2-handle depends on the framing (i.e., the above linking number) mod 2; we easily see that the spin structure extends if and only if this framing is even.

[1] Note that if a trivialization over the 1-skeleton extends to the 2-skeleton, then it extends in a homotopically unique way.

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If we attach a 1-handle to D^4, then the resulting 4-manifold is $S^1 \times D^3$, which is diffeomorphic to $(D^2 - \nu\{0\}) \times D^2$, i.e., to D^4 with an open tubular neighborhood of an embedded disk deleted. Consequently, a 1-handle can be symbolized by an unknot in $S^3 = \partial D^4$ with a dot on it, where the circle records the boundary of the 2-disk that (together with its neighborhood) has been deleted. Similarly, an m-component unlink with dots in S^3 denotes the 4-manifold obtained by adding m 1-handles to D^4. (This manifold is obtained as follows: for each component U_i of the unlink, delete from D^4 an embedded disk $(D_i, \partial D_i) \subset (D^4, \partial D^4)$ (together with its open tubular neighborhood) such that $\partial D_i = U_i$.)

Note that the attaching map of a 2-handle (which maps $S^1 \times D^2$ to the boundary of the union of the 1-handles) can now be regarded as pointing to

$$S^3 - \nu(\text{an } m\text{-component unlink}) \subset S^3$$

(ν denotes a tubular neighborhood); hence, the above framing convention (as the linking number of the attaching circle and a nearby one) extends to all cases. By the above-said, a 4-manifold obtained by adding 1- and 2-handles to D^4 is represented by a link diagram in S^3, where some knots are framed (they correspond to 2-handles) and the others (which form an unlink) are dotted (they correspond to 1-handles). Such a diagram is usually called a *link diagram* or the *Kirby diagram* of the 4-manifold considered. We easily see that the attaching region of a 2-handle passes through a 1-handle if and only if the attaching circle of the former is linked with the dotted circle representing the latter.

Turning the dot on an unknot to 0 framing, we replace the 1-handle represented by the dotted unlink (which is $S^1 \times D^3$) with a 0-framed 2-handle attached along the unknot (which is $S^2 \times D^2$), i.e., we perform a *surgery* along the 1-handle. Conversely, a 0-framed unknot gives rise to an embedded 2-sphere (as the union of the core of the handle with a spanning disk of the unknot) with trivial normal bundle; hence, we can perform surgery along this sphere. In the diagram, this means that we replace the 0 framing of the unknot by a dot, i.e., we invert the surgery described above. (In this second case, we must be careful, since the dotted circles must form an unlink.)

Definition 1.4. Let X be a compact oriented n-manifold with boundary ∂X decomposed as a disjoint union $\partial_+ X \sqcup \partial_- X$ of two compact submanifolds, either of which can be empty. (Orient $\partial_\pm X$ so that $\partial X = \partial_+ X \sqcup \overline{\partial_- X}$ in the boundary orientation.) A *handle decomposition* of X (relative to $\partial_- X$) is an identification of X with a manifold obtained from $[0,1] \times \partial_- X$ by adding handles, such that $\partial_- X$ corresponds to $\{0\} \times \partial_- X$ in the obvious way. A manifold X with a given handle decomposition is a *relative handlebody* built on $\partial_- X$, and if $\partial_- X = \emptyset$, then X is a *handlebody.*

A Morse function $f \colon X \to [0,1]$ provides a handle decomposition for X. Without loss of generality, we assume that the handles are attached in order of increase of the index. Considering $1 - f$ instead of f, we obtain a new handlebody; this operation is called *turning the handlebody upside down.*

In what follows, we restrict our attention to the 4-dimensional case. Clearly, isotoping the attaching maps of the individual handles does not change the diffeomorphism type of X. Thus, we can slide a 2-handle over any other without changing X; after the operation, the attaching circle of the 2-handle that we are sliding becomes the connected sum of the two knots (the connected sum is taken with the band along which we slid the 2-handle).

To determine the new framing, we orient the two former attaching spheres and compute their *linking number.* The new framing on the connected sum is (the sum of the framings $\pm$ twice this linking number), where the sign depends on whether the band respects or disrespects the chosen orientations. (In the first case, we *add* the corresponding handle, while in the second case we *subtract* it.)

It can be shown that if the attaching circle of a 2-handle links a dotted circle geometrically once (i.e., the dotted circle admits a genus-0 Seifert surface intersecting the attaching circle transversally at a single point), then this pair of handles can be canceled without changing the diffeomorphism type of X. In the notation, we simply erase the corresponding knots from our diagram.

However, to obtain the diagram of the new handlebody, we must slide handles. The reason for this is as follows: if a 2-handle h passes through a 1-handle k that we want to cancel with a 2-handle h', then after cancellation the attaching map of h is changed (since we change the target space of the attaching map). For the cancelling pair (k, h'), this confusion can be resolved as follows: slide the 2-handle h over h' repeatedly until the resulting 2-handle is disjoint from k. (This can be done because the attaching circle of h' links the dotted circle geometrically once.) After separating the dotted circle (representing the 1-handle k) from all 2-handles (except h'), we cancel the pair (k, h') by deleting the corresponding knots from the diagram; then the remaining (framed) knots provide a diagram of the new handlebody.

Finally, note that if we attach a 2-handle to D^4 along a (± 1)-framed unknot, then the boundary of the resulting 4-manifold is the total space of the Hopf bundle, hence diffeomorphic to S^3. Consequently, adding (or deleting) a (± 1)-framed unknot to our diagram, we change the 4-manifold but leave its boundary unchanged.

§2. Proof of Theorem 1.1

We divide the proof into several lemmas.

Lemma 2.1. *Every orientable 3-manifold M bounds a 4-manifold that can be constructed by using only one 0-handle and several 2-handles.*

Proof. Since $\Omega_3 = 0$, we know that M bounds a 4-manifold X. Fix a handle decomposition of X and cancel handles until there is a unique 0-handle and there are no 4-handles. Perform surgeries on 1-handles and then turn the handlebody upside down. Now the 3-handles of the original decomposition become 1-handles, and by surgery we turn them into 2-handles. This completes the proof. □

Lemma 2.2. *Every spin 3-manifold M is spin cobordant to $S^1 \times \Sigma_g$ for some g (and some spin structure on this latter manifold). Here, Σ_g denotes the Riemann surface of genus g.*

Proof. Consider X^4 with $\partial X = M$ which admits a handle decomposition with a single 0-handle and some 2-handles. Turn the handlebody decomposition upside down, i.e., regard X as constructed on $M \times [0, 1]$ by attaching several 2-handles and a single 4-handle. The spin structure of M extends to some 2-handles and does not extend to the others, depending on the gluing of the individual handles. Assume that h and h' are two 2-handles to which the spin structure does not extend. We easily see that in this case the spin structure extends to the 2-handle obtained by sliding h over h'.

If the spin structure extends to all 2-handles, then M is a spin boundary, and since $S^1 \times S^2$ (for example) can be equipped with a bounding spin structure, the lemma obviously follows. If there are 2-handles to which the spin structure does not extend, then sliding all 2-handles but the last one over the last one, we can assume that there is a unique such handle h. Note that if we delete the cocore of h (resulting in $(D^2 \setminus \{0\}) \times D^2$), then any spin structure of the gluing region extends to this complement (since both spin structures on $\partial D^2 \times D^2$ extend to it). Capping off the cocore of h with a Seifert surface of its belt circle in the 4-handle, we obtain an embedded surface $F \subset X$ such that the spin structure of M extends to the complement $X \setminus F$. Consequently, $X \setminus \nu F$ provides a spin cobordism between M and the boundary of νF.

Next, we show that (by changing X) it can be assumed that the D^2-bundle $\nu F \to F$ is trivial and so we have a spin cobordism between M and $S^1 \times F = S^1 \times \Sigma_g$. Take the connected sum of X with the complex projective plane $\mathbb{C}P^2$. Since $\mathbb{C}P^2$ contains an embedded sphere (a complex projective line $\mathbb{C}P^1$) having normal bundle with first Chern number 1, it follows that, tubing F to this sphere, we obtain F' having normal bundle with first Chern number 1 higher than F did. (Note that the spin structure obviously extends to the complement of this sphere in $\mathbb{C}P^2$, the complement being a disk.) Now, using $\overline{\mathbb{C}P^2}$ (which is $\mathbb{C}P^2$ with the opposite orientation) and the same argument, we lower the Chern number of the normal bundle of F. Choosing an appropriate orientation and repeating the process, we turn the Chern number of the normal D^2-bundle into 0, i.e., make it trivial. This completes the proof. □

Lemma 2.3. *$S^1 \times \Sigma_g$ (with any spin structure) is spin cobordant to the disjoint union of several 3-tori (with various spin structures on them).*

Proof. We show that Σ_g (with any spin structure) is spin cobordant to the disjoint union of some 2-dimenional tori (with various spin structures on them). This fact easily implies the above lemma: The spin structure on $S^1 \times \Sigma_g$ is the product of a spin structure on Σ_g and one on S^1. Consider this spin structure on S^1 and multiply the above spin cobordism between Σ_g and $\bigcup_{i=1}^k T^2$ with it; the resulting spin cobordism between $S^1 \times \Sigma_g$ and $\bigcup_{i=1}^k T^3$ proves the lemma.

To show that Σ_g is spin cobordant to $\bigcup_{i=1}^k T^2$, we either use the fact that the group $\Omega_2^{\mathrm{Spin}} = \mathbb{Z}_2$ is generated by the torus T^2 with a particular spin structure (so that Σ_g either is a spin boundary or is spin cobordant to a single copy of T^2), or argue as follows: Σ_g is cobordant to the disjoint union of g tori, and the cobordism can be constructed on $[0, 1] \times \Sigma_g$ by attaching 2-handles along the circles separating Σ_g into g copies of T^2. Now, no matter what the spin structure on Σ_g is, these circles inherit the bounding spin structure on S^1, and so the spin structure of $[0, 1] \times \Sigma_g$ extends to the cobordism, which proves that Σ_g with the given spin structure is spin cobordant to the disjoint union of g tori (with various spin structures on them). □

The next lemma already completes the proof of our main Theorem 1.1. Since the proof of Lemma 2.4 below consists of several steps, we first prove Theorem 1.1 and then return to the proof of the lemma.

Lemma 2.4. *The 3-dimensional torus T^3 with any of its eight spin structures is a spin boundary.*

Proof of Theorem 1.1. Once we have proved Lemma 2.4, the proof of Theorem 1.1 is obviously complete because by Lemmas 2.2 and 2.3 we have

$$[M, s] = [\Sigma_g \times S^1, \tilde{s}] = \sum_{i=1}^{k} [T^3, s_i] \quad \text{in} \quad \Omega_3^{\mathrm{Spin}}$$

(where s, $\tilde{s}$, and s_i are various spin structures on the corresponding manifolds), and by Lemma 2.4 this expression vanishes. □

Below, we prove Lemma 2.4 by using only basic handlebody theory.

§3. Proof of Lemma 2.4

First, we show that T^3 bounds the 4-dimensional handlebody obtained by gluing three 2-handles along the 0-framed Borromean rings in $S^3 = \partial D^4$. This can be verified as follows: T^2 admits a handle decomposition with one 0-, two 1-, and a 2-handle (see Fig. 1(a)).

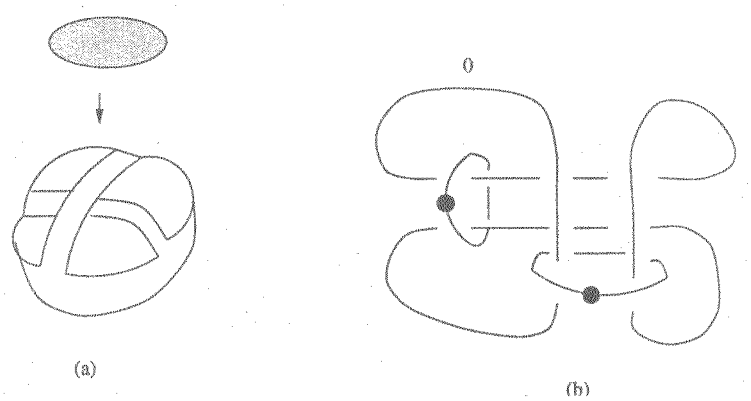

FIG. 1. (a) Handle decomposition of T^2, and (b) Kirby diagram for $T^2 \times D^2$.

Notice that the 2-handle passes through the first 1-handle, then through the second one, then returns to the first one from the opposite direction, and finally passes through the second one (from the opposite direction again). Now, multiplying everything by D^2, we obtain a handle decomposition of $T^2 \times D^2$ with boundary T^3; cf. Fig. 1(b). Surgery along the 1-handles now provides the required presentation of T^3 (see Fig. 2).

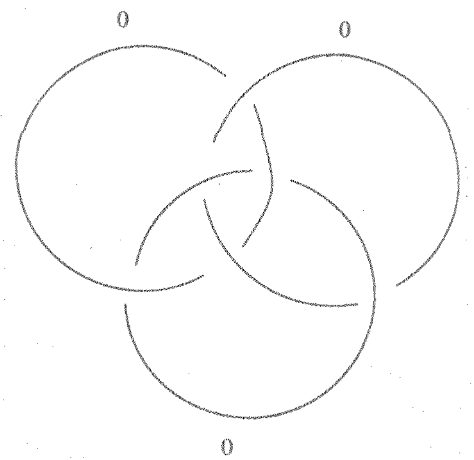

FIG. 2. Presentation of T^3.

Viewing T^3 as above, we see that the small normal circles C_1, C_2, and C_3 of the Borromean rings B_1, B_2, and B_3 represent a basis of $H_1(T^3;\mathbb{Z})$. If we glue 2-handles to $[0,1]\times T^3$ along the circles C_i, then the given spin structure of T^3 tells us whether we must use even or odd framing in order to obtain a spin cobordism. (For even framing, we always use 0, for odd one, we use -1.) Note that the eight different choices correspond to the eight possible spin structures on T^3.

Lemma 3.1. *Assume that the given spin structure on T^3 is such that one of the framings on C_1, C_2, and C_3 is 0. Then T^3 (with that given spin structure) is spin cobordant to S^3, and hence spin bounds.*

Proof. Consider the 4-manifold Y obtained by gluing 2-handles to D^4 along the 0-framed Borromean rings B_1, B_2, and B_3. As we saw above, $\partial Y = T^3$. We define X as the handlebody obtained from Y by adding 2-handles along the normal circles C_i (with the framing specified by the spin structure on T^3). Notice that $X \setminus \operatorname{int} Y$ provides a spin cobordism between T^3 and ∂X, and so we must only show that $\partial X = S^3$. (In doing this, we are free to change X as long as ∂X remains fixed.) Assume that the framing of C_1 is 0, and perform surgery on the 2-handle corresponding to C_1 (cf. the text before Definition 1.4). This surgery replaces this latter 2-handle with a 1-handle, which can be canceled by the 2-handle glued along B_1. Now B_2 and B_3 become unlinked, so that we can perform surgery along them. These surgeries result in cancelling pairs (with the 2-handles attached along C_2 and C_3), which proves that the three surgeries turn X into D^4, whence $\partial X = S^3$. Since S^3 admits a unique spin structure (which bounds the unique spin structure on D^4), we see that T^3 with the spin structure described in the lemma spin bounds a spin 4-manifold. □

§4. The end of the proof of Lemma 2.4

Finally, we analyze the case where all framings on the circles C_i are -1. First, we show that the boundary Σ of the handlebody X obtained by gluing handles along the rings B_i (with zero framing) and the circles C_i (with framing -1) is a homology sphere (hence supports a unique spin structure). Then we show that it is the boundary of a spin 4-manifold, which implies that the unique spin structure on $\Sigma = \partial X$ spin bounds. This last step completes the proof of Lemma 2.4.

Lemma 4.1. *If the framings on all circles C_i are -1, then the boundary of X is diffeomorphic to the boundary of the handlebody Z obtained by gluing a single 2-handle to D^4 along a $(+1)$-framed trefoil knot, see Fig. 3. Moreover, ∂Z is a homology 3-sphere, hence admits a unique spin structure.*

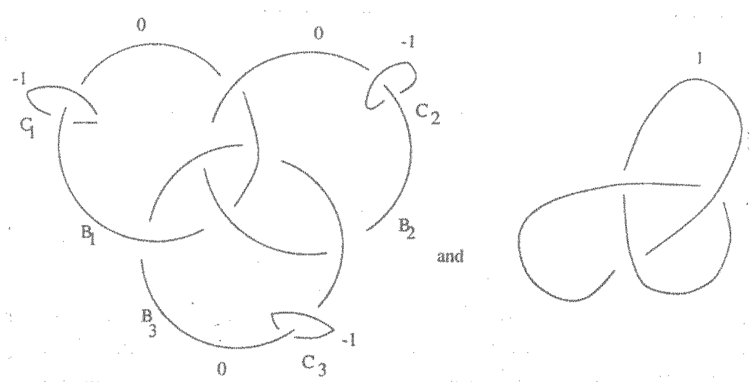

Fig. 3. Handlebodies with diffeomorphic boundaries.

Proof. Slide B_1 over C_2 twice: first use a band resulting an addition, then a subtraction of the two handles. Repeat the same process with B_1 and C_3. In this way, the resulting $\widetilde{B}_1$ is unlinked with B_2 and B_3, i.e., they can be separated by planes. (Note that algebraically B_1, B_2, and B_3 did not link each other – however, this does not mean that they could have been separated by planes.) Now surgeries on B_2 and B_3 (together with the handles glued along C_2 and C_3) create two cancelling pairs. After cancelling these pairs, we easily see that $\widetilde{B}_1$ becomes a 0-framed trefoil knot with C_1 as a (-1)-framed normal circle. Sliding C_1 off from B_1 and deleting it, we obtain the first claim of Lemma 4.1.

Finally, we show that $H_1(\partial Z;\mathbb{Z}) = 0$, i.e., ∂Z is a homology 3-sphere. To prove this, we use the fact that

$$\partial Z = (S^3 - \nu K) \cup_\varphi S^1 \times D^2,$$

where K is a knot and $\varphi\colon \partial(S^1 \times D^2) \to \partial(S^3 \setminus \nu K)$ is specified by the following property: Assume that two circles $m, l \subset S^3 \setminus \nu K$ represent a meridian and a longitude of νK, i.e., m is the boundary of a small normal disk of K, and l is a parallel copy of K with $\ell k(l, K) = 0$. Now, φ_* must satisfy $\varphi_*([\{q\} \times \partial D^2]) = [m] - [l]$ on the first homology group

$$H_1(\partial(S^1 \times D^2); \mathbb{Z}) = \langle [S^1 \times \{p\}],\ [\{q\} \times \partial D^2] \rangle$$

(because the framing of the trefoil knot in the link diagram of Z is ± 1.)

Since $H_1(S^3 \setminus \nu K; \mathbb{Z})$ is generated by the homology class $[m]$ of the meridian, an easy Mayer–Vietoris type argument concludes the proof. (Note that the above argument shows that if X^4 is given as $D^4 \cup$ (a single 2-handle with framing n), then $H_1(\partial X; \mathbb{Z})$ is the cyclic group $\mathbb{Z}_n$.) $\square$

Now we return to the proof of Lemma 2.4: The 3-manifold Σ obtained by (+1)-surgery on the trefoil knot is the link of the singularity

$$\{x^2 + y^3 + z^5 = 0\} \subset \mathbb{C}^3$$

(cf., e.g., [4]) and hence (as a submanifold of S^5) it obviously bounds a spin 4-manifold: its Seifert surface. Consequently, Σ with its unique spin structure bounds a spin 4-manifold, which completes the proof of Lemma 2.4. (See Remark 5.2 for a detailed proof of this last statement.) Here we just note here that the above 3-manifold Σ is, in fact, diffeomorphic to the famous Poincaré homology 3-sphere. (See [4] for more about this 3-manifold.) $\square$

§5. Concluding remarks

Remark 5.1. Note that in the above proof we showed that T^3 with a spin structure s is spin cobordant to a 3-manifold M admitting a unique spin structure, and then (by finding the diffeomorphism type of M) we showed that M was a spin boundary. (For seven choices of s, the corresponding 3-manifold M is S^3, while for the latter spin structure, M is the Poincaré homology sphere Σ.) However, the first step of the program easily extends to arbitrary spin 3-manifolds, and so we see that a spin 3-manifold M is spin cobordant to a 3-manifold N with $H^1(N; \mathbb{Z}_2) = 0$, i.e., N admits a unique spin structure.

Remark 5.2. For the sake of completeness, we sketch the proof of the fact that Σ (given as the boundary of a handlebody with a unique 2-handle glued along the (+1)-framed trefoil) bounds a spin 4-manifold. More precisely, we show that the boundary of the 4-manifold X obtained by gluing eight 2-handles along the (+2)-framed knots as given in Fig. 4 is diffeomorphic to Σ.

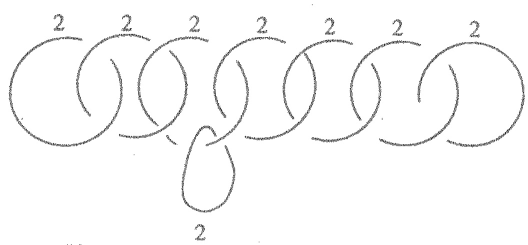

Fig. 4. Spin 4-manifold with boundary Σ.

This X is spin since all framings are even, hence the unique spin structure of D^4 extends over each 2-handle. (Essentially the same handlebody calculation can be found in [4].) Add to X a 2-handle along a (-1)-framed unknot. (Recall that this operation changes X, but leaves its boundary fixed.) Sliding the latter 2-handle over the rightmost (+2)-framed handle, we turn our presentation as it is shown by Fig. 5.

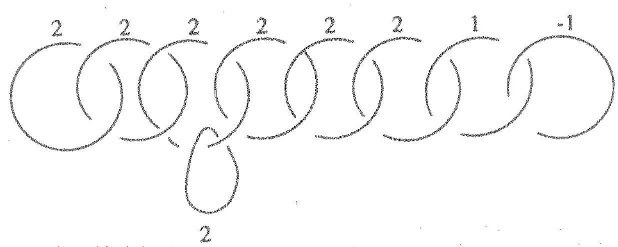

Fig. 5. Result of a handle-slide.

We observe that the framing of the (+2)-framed circle becomes +1 after this slide; consequently, it can be deleted. However, when deleting it, we change the framings of the linking circles (they decrease by 1); furthermore, the circles become linked. (Verify this by sliding both handles over the $*(+1)$-framed one and then delete the now separated (+1)-framed unknot.) This means that we can repeat this process and delete the (+1)-framed unknots (which decreases the framings of all linking circles). At the end of the process, we obtain the 4-manifold given by Fig. 6 – with the same boundary Σ.

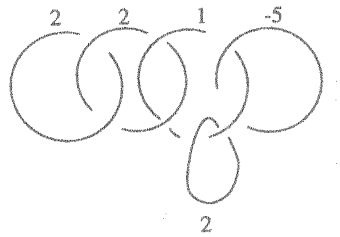

FIG. 6. An intermediate stage.

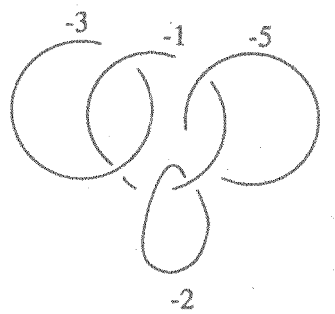

FIG. 7. An intermediate stage.

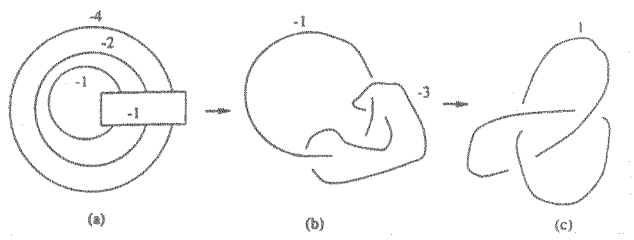

FIG. 8. The final handleslides.

After the same process has been applied to the leftmost and the bottom (+2)-circles, we obtain Fig. 7. Now, after sliding the handle corresponding to the (−1)-framed circle to all of the three remaining ones, the (−1)-circle becames separated and can be deleted without changing the boundary of the handlebody. The resulting diagram is shown in Fig. 8(a); repeating the same process with the (−1)-framed unknot of this diagram, we obtain Fig. 8(b), which represents a handlebody still with the same boundary. Now, sliding the (−1)-framed unknot of the latter diagram twice over the remaining handle and then deleting it, we arrive at Fig. 8(c), which proves the assertion.

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