FIBRATIONS OVER A CWh-BASE

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ABSTRACT. This note provides a short argument for the known fact that the total space of a fibration has the homotopy type of a CW-complex if base and fiber have.

1. NOTATION. $F \to E \to B$ is a (Hurewicz) fibration. A CWh-space is a space having the homotopy type of a CW-complex. The following result is due to Stasheff [9, Proposition (0)].

2. THEOREM. $E$ is a CWh-space if $F$ and $B$ are.

PROOF. We replace the inductive construction of [9] by the CW approximation theorem [8, p. 412] that is due to Whitehead [11]: to the topological space $E$ there exists a CW-complex $X$, called ‘CW-substitute for $E$’ in [10, p. 97], and a weak homotopy equivalence $f: X \to E$. We make $f$ into a fibration by taking the associated mapping path fibration $q: P_f \to E$, see e.g. [8, p. 99]. Then $q$ is a weak homotopy equivalence too, and $P_f$ is a CWh-space. Therefore $pq$ is a fibration with a CWh-fiber by 3 below. Hence $q$ induces a genuine homotopy equivalence between the fibers of $pq$ and $p$ and is therefore a fiber homotopy equivalence by [3, 6.3].

3. PROPOSITION. $F$ is a CWh-space if $E$ and $B$ are.

PROOF. Compare [9, Corollary (13)]. By coglueing homotopy equivalences, see [4, (1.2)] or [5, (8.7)], the pullbacks of the horizontal rows in the following diagram are homotopy equivalent.

$$
\begin{array}{ccc}
\ast & \to & B \\
\downarrow & & \downarrow \\
P_B & \to & B \\
\downarrow & & \downarrow \\
PZ_p & \to & Z_p \\
\end{array}
$$

$f_i$ is the standard factorization of $p$ over its mapping cylinder $Z_p$, $PZ_p \to Z_p$, $PB \to B$ are the fibrations of paths starting from a point $b \in Z_p$, resp.

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As it is remarked in [7, p. 27] Stasheff's proof is not correct, but can be patched.

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\( f(b) = \ast \in B \), and the other arrows are obvious. The upper pullback is the fiber \( F \) (over \( \ast \)), the lower one is the space of the paths on the CWh-space \( Z_p \) starting from \( b \) and ending in \( E \subset Z_p \), and is therefore a CWh-space by [6].

4. REMARK. If we assume that \( F \) and \( E \) are CWh-spaces, then the following is true: (a) \( B \) is not a CWh-space in general. Fiber and total spaces of Example 2.4.8 of [8, p. 77] are contractible, but the base space, the “Warsaw circle”, is not contractible, because it has the nonvanishing \( \check{\pi}_1(B) \cong \mathbb{Z} \) [2, §6]. (b) the loop space \( \Omega B \) is a CWh-space, because it is homotopy equivalent to the fiber of the inclusion \( F \to E \) [10, 2.56], and by delooping homotopy equivalences, see [1], \( B \) is a CWh-space too, if it is path-connected and has a numerable, null homotopic covering.

REFERENCES


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