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CONSTRUCTIONS OF FIBRED KNOTS AND LINKS*

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Introduction. In this paper we consider only polyhedral, that is, nonwild, situations.

In the oriented 3-sphere S^3, let T be a compact, connected, oriented surface with nonempty boundary Bd T. Let T^+ be a copy of T in $S^3 - T$ parallel to T. If the map $\pi_1(T^+) \to \pi_1(S^3 - T)$ is an isomorphism, we call T a *fibre surface*, and its boundary Bd T a *fibred link*. The reason for this language is that, given the condition on the fundamental groups, $S^3 -$ Bd T is the total space of a fibre bundle with base space the circle and fibre the interior of T [1]. A fibred link of only one component is called a fibred knot or Neuwirth knot [2].

It is known that the Alexander polynomial $A(t)$ of a fibred knot has degree equal to twice the genus of the corresponding fibre surface, and that it has leading coefficient 1 [3]; of course, also, $A(t)$ satisfies a symmetry condition. Every possible such Alexander polynomial occurs as the polynomial of some fibred knot [4]. For a fibre surface T, the translation of the fibre around the base-space circle determines an element of the mapping-class group of T, a homeomorphism $h: T \to T$ well defined up to isotopy; this element is called the *holonomy* of the fibre surface; the Alexander polynomial is the characteristic polynomial of the map the holonomy induces on $H_1(T)$. It is also known [5] that if the leading coefficient of the Alexander polynomial of an *alternating* knot is 1, then the knot is fibred.

The links which occur as isolated singularities of algebraic surfaces, certain compound torus links, are known to be fibred [6]; these are special cases of a closed positive braid, whose Alexander polynomial was found to have leading coefficient 1 [7], and which we shall show are fibred.

This paper discusses several methods of creating fibre surfaces, including plumbing, twisting, and companionization.

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Plumbing. This is a generalization of the technique described by Murasugi [5].

Consider two oriented fibre surfaces T_1 and T_2. On T_i let D_i be 2-cells, and let $h: D_1 \to D_2$ be an orientation-preserving homeomorphism such that the union of T_1 and T_2 identifying D_1 with D_2 by h is a 2-manifold T_3. That is to say:

$$h(D_1 \cap \text{Bd } T_1) \cup (D_2 \cap \text{Bd } T_2) = \text{Bd } D_2.$$

We can realize T_3 in S^3 as follows: Thicken D_1 on the positive side of T_1, to get a 3-cell, whose complementary 3-cell E_1 contains T_1 with $T_1 \cap \text{Bd } E_1 = D_1$ and with the negative side of T_1 contained in the interior of E_1. Likewise, there is a 3-cell E_2 containing T_2, with $T_2 \cap \text{Bd } E_2 = D_2$ and with the positive side of T_2 contained in the interior of E_2. The homeomorphism $h: D_1 \to D_2$ extends to $h: \text{Bd } E_1 \to \text{Bd } E_2$. The union of E_1 and E_2, identifying their boundaries by h—this is S^3—contains T_3 as $T_1 \cup T_2$. We say T_3 is obtained from T_1 and T_2 by *plumbing*.

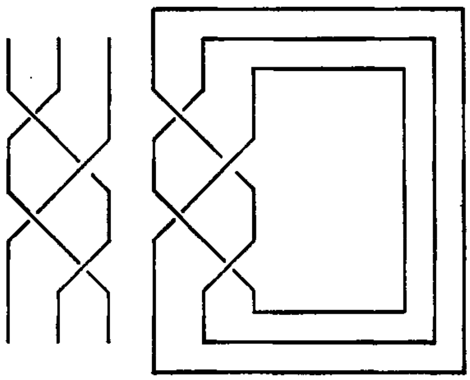
THEOREM 1. *If T_1 and T_2 are fibre surfaces, so is T_3.*

The proof can be found by examining the map on fundamental groups. We can identify

$$\pi_1(T_3) \approx \pi_1(T_1) * \pi_1(T_2),$$
$$\pi_1(S^3 - T_3) \approx \pi_1(S^3 - T_1) * \pi_1(S^3 - T_2).$$

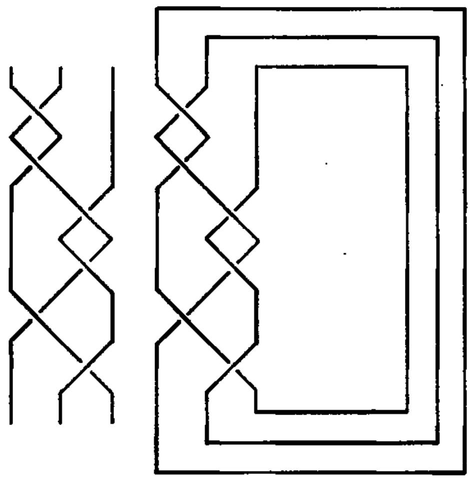
The map on the second factor is that which we would expect; on the first factor it is slightly different, the image elements of a particular basis of $\pi_1(T_1)$ being multiplied on the left and right by certain elements of $\pi_1(S^3 - T_2)$.

A special interesting case concerns braids [8]. A braid of n strands can be expressed as a word in generators $\sigma_1, \cdots, \sigma_{n-1}$, where σ_i is the braid involving a single



$\sigma_1^{-1}\sigma_2\sigma_1^{-1}\sigma_2 = \beta$ $\hat{\beta}$, the knot 4_1

FIGURE 1



$\sigma_1^{2}\sigma_2^{2}\sigma_1\sigma_2^{-1} = \beta$ β, the knot 5_2

FIGURE 2

crossing of the ith and $(i + 1)$st strands. If $\beta = \sigma_{i_1}^{\varepsilon_1} \sigma_{i_2}^{\varepsilon_2} \cdots \sigma_{i_k}^{\varepsilon_k}$, $\varepsilon_j = \pm 1$ has the two properties—(a) every σ_i occurs at least once, (b) for each i, the exponents of all occurrences of σ_i are the same—then we call β *homogeneous*. For example, if all ε_i are $+1$, we have the positive braids studied in [7]. The braid $\sigma_1\sigma_2^{-1}\sigma_1\sigma_2^{-1}$ is homogeneous; the braid $\sigma_1^2\sigma_2^2\sigma_1\sigma_2^{-1}$ is not homogeneous—see Figures 1 and 2.

Given any braid β, we can close it up to obtain a closed braid $\hat{\beta}$. There is an oriented surface T_β whose boundary is $\hat{\beta}$, obtained as the union of n disks, one for each strand, where the ith and $(i + 1)$st disks are joined by a number of half-twisted strips, one for each occurrence of σ_i in β. Then T_β is obtained by plumbing a series of surfaces $T_1, T_2, \cdots, T_{n-1}$, where T_i consists of the ith and $(i + 1)$st disks with the connecting half-twists. If β is homogeneous, the half-twists in T_i are all in the same sense, so that T_i looks like Figure 3 or its mirror image. A direct computation shows that the surfaces in this figure are fibred. Thus

THEOREM 2. *If β is a homogeneous braid, then $\hat{\beta}$ is a fibred link with fibre surface T_β.*

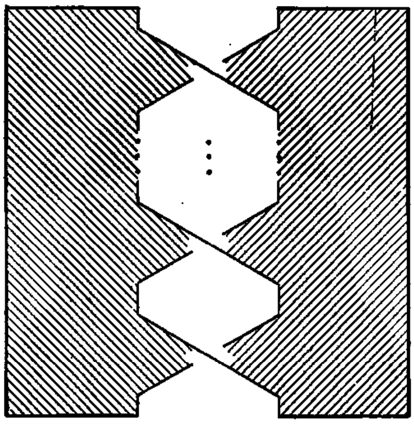


FIGURE 3

This has a curious corollary. If L is any link, it can be represented as $\hat{\beta}$ for some (nonhomogeneous) braid β. Now, by adding to the picture for β several other strands, we can isolate the positive from the negative crossings of β so that they are located on different vertical strata. The new strands can be crossed over each other, so that in the closed form they will represent a single unknot. Furthermore, we can arrange it so that this circle has arbitrarily prescribed linking numbers with the components of L. Figure 4 applies this to the braid of Figure 2.

THEOREM 3. *Given any link L in S^3, there is an unknot K disjoint from L, with arbitrarily prescribed linking numbers with the components of L, such that $K \cup L$ is a fibred link.*

By choosing the linking numbers carefully (making their sum $= 1$), we can do Dehn surgery [9] on K to obtain the 3-sphere again. This surgery will be compatible with the fibration, and thus *any link can be transformed into a fibred link by a single Dehn surgery.*

Twisting. Suppose T is a fibre surface and C is a simple closed curve on T, such that C is unknotted in S^3, and so that C bounds a disk D which is orthogonal to T along C. This latter condition is equivalent to C and C^+ having linking number

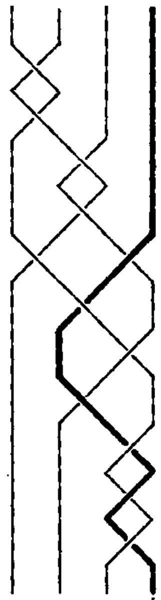


FIGURE 4

zero. Let A be a thickening of C along the side of T where D starts. The complement of A is a donut $S^1 \times D$ containing T. Let $\tau: S^1 \times D \to S^1 \times D$ be a homeomorphism, a twist along D. Look at $\tau(T)$; the fibring of $S^3 - \text{Bd } T$ contained in $S^1 \times D$ fits up, after τ has been applied, to that in A. Thus

THEOREM 4. *$\tau(T)$ is a fibre surface.*

The holonomy of $\tau(T)$ is the composition of the holonomy of T with a Lickorish twist [10] in the neighborhood of C.

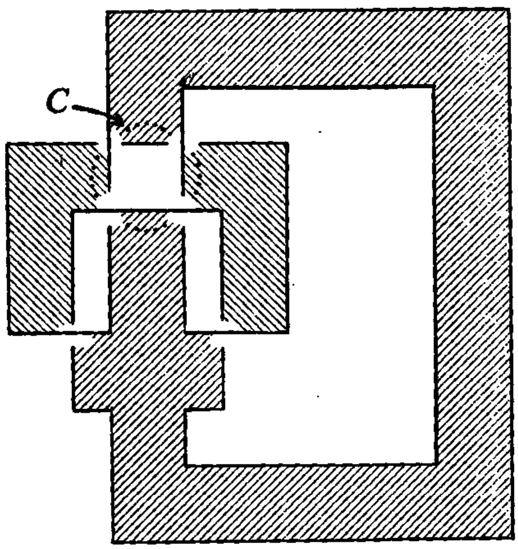


FIGURE 5

As an interesting example, the surface in Figure 5 is a fibre surface, and C is a curve along which such twists are permissible, leading to the fibre surfaces T_n described in Figure 6.

The knots K_n which are the fibred knots of Figure 6 all have the same Alexander polynomial $(t^2 - t + 1)^2$, but they can be distinguished by the fact that if M_n is

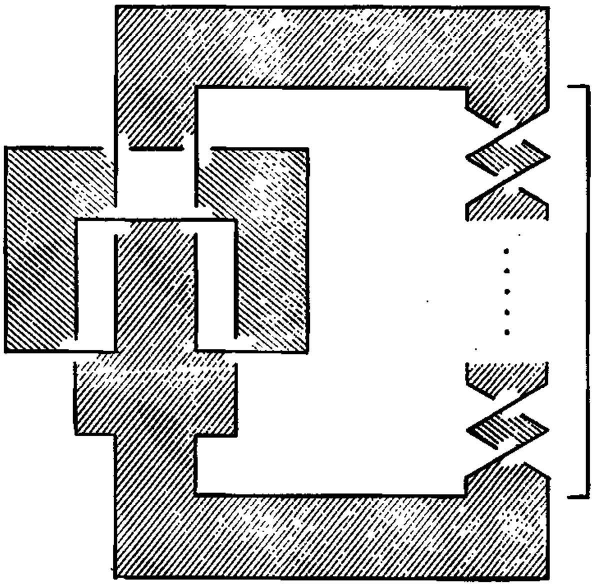


FIGURE 6

the Alexander matrix of K_n, describing the holonomy on the homology of T_n, then $M_n^2 - M_n + I$ has n as an elementary divisor. This shows M_n and M_k for $|n| \neq |k|$ are dissimilar, and so K_n and K_k are inequivalent.

Companionization. Suppose that in $S^1 \times D^2 = A$, there is a link L and an oriented surface T, such that Bd $T = L$, plus n longitudes, $n \geqq 1$, all oriented coherently (of the form $S^1 \times$ boundary point), and such that $A - L$ fibres over S^1 with fibre $T - L$. We would describe T as a fibre surface within $S^1 \times D$. Now, suppose A is embedded in S^3 via a knot K, in such a way that the longitudes in Bd T are "longitudes" of K, i.e., null-homologous in $S^3 - A$.

THEOREM 5. *If K is a fibred knot, $A = S^1 \times D$ is knotted via K, and L is, within $S^1 \times D$, a fibred link whose fibre T as above intersects* Bd A *in n longitudes to K, then L is fibred within S^3. The corresponding fibre surface consists of a fibre T of $A - L$ plus n fibre surfaces of K.*

This is geometrically obvious, but can also be shown from fundamental-group considerations. In Schubert's terminology [11], K is a companion of L. A particular case of this [12] is cabling, in which L is a torus knot on a torus parallel to the boundary of A; if we cable a fibred knot K, we obtain a new fibred knot L.

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