ON THE HOMOLOGY INVARIANTS OF KNOTS

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1. Let \( k \) be an oriented knot in 3-dimensional euclidean space \( \mathbb{R}^3 \) and \( V \) a closed tubular neighbourhood of \( k \). The boundary of \( V \) is a torus \( T \), and \( W = \mathbb{R}^3 - V + T \) is the closed complement of \( V \).

An oriented Jordan curve (i.e. a homeomorph of a circle) on \( T \) which bounds on \( V \) (on \( \mathbb{R}^3 - V + T \)) but not on \( T \) is called a meridian (longitudinal circuit). If \( m_1 \) and \( m_2 \) are any two meridians, one has \( m_1 \sim \pm m_2 \) on \( T \); likewise \( q_1 \sim \pm q_2 \) on \( T \) for any pair \( q_1, q_2 \) of longitudinal circuits.

By a topological mapping \( \phi \) one can carry \( V \) into a tubular neighbourhood \( V^* \) of an unknotted curve \( k^* \) in such a way that the longitudinal circuit of \( V \) is carried into a longitudinal circuit \( q^* \) of \( V^* \). The 3-space in which \( V^* \) lies will be designated by \( \mathbb{R}^3^* \). Fig. 1 and Fig. 2 show the situation in the case when \( k \) is a trefoil knot.

Let \( l \) be an arbitrary knot in the interior of \( V \). Then \( l \), as a 1-cycle in \( V \), is homologous in \( V \) to some multiple of \( k \), say

\[ l \sim nk \quad \text{on } V. \]  \hspace{1cm} (1)

By a suitable orientation of \( l \) we can arrange that \( n \geq 0 \). Fig. 3 shows an example in which \( k \) is a trefoil knot and \( n = 0 \).

The knot \( l \) is carried by the topological mapping \( \phi \) into a knot \( l^* \) in the interior of \( V^* \). The purpose of this paper is to prove the two following theorems:

**Theorem I.** For \( n = 0 \) the homology invariants of \( l \) and \( l^* \) are the same.

In other words: if \( M_g \) and \( M_g^* \) are the \( g \)-sheeted cyclic covering manifolds† of \( R^3 \) with the branch lines \( l \) and \( l^* \) respectively, the homology groups and linking invariants of \( M_g \) and \( M_g^* \) are equal for

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\( g = 2, 3, \ldots \), and the homology groups of \( M_\infty \) and \( M^* \), considered as groups with operators,† are isomorphic.

**Theorem II.** Between the \( L \)-polynomials \( \Delta_l(x) \), \( \Delta_{l^*}(x) \), \( \Delta_k(x) \) of the knots \( l, l^* \), and \( k \) the following equation holds

\[
\Delta_l(x) = \Delta_{l^*}(x) \Delta_k(x^n).
\]

(2)

In the case \( n = 0 \) formula (2) reduces to

\[
\Delta_l(x) = \Delta_{l^*}(x).
\]

(3)

For we then have

\[
\Delta_k(x^n) = \Delta_k(x^0) = \Delta_k(1).
\]

But it is known that \( \Delta_k(1) = 1 \) for any knot \( k \). Besides (3) is a consequence of Theorem I.

In the special case when the knot \( l \) lies on the boundary \( T \) of \( V \), formula (2) expresses a theorem due to Burau.† A theorem due to Alexander§ to the effect that the \( L \)-polynomial of a composite knot is the product of the \( L \)-polynomials of the factors is another special case of Theorem II (here \( n = 1 \)).

Theorem I illustrates the limits of the homology invariants of a knot in so far as the properties of knot \( k \) do not appear in the homology invariants of \( l \). A special case of this fact is the theorem of Whitehead's|| on the \( L \)-polynomial of a 'doubled knot'.

2. Proof of Theorem I.

The \( g \)-sheeted cyclic covering manifold \( M_g \) is the union of the complexes \( V_g \) and \( W_g \) corresponding to the decomposition of \( R^3 \) into \( V \) and \( W \). \( V_g \) is the \( g \)-sheeted cyclic covering manifold of \( V \) with branch line \( l \). \( W_g \) decomposes into \( g \) homeomorphs \( W', W'', \ldots, W^g \) of \( W \), since every closed curve of \( W \) is homologous to zero in \( R^3 - l \), \( n \) being equal to zero. The intersection of \( V_g \) and \( W^g \) is a torus \( T_g \) (\( \gamma = 1, 2, \ldots, g \)). Let \( g_x \) be the covering of the longitudinal circuit \( g \) lying on \( T_g \), and let \( a_1, a_2, \ldots, a_r \) be a set of generators of the homology

group of dimension 1 of \( V_g \). Then the homology group of \( V_g \) is defined by a set of \( m \) relations
\[
\sum_{\tau=1}^{t} \rho_{\mu \tau} a_{\tau} \sim 0 \quad \text{on} \quad V_g \quad (\mu = 1, 2, \ldots, m). \tag{4}
\]
Let
\[
g_{\gamma} \sim \sum_{\tau=1}^{t} \sigma_{\gamma \tau} a_{\tau} \quad \text{on} \quad V_g \quad (\gamma = 1, 2, \ldots, g). \tag{5}
\]
Then the homology group of \( M_g \) is defined by the relations (4) and
\[
\sum_{\tau=1}^{t} \sigma_{\gamma \tau} a_{\tau} \sim 0 \quad \text{on} \quad M_g \quad (\gamma = 1, 2, \ldots, g). \tag{6}
\]

On the other hand let us consider the \( g \)-sheeted cyclic covering manifold \( M_g^* = V_g^* + W_g^* \) of \( R^3 \) with branch line \( l^* \). Corresponding to the mapping \( \phi \) of \( V \) on \( V^* \) (cf. § 1) there exists a homeomorphic mapping \( \phi_g \) of \( V_g \) on \( V_g^* \) which carries the torus \( T \) into the torus \( T^* \), the longitudinal circuit \( q \) into the longitudinal circuit \( q^* \) and the set of generators \( a_1, a_2, \ldots, a_t \) of \( V \) into the set of generators \( a_1^*, a_2^*, \ldots, a_t^* \) of \( V_g^* \). Then we have the relations
\[
\sum_{\tau=1}^{t} \rho_{\mu \tau} a_{\tau}^* \sim 0 \quad \text{on} \quad V_g^* \quad (\mu = 1, 2, \ldots, m) \tag{4^*}
\]
and
\[
g_{\gamma}^* \sim \sum_{\tau=1}^{t} \sigma_{\gamma \tau} a_{\tau}^* \quad \text{on} \quad V_g^* \quad (\gamma = 1, 2, \ldots, g), \tag{5^*}
\]
since \( \phi_g \) is a homeomorphic mapping. It follows that the homology groups of \( M_g \) and \( M_g^* \) are isomorphic.

In order to determine the linking invariants of \( M_g \), we consider (besides \( a_1, a_2, \ldots, a_t \) \( t \) 1-dimensional chains \( a_1', a_2', \ldots, a_t' \) on \( V_g \) such that \( a_{\tau}^* \sim a_{\tau} \) on \( V_g \) and \( a_\tau \) and \( a_{\lambda} \) do not intersect for \( \tau, \lambda = 1, 2, \ldots, t \). Because of formulae (4) and (5), there are 2-chains \( A_1, A_2, \ldots, A_m \) and \( B_1, B_2, \ldots, B_g \) on \( V_g \) such that
\[
\text{boundary } A_\mu = \sum_{\tau=1}^{t} \rho_{\mu \tau} a_{\tau} \quad (\mu = 1, 2, \ldots, m)
\]
and
\[
\text{boundary } B_\gamma = \sum_{\tau=1}^{t} \sigma_{\gamma \tau} a_{\tau} - g_{\gamma} \quad (\gamma = 1, 2, \ldots, g).
\]
Then the linking invariants of \( M_g \) are determined by the \( t(t+g) \) intersection numbers
\[
S(A_\tau, a_\nu') \quad S(B_\gamma, a_\nu')
\]
\((\tau, \nu = 1, 2, \ldots, t; \gamma = 1, 2, \ldots, g)\).

If we define \( A_\tau^*, B_\gamma^* \) as the images of \( a_\tau', A_\gamma, B_\gamma \) under the mapping \( \phi_g \), it follows that
\[
S(A_\tau, a_\nu') = S(A_\tau^*, a_\nu'^*), \quad S(B_\gamma, a_\nu') = S(B_\gamma^*, a_\nu'^*),
\]
provided that the orientation of \( V_g \) is carried into the orientation of \( V_g^* \) under the mapping \( \phi \). Therefore the linking invariants of \( M_g \) and \( M_g^* \) are the same.

The assertion that the 1-dimensional homology groups of \( M_\infty \) and \( M_\infty^* \) are operator isomorphic follows from the fact that they are obtained from the operator isomorphic groups of \( V_\infty \) and \( V_\infty^* \) by adding the relations \( q_\gamma \sim 0 \) and \( q_\gamma^* \sim 0 \) \(( -\infty < \gamma < +\infty \)).

3. For the proof of formula (2) we make use of the following facts (cf. Seifert):† For any knot \( c \) there can always be found an orientable surface \( F \) without singularities whose boundary is \( c \). By cutting \( R^3 \) along \( F \) we obtain a bounded 3-dimensional manifold \( \overline{M} \) whose boundary consists of the two exposed faces of the cut, i.e. of \( F \) and a homeomorphic copy \( xF \) of \( F \). Let \( h \) be the genus of \( F \), let \( a_1, a_2, \ldots, a_{2h} \) be a (1-dimensional) homology basis of \( F \) and let \( xa_1, xa_2, \ldots, xa_{2h} \) be the corresponding basis of \( xF \). Then there are homologies of the form

\[
a_i - \sum_{j=1}^{2h} \gamma_{ij} (a_j - xa_j) \sim 0 \quad \text{in} \quad \overline{M} \quad (i = 1, 2, \ldots, 2h).
\]  

(7)

All homologies between \( a_1, a_2, \ldots, a_{2h}, xa_1, \ldots, xa_{2h} \) that exist in \( \overline{M} \) are consequences of (7).

The matrix \( \Gamma = (\gamma_{ij}) \), from which all homology invariants of \( c \) can be derived, may be called a homology matrix of \( c \). The matrix \( \Gamma \) is uniquely determined up to the choice of the spanning surface \( F \) and its homology basis \( a_1, a_2, \ldots, a_{2h} \). The \( L \)-polynomial \( \Delta_c(x) \) of \( c \) is the coefficient determinant of the system (7)

\[
\Delta_c(x) = |E - \Gamma + z\Gamma|.
\]  

(8)

where \( E \) is the unit matrix of order \( 2h \).

4. We may assume \( n > 0 \), since for \( n = 0 \) Theorem II is a consequence of Theorem I. We begin by constructing an oriented non-singular surface \( F_n^* \) bounded by \( l^* \). To this end we choose on the boundary \( T^* \) of \( V^* \) a set of \( n \) non-intersecting longitudinal circuits \( q_1^*, q_2^*, \ldots, q_n^* \) and orient them so that they all become homologous to \( k^* \) in \( V^* \). Since we have the homology \( l^* \sim \sum_{r=1}^{n} q_r^* \) in \( V^* \), it follows that there exists in \( V^* \) an oriented non-singular surface \( F_n^* \) with boundary \( l^* - \sum_{r=1}^{n} q_r^* \). From \( F_n^* \), we obtain the desired surface \( F_n \) by

adjoining \( n \) non-intersecting 2-cells \( F_{q_1}, F_{q_2}, \ldots, F_{q_n} \) which lie in \( W^* = R^3* - V^* + T^* \) and have the boundaries \( q_1^*, q_2^*, \ldots, q_n^* \) respectively.

Next we construct a surface \( F \) bounded by \( l \). The homeomorphic mapping \( \phi^{-1} \) of \( V^* \) upon \( V \) carries the surface \( F_{q_i}^* \) into a surface \( F_{q_i}^* \) whose boundary consists of \( l \) and \( n \) longitudinal circuits \( q_1, q_2, \ldots, q_n \) of \( V \), images of \( q_1^*, q_2^*, \ldots, q_n^* \) respectively. Since \( q_i \sim 0 \) in \( W \), there exists an oriented non-singular surface \( F_{q_i}^* \) in \( W \) with boundary \( q_i \). By an isotopic deformation \( F_{q_i} \) can be carried into a 'parallel' surface \( F_{q_i} \).

\[ \begin{array}{c}
q_3 \\
q_2 \\
q_1 \\
V \\
q_2 \\
q_3 \\
V \\
q_4 \\
\end{array} \]

By a second deformation \( F_{q_3} \) can be carried into a surface \( F_{q_3} \) in \( W \) with boundary \( q_3 \) such that \( F_{q_3} \) intersects neither \( F_{q_i} \) nor \( F_{q_i}^* \), and so on. \( F_{q_i}, F_{q_2}, F_{q_3}, \ldots, F_{q_n} \) form together an orientable non-singular surface \( F \) bounded by \( l \). Fig. 4 shows the situation in a schematic cross-section; \( F_{q_3}^* \) is omitted and \( n = 3 \).

The genus of \( F \) is obviously

\[ h_* + nh_k, \]

where \( h_* \) and \( h_k \) denote the genera of \( F_{q_i}^* \) and \( F_{q_i} \) respectively.

5. I shall now construct a homology basis of dimension 1 on \( F \). Let

\[ a_1^{(1)}, a_2^{(1)}, \ldots, a_n^{(1)} \]

be a homology basis on \( F_{q_i} \), and

\[ a_1^{(v)}, a_2^{(v)}, \ldots, a_{2n}^{(v)} \quad (v = 1, 2, \ldots, n) \]

the basis on \( F_{q_i}^* \) which corresponds to it with respect to the above-mentioned deformation of \( F_{q_i} \) into \( F_{q_i}^* \). On \( F_* \) we select a homology basis

\[ b_1^*, b_2^*, \ldots, b_{2n}^* \]

where \( \alpha \) and \( \alpha^* \) denote the genus.
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One can assume that these chains all lie on $F^*$, since the 2-cells $F_q^*, F_q^*, \ldots, F^*_n$ can obviously be avoided. By the homeomorphic mapping $\phi^*$ of $V^*$ on $V$ the chains (10) are carried into the chains

$$b_1, b_2, \ldots, b_{2h^*}$$  \hspace{1cm} (11)

on $F^*$. The chains (9) and (11) constitute the desired homology basis of $F^*$.

6. In order to obtain the $L$-polynomial of the knot $I$ we cut $R^3$ along the surface $F^*$ according to the general rule of § 3. I shall use the following notation. By the cutting process the complexes $R^3, V, W, T$ go into $\tilde{R}^3, \tilde{V}, \tilde{W}, \tilde{T}$. The two exposed faces of the cut are designated by $F^*$ and $x F^*$. $F^*$ consists of the $n+1$ surfaces $F_{q1}^*, F_{q2}^*, \ldots, F_{qn}^*$, and similarly $x F^*$ is the union of the surfaces $xF_{q1}^*, xF_{q2}^*, \ldots, xF_{qn}^*$. The homology basis $a^{(1)}_1, a^{(1)}_2, \ldots, a^{(1)}_{2h^*}, b_1, b_2, \ldots, b_{2h^*}$ of $F^*$ corresponds to the homology basis $xa^{(1)}_1, xa^{(1)}_2, \ldots, xa^{(1)}_{2h^*}, xb_1, xb_2, \ldots, xb_{2h^*}$ of $xF^*$. The notation used in $R^3*$ differs only in the addition of a superscript star.

7. In $\tilde{R}^3*$ we have relations of the form (cf. § 3)

$$b_i^* - \sum_{j=1}^{2h^*} \gamma_{ij}^*(b_j^* - xb_j^*) \sim 0 \hspace{1cm} (i = 1, 2, \ldots, 2h^*).$$  \hspace{1cm} (12)

The homology matrix of the knot $I^*$ is

$$\Gamma^* = (\gamma_{ij}^*).$$  \hspace{1cm} (13)

The left side of (12), being a chain in $\tilde{V}^*$ and homologous to zero in $\tilde{R}^3* = \tilde{V}^* + \tilde{W}^*$, must be homologous to a chain on $\tilde{V}^* \cap \tilde{W}^* = \tilde{T}^*$. Thus it is homologous to a linear combination of the chains $q_1^*, q_2^*, \ldots, q_{2h^*}$:

$$b_i^* - \sum_{j=1}^{2h^*} \gamma_{ij}^*(b_j^* - xb_j^*) \sim \sum_{j=1}^{2h^*} \alpha_j q_j^* \hspace{1cm} \text{on } \tilde{V}^*. \hspace{1cm} (14)$$

By the homeomorphic mapping $\phi^*$ of $V^*$ on $V$ the homology (14) corresponds to the following homology on $\tilde{V}$:

$$b_i - \sum_{j=1}^{2h^*} \gamma_{ij}^*(b_j - xb_j) \sim \sum_{j=1}^{2h^*} \alpha_j q_j \hspace{1cm} \text{on } \tilde{V}. \hspace{1cm} (15)$$

If we consider this homology in $\tilde{R}^3 = \tilde{V} + \tilde{W}$, it simplifies to

$$b_i - \sum_{j=1}^{2h^*} \gamma_{ij}^*(b_j - xb_j) \sim 0 \hspace{1cm} \text{in } \tilde{R}^3 \hspace{1cm} (i = 1, 2, \ldots, 2h^*), \hspace{1cm} (16)$$

since $q_j \sim 0$ in $\tilde{W}$. 

We still need the homologies belonging to the $a_i^{(v)}$. They have the following general form (cf. § 3):

$$a_i^{(v)} - \sum_{j=1}^{2h_k} \sum_{\mu=1}^{n} \gamma_{ij}^{(v)} (a_j^{(\mu)} - xa_j^{(\mu)}) \sim \sum_{j=1}^{2h_k} \gamma_{ij}^{(v)} (b_j - xb_j) \quad \text{in } \mathbb{R}^2. \quad (17)$$

The left side of (17) is a chain in $\overline{W}$, the right side a chain in $V$. Therefore there is a certain chain on $V \cap \overline{W} = T$ to which either side is homologous (in $V$ or $\overline{W}$ respectively). The most general such chain is a linear combination of $q_1, q_2, \ldots, q_n$, but these are $\sim 0$ in $\overline{W}$. So it follows that:

$$a_i^{(v)} - \sum_{j=1}^{2h_k} \sum_{\mu=1}^{n} \gamma_{ij}^{(v)} (a_j^{(\mu)} - xa_j^{(\mu)}) \sim 0 \quad \text{in } \overline{W}, \quad (i = 1, 2, \ldots, 2h_k; \nu = 1, 2, \ldots, n). \quad (18)$$

In order to determine the matrices

$$\Gamma^{\nu\mu} = (\gamma_{ij}^{(v)})$$

we identify in $\overline{W}$ the surfaces $F_{1q_1}$ and $xF_{1q_1}$, $F_{2q_2}$ and $xF_{2q_2}$. Hereby $\overline{W}$ goes into a complex $\overline{W}_1$, which may be described as the complex $W$ cut along $F_{1q_1}$. The chains $a_i^{(\mu)}$ and $xa_j^{(\mu)}$ are thereby identified ($\mu = 2, 3, \ldots, n$), so that (18) reduces to

$$a_i^{(1)} - \sum_{j=1}^{2h_k} \gamma_{ij}^{(1)} (a_j^{(1)} - xa_j^{(1)}) \sim 0 \quad \text{in } \overline{W}_1, \quad (i = 1, 2, \ldots, 2h_k). \quad (19)$$

But this is exactly the system of relations (7) formed for the knot $q_1$ and the surface $F_{1q_1}$. So we see from (19) that $\Gamma^{11}$ is just the homology matrix of $q_1$, or, what is the same thing, of $k$ ($k$ and $q_1$ are equivalent knots, since $k$ can be deformed in $V$ into $q_1$). If we designate the homology matrix of $k$ by $\Gamma_k$, we have the result $\Gamma^{11} = \Gamma_k$, and in the same way one proves

$$\Gamma^{\nu\nu} = \Gamma_k \quad (\nu = 1, 2, \ldots, n). \quad (20)$$

Now we note that the homologies

$$xa_i^{(\nu)} \sim a_i^{(\nu+1)} \quad \text{in } \overline{W}, \quad (i = 1, 2, \ldots, 2h_k; \nu = 1, 2, \ldots, n-1) \quad (21)$$

hold, provided that the $q_v$ have been enumerated in the right way. But we know that (16) and (18) are a complete system of homologies in $\mathbb{R}^2$ between the chains $b_i, a_i^{(v)}$, $xb_i, xa_i^{(v)}$. So (21) must be a consequence of (16) and (18) and therefore, because of the special form (16) and (18), of (18) alone.

If we write the variables $a_i^{(v)}$ and $xa_i^{(v)}$ in the order

$$a_i^{(1)}, a_i^{(2)}, \ldots, a_i^{(n)}; xa_i^{(1)}, xa_i^{(2)}, \ldots, xa_i^{(n)},$$
the coefficient matrices of (18) and (21) are

\[
\begin{array}{cccc|ccc}
E - \Gamma_k & - \Gamma^{12} & - \Gamma^{1n} & \Gamma_k & \Gamma^{12} & \Gamma^{1n} \\
- \Gamma^{21} & E - \Gamma_k & - \Gamma^{2n} & \Gamma_k & \Gamma^{21} & \Gamma^{2n} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
- \Gamma^{n1} & - \Gamma^{n2} & E - \Gamma_k & \Gamma^{n1} & \Gamma^{n2} & \Gamma_k \\
\end{array}
\]  

\text{(22)}.

and

\[
\begin{array}{ccc|ccc}
0 & -E & 0 & 0 & E & 0 & 0 \\
0 & 0 & -E & 0 & 0 & E & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & -E & 0 & E & 0 \\
\end{array}
\]

\text{(23)}.

In both matrices we add the right half to the left and obtain

\[
\begin{array}{ccc|ccc}
E & 0 & 0 & \Gamma_k & \Gamma^{12} & \Gamma^{1n} \\
0 & E & 0 & \Gamma^{21} & \Gamma_k & \Gamma^{2n} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & E & \Gamma^{n1} & \Gamma^{n2} & \Gamma_k \\
\end{array}
\]

\text{(22').}

and

\[
\begin{array}{ccc|ccc}
E & -E & 0 & 0 & 0 & 0 \\
0 & E & -E & 0 & 0 & E \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & E & -E & 0 \\
\end{array}
\]

\text{(23').}

The rows of (23') can be linear combinations of the rows of (22') only if

\[
\Gamma^{\mu} = \begin{cases} 
\Gamma_k - E & (\nu > \mu), \\
\Gamma_k & (\nu < \mu).
\end{cases}
\]

Hence we find for the homology matrix \( \Gamma_l \) of \( l \) (see (16) and (18))

\[
\Gamma_l = \begin{pmatrix}
\Gamma^{11} & \Gamma^{1n} & 0 \\
\vdots & \vdots & \vdots \\
\Gamma^{n1} & \Gamma^{nn} & 0 \\
0 & 0 & \Gamma^*_k \\
\end{pmatrix} = \begin{pmatrix}
\Gamma_k & \Gamma_k & \Gamma_k & 0 \\
0 & \Gamma_k - E & \Gamma_k & \Gamma_k \\
\vdots & \vdots & \vdots & \vdots \\
\Gamma_k - E & \Gamma_k - E & \Gamma_k & \Gamma_k & \Gamma_k \\
0 & 0 & 0 & \Gamma^*_k \\
\end{pmatrix}
\]
and for the $L$-polynomial $\Delta_l(x)$ of $l$

$$\Delta_l(x) = |E - \Gamma_l + x\Gamma_l|$$

$$= |E - \Gamma_l + x\Gamma_l| \begin{vmatrix} E - \Gamma_k + x\Gamma_k & -\Gamma_k + x\Gamma_k \\ E - \Gamma_k + x(\Gamma_k - E) & E - \Gamma_k + x\Gamma_k \end{vmatrix}.$$  \hspace{1cm} (24)

The first factor is the $L$-polynomial $\Delta_\ast(x)$ of $l^\ast$; the second factor has in the diagonal $E - \Gamma_k + x\Gamma_k$, above the diagonal $-\Gamma_k + x\Gamma_k$, and below the diagonal $E - \Gamma_k + x(\Gamma_k - E)$. This determinant can be computed as follows. Subtract successively the $(n-1)$th row from the $n$th, the $(n-2)$th from the $(n-1)$th, ..., the first row from the second. There results

$$|E - \Gamma_k + x\Gamma_k & -\Gamma_k + x\Gamma_k & -\Gamma_k + x\Gamma_k \\ -xE & E & 0 & 0 \\ 0 & -xE & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & -xE & E|$$

Next add $x$ times the $n$th column to the $(n-1)$th, $x$ times the $(n-1)$th column to the $(n-2)$th, etc. This gives

$$|E - \Gamma_k + x^n\Gamma_k & -\Gamma_k + x^{n-1}\Gamma_k & -\Gamma_k + x\Gamma_k \\ 0 & E & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & E|$$

Thus (24) becomes

$$\Delta_l(x) = \Delta_\ast(x)\Delta_k(x^n).$$

This completes the proof.