ABSTRACT. We define matrix representations of Artin groups over a 2-variable Laurent-polynomial ring and show that in the rank 2 case, the representations are faithful. In the special case of Artin’s braid group, our representation is a version of the Burau representation and our faithfulness theorem is a generalization of the well-known fact that the Burau representation of $B_3$ is faithful.

In [4], Brieskorn and Saito coined the phrase “Artin groups” to denote a certain class of groups, defined by generators and relations, which stand in relationship to arbitrary Coxeter groups much as Artin’s braid group $B_n$ [1] stands in relationship to the symmetric group $S_n$. One of the nice features of Coxeter groups is that they have “standard” representations [6] as groups of matrices over the real numbers preserving a suitably defined bilinear form and that, moreover, these representations are faithful (see [3]). Our purpose here is to show the existence of analogous matrix representations of Artin groups over Laurent-polynomial rings preserving similarly defined sequilinear forms. Unfortunately, except in the simplest cases, the question of faithfulness of these Artin group representations remains open.

In §1, we define Artin groups $G_M$ (by representation), a Hermitian form $J$, and unitary reflections for each given generator of $G_M$; these are defined using a given Coxeter matrix $M$. In §2, we show that the reflections associated to generators of $G_M$ define a matrix representation of $G_M$ (Theorem 1) and that when the presentation of $G_M$ involves 2 generators, this representation is faithful (Theorem 2). We note that in the special case of the braid groups our representation is a version of the Burau representation ([5] or see [2]). The results below are first, a generalization to arbitrary Artin groups of the author’s observation [10] that the Burau representation of $B_n$ is unitary and second, a generalization to arbitrary rank 2 Artin groups of the well-known fact (see [9 or 2]) that the Burau representation of $B_3$ is faithful.

1. Definitions. Let $n$ be a positive integer. A (rank $n$) Coxeter matrix $M$ will be an $n \times n$ symmetric matrix $M = [m(i,j)]$ each of whose entries $m(i,j)$ is a positive integer or $\infty$ such that $m(i,j) = 1$ if and only if $i = j$. Out of a Coxeter matrix $M$, we shall build some presentations and some forms.

To define the presentations, let $X = \{x_1, \ldots, x_n\}$ be a finite set. For $m$ a positive integer, define the symbol $\langle xy \rangle^m$ by the formula

$$\langle xy \rangle^m = \begin{cases} (xy)^k & \text{if } m = 2k, \\ (xy)^k x & \text{if } m = 2k + 1. \end{cases}$$

Received by the editors February 5, 1987.

1980 Mathematics Subject Classification (1985 Revision). Primary 20F36; Secondary 20H10.

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0002-9939/88 $1.00 + $.25 per page

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Let $M$ be an $n \times n$ Coxeter matrix. $G_M$ will denote the abstract group defined by generators $X = \{x_1, \ldots, x_n\}$ and relations all $\langle x_i x_j \rangle^{m(i,j)} = \langle x_j x_i \rangle^{m(i,j)}$ for $1 \leq i < j \leq n$. Throughout, the case $m(i,j) = \infty$ will stand for “no relation”. $G_M$ is the Artin group determined by $M$. $W_M$ will denote $G_M$ modulo the addition relations all $x_i^2 = 1$. Note that in the presence of the relations $x_i^2 = 1$, the defining relations of $G_M$ take the form $(x_i x_j)^{m(i,j)} = 1$. $W_M$ is called the Coxeter group determined by $M$. For the basic properties of Coxeter groups, see [3 or 6]. For a study of Artin groups and their relationship to Coxeter groups, see [4].

We define a symmetric bilinear form $J_1$ associated to $W_M$ and a Hermitian form $J$ associated to $G_M$. To motivate the definitions of $J$, we begin by recalling the (well-known—see [3]) definition of $J_1 = J_1(M)$: $J_1$ is the $n \times n$ matrix $[c_{ij}]$ where $c_{ij} = -2 \cos(\pi/m(i,j))$. Here, we adopt the convention that $\pi/\infty = 0$ so that if $m(i,j) = \infty$ then $c_{ij} = -2$. Note that each $c_{ii} = 2$. Let $V$ denote an $n$-dimensional vector space over $\mathbb{R}$ with basis $\{e_1, \ldots, e_n\}$. Identify each $v \in V$ with the column vector consisting of the coordinates of $v$ with respect to the basis $\{e_1, \ldots, e_n\}$ of $V$. With this convention, if $v \in V$, let $v'$ denote the transpose of $v$ and, for $u, v \in V$, define $\langle u, v \rangle_1 = u' J_1 v$. Thus, $J_1$ defines a symmetric bilinear form on $V$. We use $J_1$ to define a matrix representation $\rho_1$ of $W_M$ on $V$: if $v \in V$ and $x \in X$ define

$$(\rho_1(x_i))(v) = v - (e_i, v) e_i.$$ 

It is well known (again see [3]) that $\rho_1$ is a faithful linear representation of $W_M$.

To define $J$, let $\Lambda$ denote the Laurent-polynomial ring $\mathbb{R}[s, s^{-1}, t, t^{-1}]$, where $s$ and $t$ are indeterminates over $\mathbb{R}$. Define $J = J(M)$ to be the $n \times n$ matrix $[a_{ij}]$ over $\Lambda$, where

$$a_{ij} = \begin{cases} 
-2s \cos(\pi/m(i,j)), & i < j, \\
1 + st, & i = j, \\
-2t \cos(\pi/m(i,j)), & i > j.
\end{cases}$$

Note that $J_1$ may be obtained from $J$ by substituting $s = t = 1$.

To define analogues of the representation $\rho_1$ of $W_M$ defined above, we introduce an analogue of complex conjugation in the Laurent-polynomial ring $\Lambda$: if $x \in \mathbb{R}$ then, as usual, $\bar{x} = x$; also, $\bar{s} = s^{-1}$ and $\bar{t} = t^{-1}$, extended to $\Lambda$ additively and multiplicatively. Note that if complex numbers of norm 1 are substituted for $s$ and $t$ then we recover ordinary complex conjugation.

We extend the definition of conjugation to matrices entrywise and, if $A$ is a matrix over $\Lambda$, we define $A^* = \overline{A}$. For example, note that $J^* = s^{-1} t^{-1} J$.

Let $V$ denote a free $\Lambda$-module with basis $\{e_1, \ldots, e_n\}$ and, as above, identify each $v \in V$ with its column vector of coordinates. If $u, v \in V$ define $\langle u, v \rangle = u^* J v$. Finally, we define $\rho$: if $v \in V$ and $x_i \in X$ define

$$\langle \rho(x_i)(v) \rangle = v - \langle e_i, v \rangle e_i.$$ 

We shall see below that $\rho$ provides a matrix representation of the Artin group $G_M$.

Note that $\langle \rho(x_i)(v) \rangle(v - s^{-1} t^{-1} \langle e_i, v \rangle e_i) = v$. It follows that each $\rho(x_i)$ acts invertibly on $V$. In fact, each $\rho(x_i)$ is a pseudo-reflection in the sense of [3]. Also, for each $x_i \in X$ and each $u, v \in V$, we have

$$\langle \rho(x_i)(u), \rho(x_i)(v) \rangle = \langle u, v \rangle.$$ 

Combining this observation with Theorem 1 below, we conclude that $\rho$ is a representation of $G_M$ in a group of unitary matrices.
2. Theorems. In this section, we show that the function \( \rho \) defined (on generators) above extends to a representation of the Artin groups \( G_M \) and that when \( n = 2 \), this representation is faithful. (The second result includes the fact that the Burau representation of \( B_3 \) is faithful—see [9 or 2].)

To prove that \( \rho \) defines a representation of \( G_M \), we need to show that \( \rho \) respects the defining relations of \( G_M \). An important observation is the following

**Lemma.** \( \det J \neq 0 \).

**Proof.** In \( \det J \), the coefficients of \( (st)^n \) is 1, so \( \det J \neq 0 \). \( \square \)

In particular, \( \langle - , - \rangle \) is nondegenerate: if \( u \in V \) satisfies \( \langle u, v \rangle = 0 \) for all \( v \in V \), then \( u = 0 \).

At this point, it is convenient to introduce the field-of-quotients \( F \) of \( \Lambda \). \( F \) is a rational function field over \( \mathbb{R} \). Extend the definition of conjugation to \( F \). Letting \( V_F \) denote the \( F \)-vector space \( V \otimes_\mathbb{R} F \), extend \( \langle - , - \rangle \) to \( V_F \) and also view \( \rho \) as a linear transformation on \( V_F \). Note that since \( \langle - , - \rangle \) is nondegenerate, if \( u \in V_F \) satisfies \( u \neq 0 \), then \( u^\perp = \{ v \in V_F | \langle u, v \rangle = 0 \} \) is an \( (n-1) \)-dimensional subspace of \( V_F \). Also note that \( \rho(e_i) \) is the identity on \( e_i^\perp \). Given \( i, j \) satisfying \( 1 \leq i < j \leq n \), let \( V_{ij} \) denote the subspace of \( V_F \) spanned by \( e_i \) and \( e_j \), and let \( V_{ij}^\perp = e_i^\perp \cap e_j^\perp \). We need the following

**Lemma.** \( V_{ij} \cap V_{ij}^\perp = \{0\} \).

**Proof.** Let \( v = v_ie_i + v_j e_j \in V_{ij} \) where \( v_i, v_j \in \Lambda \). If \( v \in V_{ij}^\perp \), then \( \langle e_i, v \rangle = \langle e_j, v \rangle = 0 \) which leads to the following system of linear equations:

\[
\begin{align*}
v_i(1 + st) - 2v_j s \cos(\pi/m) &= 0, \\
-2v_i t \cos(\pi/m) + v_j (1 + st) &= 0,
\end{align*}
\]

where \( m \) denotes \( m(i, j) \). Since the determinant of the coefficient matrix is \( \neq 0 \) in \( \Lambda \), the only solution is \( v_i = v_j = 0 \), so \( v = 0 \), as required. \( \square \)

Noting that the defining relations of \( G_M \) each involve exactly two generators, in order to show that \( \rho \) respects the defining relations of \( G_M \), it suffices to show that each \( \langle x_i x_j \rangle^{m(i, j)} = \langle x_j x_i \rangle^{m(i, j)} \) holds under \( \rho \) on the subspace \( V_{ij} \) of \( V_F \).

Let \( a \) denote the matrix of \( x_i \) and \( b \) the matrix of \( x_j \) with respect to the basis \( e_i, e_j \) of \( V_{ij} \). Writing \( m \) for \( m(i, j) \), it follows that

\[
a = \begin{pmatrix}
-st & 2s \cos(\pi/m) \\
0 & 1
\end{pmatrix}, \\
b = \begin{pmatrix}
1 & 0 \\
2t \cos(\pi/m) & -st
\end{pmatrix}.
\]

Thus it suffices to prove

**Lemma.** The matrices \( a \) and \( b \) above satisfy \( (ab)^m = (ba)^m \).

**Proof.** Adjoin a square root \( q \) of \( st^{-1} \) to \( F \) and let

\[
R = \begin{pmatrix}
0 & q \\
q^{-1} & 0
\end{pmatrix}.
\]

It is easy to check that \( R^2 = I \) and \( b = RaR \). It follows that \( (ab)^m = (ba)^m \) if and only if \( (aR)^m = (Ra)^m \). Clearly, \( s^{-1}q(aR) \) and \( s^{-1}q(Ra) \) have determinant 1 and trace \( 2 \cos(\pi/m) \). It follows that \( (s^{-1}q(aR))^m = (s^{-1}q(Ra))^m = -I \), as required. \( \square \)

Thus we have the following theorem.
THEOREM 1. The function $\rho$ extends to a representation of $G_M$ in $GL_n(\Lambda)$.

PROOF. Each relation $(x_i x_j)^m(i,j) = (x_j x_i)^m(i,j)$ holds under $\rho$ on $V_{ij}$ by the lemma and therefore on all of $V_F$ since $x_i$ and $x_j$ are each the identity on $V_{ij}$. □

Except in the two-generator case, we do not know if the representation $\rho$ is faithful. Here is the proof in the two-generator case. Let $A$ and $B$, respectively, denote the matrices obtained by substituting $s = 1$ and $t = -1$ in $a$ and $b$ above.

LEMMA. The matrix group generated by $A$ and $B$ has presentation $(AB)^m = (BA)^m$ and

\[
(AB)^m = 1 \quad (m \text{ even}),
\]

\[
(AB)^{2m} = 1 \quad (m \text{ odd}).
\]

PROOF. View $A$ and $B$ as linear fractional transformations acting on the upper half-plane. Using the fact that the matrix $AB$ has determinant 1 and trace $2 \cos(\pi(1 - (2/m)))$, it follows that $AB$ satisfies $(AB)^m = (-1)^m I$. Thus, it suffices to prove that the group of linear fractional transformations generated by $A$ and $B$ has defining relations $(AB)^m = (BA)^m$ and $(AB)^m = 1$.

We prove this last fact by exhibiting the group generated by $A$ and $B$ as a subgroup of finite index in a suitable triangle group. Let $R_1, R_2$ and $R_3$ be transformations of the upper half-plane defined by

\[
R_1 = \text{reflection in the imaginary axis } x = 0,
\]

\[
R_2 = \text{reflection in the axis } x = \cos(\pi/m),
\]

\[
R_3 = \text{reflection in the unit circle}.
\]

Then $R_1, R_2$ and $R_3$ generate a $(2, m, \infty)$ triangle group with presentation (see [7]):

\[
R_1^2 = R_2^2 = R_3^2 = (R_1 R_3)^2 = (R_2 R_3)^m = 1.
\]

Noting that $R_1(z) = -\overline{z}$, $R_2(z) = -\overline{z} + 2 \cos(\pi/m)$ and $R_3(z) = 1/\overline{z}$, it follows that, as linear fractional transformations, $A = R_2 R_1$ and $B = R_3 R_1 R_2 R_3$. It can be checked that the subgroup of the triangle group generated by $A$ and $B$ is normal and has index 2 when $m$ is odd and index 4 when $m$ is even. A routine application of the Reidemeister-Schreier algorithm produces the required presentation of the group generated by $A$ and $B$. □

THEOREM 2. The group of matrices generated by $a$ and $b$ has presentation $(ab)^m = (ba)^m$.

PROOF. By the Lemma, the substitution produces a group with a presentation consisting of the desired relation together with a further relation $c = 1$ where $c = (ab)^m$ when $m$ is even and $c = (ab)^{2m}$ when $m$ is odd. In either case, $c$ is a central element in the group defined by $(ab)^m = (ba)^m$. It follows that any additional relation between $a$ and $b$ must be a nonzero power of $c$. But any nonzero power of $c$ has determinant a nonzero power of $-st$ and is therefore not the identity. Thus the matrix group generated by $a$ and $b$ has presentation $(ab)^m = (ba)^m$, as desired. □
3. Remarks. The (reduced) Burau representation of $B_n$ (see [2]) may be obtained by substituting $s = 1$ in the representation $\rho$ of $B_n$ that arises above. In fact, the representation $\rho$ itself is equivalent to the Burau representation: it is possible to conjugate the image of $\rho$ by a diagonal matrix that, in each $\rho(x_i)$, "moves the t's above the diagonal" and "leaves the s's alone". The matrices that result have the property that their entries depend only on the product $st$. A similar conjugation is possible whenever the Coxeter graph $\Gamma_M$ of $M$ is a forest ($\Gamma_M$ has vertices $X$ and an edge connecting $x_i$ and $x_j$ provided $m(i,j) \geq 3$). In these cases, the representations $\rho$ of $G_M$ is conjugate to a representation over the Laurent-polynomial ring $\mathbb{R}[st, (st)^{-1}] \subseteq \Lambda$. In the case of $B_n$, the representation that results is the Burau representation.

In general, the question of the faithfulness of $\rho$ remains open. The only known cases seem to be those that follow easily from Theorem 2: $G_M$ is a direct product of rank 1 or 2 Artin groups (equivalently, $\Gamma_M$ is a disjoint union of vertices and pairs of vertices connected by an edge). Much effort has been devoted (unsuccessfully) to trying to determine whether or not the Burau representation of $B_4$ is faithful. One other case that might be worth investigating is $M$ defined by each $m(i,j) = \infty$, so that $G_M$ is a free group.

REFERENCES