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THE CLASSIFICATION OF IMMERSIONS OF SPHERES IN EUCLIDEAN SPACES

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Introduction

This paper continues the theory of [6] and [7]. See these papers for further information on the problem as well as Chern [1]. See also [8] for a partial summary of the results in this paper. For the most part, however, we do not depend on our previous results.

An immersion of one differentiable manifold (all manifolds will be of class C^∞) M^k in a second V^n, $n > k$, is a regular map (a C^1 map with Jacobian of rank k) of M^k into V^n. A homotopy $f_t : M^k \to V^n$ is a *regular homotopy* if at each stage it is regular and the induced homotopy of the tangent bundle is continuous. We are concerned with the problem of classifying immersions with respect to regular homotopy. Except for a few comments, we restrict ourselves to where M^k is the k-sphere S^k and V^n is euclidean n-space E^n. Then we are able to provide a solution to this classification problem, at least in terms of homotopy groups of Stiefel manifolds. The result may be stated as follows.

Let $V_{n,k}$ be the Stiefel manifold of all k-frames of E^n (not necessarily orthonormal frames) and $F_k(S^k)$ the bundle of all k-frames of S^k. Then an immersion of $f : S^k \to E^n$ induces a map $f_* : F_k(S^k) \to V_{n,k} \times E^n$. Let $x_0 \in F_k(S^k)$ and $y_0 \in V_{n,k} \times E^n$ be fixed. We shall say an immersion $f : S^k \to E^n$ is a *based immersion* if $f_*(x_0) = y_0$. A *based regular homotopy* is a regular homotopy which at each stage is a based immersion. For any two based immersions f and g, an invariant $\Omega(f, g) \in \pi_k(V_{n,k})$ is defined as follows. Given based immersions $f, g : S^k \to E^n$, by a small regular homotopy of g, f and g can be made to agree on a neighborhood U of $q(x_0)$ which is diffeomorphic to a closed k-disk. Here $q : F_k(S^k) \to S^k$ is the bundle projection. The space $D = (S^k\text{-interior } U)$ is a topological k-disk so we can assume there is a fixed field of k-frames defined over it. From this field f and g induce maps f_* and g_* of D into $V_{n,k}$ which agree on the boundary of D. Consider D as a hemisphere of the k-sphere S^k and reflect g_* to the opposite hemisphere to obtain a map of S^k into $V_{n,k}$. The homotopy class of this map is denoted by $\Omega(f, g) \in \pi_k(V_{n,k})$.

THEOREM A. *If f and g are C^∞ based immersions of S^k in E^n they are based regularly homotopic if and only if $\Omega(f, g) = 0$. Furthermore, let*

$\Omega_0 \in \pi_k(V_{n,k})$ and let a based C^∞ immersion $f : S^k \to E^n$ be given. Then there exists a based immersion $g : S^k \to E^n$ such that $\Omega(f, g) = \Omega_0$. Thus there is a 1–1 correspondence between elements of $\pi_k(V_{n,k})$ and based regular homotopy classes of immersions of S^k in E^n.

If $n > k + 1$, $\Omega(f, g)$ can be defined for non-based immersions and Theorem A is true omitting the word based wherever it occurs. Theorem A is a direct generalization of Theorem A of [7] where $k = 2$. The case of Theorem A for $k = 1$ is included in my thesis [6]. See these papers for implications of Theorem A when $k = 1$, or 2. Many of the groups $\pi_k(V_{n,k})$ have been computed. See Paechter [5]. Since $\pi_k(V_{n,k}) = 0$ for $n \geqq 2k + 1$, Theorem A implies the following.

THEOREM B. *Two C^∞ immersions of S^k in E^n are regularly homotopic when $n \geqq 2k + 1$.*

Whitney in [9, p. 220] posed the question : Are two immersions of a manifold M^k in E^{2k} regularly homotopic if they have the same intersection number I_f ? We assume in this and the next two paragraphs that the immersions are nice enough to define an intersection number. (Whitney [9] defines I_f to be number of self-intersections of an immersion $f : M^k \to E^{2k}$ counted properly when $f(M^k)$ intersects itself only in isolated points. I_f is an integer if k is even and an integer mod 2 if k is odd, $k > 1$). We can obtain from Theorem A the following.

THEOREM C. *Two C^∞ immersions of S^k in E^{2k}, $k>1$, are regularly homotopic if and only if they have the same intersection number I_f.*

If k is even, then according to [3], $\overline{W}_k(f) = 2I_f$ where $\overline{W}_k(f)$ is the k^{th} normal Stiefel-Whitney class with integer coefficients of the immersion f. Thus in this case $\overline{W}_k(f)$ characterizes the regular homotopy class of f.

The based regular homotopy classes of based immersions of S^k in E^{k+1} correspond to the elements of $\pi_k(R_{k+1})$ where R_n is the rotation group on E^n(since $R_{k+1} = V_{k+1,k}$). To study further the situation of immersions of S^k in E^{k+1} recall Milnor's notion [4] of normal degree. If $f : M^k \to E^{k+1}$ is an immersion of closed oriented manifold let $\bar{f} : M^k \to S^k$ be the map obtained by translating the unit normal vector at a point of $f(M^k)$ in E^{k+1} to the origin. The normal degree N_f of f is the degree of $\bar{f}$.

Milnor [4] asks the question : for what k can S^k be immersed in E^{k+1} with normal degree zero ? He proved that for this to be true S^k must be parallelizable and he proved that S^3 could be immersed in E^4 with normal degree zero.

THEOREM D. *There exists an immersion of S^k in E^{k+1} with normal degree zero if S^k is parallelizable.*

The question of regular extensions is closely related to that of regular homotopy. In particular, we are interested in the following problem: Suppose $f: S^{k-1} \to E^n$ is an immersion where S^{k-1} is the boundary of the k-disk D^k. When can f be extended to an immersion of D^k ? From H. Whitney's work one obtains an affirmative answer whenever $n \geqq 2k$. The following theorems give an answer to this question under the restriction $n > k$.

THEOREM E. *If $n > k$ an immersion $f: S^{k-1} \to E^n$ can be extended to an immersion of D^k if and only if $\Omega(f, e) = 0$ where $e: S^{k-1} \to E^k \subset E^n$ is the standard unit sphere in a k-plane of E^n*

In a certain sense the results of this paper are local in nature. M. Hirsch using these results together with obstruction theory has proved theorems on the regular homotopy classification of manifolds instead of spheres. He also obtains some sufficient conditions for manifolds to be immersible in euclidean space. For example, he proves every closed 3-manifold can be immersed in E^4 [2].

The above results suggest the following questions:

(1) One problem is to replace E^n in Theorem A by an arbitrary n-manifold M^n. I believe one would get a classification of immersions of S^k in M^n in terms of $\pi_k(F_k(M^n))$ where $F_k(M^n)$ is the bundle of k-frames over M^n. I don't think this will be very difficult to prove, following the proofs in this paper.

(2) Find explicit representatives of regular homotopy classes. Whitney has essentially done this for the case $n = 2k$. What regular homotopy classes have an imbedding for a representative ?

(3) Develop an analogous theory for imbeddings. Presumably this will be quite hard. However, even partial results in this direction would be interesting.

1. The covering homotopy theorem

A triple (E, p, B) consists of topological spaces E and B with a map p from E *into* B. A triple has the CHP if it has the covering homotopy property in the sense of Serre.

Let D^k be the unit k-disk in $E^k (k \geqq 1)$ with generalized polar coordinates. That is to say, points of D^k will be pairs (t, x) where t is the distance from the origin 0 of E^k and x is a point of the boundary $\dot{D}^k$ of D^k.

Let $E_{k,n} = E$ be the set of all C^∞ immersions of D^k in E^n, $n > k$. The set E is given the C^1 topology,[1] i.e., is metrized by

[1] Added in proof: Assume all function spaces have the C^2 topology instead of the C^1 topology.

$\rho(f, g) = \max \{\bar{\rho}(f(y), g(y)), \bar{\rho}(df_y(Y), dg_y(Y)) \mid y \subset D^k, Y \subset D^k_y, |Y| = 1\}$ where $f, g \in E$, $\bar{\rho}$ is the metric on E^n, E^n being considered as its own tangent vector space, and D^k_y is the tangent space of D^k at $y \in D^k$.

Let $B_{k,n} = B$ be the set of all pairs (g, g') where g is a C^∞ immersion of $\dot{D}^k$ in E^n and g' is a C^∞ cross-section in the bundle of transversal vectors of $g(\dot{D}^k)$. Thus g' is a C^∞ map of $\dot{D}^k$ into $E^n - 0$ such that $g'(x)$ does not lie in $\dot{D}^k_{g(x)}$ the tangent plane of $g(\dot{D}^k)$ at $g(x)$. The set B is given the following metric. For $(g, g'), (h, h') \in B$ let

$$\rho'[(g, g'), (h, h')] = \max\{\rho_0(g, h), \bar{\rho}(g'(x), h'(x)) \mid x \in \dot{D}^k\}$$

where $\bar{\rho}$ is as above and ρ_0 is defined as ρ above except that y is only allowed to range over $\dot{D}^k$.

A map $\pi : E \to B$ is defined as follows. For $h \in E$ let $\pi(h) = (g, g')$ where g is the restriction of h to $\dot{D}^k$ and $g'(x) = h_t(1, x)$ (the subscript t means differentiation with respect to t). The goal of this section is to prove that (E, π, B) is a fiber space in the sense of Serre.

THEOREM 1.1. *The triple* (E, π, B) *has the* CHP.

PROOF. Some of the constructions in this proof are straightforward generalizations of those of [7]. This proof is essentially independent, however, and somewhat more detail is given here than in [7].

In the article *La classification des immersions*, Seminaire Bourbaki, December 1957, R. Thom has an interesting exposition of the proof of 1.1.

A rough account of our proof is as follows.

We are given a homotopy $h^0_v : P \to B$, hence $h^0_v(p)$ for each p and v is an immersion of a sphere $h_v(p)$ with a transversal vector field $h'_v(p)$. Furthermore, $h^0_0(p)$ for each p is covered, i.e., $h_0(p)$ is the boundary of an immersed disk $\bar{h}(p) \in E$ and $h'_0(p)$ is a transversal field induced by the immersion of the disk. The problem is to follow the homotopy $h^0_v(p)$ by an immersion of a disk $\bar{h}_v(p)$.

In our construction of $\bar{h}_v(p)$, Equation (17), the factors $\alpha(t)$, $\beta(t)$, and $M(v)$ are introduced mainly so that various boundary conditions are met and the regularity of $\bar{h}_v(p)$ is preserved.

The first and last terms of (17) roughly speaking are used to take care of the transversal field part of the homotopy. In particular, the transformation $Q_v(p)$ is the principle element here. This part of the covering homotopy could be taken care of directly by an isotopy of E^n. The latter, in fact, is what Thom does.

The second term of (17), $\alpha(t)[h_v(p)(x) - h_0(p)(x)]$, is just what makes the immersion of the disk $\bar{h}_v(p)$ project onto the immersion of the sphere $h_v(p)$. However, in general the introduction of this term will cause the map of D^k to have critical points or points where the Jacobian has rank $< k$. To counteract this, the term $\beta(t)M(v)u(v, p, x)$ is added. The effect of this term is, roughly, to blow up the immersion whenever it might have become critical.

The above construction is used in [6], but in simpler form. The reader might see the idea of this proof by looking, therefore, at that paper also.

Let $h_v^0 : P \to B$ be a given homotopy where P is a cube (recall that it is sufficient to prove the CHP for cubes) and let $\bar{h} : P \to E$ cover h_0^0. We will construct a covering homotopy $\bar{h}_v : P \to E$.

We write $h_v^0(p) = (h_v(p), h_v'(p))$ (recall the definition of B).

Let $\varepsilon_1(v, p, x)$ be the distance from $h_v'(p)(x)$ to the tangent plane of $h_v(p)(\dot{D}^k)$ at x and let $\varepsilon_1 = \min\{\varepsilon_1(v, p, x) \mid v, p, x\}$. Let

$$\varepsilon_2 = \min\{|\nabla_V h_v(p)(x)| \mid x, v, p, V \in \dot{D}_x^k, |V| = 1\}$$

and take $\varepsilon = (1/10)\min\{\varepsilon_1, \varepsilon_2, 1\}$. The symbol $\nabla_V h_v(p)(x)$ means the derivative of $h_v(p)$ with respect to V at x.

We define a linear transformation of E^n, $Q_v(p)(x)$ for $p \in P$, $x \in \dot{D}^k$ and sufficiently small v (we clarify this later) as follows. Let $V_v(p)(x)$ be the 2-plane of E^n spanned by the vectors $h_0'(p)(x)$ and $h_v'(p)(x)$, if it exists, and let $\alpha_v(p)(x)$ be the angle from the first to the second of these vectors. Let $Q_v^*(p) : \dot{D}^k \to R_n$ (the rotation group) be the rotation of E^n which takes $V_v(p)(x)$ through the angle $\alpha_v(p)(x)$ and leaves the orthogonal complement fixed (if $V_v(p)(x)$ does not exist then $Q_v^*(p)(x)$ is to be the identity rotation e). Finally, define $Q_v(p) : \dot{D}^k \to GL(n, R)$ to be the rotation $Q_v^*(p)$ multiplied by the scalar $|h_v'(p)(x)| / |h_v'(p)(x)|$. We will consider $Q_v(p)(x)$ as acting on E^n on the right. It is immediate that $Q_v(p)$ is C^∞ with respect to x and that

$$(1) \qquad h_0'(p)(x)Q_v(p)(x) = h_v'(p)(x) .$$

See [7] for the following.

LEMMA 1.2. *Let $n > k \geqq 1$, $G_{n,k}$ the Grassman manifold of oriented k-planes in E^n and S^{n-1} the unit vectors of E^n. Let a map $w : Q \to G_{n,k}$ be given which is homotopic to a constant where Q is some polyhedron. Then there is a map $u : Q \to S^{n-1}$ such that for all $q \in Q$, $u(q)$ is normal to the plane $w(q)$,*

Note that if w is C^∞ we may assume that u is also.

Now apply 1.2 taking for Q, $I \times P \times \dot{D}^k$ and for $\mathrm{w}(v, p, x)$ the plane spanned by $h_v'(p)(x)$ and the tangent plane of $h_v(p)(x)$. Because $(h_0(p), h_0'(p))$

is in the image of $\pi : E \to B$, one can show w is homotopic to a constant. Thus one obtains a map $u : I \times P \times \dot{D}^k \to S^{n-1}$.

Choose $\delta > 0$ so that for $|v - v_0| \leqq \delta$ and all $p \in P, x \in \dot{D}^k$ and $V \in \dot{D}^k_x, |V| = 1$, the following conditions are satisfied.

(2) The angle between $h'_v(p)(x)$ and $h'_{v_0}(p)$ is less than 180° (this insures that $Q_v(p)(x)$ is well-defined).

(3) $$|h'_v(p)(x) - h'_{v_0}(p)(x)| < (\varepsilon/10) \min \{1, |h'_u(p)(x)| \, | u, p, x\} .$$

(4) $$|\nabla_V h_v(p)(x) - \nabla_V h_{v_0}(p)(x)| < \varepsilon/10 .$$

(5) $$|h_v(p)(x) - h_{v_0}(p)(x)| < (\varepsilon/100) \frac{1}{\max\{ |\nabla_V u(v, p, x)| | v, p, x\}} .$$

(If quantity on right of inequality of (5) is undefined, omit (5)).

It is clear that such a choice for δ may be made and that our choice of δ depends only on $h_v(p)$ and not $\bar{h}(p)$. It is easy to check that (3) implies

(3′) $$|Q_v(p)(x) - Q_0(p)(x)| < \varepsilon/10 \min \{1, |h'_0(p)(x)| \, |p, x\} \qquad v \leqq \delta.$$

We choose now t_0, $1/2 < t_0 < 1$, so that for all $t \in [t_0, 1]$, $v \leqq \delta$, $p \in P$, $V \in \dot{D}^k_x$ and $|V| = 1$, the following conditions hold.

(6) $$\left| \frac{\bar{h}(p)(t, x) - \bar{h}(p)(1, x)}{1 - t_0} \right| < |2\bar{h}_t(p)(1, x)| .$$

(See Lemma 5.1 of [6]).

(7) $$|\bar{h}_t(p)(t, x) - \bar{h}_t(p)(1, x)| < \varepsilon/10 .$$

(8) $$|\nabla_V \bar{h}(p)(t, x) - \nabla_V \bar{h}(p)(1, x)| < \varepsilon/10 .$$

(9) $$|\bar{h}(p)(t, x) - \bar{h}(p)(1, x)| < (\varepsilon/10) \frac{1}{\max \{| \nabla_V Q_v(p)(x) | | v, p, x\}} .$$

(If quantity on right of inequality of (9) is undefined, omit (9)). It is clear such a choice for t_0 may be made. Set $t_1 = t_0 + (1/3)(1 - t_0)$.

Real C^∞ functions on I, $\alpha(t)$ and $\beta(t)$ are defined satisfying the following conditions:

(10) $\alpha(t) = 0 \qquad 0 \leqq t \leqq t_1 .$

(11) $\alpha(1) = 1 \qquad \alpha'(1) = 0 .$

(12) $|\alpha(t)| \leqq 1 \qquad |\alpha'(t)| < 2/(1 - t_0) .$

(13) $\beta(t) = 0 \qquad 0 \leqq t \leqq t_0 .$

(14) $\beta(1) = \beta'(1) = 0$

(15) $|\beta'(t)| > 10 |\alpha'(t)| \qquad t_1 \leqq t \leqq 1 .$

(16) $|\beta(t)| \leqq 20 .$

Primes in this case denote the derivatives. As in [7] we leave to the reader the task of constructing such functions.

Let

$$M(v) = \max \{|h_v(p)(x) - h_0(p)(x)| \mid p \in P, x \in \dot{D}^k\} .$$

Then for $v \leqq \delta$ the desired covering homotopy $\bar{h}_v(p)$ is defined as follows.

$$\begin{aligned} \bar{h}_v(p)(t, x) &= [\bar{h}(p)(t, x) - \bar{h}(p)(1, x)]\,[e + \alpha(t)(Q_v(p)(x) - e)] \\ &\quad + \alpha(t)\,[h_v(p)(x) - h_0(p)(x)] + \beta(t)M(v)u(v, p, x) + \bar{h}(p)(1, x) . \end{aligned} \tag{17}$$

We write down the following derivatives for reference.

$$\begin{aligned} \bar{h}_{vt}(p)(t, x) &= \bar{h}_t(p)(t, x)[e + \alpha(t)(Q_v(p)(x) - e)] \\ &\quad + [\bar{h}(p)(t, x) - \bar{h}(p)(1, x)]\alpha'(t)(Q_v(p)(x) - e) \\ &\quad + \alpha'(t)[h_v(p)(x) - h_0(p)(x)] + \beta'(t)M(v)\,u\,(v, p, x) . \end{aligned} \tag{18}$$

For $V \in \dot{D}^k_x$

$$\begin{aligned} \nabla_V \bar{h}_v(p)(t, x) &= [\nabla_V \bar{h}(t, x) - \nabla_V \bar{h}(p)(1, x)]\,[e + \alpha(t)(Q_v(p)(x) - e)] \\ &\quad + [\bar{h}(p)(t, x) - \bar{h}(p)(1, x)]\alpha(t)\nabla_V Q_v(p)(x) \\ &\quad + \alpha(t)[\nabla_V h_v(p)(x) - \nabla_V h_0(p)(x)] + \beta(t)M(v)\nabla_V u(v, p, x) \\ &\quad + \nabla_V \bar{h}(p)(1, x) . \end{aligned} \tag{19}$$

We will prove that $\bar{h}_v(p)$ has the following properties.

$$\bar{h}_v(p) \text{ is } C^\infty \text{ in } x . \tag{20}$$

$$\bar{h}_0(p) = \bar{h}(p) . \tag{21}$$

$$\bar{h}_v(p)(1, x) = h_v(p)(x) . \tag{22}$$

$$\bar{h}_{vt}(p)(1, x) = h'_v(p)(x) . \tag{23}$$

$$\bar{h}_v(p) \text{ is regular} . \tag{24}$$

First we show how 20–24 imply Theorem 1.1. Properties (20) and (24) yield that the homotopy $\bar{h}_v : P \to E$ is well-defined, (21) says that $\bar{h}_v$ is a homotopy of $\bar{h}$ and (22), (23) imply that $\bar{h}_v$ covers h^0_v. Thus it only remains to prove (20–24).

Property (20) follows from the fact that all the functions used to define $\bar{h}_v(p)$ are C^∞ in x (17).

One can check (21) immediately from (17).

One can obtain (22) from (17) noting (11), (14) and that $h_0(p)(x) = \bar{h}(p)(1, x)$.

Property (23) follows from (18) using (11), (1), (14) and the fact $\bar{h}_t(p)(1, x) = h'_0(p)(x)$.

To prove the regularity of $\bar{h}_v(p)$ it is sufficient to show $\nabla_v \bar{h}_v(p)(t,x) \neq 0$ where $V \in D^k_{(t,x)}$. Then V can be written $V = V_t + V_x$ where V_x is the projection of V into $\dot{D}^k_x$ and V_t is the projection of V into the vector space orthogonal to $\dot{D}^k_x$ in $D^k_{(t,x)}$. Then

$$\nabla_V \bar{h}_v(p)(t, x) = \rho_t \bar{h}_{vt}(p)(t, x) + \rho_x \nabla_W \bar{h}_v(p)(t, x) \tag{25}$$

where W is V_x normalized and ρ_t, ρ_x are appropriate scalars. Either $\rho_t \neq 0$ or $\rho_x \neq 0$.

LEMMA 1.3. *There is a vector b' of E^n, $|b'| < \varepsilon$ (see beginning of proof of 1.1 for definition of ε) such that*

$$\nabla_W \bar{h}_v(p)(t, x) = \nabla_W \bar{h}(p)(1, x) + b' .$$

PROOF. From (12), (3′) and (8) it follows that

$$|[\nabla_W \bar{h}(p)(t, x) - \nabla_W \bar{h}(p)(1, x)][e + \alpha(t)(Q_v(p)(x) - e)]| < 2\varepsilon/10 .$$

By (12) and (9)

$$|[\bar{h}(p)(t, x) - \bar{h}(p)(1, x)]\alpha(t)\nabla_W Q_v(p)(x)| < \varepsilon/10 .$$

By (12) and (4)

$$|\alpha(t)[\nabla_W h_v(p)(x) - \nabla_W h_0(p)(x)]| < \varepsilon/10 ,$$

and finally by (16) and (5)

$$|\beta(t)M(v)\nabla_W u(v, p, x)| < 2\varepsilon/10 .$$

Then by (19) and these four inequalities we have 1.3.

LEMMA 1.4. *There exist vectors in E^n, b, $\bar{u}$, scalars Δ, Δ' where $|b| < \varepsilon$, $\Delta > 10\,\Delta'$ and $|\bar{u}| = 1$, and $u = u(v, p, x)$ such that*

$$\bar{h}_{vt}(p)(t, x) = \bar{h}_t(p)(1, x) + b + \Delta u + \Delta' \bar{u} .$$

PROOF. We can easily obtain from (18),

$$\begin{aligned}\bar{h}_{vt}(p)(t, x) = \bar{h}_t(p)(1, x) &- \bar{h}_t(p)(1, x)\alpha(t)[Q_0(p)(x) - Q_v(p)(x)] \\ &- [\bar{h}_t(p)(1, x) - \bar{h}_t(p)(t, x)][e + \alpha(t)(Q_v(p)(x) - e)] \\ &+ [\bar{h}(p)(t, x) - \bar{h}(p)(1, x)]\alpha'(t)[Q_v(p)(x) - e] \\ &+ \alpha'(t)M(v)\bar{u} + \beta'(t)M(v)u(v, p, x)\end{aligned}$$

where $\bar{u}$ is a unit vector.

Then from (12), (3′) and (7) it follows that

$$|[\bar{h}_t(p)(1, x) - \bar{h}_t(p)(t, x)][e + \alpha(t)(Q_v(p)(x) - e)]| < 2\varepsilon/10 .$$

By (12) and (3′)

$$| h_t(p)(1, x)\alpha(t)[Q_v(p)(x) - e] | < \varepsilon/10$$

and by (12), (6) and (3′) we have

$$| \alpha'(t)[\bar{h}(p)(t, x) - \bar{h}(p)(1, x)] [Q_v(p)(x) - e] | < 4\varepsilon/10 .$$

By (15) and noting that $h_0'(p)(x) = \bar{h}_t(p)(1, x)$ and $Q_0(p)(x) = e$ the above inequalities yield 1.4.

LEMMA 1.5. *Let a, b, u, $\bar{u}$, a' and b' be vectors in E^n, Δ, Δ' scalars with the following properties*

$$| b |, | b' | < (1/10)\bar{\rho}(a, sa') \qquad \textit{all real } s$$

$$| b' | < (1/10) | a' |, | \bar{u} | = 1, | u | = 1, \Delta > 10\Delta'$$

and u normal to both a and a'. Then $a + b + \Delta u + \Delta'\bar{u}$ and $a' + b'$ are linearly independent.

PROOF. If the lemma is false then there is a scalar v such that

$$v(a + b + \Delta u + \Delta'\bar{u}) = a' + b'$$

or

$$\Delta v(u + (\Delta'/\Delta)\bar{u}) = a' + b' + v(a + b) . \tag{26}$$

Since $| (\Delta'/\Delta)\bar{u} | < (1/10) | u |$, $u + (\Delta'/\Delta)\bar{u}$ has angle less than 25° from u and thus since u is normal to a and a' the term on the left of Equation 26 is at an angle of greater than 65° from the $a - a'$ plane (the case $\Delta = 0$ offers no difficulty). On the other hand $a + b$ has an angle less than 25° from a, and $a' + b'$ has an angle less than 25° from a'. This implies that the term on the right of Equation 26 has an angle from the $a - a'$ plane less than 50°. Thus Equation 26 is false, and hence 1.5 is proved.

From the last three lemmas we are now able to prove the regularity of $\bar{h}_v(p)$. By (25) it is sufficient to show that $\bar{h}_{vt}(p)(t, x)$ and $\nabla_W\bar{h}_v(p)(t, x)$ are linearly independent. This fact follows from 1.5 making the substitutions from 1.3 and 1.4, $a = \bar{h}_t(p)(1, x)$, and $a' = \nabla_W\bar{h}_v(p)(1, x)$. One also uses the definitions of ε and u to check the hypotheses of 1.5. Thus we have proved (24).

The above construction may be repeated if $\delta < 1$ using $\bar{h}_\delta(p)$ in place of $\bar{h}$. This yields a covering homotopy for $v \leqq 2\delta$. Iteration yields a covering homotopy $\bar{h}_v$ for all $v \in I$. This proves 1.1.

2. The weak homotopy equivalence theorem

Let $f_0 : D^k \to E^n$ be a C^∞ immersion, $n > k$. Denote by $\Gamma = \Gamma_{k,n}(f_0)$ the space with the C^1 topology of all C^∞ immersions f of D^k in E^n such that f agrees with f_0 on $\dot{D}^k$ and df with df_0 on the restriction of the tangent bundle of $\dot{D}^k$ to D^k. Let $x = (x_1, \cdots, x_k)$ be rectangular coordinates on $E^k \supset D^k$ and if $f: D^k \to E^n$ is an immersion denote by $f_{x_i}(x)$ the derivative of $f(x)$ along the curve x_i. Let $\overline{f}_0 : D^k \to V_{n,k}$ (the Stiefel manifold) be the map $\overline{f}_0(x) = (f_{x_1}(x), \cdots, f_{x_k}(x))$ and $\Gamma' = \Gamma'_{k,n}(\overline{f}_0)$ the space with the compact open topology of all maps of D^k into $V_{n,k}$ which agree with $\overline{f}_0$ on $\dot{D}^k$. Let Φ be the map $\Phi(p)(x) = (f_{x_1}(x), \cdots, f_{x_k}(x))$. The proof of the following theorem is the goal of this section.

THEOREM 2.1. *If $f_0 : D^k \to E^n$ is the standard immersion of S^k in a $k+1$ plane of E^n then $\Phi : \Gamma_{k,n}(f_0) \to \Gamma'_{k,n}(\overline{f}_0)$ is a weak homotopy equivalence.*

A map $f: X \to Y$ is a weak homotopy equivalence if its restriction to each arcwise connected component of X induces an isomorphism of homotopy groups and it induces a $1-1$ correspondence between arcwise connected components of X and Y.

One could probably prove 2.1 without much trouble even if f_0: $D^k \to E^n$ is an arbitrary immersion.

Let x_∞ be the South pole of $\dot{D}^{k+1}$. Then impose a coordinate system $\bar{x} = (\bar{x}_1, \cdots, \bar{x}_k)$, $\bar{x} \in \dot{D}^{k+1} - x_\infty$, on $\dot{D}^{k+1} - x_\infty$ such that $r(\bar{x}) = (\sum_1^k \bar{x}_i^2)^{1/2} = 1$ is the equator Q of $\dot{D}^{k+1}$. From now until the end of this section we identify x and $\bar{x}$, x_i and $\bar{x}_i$; identify D^k with the upper hemisphere of of $\dot{D}^{k+1}$ and $\dot{D}^k$ with Q. Let

$$A = \{(t, x) \in D^{k+1} \mid t \leqq 1/2,\ r(x) \geqq 1 \text{ or } x = x_\infty\}$$

and let $g_0 : D^{k+1} \to E^n$ be the standard inclusion into a $k+1$ plane of E^n.

A subspace $\overline{B} = \overline{B}_{k+1,n}(g_0)$ of $B_{k+1,n}(g_0)$ is defined as follows. An element (g, g') of B belongs to $\overline{B}$ if g restricted to $\dot{D}^{k+1} \cap A$ is g_0, and g' restricted to $\dot{D}^{k+1} \cap A$ is g_{0t} (the derivative of g_0 with respect to t).

Let $A' = \{(t, x) \in D^{k+1} \mid x \neq x_\infty\}$ and define $\overline{g}_0 : A' \to V_{n,k+1}$ by $\overline{g}(y) = (g_{0x_1}(y), \cdots, g_{0x_k}(y), g_{0t}(y))$, $y \in A'$.

Let $B' = B'_{k+1,n}(\overline{g}_0)$ be the space with the compact open topology of all maps of $\dot{D}^k \cap A'$ into $V_{n,k+1}$ which agree with $\overline{g}_0$ on $\dot{D}^{k+1} \cap A' \cap A$.

A map $\Psi : \overline{B} \to B'$ is defined by

$$\Psi(g, g')(x) = (g_{x_1}(x), \cdots, g_{x_k}(x), g'(x)),\ (g, g') \in B,\ x \in A' \cap \dot{D}^{k+1}.$$

Let $f_0 : D^k \to E^n$ be the restriction of g_0.

THEOREM 2.2. *If $\Phi : \Gamma_{k,n}(f_0) \to \Gamma'_{k,n}(\overline{f}_0)$ is a weak homotopy equivalence, then $\Psi : B_{k+1,n}(g_0) \to B'_{k+1,n}(\overline{g}_0)$ is also a weak homotopy equivalence* ($n > k$).

PROOF OF 2.2. Let Γ^* be the subspace of $\Gamma = \Gamma_{k,n}(f_0)$ of those immersions which have all derivatives (of all orders) agreeing with the derivatives of f_0 on $\dot{D}^k$. The restriction of Φ to Γ^* will still be denoted by Φ.

LEMMA 2.3. *The inclusion* $i : \Gamma^* \to \Gamma$ *is a weak homotopy equivalence.*

The proof of this lemma is not difficult and will be left to the reader. The idea of the proof is that any compact subset S of Γ can be deformed so that elements of S agree with g_0 on a neighborhood of the boundary of D^k.

Define maps $p : \overline{B} \to \Gamma^*$ and $p' : B' \to \Gamma'$ as follows. For $(g, g') \in \overline{B}$ let $p(g, g')$ be g restricted to $D^k =$ upper hemisphere of $\dot{D}^{k+1}$. For $f \in B', f : \dot{D}^{k+1} \cap A' \to V_{n,k+1}, f(y) = (f_1(y), \cdots, f_k(y), f_{k+1}(y))$ let $p'(f)(x) = (f_1(x), \cdots, f_k(x))$ for $x \in D^k$. It is easily checked that the p is well-defined, continuous and that the following diagram commutes.

$$\begin{array}{ccc} \overline{B} & \xrightarrow{\Psi} & B' \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p'} \\ \Gamma^* & \xrightarrow{\Phi} & \Gamma' \end{array}$$

We claim that

(1) $(\overline{B}, p, \Gamma^*)$ has the CHP,

(2) (B', p', Γ') has the CHP, and

(3) Ψ restricted to a fiber is a homeomorphism between corresponding fibers.

PROOF OF (1). Let $h_v : P \to \Gamma^*$ be a given homotopy and $\bar{h}^0 : P \to \overline{B}$ cover h^0 where P is a polyhedron. We will construct a covering homotopy $\bar{h}_v^0 : P \to \overline{B}$.

Let $q_1 : V_{n,k+1} \to V_{n,k}$ be defined by dropping the last vector of a frame and $q_2 : V_{n,k} \to G_{n,k}$ be the map which sends a k-frame into the oriented k-plane it spans. Then $q = q_2 q_1$ has the CHP.

Define $h_v^* : P \times D^k \to G_{n,k}$ so that $h_v^*(b, x)$ is the tangent plane of $h_v(b)$ at x translated to the origin. Let $\bar{h}^* : P \times D^k \to V_{n,k+1}$ be the map

$$\bar{h}^*(b, x) = (\bar{h}_{x_1}(b)(x), \cdots, \bar{h}_{x_k}(b)(x), \bar{h}'(b)(x)), \qquad x \in D^k$$

where $\bar{h}^0(b)(x) = (\bar{h}(b)(x), \bar{h}'(b)(x))$. Then since $\bar{h}^0$ covers h_0 it follows that $q\bar{h}^* = h_0^*$. By the CHP of $(V_{n,k+1}, q, G_{n,k})$ we obtain a covering homotopy $\bar{h}_v^* : P \times D^k \to V_{n,k+1}$ from $\bar{h}^*$ and h_v^*.

Define $\bar{h}_v^0 : P \to \overline{B}$ as follows. Let

$$\bar{h}_v^0(b)(x) = (\bar{h}_v(b)(x), \bar{h}_v'(b)(x))$$

where

$$\bar{h}_v(b)(x) = h_v(b)(x) \qquad x \in D^k$$

$$\overline{h}_v(b)(x) = g_0(x) \qquad x \in \dot{D}^{k+1} \cap A$$
$$\overline{h}'_v(b)(x) = (k+1)\text{-component of } \overline{h}^*_v(b, x) \qquad x \in D^k$$
$$\overline{h}'_v(b)(x) = g_{0t}(x) \qquad x \in \dot{D}^{k+1} \cap A\,.$$

Now it can be checked that $\overline{h}^0_v(b)(x)$ is well-defined and is the desired covering homotopy. This proves (1). One proves (2) the same way.

Lastly, (3) can be seen as follows. If $g \in \Gamma^*$, $g : D^k \to E^n$ is regular, and then $p^{-1}(g)$ is the space of all $g' : D^k \to E^n$ such that for each $x \in D^k$, $g'(x)$ is transversal to the tangent plane of g at x and g' obeys a boundary condition. The fiber over $\Phi(g) \in \Gamma'$ is the same while the restriction of Ψ is a homeomorphism between these fibers. To finish the proof of 2.2 we note that Ψ maps the exact homotopy sequence of (B, p, Γ^*) into the exact homotopy sequence of (B', p', Γ'). By the five lemma the theorem follows using (3) above, the given condition on Φ and 2.3. This proves 2.2.

Let $g_0 : D^k \to E^n$ be the standard inclusion of D^k into a k-plane of E^n, and f_0 be as in 2.2.

LEMMA 2.4. *If $\Phi : \Gamma_{k,n}(g_0) \to \Gamma'_{k,n}(\overline{g}_0)$ is a weak homotopy equivalence then so is $\Phi : \Gamma_{k\,n}(f_0) \to \Gamma'_{k\,n}(\overline{f}_0)$.*

The proof of this lemma offers no trouble and we leave the proof of it to the reader. One can use for example a diffeomorphism of E^n.

THEOREM 2.5. *If $\Psi : \overline{B}_{k,n}(g_0) \to B'_{k,n}(\overline{g}_0)$ is a weak homotopy equivalence then so is $\Phi : \Gamma_{k,n}(g_0) \to \Gamma'_{k,n}(\overline{g}_0)$.*

Before we prove 2.5 we note that 2.1 follows from 2.2, 2.4 and 2.5 by induction on k keeping n fixed. The first step, that $\Psi : \overline{B}_{1,n}(g_0) \to B'_{1,n}(\overline{g}_0)$ is a weak homotopy equivalence, is trivially checked. In fact, roughly speaking, my thesis [6] contains the second step and [7] is the third step in this induction.

PROOF OF 2.5. An outline of the proof is contained in the following diagram. The spaces and maps will be defined as the proof proceeds.

$$\begin{tikzcd}
\pi_i(\overline{B}) \arrow[d, "\Psi_\#"] \arrow[r, phantom, "A", near end, shift right=6] & \pi_i(\overline{E}, F) \arrow[l, "\overline{\pi}_\#"'] \arrow[r, "\pi\Delta"] \arrow[d, "\varphi_\#"] \arrow[r, phantom, "B", shift right=6] & \pi_{i-1}(F) \arrow[r, "\alpha_{1\#}"] \arrow[d, "\overline{\varphi}_\#"] \arrow[r, phantom, "C", shift right=6] & \pi_{i-1}(\Gamma) \arrow[d, "\Phi_\#"] \\
\pi_i(B') & \pi_i(E', F') \arrow[l, "\pi'_\#"'] \arrow[r, "\Delta'"] & \pi_{i-1}(F') \arrow[r, "\alpha_{2\#}"] & \pi_{i-1}(\Gamma')
\end{tikzcd}$$

Let x_0 be a $(k-1)$-frame of $S^{k-1} = \dot{D}^k$ whose base point is a x_∞, the south pole of S^{k-1}. The $g_{0\#}(x_0)$ is a $(k-1)$-frame say y_0 of E^n (with base point) and $g_{0t}(x_\infty)$ is a vector say $\overline{y}_0$ transversal to the plane of y_0. Let B_0 be the subspace of B of elements (f, f') where $f_\#(x_0) = y_0$ and $f'(x_\infty) = \bar{y}_0$. Let $E_0 = \pi^{-1}(B_0) \subset E$. Then by 1.1

(4) (E_0, π, B_0) has the CHP.

(We sometimes denote the restriction of a map by the same symbol as the original map).

We let $\overline{E}$ be the subspace of E_0 of immersions $f : D^k \to E^n$ which agree with g_0 on A. Note that $\overline{B} \subset B_0$ and let $\bar{\pi} : \overline{E} \to \overline{B}$ be the restriction of π. Let $F = \bar{\pi}^{-1}\bar{\pi}(g_0)$. Then we will prove

(5) For all i, $\bar{\pi}_\# : \pi_i(\overline{E}, F) \to \pi_i(\overline{B}, \bar{\pi}(g_0))$ is an isomorphism onto.

For the proof of (5) consider

$$\begin{array}{ccc} \overline{E} & \xrightarrow{j} & E_0 \\ \Big\downarrow{\scriptstyle \bar{\pi}} & & \Big\downarrow{\scriptstyle \pi} \\ \overline{B} & \xrightarrow{j'} & B_0 \end{array}$$

where j and j' are inclusions. Then it is sufficient for (5) to show for all $i \geqq 0$

(6) $j_\# : \pi_i(\overline{E}, F) \to \pi_i(E_0, F)$ is $1-1$ onto,

(7) $\pi_\# : \pi_i(E_0, F) \to \pi_i(B_0)$ is $1-1$ onto, and

(8) $j'_\# : \pi_i(\overline{B}) \to \pi_i(B_0)$ is $1-1$ onto.

The truth of (7) follows from (4).

By the exact homotopy sequence of the pairs $(\overline{E}, F)$ and (E_0, F) for (6) it is sufficient to show $\pi_i(E_0) = \pi_i(\overline{E}) = 0$ for all $i \geqq 0$.

Let $g : S^j \to \overline{E}$ be given. It is sufficient to show that g is homotopic to a point. For every $\varepsilon > 0$ there is a differentiable strong deformation retraction H_t of D^k into N_ε where N_ε is diffeomorphic to D^k, $N_\varepsilon \supset A$ and for every $y \in N_\varepsilon$, $d(y, A) < \varepsilon$. Then for each such ε, there is a homotopy g_t of g defined by $g_t(p)(y) = g(p)(H_t(y))$, $p \in S^j$ and $y \in D^k$. On the other hand, for each $\varepsilon > 0$ we have the homotopy $h_t : S^j \to \overline{E}$ between g_1 and the fixed map f_0 defined by $h_t(p)(y) = (1 - t)g_1(p)(y) + tf_0(y)$. We leave it to the reader to show that $h_t(p)$ will be regular (hence h_t will be well-defined) if ε has been chosen small enough. Thus g is homotopic to a point. This proves $\pi_j(\overline{E}) = 0$.

In a similar fashion one proves $\pi_j(E_0) = 0$.

Proof of (8). We wish to show $i : \overline{B} \to B_0$ is a weak homotopy equivalence. Let $f : P \to B_0$ where P is a polyhedron. It is sufficient for the proof to show there is a homotopy $F_\lambda : P \to B_0$ such that (a) $F_1 = f$, (b) if $f(p) \in \overline{B}$ then $F_\lambda(p) = f(p)$, and (c) $F_3(p) \in \overline{B}$.

The homotopy F_λ is defined in stages with some of the details omitted. First, it can be shown that there is a neighborhood N of x_∞ and a homotopy $F_\lambda: P \to \bar{B}_0$, $0 \leqq \lambda \leqq 1$ satisfying (a) and (b) and such that $F_1(p)$ agrees with g_0 on N.

There is a number $\varepsilon > 0$ such that if $g \in B_0$ and satisfies $\rho_*(g, g_0) < \varepsilon$ where

$$\rho_*(g, g_0) = \max\{\bar{\rho}(g(y), g_0(y)), \rho(\nabla_V g(y), \nabla_V g_0(y)) \mid y \in \dot{D}^k \cap A, V \subset \dot{D}^k_{y,} \mid V \mid = 1\}$$

then $G_\lambda(y) = \lambda g(y) + (1-\lambda) g_0(y)$ is regular with $0 \leqq \lambda \leqq 1$, and $y \in \dot{D}^k \cap A$. Furthermore, if ε is small enough one can use the map G_λ to define a homotopy $H_\lambda \in B_0$ where $0 \leqq \lambda \leqq 1$ such that $H_1 = g$, $H_0 = g_0$ and if $g = g_0$, $H_\lambda = H_0$. To define H_λ from G_λ one uses a ribbon around the equator of $\dot{D}^k$. Everything in this paragraph is valid on a compact set of such g all satisfying $\rho_*(g, g_0) < \varepsilon$.

Taking ε as in the last paragraph one can define $F_\lambda: P \to B_0$ $1 \leqq \lambda \leqq 2$ such that $F_1(p)$ is the previously defined $F_1(p)$, $\rho_*(F_2(p), g_0) < \varepsilon$, and if $F_1(p) \in \bar{B}$, $F_\lambda(p) = F_1(p)$. Here F is taken as a stretching of E^n moving $F_1(p)$ except in a neighborhood of x_0. Also $F_{\lambda,}$ $1 \leqq \lambda \leqq 2$ involves a simple re-parameterization of $\dot{D}^k$. Finally, F_λ for $2 \leqq \lambda \leqq 3$ is defined directly by the H_λ of the last paragraph. This proves (8).

Let A and A' be as before and using the fixed map g_0, define another fixed map $\bar{g}_0 : A' \to V_{n,k}$ by

$$\bar{g}_0(y) = (g_{0x_1}(y), \cdots, g_{0x_{k-1}}(y), g_t(y)), \qquad y \in A'.$$

Let $E'_{k,n}(\bar{g}_0) = E'$ be the space with the compact open topology of all maps of A' into $V_{n,k}$ which agree with $\bar{g}_0$ on $A \cap A'$. Let $B'_{k,n}(\bar{g}_0) = B'$ be the space with the compact open topology of all maps of $\dot{D}^k \cap A'$ into $V_{n,k}$ which agree with $\bar{g}_0$ on $\dot{D}^k \cap A' \cap A$. Define $\pi' : E' \to B'$ by restricting a map to $\dot{D}^k \cap A'$. Let $F'_{k,n}(\bar{g}_0) = F' = \pi'^{-1}(\pi'(\bar{g}_0))$.

(9) The triple (E', π', B') has the CHP.

To prove (9), let $h_v : P \to B'$ be a given homotopy where P is a polyhedron and let $\bar{h} : P \to E'$ cover h_0. A covering homotopy $\bar{h}_v : P \to E'$ is defined by

$$\bar{h}_v(p)(t, x) = \bar{h}(p)(t, x) \qquad 0 \leqq t \leqq 1/2$$

$$\bar{h}_v(p)(t, x) = \bar{h}(p)\left(\frac{t + \tau - 1}{2\tau - 1}, x\right) \qquad 1/2 \leqq t \leqq \tau$$

and if $\tau \neq 1$

$$\bar{h}_v(p)(t, x) = h_\lambda(p)(x) \qquad \tau \leqq t \leqq 1$$

where

$$\tau = \tau(v, x) = \frac{2 + v(1 + |x|)}{2(1+v)}$$

and

$$\lambda = \lambda(v, x, t) = \frac{v(\tau - t)}{\tau - 1}, \qquad \tau \neq 1 .$$

It can be checked without much trouble that this is a good covering homotopy. We show now,
(10) $\pi_i(E') = 0, i \geqq 0$

PROOF. Let H_t be a strong deformation retraction of D^k into A, i.e., a homotopy $H_t : D^k \to D^k$ such that H_0 is the identity, $H_1(x) \in A$ for $x \in D^k$ and if $x \in A$, $H_t(x) = x$ for all t. Then define a homotopy $\bar{H}_t : E' \to E'$ by $\bar{H}_t(f)(x) = f(H_t(x))$. It is easily checked that $\bar{H}_t$ is a strong deformation retraction of E' into the point $\bar{f}_0 H_1$ of E'.

A map $\varphi : \bar{E} \to E'$ is defined as follows.

$$\varphi(g)(y) = (g_{x_1}(y), \cdots, g_{x_{k-1}}(y), g_t(y)) , \qquad g \in \bar{E}, y \in A'.$$

Then the following diagram commutes.

$$\begin{array}{ccc} \bar{E} & \xrightarrow{\varphi} & E' \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi'} \\ B & \xrightarrow{\Psi} & B' \end{array}$$

Let $\bar{\varphi} : F \to F'$ be the restriction of φ.

It is easy to check that φ is continuous and that diagrams A and B at the beginning of the proof of 2.5 commute. Then we have
(11) $\bar{\varphi}_\# : \pi_{i-1}(F) \to \pi_{i-1}(F')$ is $1-1$ onto.

For (11) first note that $\varphi_\# : \pi_i(\bar{E}, F) \to \pi_i(E', F')$ is an isomorphism onto since $\bar{\pi}_\#$ is, by (5), and π' is by (9). Then Δ and Δ' are isomorphisms onto because $\pi_j(\bar{E}) = 0$ (proof of (6)) and $\pi_j(E') = 0$ by (10). This proves (11).
(12) There are maps $\alpha_1 : F \to \Gamma$ and $\alpha_2 : F' \to \Gamma'$ which are weak homotopy equivalences and such that the following diagram commutes.

$$\begin{array}{ccc} F & \xrightarrow{\alpha_1} & \Gamma \\ \downarrow{\scriptstyle \bar{\varphi}} & & \downarrow{\scriptstyle \Phi} \\ F' & \xrightarrow{\alpha_2} & \Gamma' \end{array}$$

PROOF. Let $\alpha : N \to D^k$ be a diffeomorphism of a small closed neighborhood N of D^k-interior A into D^k which sends the $(t, x_1, \cdots, x_{k-1})$ coordinates which were introduced after the statement of 2.1 (in one dimension higher) into the $(x_1, \cdots, x_k)$ coordinates of D^k which were introduced at the beginning of Section 2. If α has been chosen properly, it induces weak homotopy equivalences α_1 and α_2 as above with the above diagram commuting.

Theorem 2.5 now follows from (11) and (12).

3. Applications

The aim of this section is to prove the theorems stated in the Introduction from the theory developed in Sections 1 and 2. Thus we prove Theorems A, C, D and E.

We first prove Theorem A.

Let $x_0 \in F_k(S^k)$ with qx_0 the South Pole of S^k, $(q : F_k(S^k) \to S^k$, the bundle map) and let $g_0 : S^k \to E^{k+1} \subset E^n$ be the standard inclusions. Let $g_{0*}(x_0) = y_0 \in V_{n,k} \times E_n$. Denote by Λ the space of C^∞ immersions of S^k based with respect to x_0 and y_0, with the C^1 topology. Let Γ_0 be the subspace of immersions of Λ which agree with g_0 on the lower hemisphere of S^k

THEOREM 3.1. *The inclusion $i : \Gamma_0 \to \Lambda$ is a weak homotopy equivalence.*

The proof of 3.1 follows from the argument used in the proof of 2.5, statement (8).

It is immediate that Γ_0 and Γ^* are naturally homeomorphic where Γ^* is as in Section 2. Thus we have by 2.1, 2.3, 2.4 and 3.1 that $\pi_0(\Lambda)$ and $\pi_0(\Gamma')$ are in a $1-1$ correspondence where $\Gamma' = \Gamma'_{k,n}(\bar{f}_0)$ is defined as in Section 2. The arcwise connected components of Γ' correspond to the elements of $\pi_k(V_{n,k})$ and the arcwise connected components of Λ are based regular homotopy classes of based immersions of S^k in E^n. The reader may check that the correspondence is that given by Theorem A. This proves Theorem A.

Theorem C is proved as follows. If $k = 1$, see [6]. Now suppose k is odd, $k > 1$. Then there exist two regular homotopy classes of S^k in E^{2k} by Theorem A since $\pi_k(V_{2k,k}) = Z_2$ if k is odd, $k > 1$. On the other hand, Whitney [9] showed that there exist immersions of S^k in E^{2k} with arbitrary intersection number I_f, I_f defined to be an integer mod 2. Whitney further showed that I_f is invariant under regular homotopy. These facts together prove Theorem C for k odd, $k > 1$.

Now suppose k is even. Let $G_{2k,k}$ be the Grassman manifold of oriented k-planes in E^{2k} and $p : V_{2k,k} \to G_{2k,k}$ be the map which sends a k-frame onto the plane it spans. Then we have the following commutative diagram.

$$\begin{array}{ccccc}
\pi_k(S^k) & \xrightarrow{\bar{f}_\#, \bar{g}_\#} & \pi_k(G_{2k,k}) & \xleftarrow{\bar{p}_\#} & \pi_k(V_{2k,k}) \\
\downarrow{h_0} & & \downarrow{h_1} & & \downarrow{h_2} \\
H_k(S^k) & \xrightarrow{\bar{f}^*, \bar{g}^*} & H_k(G_{2k,k}) & \xleftarrow{p_\#} & H_k(V_{2k,k})
\end{array} .$$

Here the vertical maps are Hurewicz maps ; f and g are immersions of S^k in E^{2k} ; $\bar{f}, \bar{g} : S^k \to G_{2k,k}$ are induced by f, g by translating tangent planes to the origin of E^{2k}. Then $\bar{f}_\#$ and $\bar{f}_*$, are induced by $\bar{f}$.

Suppose now $I_f = I_g$. We wish to prove that f and g are regularly homotopic. For this we use the result from [3] which says that for an immersion f: $M^k \to E^{2k}$ (with M^k compact oriented and I_f defined) $\overline{W}_k(f) = 2I_f$ where $\overline{W}_k(f)$ is the k^{th} Stiefel-Whitney class with integer coefficients of the normal sphere-bundle over M^k. Thus in our case $\overline{W}_k(f) = \overline{W}_k(g)$. But by [3] this implies that $\bar{f}_* = \bar{g}_*$.

It is easily checked that $p_\#\Omega(f, g) = \bar{f}_\#(S_\#) - \bar{g}_\#(S_\#)$ where $S_\#$ is a generator of $\pi_k(S^k)$. Then by the previous diagram $h_1 p_\# \Omega(f, g) = \bar{f}_*(h_0(S_\#)) - \bar{g}_*(h_0(S_\#)) = 0$ since $\bar{f}_* = \bar{g}_*$. Then by the diagram $p_* h_2 \Omega(f, g) = 0$. By [3] p_* is $1-1$ and since h_2 is $1-1$ this implies $\Omega(f, g) = 0$. Thus f and g are regularly homotopic by Theorem A. This proves Theorem C.

Theorem D is proved as follows. Let $p : V_{k+1,k} \to G_{k+1,k} = S^k$ send a k-frame into the k-plane it spans. Since p is a fiber map, we have the following exact sequence.

$$\pi_k(V_{k+1,k}) \xrightarrow{p_\#} \pi_k(S^k) \xrightarrow{\Delta} \pi_{k-1}(\text{Fiber} = R_k) .$$

The map Δ is zero since S^k is parallelizable. Hence $p_\#$ is onto. Therefore by Theorem A, there exist immersions f, g $S^k \to E^{k+1}$ such that $p_\#\Omega(f, g)$ is a generator of $\pi_k(S^k)$. Since $p_\#\Omega(f, g) = \bar{f}_\#(S_\#) - \bar{g}_\#(S_\#)$ either f or g has even normal degree, and there exists an immersion of S^k in E^{k+1} with normal degree zero [4]. This proves Theorem D.

Lastly we prove Theorem E. If $\Omega(f, e) = 0$ then f is regularly homotopic to e. Furthermore, this regular homotopy can be covered by a transversal vector field f' by the argument of Theorem 2.2 (1). By 1.1, (f, f') is in the image of π ; hence the desired extension exists.

Conversely, in order that f have an extension it must lie in the image of π (with some f'). But it follows from the proof of 2.5, Statement 6 that E is arcwise connected. This implies $\Omega(f, e) = 0$.

Addenda

Here we note that the solution of another problem posed by Milnor follows from our work. On page 284 of [4] he asks :

Let n be a dimension for which S^n is not parallelizable. Can some parallelizable n-manifold be immersed in E^{n+1} with odd degree ? Can some (necessarily parallelizable) n-manifold be immersed in E^{n+1} both with odd and with even degree ?

The answer to the first and hence to the second question is seen to be no, as follows.

As in the proof of Theorem D, Section 3, consider the homomorphism $p_\# : \pi_n(V_{n+1,n}) \to \pi_n(S^n)$. It is sufficient to consider the case of n odd with S^n not parallelizable. By Theorem A, the image of $p_\#$ consists of even elements of $\pi_n(S^n)$ since only odd degrees of immersions of S^n in E^{n+1} are possible [4]. Now if M^n is parallelizable and $f : M^n \to E^{n+1}$ is an immersion, then f induces a map $F : M^n \to V_{n+1,n}$ with $pF = f$. Then f must have even normal degree, proving our assertion.

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REFERENCES

1. S. S. CHERN, *La géométrie des sous-variétés d'un espace euclidean a plusiers dimensions*, L'Enseignement Mathematique, 40 (1955), 26-46.

2. M. W. HIRSCH, *On immersions and regular homotopies of differentiable manifolds*, Abstract, Amer. Math. Soc., 539-19, Notices Amer. Math. Soc., 5, No. 1 (1958), 62.

3. R. LASHOF AND S. SMALE, *On the immersion ofmanifolds in euclidean space*, Ann. of Math., 68 (1958), 562-583.

4. J. MILNOR, *On the immersion of n-manifolds in* $(n+1)$*-space*, Comment. Math. Helv., 30 (1956), 275-284.

5. G. F. PAECHTER, *The groups* $\pi_r(V_{n,m})$(I), Quart. J. Math., 7 (1956), 249-268.

6. S. SMALE, *Regular curves on Riemannian manifolds*, Trans. Amer. Math. Soc., 81 (1958), 492-512.

7. ———, *A classification of immersions of the two-sphere*, Trans. Amer. Math. Soc., to appear.

8. ———, On classifying immersions of S_h in euclidean space, Summer Conference on Algebraic Topology, Chicago, 1957, mimeographed notes.

9. H. WHITNEY, *The self-intersections of a smooth n-manifold in 2n-space*, Ann. of Math., 45 (1944), 220-246.