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Algebraic Topology: An Intuitive Approach

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ISŌ KIKA
(ALGEBRAIC TOPOLOGY)

by Hajime Sato

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ABSTRACT. This book develops an introduction to algebraic topology mainly through simple examples built on cell complexes. The topics covered include homeomorphisms, homotopy equivalences, the torus, the Mobius strip, closed surfaces, the Klein bottle, cell complexes, fundamental groups, homotopy groups, homology groups, cohomology groups, fiber bundles, vector bundles, spectral sequences, and characteristic classes.

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Preface

Topology forms a branch of geometry emphasizing connectedness as the most fundamental aspect of a geometrical object. In topology, therefore, one ignores virtually all geometrical traits other than connectedness, such as any form of change in a geometrical object that stretching or shrinking might cause. Classification in topology is a crude tool, but one that never fails to determine if a geometrical object is connected or not. If a geometrical object is connected then we investigate to what degree it is connected. Just as the state of connectedness characterizes the essence of many phenomena we encounter in our daily lives, it is often necessary to describe to what extent a certain object is connected or separated. Thus the terms one employs in topology are increasingly becoming important and useful in other branches of mathematics as well as in various fields in the natural sciences.

There are numerous algebraic topology books and many of them are excellent; yet we have dared to add another book on this subject. The single most difficult thing one faces when one begins to learn a new branch of mathematics is to get a feel for the mathematical sense of this subject. To somebody who has mastered the subject this essential common sense should be as familiar as the air around him. It takes a long time for a beginner to get to this point. The purpose of this book is to help an aspiring first-time reader acquire this topological atmosphere in a short period of time.

I believe that the most efficient way to fulfill this purpose is to investigate simple but meaningful examples in some concrete terms. It is important that the reader grasp a mathematical object with his or her own hands. By touching it one can feel its physical quality and then keep this as one's own. This book is a simple manual that the reader can follow, and in fact the reader who follows our instructions step by step will end up with a real working model of algebraic topology.

In order to pursue this objective we have therefore sacrificed generality and limited the objects of our discussion to the simplest but most essential cases. We did not try to expand the theory to its fullest extent to make our book an encyclopedic reference; instead, we use the easiest possible examples to help the reader see the backbone of our discussion.

We will be greatly pleased if the reader enjoys reading our book while acquiring several essential methods or approaches to discuss algebraic topology. We must await the reaction of the reader to see if our plan will succeed. We will appreciate it if the reader gives us any feedback (criticisms and comments)[1].

The basic framework of the book comes from the seminar notes "Practical Topology for Physicists" given by Akihiro Tsuchiya and compiled by Yasuhiko Yamada at the University of Nagoya in 1986. I am deeply indebted to Mr. Tsuchiya for permitting me to use his seminar notes as well as for giving me much useful advice throughout every stage of the writing. My thanks also go to Tadayoshi Mizutani, Tetsuya Ozawa, Yoshinori Machida, and Shigeo Ichiraku, who not only read the entire manuscript carefully, finding many mistakes, but also suggested various ways to improve the final product. Last but not least, I would like to thank the editors at Iwanami Shoten.

Hajime Sato

July 1996

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[1]See Preface to the English Translation

Preface to the English Translation

It is a great pleasure to me that the American Mathematical Society chose to publish my book "Algebraic Topology: An Intuitive Approach" in their translation series.

Since the publication of the original version of this book in 1996, several of my friends (including the translator) have complained that the gap between my claim that *no previous knowledge of mathematics is required...* and the actual contents of the book is too big. So I have provided the reader who has no knowledge of sets, topology, groups, *etc.* with a basic minimal list of definitions and results that may prove useful, together with readable references. This is in the Appendix at the end of the book. This does not really change my original view that the book is readable for anybody who wishes to find out about algebraic topology. I think that technical terms help both the reader and the author organize their thoughts, but they will not do much good unless both the reader and the author have "good vibes" about the subject. I have also used the book for my topology seminar (for seniors) and came to see that the reading got a little rough toward the end of the book. This is all right too, since it simply shows that good vibes alone cannot conquer everything; however, I have modified some of those troublesome spots, filling in missing links and so on.

I am grateful to the translator, Kiki Hudson, for conveying my writing style and philosophy as faithfully as possible in her translation. We discussed all the changes verbally, and consequently she had to do more writing than translating. This is especially so with the Appendix. I would also like to thank Martin Guest for valuable suggestions, Yoshinori Machida for spotting numerous typos, and the AMS editors for presenting the book in splendid style.

Hajime Sato
September 1998

Objectives

As I stated in the Preface, in topology we investigate one aspect of geometrical objects almost exclusively of the others: that is, whether a given geometrical object is connected or not connected. We classify objects according to the nature of their connectedness. One focuses on the connectivity, ignoring changes caused by stretching or shrinking.

One can measure the length of a geometrical object in meters and the weight in kilograms. How do we measure the extent to which a geometrical object is connected ? Can we develop a system with suitable units and numbered scales?

For example, we can use the number of holes in a geometrical object. But then what is a hole and how do we count the number of holes? In this book, you will find a mathematical interpretation of these concepts, termed "homotopy groups", "homology groups", and "cohomology groups". These are some of the major concerns in algebraic topology. We actually go beyond counting the number of holes and develop "characteristic classes" to describe how a geometrical object bends globally. Intuitively the "i-th homotopy group" describes the "i-dimensional *round holes*" and "i-th homology group" reveals the number of "i-dimensional rooms" in a geometrical object.

In the problem described above, which may appear to be too slippery to grasp, it would be nice if the reader would come to understand and appreciate how contemporary mathematics has constructed the theory of algebraic topology, translating geometrical concepts into algebraic terms. It has managed to express these problems cleanly and algebraically in group-theoretical terms (involving almost only the additive group of integers or cyclic groups of integers modulo prime numbers). I want the reader to spend a few minutes before beginning the book imagining the problem of classifying geometrical objects only with a yardstick that measures their connectedness. Then after finishing the book the reader should compare its contents with this original concept. If the concept and reality are far apart you will have opened a door to a brave new world, and if they are rather close your mathematical intuition will have proved to be excellent (and you will continue to go on the right track with conviction).

If you already have any familiarity with algebraic topology, you might rightly guess from the table of contents that the following are the key words in the book:

homeomorphisms, homotopy equivalences, torus, Möbius strip, closed surfaces, Klein *bottle, cell complexes,* fundamental *groups, homotopy groups, homology groups, cohomology groups, fiber bundles, vector bundles, spectral* sequences, *characteristic classes, etc.*

If you have seen some (or all) of these words somewhere before and they have vaguely interested you, then you will find upon finishing the book that they are not difficult at all but that they form some of the basic concepts in contemporary mathematics. If you have had nothing to do with them so far, I hope that the strange sound they make intrigues you enough to start the book.

Topology has developed (perhaps unintentionally) on the strength of several attractive geometrical figures which serve as characteristic examples for the theory. This pattern may not be unique in topology; we may see it repeated in other branches of mathematics and possibly in every other academic discipline.

I emphasize again that the purpose of this book is to familiarize the reader with the way to think about algebraic topology. I use the axiomatic approach to introduce homology and cohomology theories, and will later construct concrete examples such as simplicial homology groups, as I feel that this order might work better to sharpen the reader's intuitive understanding.

Needless to say, algebraic topology evolved from general topology (the theory of topological spaces). If you have already studied general topology (especially its geometrical aspects), for instance if you have read Chapters from I to XI in *Topology* of James Dugundji[2], you will be ideally prepared; however, I have tried to keep my explanation basically intuitive so that even readers with no previous knowledge of general topology will be able to follow the book.

The reader might feel a need for the theory of groups, but essentially all you need in order to read this book is to understand the following two concepts:

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[2] Topology by James Dugundji, William C. Brown. 1989

(1) The addition or subtraction of two integers gives another integer (we say that the set $\mathbb{Z}$ of the integers is an additive group).

(2) In certain situations, we regard two integers which differ by a fixed prime number p to be equal (we say that we consider integers modp). We write $\mathbb{Z}_p$ for the set of the integers modp. The addition and subtraction of integers carry over to those operations modp (we say that modp is a cyclic group of order p).

The only talent this book demands of the reader is a flexible and resilient mind.

LIST OF SYMBOLS

Symbol	Meaning	Page
$f_0 \simeq f_1$	homotopic	3
$[X, Y]$	homotopy set	4
$X \simeq Y$	X and Y have the same homotopy type	4
D^n	n-dimensional ball	9
S^{n-1}	$(n-1)$-dimensional sphere	9
I	closed unit interval $[0, 1]$	10
$P^n(\mathbb{R})$	n-dimensional real projective plane	11
e^i	(open) i-cell	13
$\bar{e}^i$	closed i-cell	13
$\pi_n(X, x_0)$	n-th homotopy group of X	25
$\pi_n(X)$	n-th homotopy group of X	25
$h_p(X)$	pth homology group of X	31
$h_*(X)$	direct sum $\sum_{p=0}^{\infty} h_p(X)$ of $h_p(X)$	31
pt	singleton set	32
$H_p(X; G)$	$h_p(X)$ for $ho(X) \cong G$	32
$H_*(X; G)$	direct sum $\sum_{p=0}^{\infty}$ of $H_p(X; G)$	32
CA	cone over A	36
$\tilde{h}_*(X)$	reduced homology of group X	38
c	chain complex	45
$Z_p(C)$	group of p-cycles	45
$B_p(C)$	group of p-boundaries	45
$\sigma^j \prec \sigma^n$	simplex σ^j belongs to the boundary of σ^n (σ^j is a face of σ^n that is different from σ^n)	48
$C_q(\mathbb{S}; \mathbb{Z})$	q-th chain group of $\mathbb{S}$ over $\mathbb{Z}$	52
$H_q(\mathbb{S}; \mathbb{Z})$	q-th homology group of $\mathbb{S}$ over $\mathbb{Z}$	53
$P^n(\mathbb{C})$	n-dimensional complex projective space	56
$h^p(X)$	p-th cohomology group of X	59
s	simplicial complex	49

Symbol	Meaning	Page
$h^*(X)$	direct sum $\sum_{p=0}^{\infty} h^p(X)$ of $h^p(X)$	59
δ^p, δ	coboundary homomorphism	60
$C^q(\mathcal{S}; G)$	q-th cohomology chain of S over G	60
$C^*(\mathcal{S}; G)$	cochain complex of S over G	61
$H^q(\mathcal{S}; G)$	q-th cohomology group of S over G	61
$Z^q(\mathcal{S}; G)$	group of q-cochains of S over G	61
$B^q(\mathcal{S}; G)$	group of q-coboundaries of S over G	61
$G_1 \otimes G_2$	tensor product	
$\mathrm{Hom}(G_1, G_2)$	abelian group of homomorphisms from G_1 to G_2	66
$\mathrm{Tor}(G_1, G_2)$	torsion	66
$\mathrm{Ext}(G_1, G_2)$	abelian group of the extensions of G_2 by G_1	67
$\times$	cross product	68
$\triangle$	diagonal map	69
$\cup$	cup product	69
(E, π, B, F)	fiber bundle	74
$F \to E \xrightarrow{\pi} B$	fiber bundle	74
$\boldsymbol{E}$	total space	74
$\boldsymbol{B}$	base space	74
$\boldsymbol{F}$	fiber	74
π	projection	74
$G^{\mathbb{R}}(m, n)$	real Grassmannian manifold	80
$G^{\mathbb{C}}(m, n)$	complex Grassmannian manifold	80
$BO(n)$	classifying space of real n-vector bundles	83
$BU(n)$	classifying space of complex n-vector bundles	83
$\mathrm{Lk}(\sigma, \mathcal{S})$	link complex of σ in S	106

CHAPTER 1

Homeomorphisms and Homotopy Equivalences

Throughout this book a map means a *continuous map.*[1]

In topology we essentially discuss the connectedness of geometrical objects called topological spaces; however, strictly speaking, we consider topological spaces and two types of continuous maps between them, which are called "homeomorphisms" and "homotopy equivalences" respectively. We might classify topological spaces up to homeomorphism, or we might do so up to homotopy equivalence. Our choice depends on how strong we want our classification to be. The classification according to homotopy equivalences is weaker (there are many spaces not "homeomorphic" to each other that are of the same "homotopy type"), but it is the one that plays the more important role in algebraic topology, because geometrical properties of homotopy equivalences translate themselves most successfully into modern algebra.

The classification of the capital letters A, B, C, . . . , Z by homeomorphisms results in the following nine classes (this also depends on the choice of font, and here we use the sans-serif style; for example we write I and not I).

{A, R}, {B}, {C, G, I, J, L, M, N, S, U, V, W, Z},
{D, O}, {E, F, T, Y}, {H, K}, {P}, {Q}, {X}.

The letters in any one of these classes are homeomorphic but no two belonging to distinct classes are.

On the other hand, homotopy classification breaks the alphabet into three distinct classes according to their "homotopy types":

{A, R, D, O, P}, {B, Q},
{C, I, L, M, N, S, U, V, W, Z, F, J, T, Y, G, H, K, X}.

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[1]See the Appendix for the definition

Two letters have the same homotopy type if and only if they belong to the same class.

We count the number of holes in each letter in the set containing the letter A as one, that of each letter in the set containing B as two, and that of each letter in the last set as zero. Have the above simple examples led you to guess the definitions of homeomorphisms and homotopy equivalences?

1.1. Homeomorphisms

DEFINITION 1.1. We say that topological spaces X and Y are *homeomorphic* if there exist continuous maps $f : X \to Y$ and $g : Y \to X$ such that the composites $g \circ f$ and $f \circ g$ are the identity maps of X and Y respectively; in short, $g \circ f = id$ and $f \circ g = id$, where *id* denotes the identity map. In this case f is a *homeomorphism* from X to Y and g is a homeomorphism of Y to X.

The fact that $g \circ f$ is the identity map implies that f is an injection and g is a surjection. Similarly the fact that $f \circ g$ is the identity map implies that f is surjective and g is injective. Altogether it follows that both f and g are continuous bijective (1–1 onto) maps.

SAMPLE PROBLEM 1.2. Consider the letters M and N. Think of them as topological spaces and construct homeomorphisms $f : M \to N$ and $g : N \to M$.

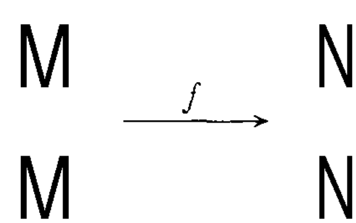


FIGURE 1.1

SOLUTION. Let f be a map which sends the left half A of the M onto the left vertical line plus the center diagonal A of the N without changing anything, while straightening the right half A of M and sending it onto the right vertical line I of N (see Figure 1.1). We want g to transfer the left vertical line and the center diagonal line of N onto the left half A of M, and to bend the right vertical line of N and map it onto the right half A of M. Then we get $g \circ f = id$ and $f \circ g = id$.

SAMPLE PROBLEM 1.3. Show that the topological spaces X and I are not homeomorphic.

SOLUTION. Suppose there existed a homeomorphism $f : X \to I$. For any point x_0 in X, the definition of a homeomorphism insures that the map $f|_{(X-x_0)}$ which is the restriction of f to the space X minus the point x_0 is a homeomorphism of $X - x_0$ onto $I - f(x_0)$. Take, in particular, the crossing point of X as x_0. Then $X - x_0$ consists of four disjoint line segments (each being half open, having one open end and one closed end), and $I - f(x_0)$ consists of two disjoint line segments (each of which is half open). These two spaces are not homeomorphic.

The basic stance of topology is to *regard all spaces homeomorphic to each other as identical.*

1.2. Homotopy equivalences

In order to define homotopy equivalences we must first say when two maps are homotopic.

DEFINITION 1.4. Two maps from a topological space X to a topological space Y,

$$f_i : X \to Y \qquad (i = 0, 1),$$

are *homotopic* if there exists a family of continuous maps

$$f_t : X \to Y \qquad (t \in [0, 1]),$$

varying continuously from f_0 to f_1. We indicate this situation by $f_0 \simeq f_1$ and say that f_t $(t \in [0, 1])$ is a *homotopy* between them.

EXAMPLE 1.5. We consider two maps f_0 and f_1 from the letter X to the letter Y: f_0 sends every point of X to the crossing point of Y, and f_1 maps the upper vee V of X onto the upper vee V of Y and the lower wedge Λ of X onto the lower vertical I of Y by closing Λ like a tweezer. Then f_0 and f_1 are homotopic because we can define f_t, $t \in [0, 1]$, to be the map sending each point x of X to the point obtained by shrinking $f_1(x)$ by t from the center crossing of Y.

EXAMPLE 1.6. Take the letter 0. Let f_0 be the map of 0 into itself which sends every point to the apex of the 0 and let f_1 be the identity map. Then f_0 and f_1 are not homotopic.

This fact is intuitively obvious (we can never change the identity map of 0 to a constant map through continuous maps: we cannot shrink the letter 0 to a point without breaking it). A precise proof, however, depends on homology theory, and we will see it in Example 4.9.

Suppose we look at the set S of the maps from a topological space X to a topological space Y. The following properties are easy to check.

1. A map $f : X \to Y$ is homotopic to itself.

2. If f is homotopic to $g : X \to Y$ then g is homotopic to f.

3. If f is homotopic to g and g is homotopic to $h : X \to Y$ then f is homotopic to h.

Therefore the relation of being homotopic is an equivalence relation on S that breaks S into equivalence classes called *homotopy* classes . We denote by

$$[X, Y]$$

the set of the homotopy classes of maps from X to Y, which we call the *homotopy* set of X to Y. In other words, we regard all homotopic maps from X to Y as identical and place them in the same homotopy class. Therefore, even if a homotopy class has a large number of maps, we need to look at only one of them. This is an algebraic simplification.

Example 1.7. Consider the letters X, Y and 0. We will discuss the following result in Chapter Three:

$$[\mathrm{X}, \mathrm{Y}] \cong \text{one point}, \quad [\mathrm{O}, \mathrm{O}] \cong \mathbb{Z} \text{ (the set of the integers)}.$$

Definition 1.8. Let X and Y be topological spaces. A map $f : X \to Y$ is a *homotopy* equivalence of X and Y if for some map $g : Y \to X$, the composites $g \circ f : X \to X$ and $f \circ g : Y \to Y$ are homotopic to the identity map of X and Y respectively.

We say that X and Y have the same *homotopy type* if there exists a homotopy equivalence between them.

In general a homotopy equivalence is neither injective nor surjective. We write $X \simeq Y$ when X and Y have the same homotopy type. We are using the same symbol for homotopic maps, but this should not cause any confusion here since both sides are topological spaces.

Problem. Show that the map $f_0 : X \to Y$ from the letter X to the letter Y in Example 1.5 is a homotopy equivalence (Hint: for a suitable $g : Y \to X$ construct a homotopy between $g \circ f_0$ and the identity map as well as a homotopy between $f_0 \circ g$ and the identity map).

From the definition we see that two topological spaces that are homeomorphic have the same homotopy type; therefore, homotopy equivalences are a looser (less strict) way of classifying topological spaces.

We have so far used only letters of the alphabet. These are one-dimensional geometrical objects (topological spaces) consisting of lines and curves; however, the definitions of homeomorphisms and homotopy equivalences carry over to geometrical objects of dimensions two or higher, including of course three-dimensional spaces.

EXAMPLE 1.9. A doughnut is homeomorphic to a coffee cup with a handle, and has the same homotopy type as the letter 0.

In later chapters we will study homology groups and cohomology groups (of topological spaces). They each offer the identical information for spaces of the same homotopy type. We will introduce other tools such as characteristic classes to determine if the given spaces are homeomorphic.

1.3. Topological pairs

In topology we frequently consider a pair of topological spaces (X, A) rather than a single space X. Passing from single spaces to pairs of spaces as objects of study was a great breakthrough in algebraic topology in the past.

By a *topological pair* (X, A) we mean a topological space X and a subspace A of X.

Given two pairs (X, A) and (Y, B), by a map of pairs $f : (X, A) \to (Y, B)$ we mean a map $f : X \to Y$ such that

$$f(A) \subset B.$$

The concept of homeomorphisms for topological pairs parallels the case for single spaces; namely, two pairs (X, A) and (Y, B) are homeomorphic if we can find maps $f : X \to Y$ and $g : Y \to X$ such that the composites $g \circ f : X \to X$ and $f \circ g : Y \to Y$ are the identity maps of X and Y respectively. The restrictions $f|_A : A \to B$ and $g|_B : B \to A$ are both homeomorphisms.

EXAMPLE 1.10. In the (x, y, z)-space $\mathbb{R}^3$, splice the ends of a string to make a simple loop A. Make a loop B in $\mathbb{R}^3$ by tying a knot in the string before splicing its ends (see Fig. 1.2).

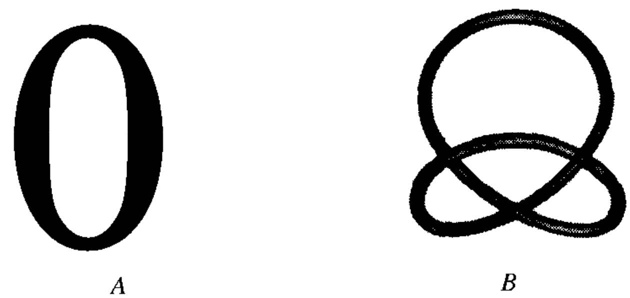


FIGURE 1.2. Knots

We can show that such pairs $(\mathbb{R}^3, A)$ and $(\mathbb{R}^3, B)$ are not homeomorphic to each other (later we will compute the "fundamental groups of the complements of A and B" using "homotopy theory").

We say that two continuous maps of pairs $f_i : (X, A) \to (Y, B)$, $i = 0, 1$, are homotopic if there exists a family of continuous maps of pairs

$$f_t : (X, A) \to (Y, B), \quad t \in [0, 1],$$

varying continuously from f_0 to f_1.

We partition the continuous maps from a pair (X, A) to another pair (Y, B) into homotopy classes; that is, we look at the set denoted by

$$[(X, A), (Y, B)]$$

in which each element is a homotopy class consisting of all homotopic maps from (X, A) to (Y, B). We say that $[(X, A), (Y, B)]$ is the homotopy set of maps from (X, A) to (Y, B). In particular, if $A = B = \emptyset$ we write X and Y in place of $(X, \emptyset)$ and $(Y, \emptyset)$. Then we have

$$[X, Y] = [(X, \emptyset), (Y, \emptyset)].$$

as the right-hand side of the equality is the homotopy set in which an element is a set of homotopic maps from X to Y.

We investigate detailed features of homotopy sets in Chapter Three.

Summary

1.1 A map from one topological space to another is a homeomorphism if it has an inverse map. Two topological spaces are homeomorphic if there exists a homeomorphism between them.

1.2 A map from one topological space to another topological space is a homotopy equivalence if it has an "inverse map" in the homotopy

sense. Two topological spaces have the same homotopy type if there exists a homotopy equivalence between them.
1.3 The same ideas carry over to homeomorphisms, homotopy equivalences and homotopy types for maps of topological pairs.

Exercises
1.1 Show that the letters W and Z are homeomorphic.
1.2 Show that the letters P and R have the same homotopy type.
1.3 The upper portion △ of the letter A is a subspace of A and the upper portion D of R is a subspace of R. Show that the pairs (A, △) and (R, D) are homeomorphic.

CHAPTER 2

Topological Spaces and Cell Complexes

There is a large selection of geometrical objects around us, ranging from basic ones such as line segments and disks to fuzzy ones whose boundaries are blurry. We must state precisely which geometrical objects are subjects of our investigation in this book. We must be able to determine if a geometrical object is connected or separated. In other words, we only consider those objects on which we can impose the concept of continuity, and we will call them topological spaces. There is a wide variety of topological spaces, among which the most basic are (solid) balls, also referred to as disks or cells. The boundary surface of a ball is a sphere. The dimensions of cells we study do not stop after one, two and three, but run up to n in general. We construct a topological space called a cell complex by splicing together finitely many cells of suitable dimensions. In this chapter we explain how to build various topological spaces and cell complexes. In the ensuing chapters we will deal with cell complexes only, unless otherwise stated.

2.1. Basic spaces

For a natural number $n \geq 1$, *we* define the *n-dimensional ball* (or n-ball) D^n by

$$D^n = \left\{ (x_1, x_2, \ldots, x_n) \in \mathbb{R}^2 | \sum_i x_i^2 \leq 1 \right\},$$

and the (n − 1)-dimensional *sphere* (or (n 1)-sphere) by

$$S^{n-1} = \left\{ (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n | \sum_i x_i^2 = 1 \right\}.$$

The O-sphere S^0 consists of two points $\{\pm 1\}$. We make the convention that the O-disk D^0 is a one-point space. The boundary ∂D^n of the n-disk D^n is the $(n-1)$-sphere S^{n-1}; that is,

$$\partial D^n = S^{n-1}.$$

The interior of a ball $D^n - \partial D^n$ is called a ball without boundary or an open ball.

Denote by I the closed interval [0, 1] between 0 and 1. Then I and D^1 are homeomorphic.

2.2. Product spaces and quotient spaces

For topological spaces X and Y the set of all ordered pairs (x, y) of points $x \in X$ and $y \in Y$, denoted by $X \times Y$, becomes a new topological space, which we call the *product space* (or the product) of X and Y.

EXAMPLE 2.1. A unit square

$$I \times I = \{ (x,y) | 0 \leq x \leq 1,\ 0 \leq y \leq 1 \}$$

is homeomorphic to D^2. We write I^2 for $I \times I$.

EXAMPLE 2.2. The product space $S^1 \times S^1$ is homeomorphic to the surface of a doughnut. We call this space the *torus* and denote it by T^2 (cf. Example 2. 15).

Consider a topological space X and an equivalence relation $\sim$ on X. We partition X into mutually disjoint subsets according to this relation; namely, elements x and y of X belong to the same subset if and only if $x \sim y$. These subsets are called equivalence classes. Denote by $\widetilde{X}$ the family of the equivalence classes of X under the equivalence relation $\sim$; then $\widetilde{X}$ is a new topological space in which each point is an equivalence class. We say that $\widetilde{X}$ is the quotient space of X (formed) under the equivalence relation $\sim$. Often we write $X/{\sim}$ for $\widetilde{X}$.

EXAMPLE 2.3. We make a circle by identifying the ends 0 and 1 of the interval I. This circle is homeomorphic to both S^1 and the letter 0. If we collapse the boundary $\partial D^n = S^{n-1}$ of D^n to one point, we get a topological space homeomorphic to S^n. Here the equivalence relation on D^n is $x \sim y \Leftrightarrow x, y \in \partial D^n$ (we regard the boundary points as a single point), and $\widetilde{D}^n = D^n/{\sim}$ is homeomorphic to S^n.

EXAMPLE 2.4. We identify the upper and lower edges of the square $I^2 = \{ (x,y) | 0 \leq x \leq 1,\ 0 \leq y \leq 1 \}$ by $(0,y) \sim (1,y)$, $0 \leq y \leq 1$, and the left and right edges by $(x,0) \sim (x,1)$, $0 \leq x \leq 1$.

In other words, we stitch the upper and lower edges together without twisting the square (we now have a cylinder) and then stitch the right and left edges (two circles) together without twisting the cylinder. Then the quotient space $\tilde{I}^2 = I^2/\sim$ is homeomorphic to the torus $T^2 = S^1 \times S^1$.

EXAMPLE 2.5. We splice the right and left edges of the square $I^2 = \{(x,y) \mid 0 \le x \le 1, 0 \le y \le 1\}$ by identifying the points on these edges symmetrically with respect to the point $(1/2, 1/2)$; that is, we regard $(0,y)$ and $(1, 1-y)$, $0 \le y \le 1$, as identical ($(0,y) \sim (1, 1-y)$, $0 \le y \le 1$). The resulting quotient space $\tilde{I}$ is the *Möbius* strip, which is well-known for its one-sidedness (see Figure 2.1).

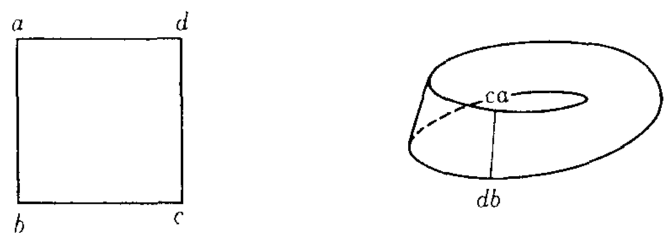


FIGURE 2.1. The Mobius strip

EXAMPLE 2.6. The quotient space $S^n/\sim$ of the n-dimensional sphere S^n with the identification of each point x of S^n with its antipodal point $-$x is the n-dimensional *real projective space* (here $x = (x_1, x_2, \ldots, x_{n+1})$, $-x = (-x_1, -x_2, \ldots, -x_{n+1}) \in S^n$). We denote this space by $P^n(\mathbb{R})$. When n = 2, in particular, $P^2(\mathbb{R})$ is the *real projective plane.*

The quotient space of D^2 under the equivalence relation identifying each pair of antipodal points on the boundary $\partial D^2 = S^1$ is homeomorphic to the real projective plane $P^2(\mathbb{R})$.

2.3. Topological sums and attaching spaces

Let X and Y be topological spaces with the intersection $A = X \cap Y$. Their union $X \cup Y$ is a topological space called the *topological* sum of X and Y, which we might also denote by $X \cup_A Y$. If $X \cap Y = \emptyset$, $X \cup Y$ is a disjoint union of X and Y (Figure 2.2).

Suppose now that $X \cap Y = \emptyset$ but that there is a homeomorphism $h : B \to A$ between some subsets $A \subset X$ and $B \subset Y$. We paste A onto B by h and pretend that $X \cap Y = A = B$ to construct the topological sum of X and Y. We indicate this sum by $X \cup_h Y$.

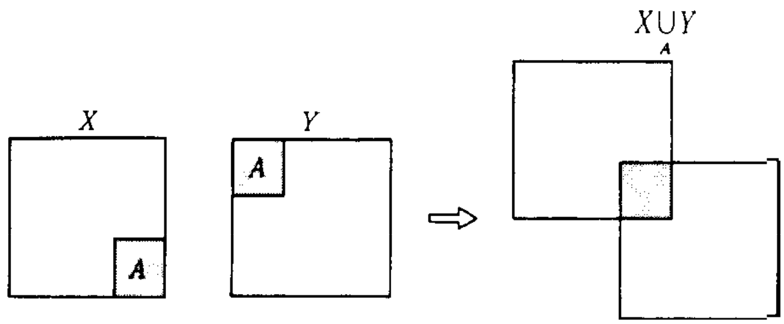


FIGURE 2.2. Topological sum

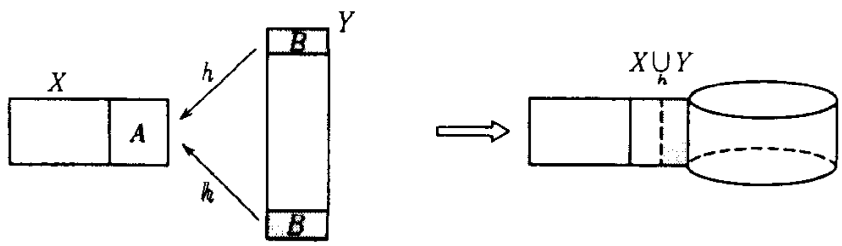


FIGURE 2.3. Attaching space

More generally, if we have a continuous map $h : A \to B$ where A c X and B c Y, we can still make a quotient space of the topological sum X ∪ Y by identifying each $b \in B$ with every a $\in A$ such that $h(u) = b$. This is the *attaching space* $X \cup_h Y$ of X and Y by the *attaching map* $h : B \to A$ (Figure 2.3).

EXAMPLE 2.7. Let X be the one-point space *pt*. Consider

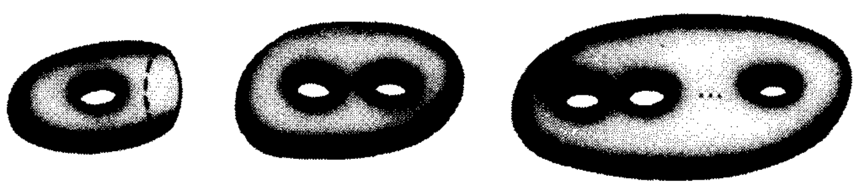
$$X = A = pt, \quad Y = D^2, \; B = \partial D^2 = S^1.$$

Then attaching map $h : B \to A$ "collapses" B to the point *pt,* and the resulting attaching space X $\cup_h$ Y is homeomorphic to the two-dimensional sphere S^2.

EXAMPLE 2.8. Let X = $A = S^1$, $Y = I \; x \; S^1$, $B = \{0\} \; x \; S^1$. Then both A and B are homeomorphic to a circle. Let $h : B \to A$ be the map that sends B twice around A, *i.e.,* $h(z) = z^2$, where we think of our circles as the set of complex numbers of modulus 1. The attaching space $X \cup_h Y$ is homeomorphic to the Mobius strip (cf. Exercise 2.2 for the proof).

EXAMPLE 2.9. $X = A = S^1$, $Y = D^2$, $B = \partial D^2 = S^1$. Here, we have the same X, A and B as above. Only Y is the new addition. If we use the above $h : B \to A$, the attaching space $X \cup_h Y$ is homeomorphic to the space that we get by attaching a two-dimensional ball to the Mobius strip along its boundary, which is in turn homeomorphic to the two-dimensional projective space $P^2(\mathbb{R})$.

EXAMPLE 2.10. We make a robot's glove T_0^2 by removing the interior of an embedded disk D^2 from the torus $S^1 \times S^1$. The boundary of T_0^2 is S^1. We join two T_0^2's along their boundaries to make a double torus M_2. We say that M_2 is a closed surface (more accurately, an orientable closed surface) of ***genus*** two. If we remove the interior of D^2 from M_2 and attach T_0 along the boundary $\partial D^2 = S^1$, we obtain a triple torus (a torus with three holes) M_3, and we continue this process to obtain an n-ple torus (a torus with n holes). In fact, every (orientable) closed surface is homeomorphic to an n-ple torus M_n for some n; we say that n is the genus of M_n. You can pretend that the n-ple torus is an inflated n-person life buoy (Figure 2.4).



T_0^2 M_2 M_n

FIGURE 2.4. Closed surfaces

2.4. Cell complexes

In this section we put together a finite number of spaces each homeomorphic to some open i-ball $D^i - \partial D^i = D^i - S^{i-1}$, $0 \leq i \leq n$, to form a topological space. It is customary to call each component (homeomorphic to an open i-ball) an i-cell and denote it by e^i (a cell in this definition has no boundary). A closed cell $\bar{e}^i$ is homeomorphic to the i-dimensional ball. The boundary, $\partial\bar{e}^i$, of $\bar{e}^i$ is homeomorphic to S^{i-1}. Thus we have that $\bar{e}^i - \partial\bar{e}^i = e^i$. We adopt the convention that $\bar{e}^0 = e^0$ is a singleton space and that $\partial\bar{e}^0 \cong S^{-1}$ is the empty set. Hence the closed 0-cell is also an open 0-cell (0-cell without boundary).

DEFINITION 2.11. We build an attaching space called a (finite) cell complex inductively according to the following recipe.

INGREDIENTS.

k_0 closed 0-cells $\bar{e}_1^0, \bar{e}_2^0, \dots, \bar{e}_{k_0}^0$,
k_1 closed 1-cells $\bar{e}_1^1, \bar{e}_2^1, \dots, \bar{e}_{k_1}^1$,
. . .
k_i closed i-cells $\bar{e}_1^i, \bar{e}_2^i, \dots, \bar{e}_{k_i}^i$,
. . . ,
k_n closed n-cells $\bar{e}_1^n, \bar{e}_2^n, \dots, \bar{e}_{k_n}^n$.

CONSTRUCTION. Our construction begins with X" = $\bar{e}_1^0 \cup \bar{e}_2^0 \cup \cdots \cup \bar{e}_{k_0}^0$, which is a disjoint sum. Set $X^{(1)} = \bar{e}_1^1 \cup \bar{e}_2^1 \cup \dots \cup \bar{e}_{k_1}^1$ (a disjoint sum) and $\partial X^{(1)} = \partial\bar{e}_1^1 \cup \partial\bar{e}_2^1 \cup \dots \cup \partial\bar{e}_{k_1}^1$. Specify an attaching map

$$h_1 : \partial X^{(1)} \to X^0,$$

and attach $X^{(1)}$ to X^0 by $h,$ and obtain the attaching space

$$X^1 \equiv X^0 \cup_{h_1} X^{(1)}.$$

Set $X^{(2)} = \bar{e}_1^2 \cup \bar{e}_2^2 \cup \cdots \cup \bar{e}_{k_2}^2$ and $\partial X^{(2)} = \partial\bar{e}_1^2 \cup \partial\bar{e}_2^2 \cup \dots \cup \partial\bar{e}_{k_2}^2$. Specify an attaching map

$$h_2 : \partial X^{(2)} \to X^1,$$

and attach $X^{(2)}$ to X^1 by h_2 to obtain the attaching space

$$X^2 \equiv X^1 \cup_{h_2} X^{(2)}.$$

We continue this process till we reach $X^{(n)}$. Set

$$X^n \equiv X^{n-1} \cup_{h_n} X^{(n)}.$$

We have now exhausted our ingredients. The final product X = X^n is an n-dimensional cell complex. For each q, $0 \le q \le n$, X^q is the *q-skeleton* of the cell complex X.

Let X be a cell complex. For each q-cell e_j^q, we have the natural inclusion map $i : \bar{e}_j^q \to X^{(q)}$, the natural identification map $\pi : X^{(q)} \to X^q = X^{q-1} \cup_{h_q} X^{(q)}$, and th e inclusion map $\iota : X^q \to X$. The composite of these maps

$$\phi_j^q \equiv \iota \circ \pi \circ i : \bar{e}_j^q \to X$$

is the *characteristic* map of the cell e_j^q.

The restriction of the characteristic map ϕ_j^q to the boundary $\partial\bar{e}_j^q$ agrees with the restriction of the attaching map $h_q : \partial X^{(q)} \to X^{q-1}$ to the boundary $\partial\bar{e}_j^q$.

REMARK. It may be that cells of some dimensions are missing in our construction, but the boundary $\partial X^{(q)}$ of $X^{(q)} = \bar{e}_1^q \cup \bar{e}_2^q . \cup \bar{e}_{i_{vq}}^q$ (a disjoint sum) must always be attached to some subcomplex X^r by an attaching map:

$$h_q : \partial X^{(q)} \to X^r, \qquad r < q.$$

The theorem below is an obvious consequence of the above definition.

THEOREM 2.12. *A cell complex X is a union of cells (without boundary); that is,*

$$X = \bigcup_{p,q} e_q^p.$$

EXAMPLE 2.13. The n-dimensional sphere S^n is a cell complex consisting of one O-cell $\bar{e}^0$ and one closed n-cell $\bar{e}^n$ with the attaching map

$$h_n : \partial X^{(n)} = \partial \bar{e}^n \to \bar{e}^0,$$

that is,

$$S^n = \bar{e}_0 \cup_{h_n} \bar{e}^n.$$

If we attach a closed n-cell $\bar{e}^n$ to $S^{n-1} = \bar{e}^0 \cup_{h_{n-1}} \bar{e}^{n-1}$ by the identity map

$$h_n : \partial X^{(n)} = S^{n-1} \to X^{n-1} = S^{n-1}$$

the resulting cell complex is an n-dimensional ball

$$D^n = \left(\bar{e}^0 \cup_{h_{n-1}} \bar{e}^{n-1}\right) \cup_{h_n} \bar{e}^n.$$

EXAMPLE 2.14. The real projective plane $P^2(\mathbb{R})$ consists of one closed O-cell $\bar{e}^0$, one closed l-cell $\bar{e}^1$ and one closed 2-cell $\bar{e}^2$:

$$P^2(\mathbb{R}) = \left(\bar{e}^0 \cup_{h_1} \bar{e}^1\right) \cup_{h_2} \bar{e}^2,$$

where $h_2 : \partial \bar{e}^2 \cong S^1 \to \left(\bar{e}^0 \cup_{h_1} \bar{e}^1\right) \cong S^1$ is the map which sends S^1 around S^1 twice.

EXAMPLE 2.15. The torus $T^2 \equiv S^1$ x S^1 consists of one O-cell $\bar{e}^0$, two closed l-cells, $\bar{e}_1^1$ and $\bar{e}_2^1$, and one closed 2-cell $\bar{e}^2$:

$$S^1 \times S^1 = \left(\bar{e}^0 \cup_{h_1} \left(\bar{e}_1^1 \cup \bar{e}_2^1\right)\right) \cup_{h_2} \bar{e}^2.$$

This structure suggests that if one opens up the torus along a suitable pair of circles intersecting at a single point, one will get a square (Figure 2.5).

FIGURE 2.5. Torus

SAMPLE PROBLEM 2.16. Show that the product $X \times Y$ of cell complexes X and Y is again a cell complex.

SOLUTION. For the cells e^i_j of the cell complex X and the cells e'^k_l of the cell complex Y we consider the cells $e^i_j \times e'^k_l$.

When X is a cell complex and $A \subset X$ is a subspace consisting of cells of X, we say that (X, A) is a pair of cell complexes. In this case A is also a cell complex: whose attaching maps are the restrictions of the attaching maps of X.

In the following chapters our topological space (a topological pair) will always be a cell complex (a pair of cell complexes) unless otherwise stated.

Summary

2.1 The basic building blocks of manifolds we use in this book are balls B^n and spheres S^{n-1} which are the boundaries of B^n.
2.2 Equivalence relations on topological spaces define quotient spaces.
2.3 We glue two spaces using an attaching map to make an attaching space.
2.4 A cell complex is a collection of a finite number of closed cells held together by characteristic maps.

Exercises

2.1 In Example 2.4, we defined the torus as a quotient space of $I^2 = \{(x, y) | 0 \le x \le 1, 0 \le y \le 1\}$. Show that the further identification of $(x, y) \in I^2$ with $(1 - x, 1 - y) \in I^2$ yields the two-dimensional sphere S^2.
2.2 Show that the attaching space $X \cup_h Y$ of Example 2.8 is homeomorphic to the Mobius strip.
2.3 Represent the double torus (torus with two holes) as a cell complex. What about the n-ple torus (torus with n holes)?

CHAPTER 3

Homotopy

Suppose that you are standing on a geometrical object (say on the Pampas). You toss a lasso and try to shrink it to a single point at your feet (the lasso must stay on dry ground). If there is no pool of water on this grassland the lasso will smoothly converge at your feet. Imagine, however, that there is a pool of water which your lasso is enclosing (We assume that the interior of the pool does not belong to our geometrical object). In this case you cannot bring the lasso to a single point without getting it wet. The fundamental group or the first homotopy group of the Pampas measures the degree of the possibility in shrinking the lasso to a point.

On the other hand the second homotopy group of a geometrical object measures the degree to which one can shrink a large piece of cloth, spread out with its border being gathered at a single point, to that point while keeping the cloth always in the object.

If the above explanation gives you some idea of what is going on, you might skip this chapter and proceed to the next one.

3.1. Homotopy sets

Recall that in Section 1.3 we denoted by

$$[(X, A), (Y, B)]$$

the set of homotopy classes of the continuous maps from a pair (X, A) to another pair (Y, B) (two maps are in a same homotopy class if they are pairwise homotopic).

For a natural number n, set $X = I^n$ and $A = \partial I^n$. Then the pair $(X, A) = (I^n, \partial I^n)$ is homeomorphic to (D^n, S^{n-1}).

By a *topological space X with a base point* x_0 (we might simply say a pointed topological space) we mean a pair whose second component is a singleton subspace $\{x_0\}$. We habitually drop the curly braces around x_0 and write (X, x_0).

A pointed topological space (X, x_0) and natural numbers n determine the homotopy sets

$$[(I^n, \partial I^n), (X, x_0)], \qquad n = 1, 2, \ldots .$$

These homotopy sets (each of which turns out to be a group; cf. §§ 3.2, 3.3) constitute one of major characteristics of the space X.

Pick a base point x_0 on the n-dimensional sphere S^n so that we have a pointed space $(S^n,\ x_0)$. We also get a pair homeomorphic to $(S^n,\ x_0)$ by squeezing the boundary ∂I^n of the n-ball I” to a point.

Since we can represent an element of [(I”, ∂I^n), (X, x_0)] by some map

$$f : (I^n, \partial I^n) \to (X, x_0),$$

where $f(\partial I^n) = x_0$, we get the following

Theorem 3.1. *There exists a natural one-to-one* correspondence

$$[(I^n, \partial I^n), (X, x_0)] \cong [(S^n, x_0), (X, x_0)] .$$

You will see that the set [(I”, ∂I^n), (X, x_0)], seemingly more complicated at the first glance, is easier to handle than [(S”, x_0), (X, x_0)] in the following section and beyond, when we give a group structure to these sets.

Example 3.2. Let X be the topological space representing the letter X and let x_0 be the crossing point of X. Then for each natural number n we have

$$[(I^n,\ \partial I^n), (X,\ x_0)] \cong \{\text{one point}\}.$$

In other words, an arbitrary map $f_1 : (I^n, \partial I^n) \to (X, x_0)$ is homotopic to the constant map

$$f_0 : (I^n, \partial I^n) \to (X, x_0), \quad f_0(x) = x_0, \quad x \in I^n.$$

To see this, consider the maps $f_t : (I^n, \partial I^n) \to (X, x_0)$, defined by

$$f_t(x) = t f_1(x), \quad x \in I^n$$

(each f_t multiplies the distance from x_0 to $f_1(x)$ by t). The family of maps f_t, $t \in [0, 1]$, gives a homotopy between f_0 and f_1.

Example 3.3. Denote by X the letter 0 = S^1 and by x_0 its apex. Then we have a one-to-one correspondence

$$[(I^1,\ \partial I^1), (S^1,\ x_0)] \cong \mathbb{Z};$$

since every element of $[(I^1, \partial I^1), (S^1, x_0)]$ has as a representative some map $f : (S^1, x_0) \to (S^1,\ x_0)$ with its rotation number (a negative integer if f changes the direction of the rotation) indicating how many times the map f makes S^1 wrap around S^1, and such maps with the same rotation number are homotopic.

Evidently taking a different base point x_1 in S^1 gives the correspondence

$$[(I^n, \partial I^n), (S^1, x_0)] \cong [(I^n, \partial I^n), (S^1, x_1)].$$

The same argument carries over to homotopy sets of topological spaces without base points, and we obtain the following results, which we saw in Example 1.7.

Proposition *3.4. For the letters X, Y and 0, we have*

$$[X, Y] \cong \{\text{one point}\}, \qquad [0, 0] \cong \mathbb{Z}.$$

More generally, we have the following theorem, whose simple proof will appear in §3.4.

Theorem *3.5. Suppose that X is a connected topological* space (note, if you are an advanced reader, that for a finite cell complex connectedness and arcwise connectedness are equivalent). *Then for any pair of points x_0 and x_1 of X and each natural* number n, we *have a one-to-one* correspondence

$$[(I^n, \partial I^n), (X, x_0)] \cong [(I^n, \partial I^n), (X, x_1)].$$

The reader might perhaps conclude from this theorem that base points are not necessary; however, in introducing a certain group structure to homotopy sets in the next section and beyond we will see that base points are very important indeed.

3.2. Fundamental groups

We give a group structure to the homotopy set [(I", ∂I^n), (X, xc)], where (X, x_0) is a pointed topological space ($3.3). We denote this group by $\pi_n(X, x_0)$ and say that $\pi_n(X, x_0)$ is the n-th homotopy group of the space (X, x_0). For a connected space X, the respective n-th homotopy groups of its pointed spaces with distinct base points are isomorphic, and so we often write $\pi_n(X)$. When X is a cell complex, π_1 (X, x_0) is finitely generated but not necessarily abelian; in particular, if $\pi_1(X, x_0) = \{1\}$, then for n $\geq$ 2, $\pi_n(X, x_0)$ is a finitely generated abelian group isomorphic to a direct sum of some copies of the infinite cyclic group $\mathbb{Z}$ and some copies of finite cyclic groups $\mathbb{Z}/(p_i)$. The first homotopy group $\pi_1(X, x_0)$ is especially important, and we have traditionally come to call it the *fundamental* group of (X, x_0).

In this section we study the case $n = 1$; fundamental groups.

We make the homotopy set $[(I, \partial I), (X, x_0)])$ determined by a pointed topological space (X, x_0) into a group. For two elements α_1 and α_2 in $[(I, \partial I), (X, x_0)]$ we define their "product"

$$\alpha_1 . \alpha_2 \in [(I, \partial I), (X, x_0)]$$

as follows: we select suitable maps

$$f_i : (I, \partial I) \to (X, x_0), \qquad i = 1, 2,$$

representing $\alpha_i, i = 1, 2$, respectively (since α_i is the homotopy class of f_i, this selection can be anything as long as it is homotopic to f_i). Divide the interval $I = [0, 1]$ into the subintervals $[0, 1/2]$ and $[1/2, 1]$ both of which are homeomorphic to $I = [0, 1]$. Since both f_1 and f_2 map ∂I to x_0, we can define a continuous map

$$f_1 \cup f_2 : (I, \partial I) \to (X, x_0)$$

by $f_1 \cup f_2 = f_1$ on $[0, 1/2]$ (identifying I with $[0, 1/2]$), and $f_1 \cup f_2 = f_2$ on $[1/2, 1]$ (identifying I with $[1/2, 1]$). Any change of f_1 or f_2 through a homotopy results only in a change of $f_1 \cup f_2$ by a homotopy; hence we may define $\alpha_1 . \alpha_2$ to be the homotopy class of $f_1 \cup f_2$.

The homotopy set $[(I, \partial I), (X, x_0)]$ with the above "product" satisfies the group axioms. We denote by $\pi_1(X, x_0)$ the resulting group, which we call the fundamental group (or the *first homotopy group)* of (X, x_0). When X is connected we often write $\pi_1(X)$.

We say that a topological space X is *simply connected* if X is connected and $\pi_1(X, x_0) \cong \{1\}$ (the unit element). You may perhaps be familiar with the fact that Cauchy's integration theorem holds for holomorphic functions defined on a simply connected domain in the complex plane.

Sample Problem 3.6. What sort of maps

$$f : (I, \partial I) \to (X, x_0)$$

represent the unit element of the group $\pi_1(X, x_0)$?

Solution. The constant map f_0 defined by $f(s) = x_0$, $s \in I$, represents the unit element of $\pi_1(X, x_0)$, the reason being that for any map $f : (I, \partial I) \to (X, x_0)$, $f_0 \cup f$ and $f \cup f_0$ are both homotopic to f. To see this, imagine the interval I as the time interval (so that $t \in I$ reads "at time t"). Then the map $f_0 \cup f$ stays still at x_0 for the first

30 seconds, say, and then hurries up to cover in the next 30 seconds what f would cover in a minute. If we decrease the interval of time during which $f_t \cup f$ remains still from 0 to $(1/2 \quad 1/2t)$ minute, then we have $f_1 = f$, and the family of maps f_t is a desired homotopy.

SAMPLE PROBLEM 3.7. Suppose that f : (I, $\partial I) \to (X, x_0)$ represents $\alpha \in \pi_1(X,\ x_0)$. Find a map that represents the inverse α^{-1}.

SOLUTION. Denote by σ the inversion of the interval I about its midpoint $1/2$, mapping 0 to 1 and 1 to 0. Then the composition $f \circ \sigma$: (I, $\partial I) \to$ (X, x_0) represents α^{-1} for the following reason: the maps $f \cup (f \circ \sigma)$ and $(f \circ \sigma) \cup f$ are both homotopic to the constant map f_0 with the homotopy $\{f_t\}$ which pulls the returning point back closer and closer to 0.

EXAMPLE 3.8. We saw in Example 1.7 that there is a bijection

$$\left[(I^1, \partial I^1),\ (S^1, x_0)\right] \cong \mathbb{Z}.$$

The rotation number of the product of two elements in $\pi_1\ (S^1)$ equals the sum of their respective rotation numbers, and hence we have a group isomorphism

$$\pi_1(S^1) \cong \mathbb{Z}.$$

EXAMPLE 3.9

$$\pi_1(D^n) \cong \{1\}, \quad n \geq 0, \qquad \pi_1(S^n) \cong \{1\}, \quad n \geq 2.$$

To show the first isomorphism, let f : (I", $\partial I^n) \to (D^n,\ *)$ represent a typical element of $\pi_1\ (D^n)$ and define f_t : (I", $\partial I^n) \to (D^n,\ *)$ by $f_t(x) = tf(x)$, $t \in [0, 1]$. Then the family of maps $\{f_t\}$, $t \in [0, 1]$, is a homotopy between f and the constant map taking the entire I^n to the center point 0 of D^n. For the second isomorphism, notice that if $n \geq 2$, then for an arbitrary map from $(I^n, \partial I^n)$ to (S^n, x_0) there is a point $x_1\ (\neq x_0)$ in S^n such that, by perturbing the image of the map by a small homotopy if necessary, we can make it avoid x_1. Since $S^n - \{x_1\}$ is homeomorphic to the interior of D^n, the proof reduces to the case for D".

EXAMPLE 3.10. We consider the figure eight: that is, two circles joined at a point, say x_0 (the choice of the intersection point is immaterial). Let $\alpha \in \pi_1$ (co) be the homotopy class of a map going once around the left circle only, and let $\beta \in \pi_1(\infty)$ be the homotopy class of a map going once around the right circle only (cf. Figure 3.1). In this situation we have

$$\pi_1 \text{ (co)} \cong \text{free group generated by } \alpha \text{ and } \beta;$$

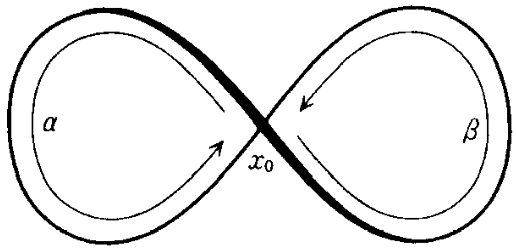


FIGURE 3.1

i. e., every element of $\pi_1(\infty)$ has some expression

$$\alpha^{a_1}\ \beta^{b_1}\ \alpha^{a_2}\beta^{b_2}\ \alpha^{a_k}\ \beta^{b_k}.$$

Note that $\alpha\beta \neq \beta\alpha$, and hence π_1 (co) is not abelian

EXAMPLE 3.11. We have the isomorphism

$$\pi_1(S^1 \text{ x } S^1) \cong \mathbb{Z} \oplus \mathbb{Z}.$$

The homotopy class α of a map of S^1 into S^1 x S^1, going around its first component S^1 once, generates the first component $\mathbb{Z}$, and the homotopy class β of a map of S^1, going around the second S^1, generates the second $\mathbb{Z}$. The following simple argument shows that $\alpha\beta = \beta\alpha$. The product S^1 x S^1 is homeomorphic to the torus (Example 2.15) that we constructed by gluing together the upper and lower edges of I^2 by $(x, 1) \sim (x, 0)$ as well as the right and left edges by $(0, y) \sim (1, y)$. Then a map of S^1 into itself going around the boundary of I^2 once represents $\alpha\beta\alpha^{-1}\beta^{-1}$, which is homotopic to the constant map that sends the entire S^1 to the center of I^2; hence $\alpha\beta\alpha^{-1}\beta^{-1}$ is the unit element of the group.

3.3. Higher homotopy groups

We adopt the same trick that we used for n = 1 to define a product structure on the homotopy set [(Γ', ∂I^n), (X, x_0)] for n $\geq$ 2; that is, we decompose the n-cube I^n *as*

$$\text{Γ'} = ([0, 1/2] \text{ u } [1/2, 1]) \text{ x } I^{-1} \cong (\text{I u I}) \text{ x } I^{n-1}.$$

Suppose two maps

$$f_i : (I^n, \partial I^n) \to (X, x_0), \quad i = 1, 2,$$

represent α_1 and $\alpha_2 \in [(I^n, \partial I^n)$, (X, x_0)] respectively. Then we define the product $\alpha_1\,\alpha_2$ to be the homotopy class of

$$f_1 \cup f_2 : (I^n, \partial I^n) \to (X, x_0).$$

We denote by $\pi_n(X, x_0)$ or simply $\pi_n(X)$ the homotopy set with this product, and say that $\pi_{(}X, x_0)$ is the n-th homotopy group of the topological space X. We show in the following theorem that the n-th homotopy group of X is abelian for n $\geq$ 2.

THEOREM 3.12. *For every* natural number n $\geq$ 2, $\pi_n(X)$ is an *abelian group; i.e.,* $\alpha_1 . \alpha_2 = \alpha_2 . \alpha_1$ for $\alpha_1, \alpha_2 \in [(I^n, \partial I^n), (X, x_0)]$.

PROOF. We prove the theorem for n = 2. The proof is exactly the same for the case $n > 2$.

We can write

$$\begin{aligned} I^2 &= ([0, 1/2] \text{ x } [1/2, 1]) \text{ x } I \\ &= ([0, 1/2] \cup [1/2, 1]) \times ([0, 1/2] \times [1/2, 1]). \end{aligned}$$

In matrix form, we have

$$I^2 = \begin{pmatrix} I_{11} & I_{12} \\ I_{21} & I_{22} \end{pmatrix},$$

where

$$\begin{aligned} I_{11} &= [0, 1/2] \times [0, 1/2], \qquad & I_{12} &= [0, 1/2] \times [1/2, 1] \\ I_{21} &= [1/2, 1] \times [0, 1/2], \qquad & I_{22} &= [1/2, 1] \times [1/2, 1], \end{aligned}$$

and we may define $\alpha_1 . \alpha_2$ by

$$f_1 \cup f_2 = \begin{cases} f_1, & \text{on } I_{11} \cup I_{21}; \\ f_2, & \text{on } I_{12} \cup I_{22}. \end{cases}$$

Through a suitable homotopy we can change f_1 and f_2 so that they map I_{21} and I_{12} respectively onto the point x_0. Denoting by $*$ the constant map taking I_{21} and I_{12} to x_0, we can write

$$f_1 \cup f_2 = (\mathrm{f1} \quad f_2) \simeq \begin{pmatrix} f_1 & * \\ * & f_2 \end{pmatrix}.$$

We think of I^2 as D^2 (since they are homeomorphic) and rotate D^2 through the angle πt. Then the composite

$$\begin{pmatrix} \cos\frac{\pi t}{2} & \sin\frac{\pi t}{2} \\ -\sin\frac{\pi t}{2} & \cos\frac{\pi t}{2} \end{pmatrix} \circ \begin{pmatrix} f_1 & * \\ * & f_2 \end{pmatrix} \circ \begin{pmatrix} \cos\frac{\pi t}{2} & \sin\frac{\pi t}{2} \\ -\sin\frac{\pi t}{2} & \cos\frac{\pi t}{2} \end{pmatrix}^{-1}$$

changes from $\begin{pmatrix} f_1 & * \\ * & f_2 \end{pmatrix}$ to $\begin{pmatrix} f_2 & * \\ * & f_1 \end{pmatrix}$ continuously as t goes from 0 to 1. The last matrix represents a map homotopic to $f_2 \cup f_1$ and hence we conclude that $f_1 \cup f_2 \simeq f_2 \cup f_1$; that is,

$$\alpha_1 . \alpha_2 = \alpha_2 . \alpha_1 \in \pi_n(X, x_0). \qquad \square$$

If a cell complex X is simply connected, then for $n \geq 2$, $\pi_n(X, x_0)$ is a finitely generated abelian group, and so it is isomorphic to a direct product of finitely many copies of the infinite cyclic group $\mathbb{Z}$ with finitely many copies of some finite cyclic groups $\mathbb{Z}/(p_i)$.

Here are several examples.

Example 3.13.

$$\pi_n(D^k) = 0, \quad k \geq 0; \qquad \pi_n(S^1) = 0, \quad n \geq 2;$$
$$\pi_n(S^n) \cong \mathbb{Z}, \quad \pi_n(S^k) = 0, \quad n < k;$$
$$\pi_n(\text{letters A, B,. . , Z}) = 0, \quad n \geq 2.$$

In order to show that $\pi_n(S^1) = 0$ $(n \geq 2)$, for instance, we regard S^1 as the quotient space of $\mathbb{R}$ under the identification of points that differ by integers. Then we can show that $\pi_n(S^1) \cong \pi_n(\mathbb{R})$, and hence our claim.

Example 3.14.

$$\pi_n(S^1 \text{ x } S^1) = 0, \quad n \geq 2.$$

We identify $S^1 \times S^1$ with the quotient space of $\mathbb{R}^2$ under the identification of points which differ componentwise by integers. Then we can show that $\pi_n(S^1 \text{ x } S^1) \cong \pi_n(\mathbb{R})$, $n \geq 2$, and the conclusion follows.

Example 3.15.

$$\pi_3(S^2) \cong \mathbb{Z}.$$

The Hopf map $f : S^3 \to S^2$ representing the generator $1 \in \mathbb{Z}$ will come aboard as the projection of some fiber bundle in Chapter Eight.

3.4. Homotopy invariance

Suppose we have a map

$$f : (X, x_0) \to (Y, y_0)$$

between two pairs (X, x_0) and (Y, ye). Then f determines naturally an induced map

$$f_* : [(I^n, \partial I^n), (X, x_0)] \to [(I^n, \partial I^n), (Y, y_0)],$$

that sends the homotopy class generated by g : $(I^n, \partial I^n) \to$ (X, x_0) to the homotopy class of f o g : $(I^n, \partial I^n) \to$ (Y, ye). It is easy to see from the definition that

$$f_* : \pi_n(X, x_0) \to \pi_n(Y, y_0),$$

is a group homomorphism.

The following theorem will become evident as we recall the definition of homotopy equivalences (cf. $1.3).

THEOREM 3.16 (homotopy invariance). *If two* pairs (X, x_0) and (Y, y_0) *have the same* homotopy *type, then for* each n we *have a group isomorphism*

$$\pi_n(X, x_0) \cong \pi_n(Y, y_0).$$

When a connected topological space X enjoys a very nice geometrical property called homogeneity (for instance, X is a manifold), there is a homeomorphism $h : (X, x_0) \to (X, x_1)$ for any pair of points x_0 and x_1 of X. This implies that there exists a homotopy equivalence between (X, x_0) and (X, x_1), and so we have group isomorphisms

$$\pi_n(X, x_0) \cong \pi_n(X, x_1)$$

for all n.

Actually, for a connected cell complex X, not necessarily homogeneous, the pointed spaces (X, x_0) and (X, x_1) have the same homotopy type for any pair of points x_0, x_1, and hence we have an isomorphism

$$\pi_n(X, x_0) \cong \pi_n(X, x_1)$$

for every n. The proof of homotopy type is not straightforward, however, and so we will directly establish isomorphisms of homotopy groups in the following

THEOREM 3.17. *If X is a connected* topological *space, then for* any *two* points x_1 *and* x_1 *we have a group* isomorphism

$$\pi_n(X, x_0) \cong \pi_n(X, x_1)$$

for *each natural number n.*

PROOF. Consider a continuous curve $x_t \in X$, $t \in [0, 1]$, which connects x_0 and x_1. We define a family of maps from I" to X taking the boundary of *I"* along this curve, which naturally defines a family of homomorphisms

$$ht : \pi_n(X, x_0) \to \pi_n(X, x_t),$$

where $t \in [0, 1]$. Similarly we get homomorphisms

$$\hat{h}_t : \pi_n(X, x_t) \to \pi_n(X, x_0),$$

so that we have

$$h_1 \, o \, h_1 = id, \quad h_1 \, o \, h_1 = id.$$

□

Summary

3.1 The set of homotopy classes of the maps from the pair consisting of the unit interval and its boundary to a pointed space has a (natural) group structure, and it is called the fundamental group of this pointed space.

3.2 We say that a topological space is simply connected if its fundamental group is trivial.

3.3 The set of homotopy classes of the maps from the topological pair of I" and its boundary to a pointed topological space becomes an abelian group for n $\geq$ 2, and we call it the n-th homotopy group of the pointed space.

Exercises

3.1 Show that $\pi_n(S^k) = 0$, n $< k$.

3.2 Determine the fundamental group $\pi_1(P^2(\mathbb{R}))$ of the real projective plane $P^2(\mathbb{R})$.

3.3 Compute the fundamental group $\pi_1(M_2)$ of the two-holed torus M_2.

CHAPTER 4

Homology

We are now ready to discuss homology. We start with the axiomatic treatment of homology. Our approach in this chapter will be neat and tidy for the reader who is inclined towards abstraction. If you prefer a more concrete approach to the subject, you may simply skim through this chapter; however, while actually computing the homology groups of simplicial complexes in the next chapter, you might come to appreciate these axioms. In fact, the history of homology reveals how great mathematicians in the present and past struggled with computations before finally formulating suitable axioms to ease their work (or sharpen their results or whatever); therefore, rest assured that a complete homology theory that satisfies homology axioms does exist, although we will not give a direct proof.

4.1. Homology groups

A homology theory h,(X) = $\sum_{p=0}^{\infty} h_p(X)$ assigns to a topological space X a family of abelian groups $h_p(X)$, $p = 0, 1, 2, \ldots$, and hence their direct sum h,(X) = $\sum_{p=0}^{\infty} h_p(X)$ with the property that if X and X' have the same homotopy type (this is of course the case if they are homeomorphic) then

$$h_p(X) \cong h_p(X') \quad \text{(isomorphic as groups)},$$

for each p. The direct sum *h,(X)* is a *homology* of X, and each summand $h_p(X_p)$ is the pth homology group of X. Thus if for some p the p-th homology groups $h_p(X)$ and $h_p(X')$ are not isomorphic, then X and X' are not of the same homotopy type, and so we conclude that they are not homeomorphic. In this sense we might say that homology groups are basic topological invariants.

Homology theories are not unique. In particular, any abelian group G can appear as the 0-th homology group $h_0(pt)$ of a one-point space *pt*. When $h_0(pt)$ = G, we denote by $H_p(X; G)$ the p-th homology group $h_p(X)$ of X and by $H_*(X; G) = \sum_{p=0}^{\infty} H_p(X; G)$

the homology $h_*(X) = \sum_{p=0}^{\infty} h_p(X)$. We say that $H_*(X;G)$ is the homology of X over the coefficient group G (or simply over G). Conversely for an arbitrary abelian group G there exists a homology theory $h_* = \sum_{p=0}^{\infty} h_p$ with $h_0(pt) \cong G$.

What sort of abelian groups do we know? We have a famous theorem (the fundamental theorem of abelian groups) which says that a finitely generated abelian group is a direct sum of finitely many copies of the infinite cyclic group $\mathbb{Z}$ and finite cyclic groups $\mathbb{Z}_q$ of period q, where q is a prime number. There is a wide variety of abelian groups not finitely generated, among which are the real numbers $\mathbb{R}$, the rational numbers $\mathbb{Q}$, the complex numbers $\mathbb{C}$, and so on. Accordingly, we can conceive homologies such as $H_*(X;\mathbb{Z})$, $H_*(X;\mathbb{Z}_q)$, $H_*(X;\mathbb{R})$, *etc.* In a later chapter we will show that for any abelian group G, we can calculate $H_*(X;G)$ algebraically from $H_*(X;\mathbb{Z})$ (the universal coefficient theorem). Thus $H_*(X;\mathbb{Z})$ is the most basic homology here, but then in many cases $H_*(X;\mathbb{R})$ is easier to compute.

If G is abelian, the pth homology group $H_p(X;G)$ is also abelian. In this book we use only (finite) cell complexes, and this means that if G is finitely generated so is $H_p(X;G)$, and if G is $\mathbb{Z}$ then $H_p(X;\mathbb{Z})$ is a direct sum of a suitable number of copies of $\mathbb{Z}$ and $\mathbb{Z}_q$'s. The numbers of copies of the respective distinct summands dictate the topology of the space X. We mention in passing that the homology group $H_p(X;\mathbb{R})$ with an infinitely generated coefficient group $\mathbb{R}$ turns out to be the direct sum of a certain number of copies of $\mathbb{R}$.

One can indeed define a homology $h_*(X) = \sum_{p=0}^{\infty} h_p(X)$ in many different ways; however, for the following two reasons we will postpone giving a certain simple definition till later and start here by taking the basic properties of homology theories as axioms which hold no matter how we define h_*:

(a) Two homology theories with the identical $h_0(pt)$ agree on every $h_*(X)$ for any cell complex X.

(b) Computing homologies from the definition is difficult even if we define them first.

4.2. Homology axioms

Let X be a topological space. Recall that we defined a pair to be an ordered pair (X, A), where A is a subspace of X (cf. 1.3). If A is the empty set, we simply write X in place of $(X, \emptyset)$. When we have determined a homology theory for every pair (X, A) we will certainly have one for $(X, \emptyset) = X$, and besides we can understand how to define a homology of X more clearly when we include pairs, (X, A)'s, in our consideration.

Given two topological spaces A and B, we form pairs $(B, B \cap A)$ and $(A \cup B, A)$. The inclusion map $i : B \to (A \cup B)$ determines the map of pairs

$$i: (B, B \cap A) \to (A \cup B, A),$$

since $i(B \cap A) \subset A$.

A sequence of abelian groups and homomorphisms

$$\to G_{p+1} \xrightarrow{f_{p+1}} G_p \xrightarrow{f_p} G_{p-1} \xrightarrow{f_{p-1}} G_{p-2} \to \cdots$$

is *exact* if we have

$$\ker f_p = \operatorname{im} f_{p+1},$$

where $\ker f_p$ is the subset of G_p mapped to 0 by f_p and $\operatorname{im} f_{p+1}$ is the image of G_{p+1} by f_{p+1}.

In order to state the properties of homology groups inclusive of all conceivable coefficients groups, we denote by h_p the p-th homology group, where p is 0, 1, 2, ... By setting $h_p(X) = 0$ for $p < 0$, however, we may even assume that $p \in \mathbb{Z}$.

Once again recall that our pairs are always pairs of cell complexes.

AXIOMS 4.1 (HOMOLOGY AXIOMS). We say that $h_* = \sum_{p=0}^{\infty} h_p$ is a homology theory if h_* assigns to each topological pair (X, A) abelian groups $h_p(X, A)$, $p = 0, 1, 2 \ldots$, which satisfy the following properties.

(1) To an arbitrary continuous map $f : (X, A) \to (X', A')$, there corresponds for each p a homomorphism of abelian groups

$$f_* : h_p(X, A) \to h_p(X', A'),$$

satisfying the following three properties:

(a) The identity map $i : (X, A) \to (X, A)$ determines the identity homomorphism

$$id_* : h_p(X, A) \to h_p(X, A).$$

(b) If we have a second map $g : (X', A') \to (X'', A'')$ then the composition $(g \circ f)_*$ determined by $g \circ f : (X, A) \to (X'', A'')$ satisfies

$$(g \circ f)_* = g_* \circ f_* : h_p(X, A) \to h_p(X'', A'').$$

(c) HOMOTOPY AXIOM. If two maps f and f' of (X, A) into (X', A') are homotopic $(f \simeq f' : (X, A) \to (X', A'))$, then

$$f_* = f'_* : h_p(X, A) \to h_p(X', A')$$

(2) **Boundary Axiom.** To a pair (X, A) there corresponds a homomorphism, called the *boundary homomorphism (connecting homomorphism* or *differential),* for each p:

$$\partial_p : h_p(X, A) \to h_{p-1}(A)$$

(we often drop the subscript p and simply write ∂), such that for any continuous map $f : (X, A) \to (X', A')$ and for each p *we* have $\partial \circ f_* = (f|_A)_* \circ \partial$. In short, the following diagram commutes:

$$\begin{array}{ccc} h_p(X,A) & \xrightarrow{f_*} & h_p(X', A') \\ \downarrow a & & \downarrow a \\ h_{p-1}(A) & \xrightarrow{(f|_A)_*} & h_{p-1}(A'). \end{array}$$

(3) **Excision Axiom.** For each p, the inclusion map i : $(B, B \cap A) \to (A \cup B, A)$ induces an isomorphism

$$i_* : h_p(B, B \cap A) \to h_p(A \cup B, A).$$

(4) **Exactness Axiom.** For a topological pair (X, A) with the natural inclusion maps $i : A \to X$ and $j : X = (X, \emptyset) \to (X, A)$ there is an exact sequence

$$\cdots \to h_{p+1}(A) \xrightarrow{i_*} h_{p+1}(X) \xrightarrow{j_*} h_{p+1}(X, A)$$
$$\xrightarrow{\partial_{p+1}} h_{,}(A) \xrightarrow{i_*} h_p(X) \to \cdots.$$

(5) **Dimension Axiom.** For each $p \geq I$

$$h_p(pt) = 0,$$

where pt is a one-point space.

This completes Axiom 4.1.

We say that $G = h_0(pt)$ is the *coefficient* group G of the homology theory $h_,$.

Suppose that G is an abelian group. We can prove that for cell complex pairs (X, A) there is a unique homology group over the coefficient group G, $h_*(X, A) = \sum_{p=0}^{\infty} h_p(X, A)$, satisfying the homology axioms. The computations of homologies of various spaces that you will read in the following sections will shed some light upon this fact.

When G is the coefficient group of h we write

$$h_p(X, A) = H_p(X, A; G),$$

$$h_*(X, A) = H_*(X, A; G) = \sum_{p=1}^{\infty} H_p(X, A; G).$$

Suppose h_* satisfies the axioms (1) through (4) but not (5); for instance, $h_p(pt) \neq 0$ for some $p \geq 1$, or $h_p(X, A) \neq 0$ for some $p < 0$. In this case we say that h_* is a generalized *homology theory*. We do not discuss this type of homology in this book; however, there are some important ones, such as K-theory (that comprise one of the main topics in the Iwanami series: Developments in Contemporary Mathematics).

EXAMPLE 4.2. For any X, we have that

$$h_p(X, X) = 0, \quad p = 0, 1, 2, \ldots,$$

for the following reason. By Axiom (4) the long exact sequence

$$\cdots \to h_{p+1}(X) \xrightarrow{i_*} h_{p+1}(X) \xrightarrow{j_*} h_{p+1}(X, X) \xrightarrow{\partial_{p+1}} h_p(X) \xrightarrow{i_*} h_p(X) \to \cdots$$

of the pair (X, X) is exact. The inclusion map $i : X \to X$ is the same as the identity map $id : X \to X$, and so $i_* = id : h_p(X) \to h_p(X)$. The exactness of the above sequence implies that $j_* : h_{p+1}(X) \to h_{p+1}(X, X)$ is a zero map and $\partial_{p+1} : h_{p+1}(X, X) \to h_p(X)$ is also a zero map. Hence we have $h_p(X, X) = 0$.

4.3. Immediate consequences of the axioms

(a) **HOMOTOPY INVARIANCE.** The homology axioms lead to the following theorem, known as the *homotopy* invariance of homology groups.

THEOREM 4.3. *If $f : (X, A) \to (X', A')$ is an homotopy equivalence, then for each natural number p, $f_* : H_p(X, A) \to H_p(X', A')$ is an isomorphism.*

PROOF. Since f is an homotopy equivalence, there exists a map $g : (X', A') \to (X, A)$ such that $g \circ f \cong id$ and $f \circ g \cong id$. Then from Axiom 4.3 (1) we get the equalities

$$g_* \circ f_* = id : H_p(X, A; G) \to H_p(X', A'; G),$$

$$f_* \circ g_* = id : H_p(X', A'; G) \to H_{(}X, A; G);$$

therefore, we have the isomorphisms f_* and g_* of abelian groups. □

(b) Homology of Quotient Spaces. Let X be a topological space and A a subspace of X. We define the quotient space X/A to be the space where the set A is regarded as a single point denoted by pt. When X is a cell complex the excision axiom (Axiom 4.1 (3)) gives the following result.

THEOREM 4.4. *Let* (X, A) *be a cell complex pair. For each natural number* p, *there exists an isomorphism*

$$h_p(X, A) \cong h_p(X/A, pt)$$

PROOF. Let $f : A \times \{0\} \to *$ be the constant map of the product space $A \times I$ onto a one-point space $*$. The attaching space

$$CA = * \cup_f (A \times I)$$

has the homotopy type of a one-point space. We say that CA is the cone over A, and we regard A as $A \subset CA$. We can extend the homotopy which collapses CA to a point to a homotopy of the sum $X \cup_A CA$ still collapsing CA to the point (this is possible because X is a topological pair of cell complexes); hence, we have a homotopy equivalence

$$(X \cup_A CA, CA) \simeq (X/A, pt).$$

The excision axiom implies that

$$h_p(X, X \cap CA) \cong h_p(X \cup_A CA, CA),$$

but we have $X \cap CA = A$, and so $(X, X \cap CA) = (X, A)$. The conclusion follows from homotopy invariance. □

(c) Reduced Homology Groups. The homology groups (unlike the homotopy groups) $h_*(X)$ and $h_*(X, x_0)$ of a topological space X and a pointed space (X, x_0) $(x_0 \in X)$, respectively, are different from each other.

THEOREM 4.5.

$$h_0(X) \cong h_0(X, x_0) \oplus h_0(x_0),$$
$$h_p(X) \cong h_p(X, x_0), \quad p > 0.$$

PROOF. Since $h_p(x_0) = 0$ for $p > 0$ and $h_0(x_0) = G$, the exact sequence

$$\cdots \to h_p(x_0) \xrightarrow{i_*} h_p(X) \xrightarrow{j_*} h_p(X, x_0) \xrightarrow{\partial_p} h_{p-1}(x_0) \to \cdots$$

yields $h_p(X) \cong h_p(X, x_0)$, $p \geq 2$. Furthermore, the constant map $f : X \to x_0$ induces the homomorphism $f_* : h_0(X) \to h_0(x_0)$, and we get

$$f_* \circ i_* = id : h_0(x_0) \to h_0(x_0),$$

as $f \circ i = id$: $x_0 \to x_0$; hence, i_* is injective and we have the short exact sequence

$$0 \to h_0(x_0) \xrightarrow{i_*} h_0(X) \xrightarrow{j_*} h_0(X, x_0) \to 0.$$

Because f_* is the left inverse of i_*, the middle of the above short exact sequence splits:

$$h_0(X) \cong h_0(X, x_0) \oplus h_0(x_0).$$

It follows from $h_p(x_0) = 0$, $p \geq 1$, that $h_1(X) \cong h_1(X, x_0)$. □

Set $\tilde{h}_p(X) \equiv h_p(X, x_0)$. The expression

$$h_*(X, x_0) = \sum_{p=0}^{\infty} h_p(X, x_0)$$

becomes

$$\tilde{h}_*(X) = \sum_{p=0}^{\infty} \tilde{h}_p(X).$$

We say that $\tilde{h}_*(X)$ is the reduced *homology* group of X. By the above theorem these groups are related by

$$\tilde{h}_0(X) \oplus h_0(x_0) \cong h_0(X), \qquad \tilde{h}_p(X) \cong h_p(X), \quad p > 0.$$

Example 4.6. The reduced homology group of a one-point space is always zero:

$$\tilde{h}_*(x_0) = 0;$$

that is, $\tilde{h}_p(x_0) = 0$ for every natural number p.

We may also define the reduced homology group $\tilde{h}_p(X)$ to be the kernel of the homomorphism $f_* : h_p(X) \to h_p(x_0)$ induced by the constant map $f : X \to x_0$.

We define the reduced homology group $\tilde{h}_*(X, A)$ of a pair (X, A) simply by

$$\tilde{h}_*(X, A) \equiv h_*(X, A).$$

For an arbitrary map $f : (X, A) \to (X', A')$ we may define the reduced homomorphism

$$f_* : \tilde{h}_p : (X, A) \to \tilde{h}_p(X', A')$$

to be the restriction of f_*, and the boundary homomorphism

$$\tilde{\partial} : \tilde{h}_p(X, A) \to \tilde{h}_{p-1}(A)$$

to be the restriction of the boundary homomorphism ∂ of the homology groups. If one examines the zero-dimensional situation carefully one can immediately establish the following.

PROPOSITION 4.7. *We have a long exact sequence of reduced homology groups:*

$$\cdots \to \tilde{h}_{p+1}(A) \xrightarrow{\tilde{i}_*} \tilde{h}_{p+1}(X) \xrightarrow{\tilde{j}_*} \tilde{h}_{p+1}(X, A) \xrightarrow{\partial_{p+1}} \tilde{h}_p(A) \xrightarrow{\tilde{i}_*} \tilde{h}_p(X) \to \cdots$$

(d) HOMOLOGY GROUPS OF SPHERES. Since the n-dimensional ball D^n has the homotopy type of a singleton set, we have for every natural number p

$$\tilde{h}_p(D^n) = 0;$$

in other words,

$$H_p(D^n; G) = 0, \quad p > 0, \qquad H_0(D^n; G) \cong G.$$

The n-sphere S^n is also easy to calculate.

PROPOSITION 4.8. *For any* $n \geq 0$, *we have*

$$\tilde{h}_p(S^n) \cong \begin{cases} G, & p = n, \\ 0, & p \neq n. \end{cases}$$

PROOF. We first compute the homology groups of $S^0 = \{x_0, x_1\}$. From the excision axiom (AXIOM 4.1 (3)) we get

$$h_p(x_1, \emptyset) \cong h_p(S^0, x_0), \quad p \geq 0.$$

The long exact sequence for the pair (S^0, x_0)

$$\cdots \longrightarrow h_p(x_0) \xrightarrow{i_*} h_p(S^0) \xrightarrow{j_*} h_p(S^0, x_0) \xrightarrow{\partial_p} h_{p-1}(x_0) \longrightarrow$$

together with $h_p(x_0) = 0$ $(p > 0)$ and $h_0(x_0) \cong G$ implies that

$$h_0(S^0) \cong G \oplus G, \qquad h_p(S^0) = 0, \quad p > 0,$$

or equivalently that

$$\tilde{h}_0(S^0) \cong G, \qquad \tilde{h}_p(S^0) = 0, \quad p > 0.$$

We now note that, since the quotient space $D^n/\partial D^n$ is homeomorphic to S^n,

$$\tilde{h}_p(S^n) = h_p(S^n, pt) \cong h_p(D^n, \partial D^n) \cong \tilde{h}_p(D^n, \partial D^n).$$

We also have the exact sequence

$$\cdots \to \tilde{h}_p(D^n) \xrightarrow{j_*} \tilde{h}_p(D^n, \partial D^n) \xrightarrow{\partial_p} \tilde{h}_{p-1}(S^{n-1}) \xrightarrow{i_*} \tilde{h}_{p-1}(D^n) \to \cdot$$

for the pair $(D^n, \partial D^n) = (D^n, S^{n-1})$. Hence we have that

$$\tilde{h}_p(S^n) \cong \tilde{h}_{p-1}(S^{n-1}),$$

and proceeding inductively with $p = 1, 2, \ldots,$ we get what we wanted. □

EXAMPLE 4.9. Let f_0 be the map of the letter 0 into itself which send every point of 0 to the apex x_0 of 0, and let f_1 be the identity map of 0. Then by AXIOM 4.1 (1), which says $id_* = id$, we have that

$$(f_1)_* = \text{id} : H_p(\text{O}; \text{G}) \to H_p(\text{O}; \text{G}).$$

On the other hand, let $i : x_0 \to 0$ be the natural embedding. Then as $f_0 = i \circ f_0$, by Axiom 4.1 (1) ($(g \circ f)_* = g_* \circ f_*$), the image of the map

$$(f_0)_* : H_p(\text{O}; G) \to H_p(\text{O}; G)$$

is contained in the image of the map

$$i_* : H_p(x_0; \text{G}) \to H_p(\text{O}; \text{G}).$$

By the dimension axiom we get

$$H_1(x_0; \text{G}) = 0.$$

Since 0 and S^1 are homeomorphic, we also have

$$H_1(\text{O}; \text{G}) \cong \text{G}.$$

Thus we have that

$$(f_0)_* \neq (f_1)_*; H_1(\text{O}; G) \to H_1(\text{O}; G),$$

and we conclude by AXIOM 4.1 (1) that f_0 and f_1 are not homotopic.

OBSERVATION. Let us review how we used the homology axioms to compute the homology groups of a general n-dimensional sphere. Our starting point was the homology group of a singleton space. The zero-dimensional sphere S^0 consists of two points x_0 and x_1, and we used the exact sequence of the pair (S^0, x_0) and the excision axiom for S^0 to determine its homology groups. The homotopy invariance enabled us to compute the homology groups of the n-ball D^n. Then, using the exact sequence of the pair (D^n, S^{n-1}), we computed the homology groups of $S^n = D^n/S^{n-1}$. This example suggests that exact sequences of suitable pairs would help us calculate the homology groups of a variety of new spaces.

(e) HOMOLOGY EXACT SEQUENCES OF TRIPLES. We can obtain homology exact sequences of topological triples using only the homology axioms, though we need some lengthy but fun diagram-chasing. Exact sequences of triples will be quite useful in various ways in the later chapters.

Consider three topological spaces X, A and B, $B \subset A \subset X$. By the homology axioms, we have the following exact sequences:

$$\cdots \to h_{p+1}(X) \xrightarrow{j_*} h_{p+1}(X, A) \xrightarrow{\partial_{p+1}} h_p(A) \xrightarrow{i_*} h_p(X) \to \cdots,$$

$$\cdots \to h_{p+1}(X) \xrightarrow{j_*} h_{p+1}(X, B) \xrightarrow{\partial'_{p+1}} h_p(B) \xrightarrow{i_*} h_p(X) \to \cdots,$$

$$\cdots \to h_{p+1}(A) \xrightarrow{j''_*} h_{p+1}(A, B) \xrightarrow{\partial''_{p+1}} h_p(B) \xrightarrow{i''_*} h_p(A) \to \cdots.$$

The natural inclusion maps

$$\bar{i}: (A, B) \to (X, B) \quad \text{and} \quad \bar{j}: (X, B) \to (X, A)$$

determine the group homomorphism

$$\bar{i}_*: h_*(A, B) \to h_*(X, B) \quad \text{and} \quad \bar{j}_*: h_*(X, B) \to h_*(X, A).$$

We define the boundary homomorphism

$$\bar{\partial}_{p+1}: h_{p+1}(X, A) \to h_p(A, B)$$

by

$$\bar{\partial}_{p+1} = j''_* \circ \partial_{p+1}: h_{p+1}(X, A) \xrightarrow{\partial_{p+1}} h_p(A) \xrightarrow{j''_*} h_p(A, B).$$

We then get the following exact sequence of of the triple (X, A, B).

THEOREM 4.10. *The long* sequence

$$\cdots \to h_{p+1}(X, B) \xrightarrow{\bar{j}_*} h_{p+1}(X, A) \xrightarrow{\bar{\partial}_{p+1}} h_p(A, B) \xrightarrow{\bar{i}_*} h_p(X, B) \to \cdots$$

is exact.

PROOF. Enjoy the fun game of a diagram chase applying the homology axioms repeatedly. Make the use of the diagram shown in Figure 4.1.

□

If you are one of those who find diagram-chasing too tedious to go through, you may instead adopt this theorem as an axiom.

REMARK. For more general spaces one can count singular homology, Čech homology, and so forth as homology theories which satisfy the homology axioms.

$$
\begin{array}{ccccccc}
h_{p+1}(A) & \xrightarrow{i_*} & h_{p+1}(X) & & & & \\
\downarrow{\scriptstyle j''_*} & & \downarrow{\scriptstyle j'_*} & & & & \\
h_{p+1}(A,B) & \xrightarrow{i_*} & h_{p+1}(X,B) & \xrightarrow{j_*} & h_{p+1}(X,A) & & \\
 & & \downarrow{\scriptstyle \partial'_{p+1}} & & \downarrow{\scriptstyle \partial_{p+1}} & & \\
 & & h_p(B) & \xrightarrow{i''_*} & h_p(A) & \xrightarrow{i_*} & h_p(X) \\
 & & & & \downarrow{\scriptstyle j''_*} & & \downarrow{\scriptstyle j'_*} \\
 & & & & h_p(A,B) & \xrightarrow{i_*} & h_p(X,B).
\end{array}
$$

FIGURE 4.1

Summary

4.1 There are five homology axioms.

4.2 For a fixed homology theory, the homology group of a pair is equal to the reduced homology of the corresponding quotient space.

4.3 One can compute the homology groups of spheres straight from the homology axioms.

4.4 The homology exact sequence of a triple will be useful later.

Exercises

4.1 MORE CHALLENGING. Let X, X_1, X_2 and A be topological spaces, where $X_i \subset X$ $(i = 1, 2)$, $X = X_1 \cup X_2$ and $A = X_1 \cap X_2$. Denote by $h_1 : A \to X_i$ and $m_i : X_i \to X$ the natural embeddings of A and X_i in X_1 and X respectively. Set $\phi = (h_{1*}, h_{2*})$ and $\psi = m_{1*} \quad m_{2*}$. Show that there exists a long exact sequence (called the *Mayer- Vietoris exact sequence)*:

$$\cdots \to h_q(A) \xrightarrow{\phi} h_q(X_1) \oplus h_q(X_2) \xrightarrow{\psi} h_q(X) \to h_{q-1}(A) \to \cdots$$

4.2 We identify a point on the p-dimensional sphere S^p with a point on the q-dimensional sphere S^q and denote by $S^p \vee S^q$ the resulting quotient space. Calculate the homology groups of $S^p \vee S^q$ using the Mayer-Vietoris exact sequence.

CHAPTER 5

Homology Groups of Cell Complexes

We are going to compute the homology groups (for homologies satisfying the axioms) of some concrete cell complexes. First we outline our plans using a lot of technical jargon. This will help you organize your strategy if you are half-way familiar with these words. If you have never heard of them, just skip this introductory gibberish. We will define everything in the coming sections.

We will compute the i-th homology group as the quotient group of a subgroup of i-cycles (i-th chains which get sent to the zero element by the boundary operator) by a subgroup of i-boundaries (the image of the i + 1-chains by the boundary operator). The boundary operator will become quite easy to see once you get acclimatized to your environment. We will favor a more theoretical approach in this portion of our discussion (though this may contradict our policy of introducing you to the theory with concrete examples), and we first explain how to compute the homology groups of simplicial complexes (that is, triangulated spaces), which correspond directly to the homology axioms.

In short, one can surmise the situation by saying that the calculation of the homology groups of simplicial complexes is clear in its direction but a bit complicated in the actual manipulation. The calculation of cell complexes requires that you know the behavior of the boundary operator, which is not immediate from the homology axioms, but becomes quite simple to handle once you get to know it. You decide which is more palatable to you.

Once you finish this chapter you will be able to calculate the homology groups of all sorts of topological spaces which you treat as cell complexes.

5.1. Calculation of homology groups of cell complexes

Let X be a k-dimensional cell complex. We denoted by X^q the q-skeleton of X ($2.4). Recall that for $n \leq k$, there is an exact sequence of the triple (X^n, X^{n-1}, X^{n-2}):

$$\to h_n(X^n, X^{n-2}) \xrightarrow{j_*} h_n(X^n, X^{n-1}) \xrightarrow{\partial_n} h_{n-1}(X^{n-1}, X^{n-2}) \xrightarrow{i_*} \cdots,$$

For $n = 0, 1, 2, \ldots$, we define abelian groups $\bar{C}_n(X)$ by

$$\bar{C}_n(X) \equiv h_n(X^n, X^{n-1}).$$

Then $\bar{C}_n(X) = 0$ for $n < 0$. For $n > k = \dim X$ we have $X^n = X^{n-1}$, and so $\bar{C}_n(X) = 0$. We regard $\bar{\partial}_n$ as the map

$$\bar{\partial}_n : \bar{C}_n(X) \to \bar{C}_{n-1}(X).$$

THEOREM 5.1. *Let* X be a cell complex. Then *the composite*

$$\bar{\partial}_{n-1} \circ \bar{\partial}_n : \bar{C}_n(X) \to \bar{C}_{n-2}(X)$$

satisfies $\bar{\partial}_{n-1} 0\, \bar{\partial}_n = 0$ *for* every *integer* $n \geq 0$.

PROOF. Recall that we defined $\bar{\partial}_n$ as $j_* \circ \partial_n$, and so we get

$$\bar{\partial}_{n-1} \circ \bar{\partial}_n = j_* \circ \partial_{n-1} \circ j_* \circ \partial_n = j_* \circ (\partial_{n-1} \circ j_*) \circ \partial_n = 0.$$

□

The next proposition computes $\bar{C}_n(X)$.

PROPOSITION 5.2. *Let X be a cell complex and let q_n be the number of n-cells in X. Let $h_0(pt) \cong G$. Then*

$$\bar{C}_n(X) \cong G^{q_n} = \overbrace{G \oplus \cdots \oplus G}^{q_n}.$$

PROOF. Choose a point x_0 on the n-sphere S^n. Take q_n copies of this sphere and let $S^n \vee \ldots \vee S^n$ be the quotient space, where q_n copies of the point x_0 are regarded as a single point. We call this quotient space a bouquet of S". It follows from the excision axiom that

$$h_n(X^n, X^{n-1}) \cong h_n\left(\bigcup \bar{e}_i^n, \bigcup \partial(\bar{e}_i^n)\right) \cong \tilde{h}_n(S^n \vee \cdots \vee S^n).$$

By induction starting with $q_n = 1$, we obtain the result

$$\tilde{h}_n(S^n \vee \cdots \vee S^n) \cong G^{q_n}.$$

□

Similarly we can show the following result.

PROPOSITION 5.3. *For an arbitrary cell complex X,*

$$h_p(X^n,\ X^{n-1}) = 0, \quad p \neq n.$$

We use Proposition 5.3 and the exact sequence of (X^n, X^{n-1}) to obtain

PROPOSITION 5.4. *For a cell complex X and the natural inclusion map* $i : X^n \to X$, *we* have *the following:*

$$h_q(X^n) = 0, \quad q > n,$$

$$i_* : h_q(X^n) \to h_q(X) \text{ is an isomorphism}, \quad q < n.$$

OBSERVATION. We use only the homology axioms to prove the last proposition, but it nicely exhibits one basic property of the homology groups of a cell complex; that is, for the q-th homology group we need only to consider the p-dimensional skeleton X^p, $p \geq q + 1$. The q-th homology group vanishes on any skeleton X^n if $n \leq q - 1$.

We shall now move to a general theory: homology groups of chain complexes. Here we do not build axioms but compute algebraically.

DEFINITION 5.5. A *chain complex C* consists of a long sequence of abelian groups G_p and homomorphisms f_p

$$\cdots \to G_{p+1} \xrightarrow{f_{p+1}} G_p \xrightarrow{f_p} G_{p-1} \xrightarrow{f_{p-1}} G_{p-2} \cdots.$$

such that $f_p \circ f_{p+1} = 0$ (i.e. ker $f_p \supset$ im f_{p+1}) for every p. The homomorphisms f_p are *boundary operators* or boundary homomorphisms. Note that this condition is a bit weaker than the exactness condition, ker f_p = im f_{p+1}.

SIDETRACK. The term *complex of a cell complex* refers to a combined object, whereas that of a chain complex refers to the existence of boundary operators which satisfy $f_p \circ f_{p+1} = 0$.

DEFINITION 5.6. Let C be a chain complex. We define the pth homology group $h_p(C)$ of C by

$$h_p(C) = \ker f_p / \operatorname{Im} f_{p+1}.$$

We can say that an exact sequence is a chain complex whose homology groups all vanish.

For a chain complex C, we define a subgroup $Z_p(C)$ ($\subset G_p$) of *p-cycles* to be the kernel of f_p, and a subgroup $B_p(C)$ ($\subset G_p$) of *p-boundaries* to be the image of G_{p+1} under f_{p+1}. With this notation we can express the p-th homology group of the chain complex C as

$$h_p(C) = Z_p(C)/B_p(C).$$

The boundary operators $\bar{\partial}$ of the long sequence C(X) of a cell complex X,

$$\cdots \to \bar{C}_{p+1}(X) \xrightarrow{\partial_{p+1}} \bar{C}_p(X) \xrightarrow{\bar{\partial}_p} \bar{C}_{p-1}(X) \xrightarrow{\partial_{p-1}} \bar{C}_{p-2}(X) \to \cdots$$

satisfy $\bar{\partial}_{p+1}$ 0 $\bar{\partial}_p = 0$. Hence C(X) is a chain complex and so we can define the homology groups $h_p(\bar{C}(X))$ of C(X).

We give the fundamental theorem on the homology groups of a cell complex as follows.

THEOREM *5.7. For each p, there exists* a natural *isomorphism*

$$h_p(X) \cong h_p(\bar{C}(X)),$$

where the left-hand side is the axiomatic homology of *the cell* complex *X which gives rise to the chain complex c(X). The right&hand side is the homology of $\bar{C}(X)$ computed algebraically.*

PROOF. We make a commutative diagram by arranging the exact sequence of a triple (X^{p+1}, X^p, X^{p-1}) horizontally and that of (X^p, X^{p-1}, X^{p-2}) vertically:

$$\begin{array}{ccccccc}
 & & 0 = h_p(X^{p-1}, X^{p-2}) & & & & \\
 & & \downarrow & & & & \\
h_{p+1}(X^{p+1}, X^p) & \xrightarrow{\partial_{p+1}} & h_p(X^p, X^{p-2}) & \xrightarrow{i_*} & h_p(X^{p+1}, X^{p-2}) & \longrightarrow & h_p(X^{p+1}, X^p) \\
\| & & \downarrow j_* & & & & \| \\
\bar{C}_{p+1}(X) & & \bar{C}_p(X) = h_p(X^p, X^{p-1}) & & & & 0 \\
 & & \downarrow \bar{\partial}_p & & & & \\
 & & \bar{C}_{p-1}(X) = h_{p-1}(X^{p-1}, X^{p-2}) & & & &
\end{array}$$

From the vertical sequence we get

$$\ker \bar{\partial}_p \cong \operatorname{im} j_* \cong h_p(X^p, X^{p-2}),$$

while the identity $\bar{\partial}_{p+1} = j_* \circ \partial_{p+1}$ and the fact that j_* is injective imply

$$\operatorname{im} \bar{\partial}_{p+1} \cong \operatorname{im} \partial_{p+1}.$$

Hence it follows that

$$h_p(\bar{C}(X)) \equiv \ker \bar{\partial}_p / \operatorname{im} \bar{\partial}_{p+1} \cong h_p(X^p, X^{p-2}) / \operatorname{im} \partial_* \\ \cong h_p(X^{p+1}, X^{p-2}).$$

But from Proposition 5.4 one gets

$$h_p(X^{p+1}, X^{p-2}) \cong h_p(X).$$

□

You might ask if you can actually calculate the homology groups of a cell complex from the right-hand side of the equality in the theorem. Surprisingly, the answer is YES. You might get the impression that you need to calculate some homology groups (so far unknown to you) in order to build the chain complex on the right-hand side. Recall, however, that each p–chain $\bar{C}_p(X)$ is isomorphic to the direct sum of the-number-of-the-cells many copies of G. How should we interpret the boundary operator $\bar{\partial} : \bar{C}_p(X) \to \bar{C}_{p-1}(X)$?

For simplicity we discuss the case G = $\mathbb{Z}$. Denote by $\langle e_\lambda \rangle$ the generator $1 \in \mathbb{Z}$ of $\bar{C}_p(X)$ corresponding to a p-cell e_λ. Then we can write

$$\bar{\partial}\langle e_\lambda \rangle = \sum_{e_\mu \in X^{(p-1)}} [e_\lambda, e_\mu]\langle e_\mu \rangle, \quad [e_\lambda, e_\mu] \in \mathbb{Z}. \tag{5.1}$$

We call $[e_\lambda, e_\mu]$ the incidence number of e_λ and e_μ. When we have calculated all incidence numbers we will have the homology groups of the cell complex. So let us investigate the incidence number $[e_\lambda, e_\mu]$. Let q_{p-1} be the number of $p-1$-cells of X. We use the attaching map $h_\lambda : \partial\bar{e}_\lambda \to X^{p-1}$ of e_λ and the boundary isomorphism

$$\partial : H_p(\bar{e}_\lambda, \partial\bar{e}_\lambda; \mathbb{Z}) \cong \mathbb{Z} \to H_{p-1}(\partial\bar{e}_\lambda; \mathbb{Z}) \cong \mathbb{Z}$$

to define a homomorphism $\tilde{h}_{\lambda_*}$ of $H_p(\bar{e}_\lambda, \partial\bar{e}_\lambda; \mathbb{Z})$ by

$$\tilde{h}_{\lambda_*} \equiv h_{\lambda_*} \circ \partial : H_p(\bar{e}_\lambda, \partial\bar{e}_\lambda; \mathbb{Z}) \cong \mathbb{Z} \to H_{p-1}(X^{p-1}; \mathbb{Z})$$

Furthermore, the characteristic map of e_μ, $\phi_\mu : \bar{e}_\mu \to X^{p-1}$ induces the injective homomorphism

$$\phi_{\mu_*} : H_{p-1}(\bar{e}_\mu, \text{de,}; \mathbb{Z}) \cong \mathbb{Z} \to H_{p-1}(X^{p-1}, X^{p-2}; \mathbb{Z}) \cong \mathbb{Z}^{q_p - 1}.$$

We also have a well-defined homomorphism

$$(\phi_{\mu_*})^{-1} : H_{p-1}(X^{p-1}, X^{p-2}; \mathbb{Z}) \cong \mathbb{Z}^{q_p - 1} \to H_{p-1}(\bar{e}_\mu, \text{de,}; \mathbb{Z}) \cong \mathbb{Z}$$

such that $(\phi_{\mu_*})^{-1} \circ \phi_{\mu_*} = 1$ on $H_{p-1}(\bar{e}_\mu, \partial\bar{e}_\mu; \mathbb{Z})$.

Consider the induced homomorphism

$$j_*: H_{p-1}(X^{p-1};\mathbb{Z}) \to H_{p-1}(X^{p-1}, X^{p-2};\mathbb{Z})$$

of the inclusion $(X^{p-1}, \emptyset) \to (X^{p-1}, X^{p-2})$.

The following is an easy consequence of definitionchasing.

Theorem 5.8. *The incidence number $[e_\lambda, e_\mu] \in \mathbb{Z}$ of the homology* group *with $\mathbb{Z}$ coefficients of a cell complex X is well-defined by*

$$[e_\lambda, e_\mu] = \left((\phi_{\mu*})^{-1} \circ j_* \circ \tilde{h}_{\lambda*}\right)1) \in \mathbb{Z} \cong H_{p-1}(\bar{e}_\mu, \partial\bar{e}_\mu;\mathbb{Z}),$$
$$1 \in \mathbb{Z} \cong H_p(\bar{e}_\lambda, \partial\bar{e}_\lambda;\mathbb{Z}).$$

By *a regular cell complex* we mean a cell complex whose characteristic maps are homeomorphisms. Simplicial complexes are regular, and we will look into them in the following sections. If X is a regular cell complex, then we can determine directly from the homology axioms that

$$[e_\lambda, e_\mu] = 0 \text{ or } \pm 1.$$

Hence, the homology groups of a simplicial complex are determined uniquely from homology axioms. In the coming sections, we will give a direct definition of a homology of simplicial complexes which agrees with the homology determined by the homology axioms. Then we can calculate the incidence numbei tnd hence the boundary operator $\bar{\partial}$ given by (5.1) of an arbitrary cell complex.

5.2. Homology of simplicial complexes

(a) **Definition of Simplicial Complex.** Calculating the homology groups of a two-dimensional sphere is easy if we treat it as the surface of a regular cube. We will investigate the possibility of regarding more general spaces (including spaces of higher dimensions) as collections of "triangles". We first define a simplex as a higher-dimensional triangle.

Definition 5.9 **(simplex)**. Suppose N is a sufficiently large natural number. Let p_0, pi, . , p_n be n + 1 points of $\mathbb{R}^N$ in general position (i.e., the vectors $\overrightarrow{p_0p_1}$, . . . , $\overrightarrow{p_0p_n}$ are linearly independent), and denote by $\sigma^n = (p_0, \ldots, p_n)$ the smallest convex set spanned by these points. We say that σ^n is the *n-simplex,* and that p_0, . . . ,p_n are the vertices and n is the *dimension* of σ^n. The j-simplex spanned by a subset $\{p_{i_1}, \ldots, p_{i_j}\}$ of $\{p_0, \ldots, p_n\}$ is called a face of σ^n. In particular, σ^n is its own face. Denote by da" the collection of the faces of σ^n of dimension less than n. We say that *da"* is the *boundary* of σ^n. We indicate that σ^j is a simplex belonging to the boundary of σ^n by writing $\sigma^j \prec \sigma^n$.

EXAMPLE 5.10. In the (x, y, z)–space $\mathbb{R}^3$ the points $p_0 = (0, 0, 0)$, $p_1 = (1, 0, 0)$ and $p_2 = (1, 2, 0)$ span a two-simplex. Any set of three points in general position (i.e., not all on the same line) defines a two-simplex. Similarly, any set of four points not on the same plane determines a three-simplex.

DEFINITION 5.11 (SIMPLICIAL COMPLEX). Let $S = \{\sigma^n\}$ be a finite family of simplexes of various dimensions in $\mathbb{R}^N$, where N is large, satisfying the following conditions:

(i) If $\sigma^n \in S$, then every face of σ^n is also in S.

(ii) If $\sigma_1^m \sigma_2^n \in S$, then $\sigma_1^m \cap \sigma_2^n$ is a face of each of σ_1^m and σ_2^n.

We say that S is a simplicial complex. The highest dimension of the simplexes in S is the dimension of S. A zero-simplex in S is called a vertex of S.

In short, a simplicial complex S is a collection of simplexes spliced together along some faces such that any face of a simplex in S is again a simplex in S.

A simplicial complex is a regular cell complex; that is, its characteristic maps are homeomorphisms.

EXAMPLE 5.12. Here is an example of a simplex σ^n whose boundary is a simplicial complex. Consider four points in $\mathbb{R}^3$:

$$p_0 = (0,0,0),\ p_1 = (1,0,0),\ p_2 = (1,2,0),\ p_3 = (2,3,4),$$

four two-simplexes:

$$\sigma_1^2 = (p_0, p_1, p_2), \quad \sigma_2^2 = (p_1, p_2, p_3),$$
$$\sigma_3^2 = (p_0, p_2, p_3), \quad \sigma_4^2 = (p_0, p_1, p_3),$$

six one-simplexes:

$$\sigma_1^1 = (p_0, p_1),\ \sigma_2^1 = (p_0, p_2),\ \sigma_3^1 = (p_0, p_3),$$
$$\sigma_4^1 = (p_1, p_2),\ \sigma_5^1 = (p_1, p_3),\ \sigma_6^1 = (p_2, p_3),$$

and four zero-simplexes:

$$\sigma_1^0 = (p_0),\ \sigma_2^0 = (p_1),\ \sigma_3^0 = (p_2),\ \sigma_4^0 = (p_3).$$

The set S consisting of these fourteen simplexes satisfy the conditions (i) and (ii), and so it is a two-dimensional simplicial complex which is the boundary of the three-dimensional simplex spanned by the p_i's (Figure 5.1).

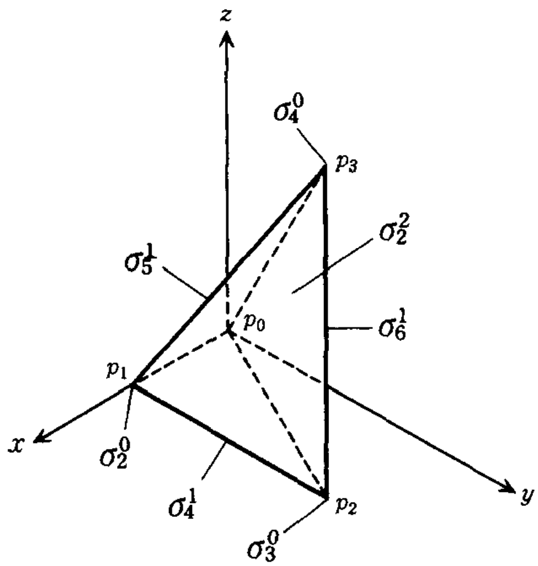


FIGURE 5.1. Simplicial complex

EXAMPLE 5.13. In the (x, y, z)–space $\mathbb{R}^3$, add $p_3' = (-2, -3, -4)$ to the points in Example 5.12. Then the boundary S of the simplex spanned by four points $p_0 = (0,0,0)$, $p_1 = (1,0,0)$, $p_2 = (1,2,0)$ and $p_4' = (-2, -3, -4)$ consisting of fourteen simplexes is also a simplicial complex. The union

$$\mathcal{S} \cup \mathcal{S}'$$

is another simplicial complex of dimension two which consists of twenty-one simplexes (the number of simplexes in $\mathcal{S} \cap \mathcal{S}'$ being seven).

DEFINITION 5.14. Let S be a simplicial complex and denote by $|\mathcal{S}|$ the subset of $\mathbb{R}^N$ consisting of the points belonging to simplexes of S. Then $|\mathcal{S}|$ has the relative topology as a subset of $\mathbb{R}^N$. Since we can regard each simplex as a cell, $|\mathcal{S}|$ is a cell complex.

EXAMPLE 5.15. Look at S, S', and $\mathcal{S} \cup \mathcal{S}'$ of Example 5.12 and Example 5.13. The spaces $|\mathcal{S}|$ and $|\mathcal{S}'|$ are both homeomorphic to the two-dimensional sphere S^2, while $|\mathcal{S} \cup \mathcal{S}'|$ is homeomorphic to the space which one obtains from the sphere S^2 by putting the divider

$$\{(0, x_2, x_3) \mid x_2^2 + x_3^2 \leq 1\} \subset S^2$$

in it. The space $|\mathcal{S} \cup \mathcal{S}'|$ has the homotopy type of $S^2 \vee S^2$, which is the quotient space of two copies of S^2 attached at a single point.

(b) HOMOLOGY OF SIMPLICIAL COMPLEXES.

ORIENTATIONS OF SIMPLEXES. If we reorder the vertices in a simplex $\sigma^n = (p_0, \ldots, p_n)$, we still have the same simplex. A specific order of the vertices defines an *orientation* of the simplex σ^n, and we say that we have oriented σ^n. Those orientations which can reach each other through an even number of permutations are regarded as equal, and those differing by an odd number of permutations are regarded as unequal, so that there are essentially two distinct orientations.

When we orient σ^n by $p_0, \ldots, p_n$ in this order we write $\langle \sigma^n \rangle =$ (pa,. . . , $p_n\rangle$. If we change the order through an odd permutation we add the minus sign; for example,

$$\langle p_0, p_1, p_2, \ldots, p_n \rangle = -\langle p_0, p_1, p_2, p_2, \ldots, p_n \rangle.$$

EXAMPLE 5.16. We can orient the one-simplex with vertices p_0 and p_1 by:

$$\langle p_0, p_1 \rangle = -\langle p_1, p_0 \rangle,$$

which we show by an arrow as in Figure 5.2.

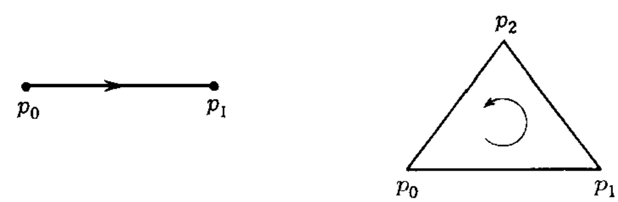


FIGURE 5.2. Oriented 1-simplex and 2-simplex

EXAMPLE 5.17. We orient the two-simplex spanned by vertices p_0, p_1 and p_2 by

$$\begin{aligned}(5.2)\qquad \langle p_0, p_1, p_2 \rangle &= \langle p_1, p_2, p_0 \rangle = \langle p_2, p_0, p_1 \rangle \\ &= -\langle p_0, p_2, p_1 \rangle = -\langle p_1, p_0, p_2 \rangle = -\langle p_2, p_1, p_0 \rangle,\end{aligned}$$

which we can indicate by an arrow as in Figure 5.2.

INTEGRAL HOMOLOGY OF SIMPLICIAL COMPLEXES.

DEFINITION 5.18. Let S be a simplicial complex. Denote by k_q the number of q-simplexes in S. Then we say that the abelian group

$$C_q(\mathcal{S};\ \mathbb{Z}) = \mathbb{Z}^{k_q} = \overbrace{\mathbb{Z} \oplus \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}}^{k_q \text{ copies}}$$

is the *q-chain group* of S over the coefficient group $\mathbb{Z}$; when $q < 0$ or $q > \dim \mathcal{S}$, we set $C_q(\mathcal{S}; \mathbb{Z}) = 0$.

We now construct a chain complex by defining a boundary operator $\partial_q : C_q(\mathcal{S}; \mathbb{Z}) \to C_{q-1}(\mathcal{S}; \mathbb{Z})$.

DEFINITION 5.19. Orient each simplex in S in any way you wish. Label the q-simplexes by 1 through k_q, $\sigma_1^q, \sigma_2^q, \ldots, \sigma_{k_4}^q$. Then form a formal set:

$$A_q(\mathcal{S};\mathbb{Z}) = \left\{ a_1\langle\sigma_1^q\rangle + a_2\langle\sigma_2^q\rangle + \cdots + a_{k_q}\langle\sigma_{k_q}^q\rangle \mid a_i \in \mathbb{Z} \right\}.$$

We make this set into an abelian group isomorphic to the q-chain group $C_q(\mathcal{S};\mathbb{Z}) = \mathbb{Z}^{k_q} = \mathbb{Z}\oplus\mathbb{Z}\oplus\cdots\oplus\mathbb{Z}$ by

$$\left(a_1\langle\sigma_1^q\rangle + \cdots + a_{k_q}\langle\sigma_{k_q}^q\rangle\right) + \left(b_1\langle\sigma_1^q\rangle + \cdots + b_{k_q}\langle\sigma_{k_q}^q\rangle\right)$$
$$= (a_1+b_1)\langle\sigma_1^q\rangle + \cdots + (a_{k_q}+b_{k_q})\langle\sigma_{k_4}^q\rangle$$

(technically we say that $A_q(\mathcal{S};\mathbb{Z})$ is the free abelian group generated by $\langle\sigma_1^q\rangle, \ldots, \langle\sigma_{k_q}^q\rangle$).

If $\langle\sigma_i^q\rangle = \langle p_0, \ldots, p_q\rangle$ we define $\partial_q(a_i\langle\sigma_i^q\rangle = \sigma_q(a_i\langle p_0, \ldots, p_q\rangle)$ by

$$\partial\,(a_i\langle\sigma_i^q\rangle) = \sum_{j=0}^{k_q}(-1)^j a_i\langle p_0, p_1, \ldots, p_{j-1}, \check{p}_j, p_{j+1}, \ldots, p_q\rangle,$$

where $\check{p}_j$ means p_j is being deleted. Hence $\partial_q(a_i\langle\sigma_i^q\rangle)$ is an element of $A_{q-1}(S;\mathbb{Z})$, and so we can define a boundary homomorphism ∂_q : $A_q(\mathcal{S};\mathbb{Z}) \to A_{q-1}(\mathcal{S};\mathbb{Z})$ by

$$\partial_q\left(a_1\langle\sigma_1^q\rangle + a_2\langle\sigma_2^q\rangle + \cdots + a_{k_q}\langle\sigma_{k_q}^q\rangle\right)$$
$$= \partial_q(a_1\langle\sigma_1^q\rangle) + \partial_q(a_2\langle\sigma_2^q\rangle) + \cdots + \partial_q(a_{k_q}\langle\sigma_{k_q}^q\rangle).$$

Thus we have defined a boundary homomorphism

$$\partial_q : C_q(\mathcal{S};\mathbb{Z}) \to C_{q-1}(\mathcal{S};\mathbb{Z})$$

(which depends on the choice of the orientation on each simplex and the order in which we arranged the simplexes). We also set $\partial_q = 0$ for $q > \dim\mathcal{S}$ or $q \leq 0$.

A simple calculation leads to

PROPOSITION 5.20. *For every q, the composition*

$$\partial_q \circ \partial_{q+1} : C_{q+1}(\mathcal{S};\mathbb{Z}) \to C_{q-1}(\mathcal{S};\mathbb{Z})$$

satisfies $\partial_q \circ \partial_{q+1} = 0$.

PROBLEM. Prove Proposition 5.20.

Thus if dim S = n, we get a chain complex $C_*(\mathbb{S};\mathbb{Z})$:

$$0 \xrightarrow{\partial_{n+1}} C_n(\mathbb{S};\mathbb{Z}) \xrightarrow{\partial_n} C_{n-1}(\mathbb{S};\mathbb{Z}) \xrightarrow{\partial_{n-1}} \cdots$$
$$\xrightarrow{\partial_2} C_1(\mathbb{S};\mathbb{Z}) \xrightarrow{\partial_1} C_0(\mathbb{S};\mathbb{Z}) \xrightarrow{\partial_0} 0.$$

DEFINITION 5.21. We say that the q-th homology group of the chain complex $C_*(\mathbb{S};\mathbb{Z})$, *i.e.*,

$$H_q(C_*(\mathbb{S};\mathbb{Z})) = \ker\partial_q/\operatorname{im}\partial_{q+1},$$

is the *q-th homology group of the simplicial complex S* over the coefficients $\mathbb{Z}$. We will write $H_q(\mathbb{S};\mathbb{Z})$ instead of $H_q(C_*(\mathbb{S};\mathbb{Z}))$. Also, we denote by $Z_q(\mathbb{S};\mathbb{Z})$ the group of cycles, $\ker\partial_q$, and by $B_q(\mathbb{S};\mathbb{Z})$ the group of boundaries, $\operatorname{im}\partial_{q+1}$ (so $H_q(\mathbb{S};\mathbb{Z}) = Z_q(\mathbb{S};\mathbb{Z})/B_q(\mathbb{S};\mathbb{Z})$).

By a simple calculation one can show that $H_q(\mathbb{S};\mathbb{Z})$ depends only on the simplicial complex S and does not depend either on the choice of the orientation of simplexes or on the way the simplexes are indexed.

EXAMPLE 5.22. Here we compute the homology groups over $\mathbb{Z}$ of the simplicial complex in Example 5.12. We use the orientation of each simplex and the indexing of the simplexes as in Example 5.12. Then the boundary operators $\partial_2 : C_2(\mathbb{S};\mathbb{Z}) \cong \mathbb{Z}^4 \to C_1(\mathbb{S};\mathbb{Z}) \cong \mathbb{Z}^6$ and $\partial_1 : C_1(\mathbb{S};\mathbb{Z}) \cong \mathbb{Z}^6 \to C_0(\mathbb{S};\mathbb{Z}) \cong \mathbb{Z}^4$ correspond to the following 6 x 4 and 4 x 6 matrices respectively:

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & -1 \\ 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} -1 & -1 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}.$$

Hence we have the following:

(i) $Z_2(\mathbb{S};\mathbb{Z}) = \{\, a\langle\sigma_1^2\rangle - a\langle\sigma_2^2\rangle + a\langle\sigma_3^2\rangle \quad a\langle\sigma_4^2\rangle \;\; a \in \mathbb{Z}\,\} \cong \mathbb{Z}$,
$B_2(\mathbb{S};\mathbb{Z}) = 0$.

(ii) $Z_1(\mathbb{S};\mathbb{Z}) = B_1(\mathbb{S};\mathbb{Z})$
$= \{\, a_1\langle\sigma_1^1\rangle + a_2\langle\sigma_2^1\rangle - (a_1 + a_2)\langle\sigma_3^1\rangle + (a_1 - a_3)\langle\sigma_4^1\rangle$
$+(a_3\langle\sigma_5^1\rangle + (a_1 + a_2 - a_3)\langle\sigma_6^1\rangle \mid a_1 \in \mathbb{Z}\,\} \cong \mathbb{Z}^3$.

(iii) $Z_0(\mathbb{S};\mathbb{Z}) = C_0(\mathbb{S};\mathbb{Z}) \cong \mathbb{Z}^4$,
$B_0(\mathbb{S};\mathbb{Z}) = \{\, a_1\langle\sigma_1^0\rangle + a_2\langle\sigma_2^0\rangle + a_3\langle\sigma_3^0\rangle + a_4\langle\sigma_4^0\rangle \mid \sum a_i = 0\,\} \cong \mathbb{Z}^3$.

Thus we get the result:

(i) $H_2(\mathcal{S};\mathbb{Z}) \cong \mathbb{Z}$, (ii) $H_1(\mathcal{S};\mathbb{Z}) = 0$, (iii) $H_0(\mathcal{S};\mathbb{Z}) \cong \mathbb{Z}$.

The generator of $H_2(\mathcal{S};\mathbb{Z}) \cong \mathbb{Z}$ (corresponding to $\pm 1 \in \mathbb{Z}$) is generated by $\langle\sigma_1^2\rangle - \langle\sigma_2^2\rangle + \langle\sigma_3^2\rangle - \langle\sigma_4^2\rangle$. The reader should examine the figure (Figure 5.1) to confirm that this is a two-cycle which is not a geometrical boundary.

The topological space $|\mathcal{S}|$ is homeomorphic to the two-dimensional sphere S^2, and $H_*(\mathcal{S};\mathbb{Z})$ is exactly the same as the homology of S^2, which we derive from the homology axioms.

To define homology groups of simplicial (complex) pairs $(\mathcal{S}, \mathcal{T})$ (relative homology) we define the q-chain $C_q(\mathcal{S}, \mathcal{T})$ by

$$C_q(\mathcal{S},\mathcal{T}) = C_q(\mathcal{S})/C_q(\mathcal{T}),$$

and carry out the computation.

The homology theory over $\mathbb{Z}$ is called the *integral homology*. We can develop a homology theory over a more general coefficient group G as follows. For an abelian group G, define the chain as

$$C_q(\mathcal{S};\mathbb{Z}) \otimes G = C_q(\mathcal{S};\mathbb{Z})$$

and extend the definitions.

In order to identify a topological space as a simplicial complex we must visualize the space as a collection of warped simplexes, which we can choose to be homeomorphic to a simplicial complex in some real space $\mathbb{R}^N$ of high enough dimension. Given a topological space X, the process of finding a simplicial complex S homeomorphic to X is called a *triangulation* of X (or a simplicial decomposition). Homology groups for simplicial complexes are called simplicial *homology* groups

EXAMPLE 5.23. REAL PROJECTIVE PLANE $P^2(\mathbb{R})$. Let us compute $H_*(P^2(\mathbb{R});\mathbb{Z})$. The projective plane $P^2(\mathbb{R})$ is a quotient space of S^2 with antipodal points identified; in other words, it is a quotient space of the two-dimensional ball D^2 with antipodal points of its boundary S^1 identified. In Figure 5.3 one sees its subdivision with curved simplexes. The integral homology and &-homology of $P^2(\mathbb{R})$ are as follows:

$$H_0(P^2(\mathbb{R});\mathbb{Z}) \cong \mathbb{Z}, \quad H_1(P^2(\mathbb{R});\mathbb{Z}) \cong \mathbb{Z}_2, \quad H_2(P^2(\mathbb{R});\mathbb{Z}) = 0,$$
$$H_0(P^2(\mathbb{R});\mathbb{Z}_2) \cong \mathbb{Z}_2, \quad H_1(P^2(\mathbb{R});\mathbb{Z}_2) \cong \mathbb{Z}_2, \quad H_2(\mathbb{R};\mathbb{Z}_2) \cong \mathbb{Z}_2.$$

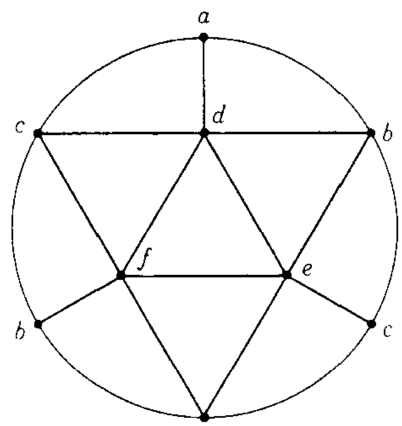


FIGURE 5.3. Real projective plane

Recall that a triangulation of a topological space meant identifying the space with a simplicial complex. If you triangulate triangulable spaces finely enough, you can approximate any map between them by a simplicial map (sending a simplex to a simplex), which induces homomorphisms between their simplicial homology groups.

The following theorem will shed some light on the validity of our axiomatic homology theory. We do not give its (easy) proof here, as it is somewhat lengthy.

THEOREM 5.24. *The simplicial homology of a triangulable space satisfies the homology axioms. Any two homologies of a triangulable space are equal over the same coefficient group (and so they agree with its simplicial homology too).*

You will have fun proving the exactness axiom (Axiom 4.1(4)) for simplicial homology. It is also easy to prove the excision axiom.

5.3. Homology calculation of cell complexes

Now that we know the homology groups of simplicial complexes, we can proceed to compute the incidence numbers of a cell complex and then its boundary operators

$$\bar{\partial}_n : \bar{C}_n(X) \to \bar{C}_{n-1}(X).$$

We discuss this procedure by examples.

EXAMPLE 5.25. In Example 2.14, we gave a cell division of the real projective plane:

$$P^2(\mathbb{R}) = (e^0 \cup_{h_1} \bar{e}^1) \cup_{h_2} \bar{e}^2.$$

The incidence numbers in the integral coefficients are $[e^1, e^0] = 0$ and $[e^2, e^1] = 2$; hence we have

$$\partial\langle e^1\rangle = 0, \quad \partial\langle e^2\rangle = 2\langle e^1\rangle.$$

Hence we get the following groups:

$$H_0(P^2(\mathbb{R});\ \mathbb{Z}) \cong \mathbb{Z}, \quad H_1(P^2(\mathbb{R});\ \mathbb{Z}) \cong \mathbb{Z}_2, \quad H_2(P^2(\mathbb{R});\ \mathbb{Z}) = 0.$$

EXAMPLE 5.26. The torus in Example 2.15 has a cell division

$$S^1 \times S^1 = (e^0 \cup_{h_1} \bar{e}_1^1 \cup \bar{e}_2^1)) \cup_{h_2} \bar{e}^2.$$

The integral incidence numbers are $[e_j^1, e^0] = 0$ and $[e^2, e_j^1] = 0$ for $j = 1, 2$, and so we get

$$\partial\langle e_1^1\rangle = \partial\langle e_2^1\rangle = 0, \quad \partial\langle e^2\rangle = 0,$$

and hence

$$H_0(T^2;\ \mathbb{Z}) \cong \mathbb{Z}, \quad H_1(T^2; \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}, \ H_2(T^2; \mathbb{Z}) = \mathbb{Z}.$$

EXAMPLE 5.27. The n-dimensional complex projective space

$$P^n(\mathbb{C}) = U(n+1)/U(n) \times U(1)$$

has a cell division:

$$P^n(\mathbb{C}) = \bar{e}^0 \cup_{h_2} \bar{e}^2 \cup \cdots \cup_{h_{2n}} \bar{e}^{2n}.$$

Hence $\bar{\partial}_i : \bar{C}_i(X) \to \bar{C}_{i-1}(X) = 0$ for each i; therefore,

$$H_i(P^n(\mathbb{C});\ \mathbb{Z}) \cong \begin{cases} \mathbb{Z}, & i = 0, 2, \ldots, 2n; \\ 0,0 & \text{otherwise} \end{cases}$$

Summary

5.1 The homology of a cell-complex is the same as the homology of its chain complex.

5.2 The n-th chain group (group of n-chains) of a cell-complex is a direct sum of n_k copies of its coefficient group G, where n_k is the number of n-cells.

5.3 We define the chain complex, and the boundary operators of a simplicial complex, and thereby compute its homology.

5.4 We compute the homology of a cell complex by investigating its boundary operators.

Exercises

5.1 Using cell divisions of S^n and D^n, compute their respective integral homology groups.

5.2 Derive the integral homology groups of the double torus M_2 from the result on the torus $T^2 = S^1 \times S^1$ together with the Mayer-Vietoris exact sequence.

5.3 Calculate the integral homology of the torus with n holes using a cell division of M_2.

CHAPTER 6

Cohomology

We obtain the cohomology axioms by reversing the direction of arrows for induced homomorphisms and boundary homomorphisms of maps used in developing homology theory. We have a unique cohomology theory once we specify the coefficient group. We can also compute cohomology groups directly from the corresponding homology groups with the aid of the universal coefficient theorem, which we study in the next chapter; however, in many instances cohomology is much easier to handle. In any case, as cohomology and homology are on parallel tracks, we will just state the cohomology axioms and give a direct definition of simplicial cohomology and its calculation.

6.1. Cohomology axioms

We remind the reader that our topological space is a cell complex.

AXIOMS 6.1. A *cohomology theory* $h^* = \sum_{p=0}^{\infty} h^p$ assigns to each pair (X, A) a direct sum $\sum_{p=0}^{\infty} h^p(X,A)$ of abelian groups $h^p(X, A)$, $p = 0, 1, 2, \ldots$, which satisfy the following properties.

(1) **HOMOTOPY AXIOM.** To an arbitrary map $f : (X, A) \to (X', A')$ and each natural number p, there corresponds a homomorphism

$$f^* : h^p(X', A') \to h^p(X, A)$$

of abelian groups such that

$$id^* : h^p(X,A) \to h^p(X,A),$$

where $id : (X, A) \to (X, A)$ is the identity map, is always equal to the identity homomorphism. If $g(X', A') \to (X'', A'')$ is another continuous map, then

$$(g \circ f)^* = f^* \circ g^* : h^p(X'', A'') \to h^p(X, A).$$

If $f \simeq f' : (X, A) \to (X', A')$, then

$$f^* = f'^* : h^p(X', A') \to h^p(X, A).$$

(2) **Coboundary Axiom.** To a pair (X, A) and each natural number p there corresponds a homomorphism, called the *coboundary homomorphism,*

$$\delta^p : h^p(A) \to h^{p+1}(X, A),$$

such that for any continuous map $f : (X, A) \to (X', A')$

$$\delta^p \circ (f|_A)^* = f^* \circ \delta^p.$$

We often write δ for δ^p.

(3) **Excision Axiom.** The inclusion map $i : (B, B \cap A) \to (A \cup B, A)$ induces an isomorphism

$$i^* : h^p(A \cup B, A) \to h^p(B, B \cap A)$$

for every p.

(4) **Exactness Axiom.** For a pair (X, A) and the natural inclusion maps $i : A \to X$, $j : X = (X, \emptyset) \to (X, A)$ we have a long exact sequence

$$\cdots \to h^{p-1}(X) \xrightarrow{i^*} h^{p-1}(A) \xrightarrow{\delta^{p-1}} h^p(X, A) \xrightarrow{j^*} h^p(X) \xrightarrow{i_*} h^p(A) \to \cdots$$

(5) **Dimension Axiom.** If $p > 0$, then

$$h^p(pt) = 0,$$

where pt is a singleton space.

6.2. Cohomology of simplicial complexes

Let $\mathcal{S}$ be a simplicial complex. We orient each simplex in $\mathcal{S}$ in any old way. For each q, we label the q-simplexes with numbers 1 through k_q: $\sigma_1^q, \sigma_2^q, \cdots, \sigma_{k_q}^q$, where k_q is the number of q-simplexes in $\mathcal{S}$. We take

$$A_q(\mathcal{S}) = \{ a_1\langle\sigma_1^q\rangle + a_2\langle\sigma_2^q\rangle + \cdots + a_{k_q}\langle\sigma_{k_q}^q\rangle \mid a_i \in \mathbb{Z} \}$$

for the q-th integral chain group (the group of q-chains over $\mathbb{Z}$) and define the *q-cochain group over G,* $C^q(\mathcal{S}; G)$, by

$$C^q(\mathcal{S}; G) = \mathrm{Hom}(C^q(\mathcal{S}; \mathbb{Z})), G).$$

In the case $G = \mathbb{Z}$, we have $C^q(\mathcal{S}; \mathbb{Z}) = \mathrm{Hom}(C^q(\mathcal{S}; \mathbb{Z}), \mathbb{Z})$. We define *coboundary operators* $\delta^p : C^q(\mathcal{S}; G) \to C^{p+1}(\mathcal{S}; G)$ by

$$\delta^q(x)(a) = x\,(\partial(a)), \quad x \in C^q(\mathcal{S}; G),\ a \in C^{p+1}(\mathcal{S}; G).$$

Then we get a chain complex

$$0 \leftarrow C^n(\mathcal{S}; G) \overset{\delta^{n-1}}{\longleftarrow} C^{(n-1)}(\mathcal{S}; G) \overset{\delta^{n-2}}{\longleftarrow}$$

$$\ldots \overset{\delta^1}{\longleftarrow} C^1(\mathcal{S}; G) \overset{\delta^0}{\longleftarrow} C^0(\mathcal{S}; G) \leftarrow 0,$$

which we denote by $C^*(\mathcal{S}; G)$. We say that $C^*(\mathcal{S}; G)$ is the cochain complex of S over G (G-cochain complex for short).

DEFINITION 6.2. By the q-th cohomology group of a simplicial complex S over coefficient group G we mean the q-th homology group

$$H_q(C^*(\mathcal{S}; G)) = \ker \delta^q / \operatorname{im} \delta^{q-1}$$

of the cochain complex $C^*(\mathcal{S}; G)$ over G of S, which we shall denote by $H^q(\mathcal{S}; G)$.

DEFINITION 6.3. We denote by $Z^q(\mathcal{S}; G)$ the group of q-cycles in the G-cochain complex $C^*(\mathcal{S}; G)$. Elements of $Z^q(\mathcal{S}; G)$ are *q-cocycles* of the simplicial complex S over the group G. Similarly, $B^q(\mathcal{S}; G)$ denotes the group of q-boundaries of $C^*(\mathcal{S}; G)$. The elements of $B^q(\mathcal{S}; G)$ are called the *q-coboundaries* of S. We can now write

$$H^q(\mathcal{S}; G) = Z^q(\mathcal{S}; G)/B^q(\mathcal{S}; G).$$

EXAMPLE 6.4. In Example 5.22 we calculated the integral homology of the simplicial complex of Example 5.12. Here we calculate its integral cohomology. For each oriented simplex $\langle \sigma_i^q \rangle$, we define $\langle \sigma_i^q \rangle^* \in C^q(\mathcal{S}; \mathbb{Z})$ by

$$\langle \sigma_i^q \rangle^* \left(\langle \sigma_j^q \rangle \right) = \delta_j^i,$$

where δ_j^i is the Kronecker delta,

$$\delta_j^i = \begin{cases} 1 \ , & i = j; \\ 0 \ , & i \neq j. \end{cases}$$

Then $C^q(\mathcal{S}; \mathbb{Z})$ is isomorphic to $\{\sum_{i=1}^{k_q} a_i \langle \sigma_i^q \rangle^* \ \ a_i \in \mathbb{Z}\}$, a free abelian group generated by $\langle \sigma_i^q \rangle^*$, $i = 1, \ldots, k_q$, and we get the following:

$$C^0(\mathcal{S}; \mathbb{Z}) \cong \mathbb{Z}^4, \qquad C^1(\mathcal{S}; \mathbb{Z}) \cong \mathbb{Z}^6, \qquad C^2(\mathcal{S}; \mathbb{Z}) \cong \mathbb{Z}^4.$$

The calculation of the coboundary homomorphism $\delta^0 : C^0(\mathcal{S}; \mathbb{Z}) \to C^1(\mathcal{S}; \mathbb{Z})$ goes as follows. First we notice that

$$\delta^0 \ (\langle p_i \rangle^*) \ (\langle p_j, p_k \rangle) = \langle p_i \rangle^* \ (\partial \langle p_j, p_k \rangle) = \langle p_i \rangle^* \ (\langle p_k \rangle \quad \langle p_j \rangle) = \delta_k^i - \delta_j^i.$$

Then, we get $\delta^0(\langle p_0 \rangle^*)$ for example as

$$\delta^0 \ (\langle p_0 \rangle^*) = -\langle p_0, p_1 \rangle^* - \langle p_0, p_2 \rangle^* - \langle p_0, p_3 \rangle^*.$$

Similarly we can compute the coboundary operator $\delta^1 : C^1(\mathcal{S};\mathbb{Z}) \to C^2(\mathcal{S};\mathbb{Z})$. Thus we have the following 6 x 4 and 4 x 6 matrices representing δ^0 and δ^1 respectively:

$$\begin{pmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 \\ 0 & 1 & -1 & 0 & 0 & 1 \\ 1 & 0 & -1 & 0 & 1 & 0 \end{pmatrix}.$$

These matrices are the transposes of the matrices for the boundary operators in the homology calculation. Hence we get

(i) $Z^0(\mathcal{S};\mathbb{Z}) = \{a\langle\sigma_1^0\rangle^* \quad a\langle\sigma_2^0\rangle^* + a\langle\sigma_3^0\rangle^* \quad a\langle\sigma_4^0\rangle^* \ \ a \in \mathbb{Z}\} \cong \mathbb{Z}$,
$B^0(\mathcal{S};\mathbb{Z}) = 0$,

(ii) $Z^1(\mathcal{S};\mathbb{Z}) = B^1(\mathcal{S};\mathbb{Z})$

$$= \{a_1\langle\sigma_1^1\rangle^* + a_2\langle\sigma_2^1\rangle^* + a_3\langle\sigma_3^1\rangle^* + (-a_1 + a_2)\langle\sigma_4^1\rangle^* + (-a_1 + a_3)\langle\sigma_5^1\rangle^* + (-a_2 + a_3)\langle\sigma_6^1\rangle^* \ \ a_i \in \mathbb{Z}\} \cong \mathbb{Z}^3,$$

(iii) $Z^2(\mathcal{S};\mathbb{Z}) = C^2(\mathcal{S};\mathbb{Z}) \cong \mathbb{Z}^4$,

$$B^2(\mathcal{S};\mathbb{Z}) = \left\{ \sum_{1=1}^{4} a_i\langle\sigma_0^i\rangle^* \,|\, a_1 - a_2 + a_3 - a_4 = 0 \right\} \cong \mathbb{Z}^3.$$

Hence we get the following result:

(i) $H^0(\mathcal{S};\mathbb{Z}) \cong \mathbb{Z}$, (ii) $H^1(\mathcal{S};\mathbb{Z}) \cong 0$, (iii) $H^2(\mathcal{S};\mathbb{Z}) \cong \mathbb{Z}$.

Now we realize the correspondence from $Z^2(\mathcal{S};\mathbb{Z}) = C^2(\mathcal{S};\mathbb{Z}) \cong \mathbb{Z}^4$ to $H^2(\mathcal{S};\mathbb{Z}) \cong Z$ by sending an element

$$a_1\langle\sigma_1^2\rangle^* + a_2\langle\sigma_2^2\rangle^* + a_3\langle\sigma_3^2\rangle^* + a_4\langle\sigma_4^2\rangle^*$$

to the element

$$a_1 - a_2 + a_3 - a_4 \in \mathbb{Z}.$$

Recall that

$$\langle\sigma_1^2\rangle - \langle\sigma_2^2\rangle + \langle\sigma_3^2\rangle - \langle\sigma_4^2\rangle$$

is the fundamental cycle which generates $H_2(\mathcal{S};\mathbb{Z})$. We can say that we realize the correspondence to the cohomology group $H^2(\mathcal{S};\mathbb{Z}) \cong \mathbb{Z}$ by "integrating" the two-cochain at this fundamental cycle (do not fret about this, since most likely it makes no sense to you yet).

Summary

6.1 One obtains the cohomology axioms from the homology axioms by reversing the arrow in every homomorphism appearing in the homology axioms.

6.2 We can define and compute any simplicial cohomology directly.

Exercises

6.1 Use a triangulation of the torus $T^2 \equiv$ S x S^1 to compute its integral cohomology groups.

6.2 Triangulate the real projective plane $P^2(\mathbb{R})$ to compute its integral cohomology groups.

CHAPTER 7

The Universal Coefficient Theorem

In this chapter we state without proof how to calculate homology and cohomology groups (we will write (co)homology groups) of product spaces as well as (co)homology groups with a change of coefficient groups. This will be handy for computing (co)homology groups over various coefficient groups of concrete geometrical objects. We also define the cup product of cohomology groups. In Chapter Nine we will deal with the calculation of the (co)homology groups of fiber spaces, which are generalized version of product spaces, by using spectral sequences.

7.1. Products of abelian groups

We will discuss four types of products of abelian groups; however, we do no more than state some properties of each of these products, which are enough for the calculation of (co)homology groups. We hope that you consult the recommended reading at the end of the book if you are interested in knowing their precise definitions.

(a) **TENSOR PRODUCTS.** For two abelian groups G_1 and G_2, there is a well-defined abelian group called their *tensor product* (over $\mathbb{Z}$),

$$G_1 \otimes G_2,$$

such that if G_1 and G_2 are direct sums, $G_1 = \sum_i G_1^i$ and $G_2 = \sum_j G_2^j$, then there is an isomorphism

$$G_1 \otimes G_2 \cong \sum_{i,j} G_1^i \otimes G_2^j.$$

The tensor product also satisfies

$$G_1 \otimes G_2 \cong G_2 \otimes G_1.$$

Moreover, for every abelian group G we have that

$$\mathbb{Z} \otimes G \cong G \otimes \mathbb{Z} \cong G,$$

and for every pair of natural numbers m, n and their greatest common divisor (m, n), the following relations hold:

$$\mathbb{Z} \otimes \mathbb{Z} \cong \mathbb{Z}, \qquad \mathbb{Z} \otimes \mathbb{Z}_m \cong \mathbb{Z}_m \otimes \mathbb{Z} \cong \mathbb{Z}_m, \qquad \mathbb{Z}_m \otimes \mathbb{Z}_n \cong \mathbb{Z}_{(m,n)}.$$

Note that these relations completely determine the tensor product of two finitely generated abelian groups.

(b) Horn. For any two abelian groups G_1 and G_2 we have the abelian group $\mathrm{Hom}(G_1, G_2)$ of the homomorphisms from G_1 to G_2. In particular, if $G_1 = \sum_i G_1^i$ and $G_2 = \sum_j G_2^j$ are direct sums, we have

$$\mathrm{Hom}(G_1, G_2) \cong \sum_{i,j} \mathrm{Hom}(G_1^i, G_2^j).$$

An abelian group G satisfies

$$\mathrm{Hom}(\mathbb{Z}, G) \cong G.$$

Finally, the following relations hold:

$$\mathrm{Hom}(\mathbb{Z}, \mathbb{Z}) \cong \mathbb{Z}, \qquad \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}_m) \cong \mathbb{Z}_m, \qquad \mathrm{Hom}(\mathbb{Z}_m, \mathbb{Z}) = 0,$$
$$\mathrm{Hom}(\mathbb{Z}_m, \mathbb{Z}_n) \cong \mathbb{Z}_{(m,n)}.$$

Note again that these properties completely determine the abelian group $\mathrm{Hom}(G_1, G_2)$ of any two finitely generated abelian groups G_1 and G_2.

(c) TORSION PRODUCTS. Two abelian groups G_1 and G_2 determine an abelian group, called their torsion product (over $\mathbb{Z}$),

$$\mathrm{Tor}(G_1, G_2),$$

which depends only on the torsion parts of G_1 and G_2 (their respective subgroups consisting of the elements whose integral multiples become 0 for some integers). If G_1 and G_2 are direct sums, $G_1 = \sum_i G_1^i$ and $G_2 = \sum_j G_2^j$, then we have

$$\mathrm{Tor}(G_1, G_2) \cong \sum_{i,j} \mathrm{Tor}(G_1^i, G_2^j).$$

We also have

$$\mathrm{Tor}(G_1, G_2) \cong \mathrm{Tor}(G_2, G_1)$$

for any abelian groups G_1 and G_2, and for any abelian group G we have that

$$\mathrm{Tor}(\mathbb{Z}, G) \cong \mathrm{Tor}(G, \mathbb{Z}) = 0.$$

In addition, we have the following:

$$\mathrm{Tor}(\mathbb{Z}, \mathbb{Z}) \cong \mathrm{Tor}(\mathbb{Z}, \mathbb{Z}_m) \cong \mathrm{Tor}(\mathbb{Z}_m, \mathbb{Z}) = 0, \ \mathrm{Tor}(\mathbb{Z}_m, \mathbb{Z}_n) \cong \mathbb{Z}_{(m,n)}.$$

Sometimes we write Tori (G_1, G_2) in place of $\mathrm{Tor}(G_1, G_2)$; in this case we set $\mathrm{Tor}_0(G_1, G_2) \equiv G_1 \otimes G_2$.

Note that we can determine the torsion product of two finitely generated abelian groups from these properties.

(d) Ext. For two abelian groups G_1 and G_2, an *extension* of G_2 by G_1 is a group G together with an exact sequence

$$0 \to G_2 \to \mathrm{G} \to G_1 \to 0.$$

Equivalence classes of extensions of G_2 by G_1 determine an abelian group, denoted by

$$\mathrm{Ext}(G_1, G_2).$$

If G_1 and G_2 are direct sums, $G_1 = \sum_i G_1^i$, $G_2 = \sum_j G_2^j$, then there is an isomorphism

$$\mathrm{Ext}(G_1, G_2) \cong \sum_{i,j} \mathrm{Ext}(G_1^i, G_2^j).$$

For any abelian group G, we have that

$$\mathrm{Ext}(\mathbb{Z}, \mathrm{G}) = 0.$$

We also have the following:

$$\mathrm{Ext}(\mathbb{Z}, \mathbb{Z}) \cong \mathrm{Ext}(\mathbb{Z}, \mathbb{Z}_m) = 0,$$
$$\mathrm{Ext}(\mathbb{Z}_m, \mathbb{Z}) \cong \mathbb{Z}_m, \quad \mathrm{Ext}(\mathbb{Z}_m, \mathbb{Z}_n) \cong \mathbb{Z}_{(m,n)}.$$

Sometimes we write $\mathrm{Ext}^1(G_1, G_2)$ in place of $\mathrm{Ext}(G_1, G_2)$, and in this case we set $\mathrm{Ext}^0(G_1, G_2) \equiv \mathrm{Hom}(G_1, G_2)$.

Note that these properties completely determine $\mathrm{Ext}(G_1, G_2)$ for two finitely generated abelian groups G_1 and G_2.

We list the similar relations using $\mathbb{R}$:

$$\mathbb{R} \otimes \mathbb{Z} \cong \mathbb{Z} \otimes \mathbb{R} \cong \mathbb{R}, \ \mathbb{R} \otimes \mathbb{Z}_m \cong \mathbb{Z}_m \otimes \mathbb{R} = 0,$$
$$\mathrm{Hom}(\mathbb{R}, \mathbb{Z}) = 0, \mathrm{Hom}(\mathbb{Z}, \mathbb{R}) \cong \mathbb{R}, \mathrm{Hom}(\mathbb{R}, \mathbb{Z}_m) = 0, \mathrm{Hom}(\mathbb{Z}_m, \mathbb{R}) = 0,$$
$$\mathrm{Tor}(\mathbb{R}, \mathbb{Z}) \cong \mathrm{Tor}(\mathbb{Z}, \mathbb{R}) = 0, \ \mathrm{Tor}(\mathbb{R}, \mathbb{Z}_m) \cong \mathrm{Tor}(\mathbb{Z}_m . \mathbb{R}) = 0,$$
$$\mathrm{Ext}(\mathbb{Z}, \mathbb{R}) = 0, \ \mathrm{Ext}(\mathbb{Z}_m, \mathbb{R}) = 0.$$

Note that all *these* groups *are regarded as Z-modules.*

7.2. The Kiinneth formula

We will now state a formuia to compute the (co)homology groups of the product spaces X x Y of topological spaces X and Y given their respective (co)homology groups.

We calculated the homology groups of a cell complex from its chain complex (Theorem 5.7). We can regard the tensor product of the respective chains of X and Y naturally as a chain on X x Y, which induces a homomorphism

$$\times : H_p(X;\mathbb{Z}) \otimes H_q(Y;\mathbb{Z}) \to H_{p+q}(X \times Y;\mathbb{Z}).$$

Similarly we get the induced homomorphism

$$\times : H^p(X;\mathbb{Z}) \otimes H^q(Y;\mathbb{Z}) \to H^{p+q}(X \times Y;\mathbb{Z}).$$

We say that these maps are induced by the cross product.

We have a very strong result in the following theorems, which imply that the map induced by the cross product is injective.

THEOREM 7.1 (the homology Kiinneth formula).

$$H_n(X \times Y;\mathbb{Z}) \cong \sum_{p+q=n} H_p(X;\mathbb{Z}) \otimes H_q(Y;\mathbb{Z}) \oplus \sum_{p+q=n-1} \operatorname{Tor}(H_p(X;\mathbb{Z}), H_q(Y;\mathbb{Z}))$$

THEOREM 7.2 (the cohomology Kiinneth formula).

$$H^n(X \times Y;\mathbb{Z}) \cong \sum_{p+q=n} H^p(X;\mathbb{Z}) \otimes H^q(Y;\mathbb{Z}) \oplus \sum_{p+q=n+1} \operatorname{Tor}(H^p(X;\mathbb{Z}), H^q(Y;\mathbb{Z}))$$

7.3. Cup products

We define: in this section, a cup product which gives a product structure to cohomology groups. For a topological space X *the diagonal* map

$$\triangle : X \to X \times X,$$

sending $x \in X$ to $(x, x) \in X \times X$, is continuous. Hence the composition of the cross product and the induced map $\triangle^*$,

$$H^p(X;G) \times H^q(X;G) \xrightarrow{\times} H^{p+q}(X \times X;G) \xrightarrow{\triangle^*} H^{p+q}(X;G),$$

defines a homomorphism

$$\cup : H^p(X;G) \times H^q(X;G) \to H^{p+q}(X;G).$$

For $a \in H^p(X;G)$ and $b \in H^q(X;G)$, we define their *cup product* $a \cup b$ by

$$a \cup b = \triangle^*(a \times b) \in H^{p+q}(X;G).$$

The definition implies that the structure induced on a cohomology theory by the cup product is homotopy invariant. The cup products satisfy the following properties (which follow naturally from the properties of tensor products together with the trick of regarding the tensor product of chains as a chain):

For $a \in H^p(X;G)$, $b \in H^q(X;G)$, $c \in H^r(X;G)$,

$$(a \cup b) \cup c = a \cup (b \cup c), \quad a \cup b = (-1)^{pq}(b \cup a);$$

For a map $f : X \to Y$, $f^*(a \cup b) = f^*(a) \cup f^*(b)$.

The cohomology group $H^*(X;G) = \sum_p H^p(X;G)$ thus equipped with a product structure has become a ring, and f^* is a product–preserving homomorphism (ring homomorphism).

Example 7.3. The topological space $S^2 \vee S^4$, which is the quotient space of $S^2 \cup S^4$ under the identification of one point on S^2 with another on S^4, has the integral cohomology

$$H^j(S^2 \vee S^4; \mathbb{Z}) \cong \begin{cases} \mathbb{Z}, & j = 0, 2, 4, \\ 0, & \text{otherwise.} \end{cases}$$

Hence one can show that the cup product homomorphism

$$\cup : H^2(S^2 \vee S^4; \mathbb{Z}) \times H^2(S^2 \vee S^4; \mathbb{Z}) \to H^4(S^2 \vee S^4; \mathbb{Z})$$

is the zero map (cf. Exercise 7.4 with its hint).

On the other hand, although the complex projective plane $P^2(\mathbb{C})$ has the same integral cohomology

$$H^j(P^2(\mathbb{C}); \mathbb{Z}) \cong \begin{cases} \mathbb{Z}, & j = 0, 2, 4, \\ 0, & \text{otherwise,} \end{cases}$$

the cup product

$$\cup : H^2(P^2(\mathbb{C}); \mathbb{Z}) \times H^2(P^2(\mathbb{C}); \mathbb{Z}) \to H^4(P^2(\mathbb{C}; \mathbb{Z})$$

satisfies $\cup(1, 1) = 1$ if we look at $\cup$ as $\cup : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$.

The above example shows that the cup product observes topological spaces in finer detail than cohomology does; nevertheless, the geometric implication of the cup product is in general hard to grasp. Cup products of manifolds are closely related to intersection theory.

7.4. The universal coefficient theorem

We compute homology groups over a general group from the corresponding integral homology groups, and we obtain cohomology groups over a general group from the integral homology or integral cohomology groups. We list four formulae known as the universal *coefficient theorem.*

THEOREM 7.4. *We can* calculate homology *over* a *general coefficient* group G *by* using *the* corresponding integral homology and *the torsion product:*

$$H_n(X; G) \cong H_n(X; \mathbb{Z}) \otimes G \oplus \mathrm{Tor}(H_{n-1}(X; \mathbb{Z}), G).$$

THEOREM *7.5. We can calculate cohomology over a general coefficient group G* using *the* corresponding *integral homology and the* extension *product:*

$$H^n(X; G) \cong \mathrm{Hom}(H_n(X; \mathbb{Z}), G) \oplus \mathrm{Ext}(H_{n-1}(X; \mathbb{Z}), G).$$

THEOREM *7.6. We* can *also compute cohomology over* a general *coefficient* group G from *the* integral *cohomology and the torsion product:*

$$H^n(X; G) \cong H^n(X; \mathbb{Z}) \otimes G \oplus \mathrm{Tor}(H^{n+1}(X; \mathbb{Z}), G).$$

THEOREM *7.7. We can calculate* homology *over a general coefficient* group G from *the* integral *cohomology and the extension product:*

$$H_n(X; G) \cong \mathrm{Hom}(H^n(X; \mathbb{Z}), G) \oplus \mathrm{Ext}(H^{n+1}(X; \mathbb{Z}), G).$$

Summary

7.1 For abelian groups G_1 and G_2, one defines $G_1 \otimes G_2, \mathrm{Hom}(G_1, G_2)$, $\mathrm{Tor}(G_1, G_2)$ and $\mathrm{Ext}(G_1, G_2)$.

7.2 One defines the cross product by identifying the tensor products of the respective (co)homology groups of two topological spaces with the (co)homology groups of the product space.

7.3 The cross product homomorphism is injective, and the (co)homology groups of a product space fall out from the Künneth formula.

7.4 With the structure of a cup product, cohomology groups become rings.

7.5 One computes (co)homology groups over a general coefficient group using the integral (co)homology groups together with the universal coefficient theorem.

Exercises

7.1 Use the integral homology of the real projective plane,

$$H_0(P^2(\mathbb{R});\mathbb{Z}) \cong \mathbb{Z}, \; H_1(P^2(\mathbb{R});\mathbb{Z}) \cong \mathbb{Z}_2, \; H_2(P^2(\mathbb{R});\mathbb{Z}) = 0,$$

to compute the homology group $H_*(P^2(\mathbb{R}) \times P^2(\mathbb{R});\mathbb{Z})$ of the product space P"(R) x $P^2(\mathbb{R})$.

7.2 Determine the integral cohomology groups of the real projective plane $P^2(\mathbb{R})$ using the results on the corresponding homology groups.

7.3 Compute the integral cohomology groups of the product space $P^2(\mathbb{R})$ x P"(R).

7.4 Show that the map

$$u : H^2(S^2 \vee S^4;\mathbb{Z}) \times H^2(S^2 \vee S^4;\mathbb{Z}) \to H^4(S^2 \vee S^4;\mathbb{Z})$$

is trivial (Hint: consider an onto map $F : S^2 \vee S^4 \to S^2$ which collapses S^4 to a point).

7.5 Compute the homology groups of the real projective plane $P^2(\mathbb{R})$ over $\mathbb{Z}_2$ using its integral homology groups.

7.6 Derive the cohomology groups of the real projective plane $P^2(\mathbb{R})$ over $\mathbb{Z}_2$ from its integral homology groups.

CHAPTER 8

Fiber Bundles and Vector Bundles

In order to investigate a curve on the plane we take its derivative at each point. We get a rough idea of the shape of the curve from the distribution of the signs of the derivatives. Similarly, if we look at the tangent plane at each point of a surface and observe how these planes change as we move the points, we get some global information about the surface. We can do the same for geometrical objects in higher-dimensional spaces; therefore, in investigating a higher-dimensional smooth geometrical object (differentiable manifold, to be precise) we must consider the linear space tangent to it at each of its points. This leads us to tangent bundles of manifolds, general vector bundles and fiber bundles, and to their firm establishment as powerful tools for the exploration of manifolds. Meanwhile physicists too established gauge theories, while building field theories for elementary particle physics; this turns out to be essentially equivalent to the concept of fiber bundles.

In this chapter we discuss fiber bundles and vector bundles, and then we define Grassmann manifolds, which have absolute control over the equivalence classes of vector bundles, and investigate their roles.

8.1. Fiber bundles

Let S^1 be the unit circle and $D^1 = [-1, 1]$ the unit interval. We first consider the product space $\boldsymbol{E} = S^1 \times D^1$. Evidently there exists a natural projection $\pi : \boldsymbol{E} \to S^1$ such that $\pi^{-1}(U) = U \times D^1$ holds for any open interval U in S^1 (Figure 8.1).

Next we look at the Mobius band M. In this example, too, the center of the band is a circle S^1 and there is a projection π from M to S^1. We see that $\pi^{-1}(U) = U \times D^1$ for any open interval U of S^1 (Figure 8.2).

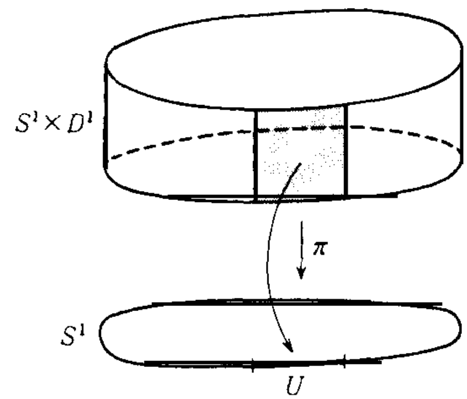


FIGURE 8.1. S^1 x D^1

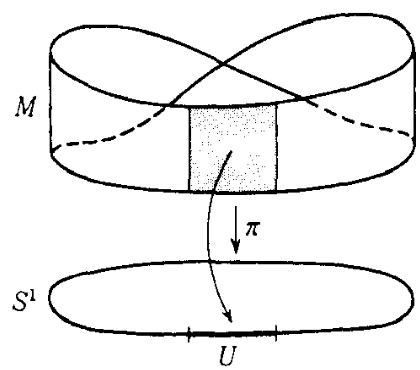


FIGURE 8.2. The Mobius strip

Keep the above examples in mind for the definition of a *fiber bundle*. Suppose we have topological spaces E, B and F, and a map $\pi : E \to B$ such that for each $b \in B$, $\pi^{-1}(b)$ is homeomorphic to F and there is a neighborhood U of b with a homeomorphism $h : \pi^{-1}(U) \to U \times F$ making the following diagram commute, where $p_1 : U \times F \to U$ is the projection of U onto the first component:

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{h} & U \times F \\ & {}_{\pi}\searrow \quad \swarrow_{p_1} & \\ & U. & \end{array}$$

Then we say that the quadruple (E, π, B, F) is a *fiber bundle* with the *total space* E, the base space B, the fiber F and the projection π. We also denote this fiber bundle by

$$F \to E \xrightarrow{\pi} B.$$

We say that the homeomorphism h between $\pi^{-1}(U)$ and $U \times F$ is a *local trivialization*. We say that $\pi^{-1}(b)$ is the *fiber over b*.

Remark. If you have read other books on the subject, you might be concerned with the structural group. Our structural group in this book is always the group of self-homeomorphisms $\mathrm{Homeo}(F)$.

Example 8.1. The product space $E = B \times F$ is the simplest fiber bundle; it is called a **trivial** *bundle* (over B).

Example 8.2. Let G be a Lie group, and let $H \subset G$ and $K \subset H$ be closed subgroups of G. Then we have a fiber bundle

$$H/K \to G/K \xrightarrow{\pi} G/H.$$

In particular, consider $G = SO(3)$, $H = SO(2)$ and $K = \{e\}$ (the trivial group consisting of the unit element). Then we have a fiber bundle

$$S^1 = SO(2) \to SO(3) \to S^2 = SO(3)/SO(2),$$

with the fiber S^1 and the base space S^2. More generally, if we take the groups $SO(n+1)$, $SO(n)$ and $SO(n-1)$ for G, H and K respectively, we obtain a fiber bundle over the base space S^n with the fiber S^{n-1}:

$$S^{n-1} = \frac{SO(n)}{SO(n-1)} \to \frac{SO(n+1)}{SO(n-1)} \to S^n = \frac{SO(n+1)}{SO(n)}.$$

This is a fiber bundle known as the (unit) *tangent sphere bundle.*

Let $G = SU(2)$, $H = U(1)$ and $K = \{e\}$. With the natural inclusion map $U(1) \to SU(2)$ we get a fiber bundle

$$U(1) \to SU(2) = S^3 \to SU(2)/U(1) = S^2.$$

The projection map $\pi : SU(2) = S^3 \to SU(2)/U(1) = S^2$ is the so-called *Hopf map* and induces a generator of $\pi_3(S^2) \cong \mathbb{Z}$ (cf. Example 3.5).

Now that we have defined fiber bundles, we can define bundle maps between any two of them, and settle the question of bundle equivalences.

Definition 8.3. Suppose that we have two fiber bundles (with the same fiber), (E, π, B, F) and (E', π', B', F). Then a *bundle* map

$$f : (E, \pi, B, F) \to (E', \pi', B', F)$$

consists of a pair of maps

$$f = (f_1, f_2), \quad f_1 : E \to E', \quad f_2 : B \to B',$$

subject to the condition that the diagram

$$\begin{array}{ccc} E & \xrightarrow{f_1} & E' \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi'} \\ B & \xrightarrow[f_2]{} & B' \end{array}$$

commutes (that is to say, $f_2 \circ \pi = \pi' \circ f_1$), and that for every $b \in B$ the restriction of f_1 to the fiber $\pi^{-1}(b)$ over b,

$$f_1 : \pi^{-1}(b) \to \pi'^{-1}(f_2(b)),$$

is a homeomorphism.

Let (E, π, B, F) be a fiber bundle and let A be an arbitrary topological space. For any map $g : A \to B$ we have the following natural construction of a fiber bundle (D, ρ, A, F) over A and a bundle map $\tilde{g} = (\hat{g}, g) : (D, \rho, A, F) \to (E, \pi, B, F)$.

CONSTRUCTION. Put

$$D = \{(a, e) \in A \times E \mid g(a) = \pi(e)\},$$
$$\rho(a, e) = a,$$
$$\hat{g}(a, e) = e.$$

You will see immediately that (D, ρ, A, F) is a fiber bundle and that $(\hat{g}, g)$ is a bundle map. We say that (D, ρ, A, F) is the *induced bundle* or the *pullback* of (E, π, B, F) by the map $g : A \to B$, and we denote it by $g^*(E, \pi, B, F)$ (or $g^*(\xi)$, where $\xi = (E, \pi, B, F)$).

EXAMPLE 8.4. Let $g : A \to B$ be a constant map. In this case for any fiber bundle $\xi = (E, \pi, B, F)$ over B, the induced bundle $g^*(\xi)$ is the trivial bundle (over A).

The definition of fiber-bundle equivalences follows naturally from the definition of bundle maps.

DEFINITION 8.5. We say that two fiber bundles (E, π, B, F) and (E', π', B', F) over F are *bundle equivalent* if there exist bundle maps

$$f = (f_1, f_2) : (E, \pi, B, F) \to (E', \pi', B', F),$$
$$g = (g_1, g_2) : (E', \pi', B', F) \to (E, \pi, B, F),$$

such that

$$g_1 \circ f_1 = id_E, \quad f_1 \circ g_1 = id_{E'}$$

Notice that in the above definition, we have

$$g_2 \circ f_2 = id_B, \qquad f_2 \circ g_2 = id_{B'},$$

and there are homeomorphisms between E and E', and B and B' respectively.

EXAMPLE 8.6. Let $f = (f_1, f_2) : (E, \pi, B, F) \to (E', \pi', B', F)$ be a bundle map of $\xi = (E, \pi, B, F)$ to $\xi' = (E', \pi', B', F)$. Then ξ is bundle equivalent to the induced bundle $f_2^*(\xi')$.

EXAMPLE 8.7. This example may be a bit hard to understand. For a trivial bundle $\xi = (E, \pi, B, F)$ there are infinitely many bundle equivalences of $B \times F$ inducing the identity map $B \to B$ (since we can rotate the fibers). Physicists use a simple expression, "choosing a gauge" , when they pick one fiber bundle which is bundle-equivalent to the trivial bundle.

8.2. Vector bundles

Perhaps you are not yet familiar with manifolds, but they form a family of topological spaces which share extremely beautiful properties, and you will find an abundance of literature concerning them (see the References) everywhere you look. In the most pithy way, one might, characterize a manifold as a topological space each point of which has a neighborhood homeomorphic to some fixed n-dimensional Euclidean space $\mathbb{R}^n$. Vector bundles are linear approximations of manifolds and are essential tools for the investigation of manifolds. In order to study manifolds, we look at their tangent bundles, which are easier to handle.

Tangent bundles form a special class among vector bundles, which in turn form a special class among fiber bundles. So we first define vector bundles. The underlying vector space is either real or complex, and we will use the notation V^n for either $\mathbb{R}^n$ or $\mathbb{C}^n$.

DEFINITION 8.8. By an n-dimensional *vector bundle* we mean a map $\pi : E \to B$ of topological spaces E and B such that for any $b \in B$, the inverse image $\pi^{-1}(b)$ of b has the structure of the n-dimensional vector space V^n (that is, the addition and the scalar product are defined on $\pi^{-1}(b)$ and satisfy the vector space axioms), having the following *property of locally triviality:*

For each $b \in B$, we can find a neighborhood U of b in B and a homeomorphism

$$h : \pi^{-1}(U) \to U \times V^n$$

such that for every $b' \in U$ the assignment of $x \in \pi^{-1}(b')$ to $h(b', x)$ is an isomorphism of vector spaces $\pi^{-1}(b')$ and V^n.

So you see that an n-vector bundle is a fiber bundle whose fiber is the n-dimensional vector space, and locally trivial maps preserve the vector structure.

We often write (E, π, B) for an n-vector bundle, since its fiber is V^n. The direct product $E = B \times V^n$ is the simplest vector bundle, which we call a trivial vector bundle.

EXAMPLE 8.9. $E = B \times \mathbb{R}$ is a one-dimensional trivial real vector bundle. The Mobius strip is a one-dimensional vector bundle whose fiber is the real one-dimensional vector space.

You will some day come to see that the following is an example of ultimate importance.

EXAMPLE 8.10. Let M be an n-dimensional differentiable manifold. Then the space TM consisting of the tangent vectors of M becomes, in a natural way, the total space of a real n-vector bundle whose base space is M.

We can also define bundle maps and bundle isomorphisms for vector bundles in a natural way.

DEFINITION 8.11. By *a (vector) bundle map* f between two n-dimensional vector bundles (E, π, B) and (E', π', B') we mean a bundle map $f = (f_1, f_2)$ with the property that the restriction of f_1 to each fiber $\pi^{-1}(b)$, $b \in B$,

$$f_1|_{\pi^{-1}(b)} : \pi^{-1}(b) \to \pi'^{-1}(f_2(b)),$$

is an isomorphism of vector spaces.

DEFINITION 8.12. Vector bundles (E, π, B) and (E', π', B') are *bundle isomorphic* (isomorphic *as vector bundles)* when we can find (vector) bundle maps

$$f = (f_1, f_2) : (E, \pi, B) \to (E', \pi', B'),$$
$$g = (g_1, g_2) : (E', \pi', B') \to (E, \pi, B)$$

such that

$$g_1 \circ f_1 = id_E, \quad f_1 \circ g_1 = id_{E'}.$$

A FUNDAMENTAL PROBLEM. For a fixed natural number n, how many n-dimensional vector bundles with a given base space (of course, here we count isomorphic vector bundles as one) are there?

Example 8.13. There are exactly two distinct one-dimensional real vector bundles over S^1. These are the trivial bundle and the Mobius band (cf. 8.21).

A Bit of History. There exist infinitely many four-dimensional real vector bundles over the four-dimensional sphere. In 1957, J. Milnor showed that among these bundles there are some which are not vector-bundle isomorphic despite the fact that their total spaces are homeomorphic; and using this, he came up with several manifolds that are homeomorphic but not diffeomorphic to the seven-dimensional sphere S^7. His discovery made a big splash in the mathematical community.

The fundamental problem above boils down to a question of classifying homotopy classes of maps from the base space to a Grassmann manifold, where we utilize the remarkable result that characteristic classes which are elements in some suitable cohomology groups can detect differences among vector bundles. We will try to give a lucid explanation of this result in the rest of this chapter (and beyond).

8.3. Grassmann manifolds

(a) **Definition of Grassmann Manifold.** For natural numbers m and n with m > n, we define the Grassmann manifold $G^{\mathbb{R}}(m,n)$ to be the space of all n-dimensional linear subspaces of the real vector space $\mathbb{R}^m$ of dimension m. Similarly we define the complex Grassmann manifold $G^{\mathbb{C}}(m,n)$.

Can you visualize the space of the n-dimensional subspaces (each point is an n-dimensional linear subspace)?

Example 8.14. Let m = 2 and n = 1. A one-dimensional subspace of the plane $\mathbb{R}^2$ is a line through the origin with slope θ, $-\infty \le \theta \le +\infty$. The lines with slopes $-\infty$ and $+\infty$ respectively are identical (the y-axis). Hence $G^{\mathbb{R}}(2,1)$ is homeomorphic to the circle which is the straight line (in θ) with $-\infty$ and $+\infty$ identified; that is,

$$G^{\mathbb{R}}(2,1) = S^1.$$

Example 8.15. Let $m = 3$ and n = 1. Each line through the origin in $\mathbb{R}^3$ intersects the unit sphere S^2 in two antipodal points, so that the correspondence of a one-dimensional linear subspace to the corresponding pair of antipodal points on S^2 is one-to-one. Hence, the Grassmann manifold $G^{\mathbb{R}}(3,1)$ is the quotient space of S^2 where the antipodal points are identified; in other words, it is the real projective plane $P^2(\mathbb{R})$:

$$G^{\mathbb{R}}(3,1) = P^2(\mathbb{R}).$$

It is evident, further, that $G^{\mathbb{R}}(3,2)$ is homeomorphic to $G^{\mathbb{R}}(3,1)$ (consider the line intersecting orthogonally with a two-dimensional linear subspace), and in general,

$$G^{\mathbb{R}}(m, n) = G^{\mathbb{R}}(m, m - n).$$

We should mention further that $G^{\mathbb{R}}(m,n)$ is an n x (m $- n$)-dimensional manifold (i.e., $G^{\mathbb{R}}(m, n)$ is a compact topological space such that each of its points has some neighborhood homeomorphic to $\mathbb{R}^{n\times(m-n)}$). If you are familiar with the definition of the k-th orthogonal group O(k), you will probably guess that

$$G^{\mathbb{R}}(m, n) = O(m)/O(m \quad n) \text{ x } O(n)$$

So you see that Grassmann manifolds have many fascinating facets, but in fact, beyond just being amusing, they dominate the universe of the equivalence classes of all n-dimensional vector bundles over all possible base spaces, with the limit $\lim_{m\to\infty} G^{\mathbb{R}}(m,n)$ as their tsar. We pursue this topic next.

(b) CANONICAL BUNDLES OVER GRASSMANN MANIFOLDS. We look at the subset

$$E = \left\{ (X,x) \in G^{\mathbb{R}}(m,n) \times \mathbb{R}^m \mid x \in X \right\}$$

of the product space $G^{\mathbb{R}}$ (m, n) x $\mathbb{R}^m$. This is a topological space where every point X (an n-linear subspace of R^m) in $G^{\mathbb{R}}(m, n)$ appears with a train (homeomorphic to R") of attendants (points of R^m belonging to X). If we define $\pi : E \to G^{\mathbb{R}}(m, n)$ by $\pi(X, x)$ = X it is obvious that

$$\gamma^n \left(G^{\mathbb{R}}(m, n)\right) \equiv \left(E, \pi, G^{\mathbb{R}}(m, n)\right)$$

is an n-dimensional vector bundle over the base space $G^{\mathbb{R}}(m, n)$ with the total space E. We say that $\gamma^n(G^{\mathbb{R}}(m,n))$ the canonical vector *bundle* over the Grassmann manifold $G^{\mathbb{R}}(m, n)$.

EXAMPLE 8.16. The Grassmann manifold $\gamma^1(G^{\mathbb{R}}(2, 1))$ is homeomorphic to the Mobius strip.

EXAMPLE 8.17. It is a bit hard to visualize $\gamma^1(G^{\mathbb{R}}(3, 1))$. You might look at $G^{\mathbb{R}}(m, n)$ in the following way:

$$G^{\mathbb{R}}(3,1) = P^2(\mathbb{R}) = (\text{Mobius strip}) \cup D^2.$$

First you think of the one-dimensional vector bundle over the circle S^1 whose total space is the Mobius strip (this can be a different copy from the Mobius strip in the above decomposition). If you think of the base space S^1 as the center circle of the Mobius band and extend this bundle over the entire Mobius band, you have the trivial one-dimensional vector bundle along its boundary edge (which is also the boundary of D^2). Now extend this bundle over all of D^2. The resulting total space is E.

(c) Grassmann Manifolds as Classifying Spaces. Since we may think of n-dimensional subspaces of $\mathbb{R}^m$ as those of $\mathbb{R}^{m+1}$, we have the following inclusion sequence:

$$G^{\mathbb{R}}(n+1,n) \subset G^{\mathbb{R}}(n+2,n) \subset \cdots \subset G^{\mathbb{R}}(n+N,n) \subset \cdots .$$

We also have the inclusion sequence of the corresponding canonical vector bundles:

$$\gamma^n\left(G^{\mathbb{R}}(n+1,n)\right) \subset \gamma^n\left(G^{\mathbb{R}}(n+2,2)\right) \subset \cdots \subset \gamma^n\left(G^{\mathbb{R}}(n+N,n)\right) \subset \cdots .$$

We have a very important theorem, whose proof we will only outline.

Theorem 8.18. *Let X be a cell complex and let $\xi = (E,\pi,X)$ be an arbitrary n-dimensional vector bundle. Then for a sufficiently large number* N *(in fact,* $N \geq \dim X + 2$*), there exists a vector bundle map from ξ to the canonical vector bundle $\gamma^n(G^{\mathbb{R}}(n+N,n))$; thus, ξ is an induced bundle of γ^n.*

Outline of Proof. To show that there exists a vector bundle map from ξ to the canonical bundle $\gamma^n(G^{\mathbb{R}}(n+N,n))$, we need a map from E to $\mathbb{R}^{n+N}$ such that the image of its restriction to each fiber is always an n-dimensional linear subspace of $\mathbb{R}^{n+N}$. We can show that such a map from E to $\mathbb{R}^{n+N}$ exists if N is large enough. For instance, for a trivial vector bundle $\xi = (E, \pi, B)$ all you need is $N = 0$. □

Recall (Example 8.6) that if a vector bundle map of ξ to γ exists then its pullback is equivalent to ξ. Denote by $\tilde{g} = (\hat{g}, g)$ a bundle map of the above theorem. Then $g : X \to G^{\mathbb{R}}(n+N,n)$ and

$$\xi = g^*\left(\gamma^n(G^{\mathbb{R}}(n+N,n))\right).$$

So the gist of the theorem is that the bundle $\gamma^n = \gamma^n(G^{\mathbb{R}}(n+N,n))$ is more complicated than any other n-dimensional vector bundle ξ

and hence that ξ is induced from γ^n (actually reduced from γ^n), with no exceptions.

We say that two vector bundle maps are *bundle homotopic* if they change continuously from one to the other through vector bundle maps. Under this circumstance, their induced maps of the base spaces are homotopic (as ordinary maps).

The proof of the following theorem is analogous to that of Theorem 8.18.

THEOREM 8.19. *Let X be a cell complex and let* $\xi = (E, \pi, X)$ *be an n-dimensional vector bundle. Then for a large enough N (for instance,* $N \geq \dim + 3$), *any two vector bundle maps from* ξ *to the canonical vector bundle* $\gamma^n(G^{\mathbb{R}}(n + N, n))$ *are bundleehomotopic.*

Since an arbitrary map $f : X \to G^{\mathbb{R}}(n + N, n)$ induces a pullback $f^*\gamma^n$ over X, as a corollary to Theorem 8.18 we get the following result.

THEOREM 8.20. *There is a one-to-one correspondence between the set of all homotopy classes of maps from X to* $G^{\mathbb{R}}(n + N, n)$, *where N is at least* $dim X + 3$, *and the set of all isomorphism classes of n-dimensional vector bundles over X (two bundles belong to the same isomorphism class if they are vector-bundleeisomorphic).*

EXAMPLE 8.21. Take $X = S^1$ and consider the set of all (isomorphism classes of) one-dimensional vector bundles over S^1, which consists of two points (the trivial vector bundle and the Mobius band, cf. Example 8.13). On the other hand, $G^{\mathbb{R}}(1 + N, 1)$ is homeomorphic to the N-dimensional real projective space $P^N(\mathbb{R})$. It is evident that the set of homotopy classes of maps from S^1 to $P^N(\mathbb{R})$ also consists of two points (the homotopy class of a constant map and the homotopy class of a homeomorphism of S^1 to the oneedimensional real projective line $P^1(\mathbb{R}) = S^1 \subset P^N(\mathbb{R})$).

We denote by $\lim_{N\to\infty} G^{\mathbb{R}}(n + N, n)$ any Grassmann manifold $G^{\mathbb{R}}(n + N, n)$ with a large N, and write

$$BO(n) = \lim_{N\to\infty} G^{\mathbb{R}}(n + N, n).$$

Since there is a one-to-one correspondence between the set of all vector bundles over X and the set of all homotopy classes of X to $BO(n)$, we say that $BO(n)$ is the *classifying space* for the n-dimensional real vector bundles. Now we must refer you to some results concerning

Lie groups; that is, the Grassmann manifold $G^{\mathbb{R}}(n+N, n)$ has a representation as a homogeneous space (quotient space of a Lie group by its Lie subgroup) $G^{\mathbb{R}}(n+N, n) = O(n+n)/O(n) \times O(N)$. Hence we can write

$$BO(n) = \lim_{N\to\infty} O(n+N)/O(n) \times O(N)$$
$$= O(n+\infty)/O(n) \times O(\infty).$$

In a similar way, we can classify complex vector bundles using the complex Grassmann manifold $G^{\mathbb{C}}(n+N, n)$ and the classifying space

$$BU(n) = \lim_{N\to\infty} U(n+N)/U(n) \times U(N)$$
$$= U(n+\infty)/U(n) \times U(\infty).$$

The above argument has reduced the fundamental problem of classifying the n-dimensional real (complex) vector bundles to that of investigating the set of homotopy classes of maps into the classifying space $BO(n)$ $(BU(n))$.

Have we simplified the problem? Now cohomology plays a useful role. A map f from X to $BO(n)$ induces the cohomology homomorphism

$$f^* : H^*(BO(n); G) \to H^*(X; G)$$

for any coefficient group G. Recall that homotopic maps induce the identical homomorphism. If a map is constant then its induced homomorphism maps every element of $H^*(BO(n); G)$ to 0. But a constant map corresponds to the trivial vector bundle, and so if $f^*(c) \in H^*(X;G)$ does not vanish for some non-zero element $c \in H^*(BO(n);G)$, then the corresponding pullback $f^*(\gamma^n)$ over X is guaranteed to be non-trivial.

For a non-zero $c \in H^*(BO(n); G)$, we say that $f^*(c) \in H^*(X; G)$ is a characteristic *class* of the vector bundle $f^*(\gamma^n)$ over X.

We can thus rephrase our problem: how many non-zero elements are there in $H^*(BO(n);G)$? In the next chapter we will investigate the cohomology $H^*(BU(n);\mathbb{R})$ of the classifying space of n-dimensional complex vector bundles while limiting ourselves to stating only the results for $H^*(BO(n); G)$.

Summary

8.1 A fiber bundle is locally a product, but not globally a product in general.

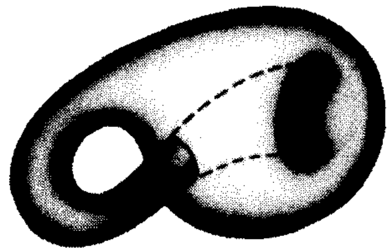


FIGURE 8.3. The Klein bottle

8.2 We say that a fiber bundle is a vector bundle if each of its fibers has the structure of a vector space.
8.3 A Grassmann manifold consists of all subspaces of a certain dimension in a vector space, over which we have the canonical vector bundle.
8.4 There is a one-to-one correspondence between the set of all isomorphism classes of vector bundles (of a fixed dimension, say n) and the set of homotopy classes of maps into the classifying space which is the limit of Grassmann manifolds $G^{\mathbb{R}}(n+N, n)$ as N becomes large.

Exercises
8.1 Define a homeomorphism $r : S^1 \to S^1$ of the circle to be the reflection along the plumb line dividing S^1 symmetrically right and left. We define the *Klein bottle* (Figure 8.3) as the quotient space of the product space $I \times S^1$, where I is the interval [0, 1], identifying the point $\{0\} \times x$ on $\{0\} \times S^1$ with the point $\{1\} \times r(x)$ on $\{1\} \times S^1$ for every $x \in S^1$. Show that the Klein bottle is the total space of a certain fiber bundle.
8.2 Show that the complex Grassmann manifold $G^{\mathbb{C}}(2, 1)$ is homeomorphic to the two-dimensional sphere S^2.

CHAPTER 9

Spectral Sequences

We can compute homology groups of product spaces using the Kiinneth formula that we discussed in an earlier chapter, but we can do better. We now introduce the theory of spectral sequences, which relate homology groups of the total spaces of fiber bundles to those of the base spaces and fibers. This theory, at a glance, looks somewhat complicated; however, once one masters its usages one will find it to be quite an attractive and practical computational tool. Spectral sequences enable us to show the existence of the Chern classes of vector bundles which occupy an important space in current mathematics.

9.1. Exact couples and spectral sequences

By an *exact couple* we mean two abelian groups D and F, and homomorphisms $i : D \to D$, $j : D \to E$ and $k : E \to D$ between them with the exact triangle:

$$\begin{array}{ccc} D & \xrightarrow{i} & D \\ & {}^{k}\nwarrow \quad \swarrow_{j} & \\ & E, & \end{array}$$

that is,

$$\begin{aligned} \ker k &= \operatorname{im} j, \quad \text{at } E, \\ \ker i &= \operatorname{im} k, \quad \text{at } D, \\ \ker j &= \operatorname{im} i, \quad \text{at } D. \end{aligned}$$

Given an exact couple as above, set

$$d \equiv jk : E \to E.$$

Then d satisfies

$$d^2 = dd = (jk)(jk) = j(kj)k = 0.$$

From a given exact couple we derive a new exact couple

$$\begin{array}{ccc} D' & \xrightarrow{i'} & D' \\ & {\scriptstyle k'}\nwarrow \quad \swarrow{\scriptstyle j'} & \\ & E' & \end{array}$$

as follows. Set

$$\text{D'} \equiv \text{i(D)} \ (= \ \ker j) \subset D,$$
$$E' \equiv H(E, d) \ (= \ker d/\operatorname{im} d),$$

and define

$$i' : D' \to D', \qquad j' : D' \to E', \qquad k' : E' \to D'$$

by

$$i' \equiv i|_{D'} : D' \to D', \quad \text{restriction of i to } D',$$
$$j' \equiv ji^{-1} : D' \equiv i(D) \to \ker(dj) \to \ker d/\operatorname{im} d \equiv E';$$

that is, for $i(x) \in D'$ we have $d(j(x)) = (jk)j(x) = 0$, and so $j(x)$ determines the homology class $[j(x)] \in H(E, d) \equiv E'$, where we set $j'(i(x)) \equiv [j(x)]$ (the definition of j' is good, since if we take x' with $i(x') = i(x) \in D'$, the fact keri = imk implies $x' \quad x = k(y)$, and $j(x') - j(x) = jk(y) = d(y)$, and so we get $[j(x)] = [j(x')]$).

We define k' by

$$k'[y] \equiv k(y);$$

that is, for [y] $\in E'$, y $\in$ kerd, the fact $jk(y)$ = d(y) = 0 implies that $k(y) \in$ kerj = imi. Hence, $k(y) \in D'$, and so we can determine $k'[y]$ as an element of D'. (This k' is well-defined. Take [y'] = [y]; then from y' − y = d(z) we get $k(y') - k(y) = kd(z) = (kj)k(z) = 0$.) We have thus constructed the *derived couple* as shown in the above diagram.

SAMPLE PROBLEM 9.1. The derived couple gives an exact triangle; that is,

$$\ker i' = \operatorname{im} k',$$
$$\ker k' = \operatorname{im} j',$$
$$\ker j' = \operatorname{im} i'.$$

SOLUTION. Take your time and enjoy.

If we continue deriving a new exact couple out of the old, we will arrive at the n-th derived couple

$$\begin{array}{ccc} D^n & \xrightarrow{i^n} & D^n \\ & {\scriptstyle k^n}\nwarrow \quad \swarrow {\scriptstyle j^n} & \\ & E^n & \end{array}$$

in a natural way.

DEFINITION 9.2. Let (D, E, i, j, k) be an exact couple and consider the sequence of the derived exact couples $(D^n, E^n, i^n, j^n, k^n)$. Set $d^n = j^n k^n$. Then $d^n : E^n \to E^n$ is an endomorphism of E^n ($d^n d^n = 0$). We say that the sequence (E^n, d^n), $n = 1, 2, \ldots, n$, is the *spectral sequence* of (D, E, i, j, k) $(E^{n+1} = H(E^n, d^n))$. The spectral sequence converges if for some integer $k > 0$, we have $E^n = 0$ for every $n \geq k$. In this case ker $d^n = E^n$ and im $d^n = 0$; hence

$$E^k = E^{k+1} = E^{k+2} = \cdot$$

We denote by E^∞ these mutually identical abelian groups.

We need some new terminology.

DEFINITION 9.3. We say that an abelian group A is *bigraded* if it has a direct sum representation

$$A = \sum_{p \in \mathbb{Z}} \sum_{q \in \mathbb{Z}} A_{p,q}.$$

The abelian group A is *first quadrant bigraded* if it has the expression

$$A = \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} A_{p,q}.$$

We say that a homomorphism

$$f : A = \sum_{p \in \mathbb{Z}} \sum_{q \in \mathbb{Z}} A_{p,q} \to A = \sum_{p \in \mathbb{Z}} \sum_{q \in \mathbb{Z}} A_{p,q}$$

has *bidegree* (a, b) if for every $p, q \in \mathbb{Z}$ we have

$$f(A_{p,q}) \subset A_{p+a,q+b}.$$

For the same A and $A_{p,q}$ as above, suppose we attach $A_{p,q} \neq 0$ to the lattice point (p, q), and suppose that A is first quadrant bigraded. Then the dot to represent (p, q) in the plane always stays in the first quadrant.

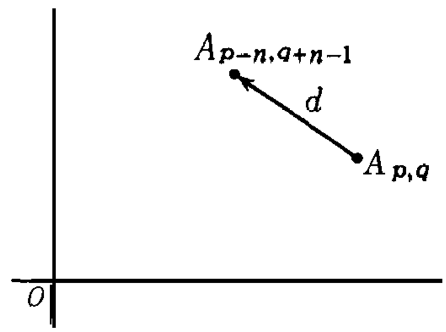


FIGURE 9.1. Differential

DEFINITION 9.4. We say that a homomorphism $d : A \to A$ is a *differential* when it satisfies

$$dd = 0.$$

Suppose a differential d $: A = \sum_{p\in\mathbb{Z}}\sum_{q\in\mathbb{Z}} A_{p,q} \to A = \sum_{p\in\mathbb{Z}}\sum_{q\in\mathbb{Z}} A_{p,q}$ has the bidegree $(-n, n-1)$. Put

$$A_i \equiv \sum_{p+q=i} A_{p.q},$$

$$d_i = d|_{A_i} : A_i \to A_{i-1}.$$

Then we have the chain complex

$$\to A_{i+1} \xrightarrow{d_{i+1}} A_i \xrightarrow{d_i} A_{i-1} \xrightarrow{d_{i-1}} A_{i-2} \to \cdots.$$

See Figure 9.1.

If we start out with a bigraded exact couple, we will end up with a bigraded spectral sequence.

9.2. Spectral sequences of fiber bundles

In Chapter Seven we computed homology groups of a product space B x F via the Künneth formula. Serre developed a method to calculate the homology groups of the total space E of a fiber bundle (E, π, B, F) from those of B and F. In the latter case it is more complicated than passing to the tensor products and so on. In fact one looks at the limit E_∞ of a certain spectral sequence to obtain some information about $H_*(E;$ G). To be more explicit, we can prove the following Theorems 9.6 and 9.9.

Let B be a simply connected cell complex. For $q = 0, 1, 2, \ldots$, denote by B^q the q-skeleton of B, which consists of all cells of dimension less than or equal to q. Thus we have that

$$\emptyset = B^{-1} \subset B^0 \subset B^1 \subset \ldots \subset B^b = B, \quad b = \dim B.$$

Putting

$$E^q = \pi^{-1}(B^q) \subset E,$$

we get

$$\emptyset = E^{-1} \subset E^0 \subset E^1 \subset \cdots \subset E^b = E.$$

We choose an abelian group G and set

$$F_{p,q} \equiv \operatorname{im}\left(H_{p+q}(E^p; G) \to H_{p+q}(E; G)\right).$$

We can reduce the calculation of the n-th homology group of E to its calculation over the n-cells of the base space B. Thus we can prove the next proposition by an argument similar to the one we used for Theorem 5.7.

PROPOSITION 9.5. *For any n, n = 0,* $1, 2, \ldots$, *we have*

$$H_n(E; G) = F_{n,0} \equiv \operatorname{im}\left(H_n(E^n; G) \to H_n(E; G)\right)$$

Hence we have the following sequence of abelian subgroups:

$$0 = F_{-1,n+1} \subset F_{0,n} \subset \cdots \subset F_{n-1,1} \subset F_{n,0} = H_n(E; G).$$

THEOREM 9.6 (Serre's homology spectral sequence). *Let G be an abelian group. Let (E, π, B, F) be a fiber bundle, where B a simply connected cell complex. Then we can define in a natural way an exact couple which has the convergent spectral sequence (E^n, d^n) with the following properties.*

(1) $E^n = \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} E^n_{p,q}$ *(first quadrant bigraded), and d^n is a differential of bidegree $(-n, n-1)$; $d^n(E^n_{p,q}) \subset E^n_{p-n,q+n-1}$.*

(2) $E^2_{p,q} \cong H_p\left(B; H_q(F; G)\right)$.

(3) $E^\infty_{p,q} \cong F_{p,q}/F_{p-1,q+1}$

This is the *Serre homology spectral sequence* of the fiber bundle (E, π, B, F) over G.

We obtain the initial exact couple for the theorem as follows. Consider the chain complexes $\bar{C}_*(E)$, $\bar{C}_*(E^p)$ and $\bar{C}_*(E^{p-1})$ of the cell complexes E, E^p and E^{p-1} respectively. They induce the homology exact sequence that we can use to define the initial exact couple:

$$\begin{array}{ccc} D^1 = \sum_{p,q} H_{p+q}(\bar{C}_*(E)/\bar{C}_*(E^{p-1})) & \xrightarrow{i} & D^1 = \sum_{p,q} H_{p+q}(\bar{C}_*(E)/\bar{C}_*(E^{p-1})) \\ {}_{k}\nwarrow & & \swarrow_{j} \\ & E^1 = \sum_{p,q} H_{p+q}(\bar{C}_*(E^p)/\bar{C}_*(E^{p-1})). & \end{array}$$

Observation Do you understand the implication of the theorem? What the theorem says is the following. First we compute the $E^2_{p,q}$ from the homologies of B and F. From this we keep taking homologies (compute inductively $E^3_{p,q}, E^4_{p,q}, \ldots$). Then the sequence converges to $E^\infty_{p,q}$, which is the quotient $F_{p,q}/F_{p-1,q+1}$ of the adjacent groups in the sequence $0 = F_{-1,n+1} \subset F_{0,n} \subset \cdots \subset F_{n-1,1} \subset F_{n,0} = H_n(E; G)$ of subgroups of $H_n(E)$.

For instance, $E^2_{p,q} = H_p(B; H_q(F; G))$ is isomorphic to

$$H_p(B;\mathbb{Z}) \otimes H_q(F;G) \oplus \operatorname{Tor}(H_{p-1}(B;\mathbb{Z}), H_q(F;G))$$

by the universal coefficient theorem.

Example 9.7. For a trivial bundle $(E, \pi, B, F) = B \times F$, we have $d^n = 0$ for $n \geq 2$, and so $E^2 = E^3 = \ldots = E^\infty$. Moreover, we have the equality

$$\begin{aligned} & H_n(B \times F;\mathbb{Z}) \\ = & \sum_{p+q=n} H_p(B;\mathbb{Z}) \otimes H_q(F;\mathbb{Z}) \oplus \sum_{p+q=n-1} \operatorname{Tor}(H_p(B;\mathbb{Z}), H_q(F;\mathbb{Z})), \end{aligned}$$

which is nothing but the Künneth formula.

Definition 9.8. We say that a spectral sequence (E^n, d^n) *collapses* when $d^n = 0$, $n \geq 2$, and hence when

$$E^2 = E^3 = \cdots = E^\infty.$$

In §9.6 and beyond we will encounter some non-trivial collapsing spectral sequences which will work a miracle in the computation of cohomologies of classifying spaces.

In the above spectral sequence, we see that the points $E^r_{0,q}$ on the y-axis is a cycle for $r > 0$ (since d^r sends $E^r_{0,q}$ to $E^r_{-r,q+r-1}$, which is in the second quadrant and hence is zero). Hence a natural map

$$E^r_{0,q} \to E^{r+1}_{0,q}$$

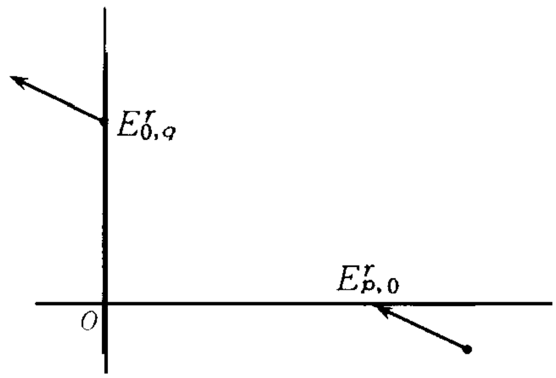


FIGURE 9.2

onto the homology classes exists. Furthermore, the fact $F_{-1,q+1} = 0$ implies that $E^\infty_{0,q} \cong F_{0,q}/F_{-1,q+1} = F_{0,q} \subset H_q(E; G)$. By Theorem 9.6 (2) we have that $E^2_{0,q} \cong H_0(B; H_q(F; G))$, but then the universal coefficient theorem and the fact $H_0(B; \mathbb{Z}) \cong \mathbb{Z}$ imply that $E^2_{0,q} \cong F_q(F; G)$. Hence we have a natural map

$$\hat{i} : H_q(F; G) \cong E^2_{0,q} \to E^\infty_{0,q} \to H_q(E; G).$$

Similarly, the boundary of a point $E^r_{p,0}$ on the x-axis comes from the fourth quadrant and so it is zero (Figure 9.2). Hence the cycles and homology classes agree, and there is a natural one-to-one map

$$E^{r+1}_{p,0} \to E^r_{p,0}.$$

We also have a projection $H_p(E) = F_{p,0} \to E^\infty_{p,0}$, because $E^\infty_{p,0} \cong F_{p,0}/F_{p-1,1}$. It follows from Theorem 9.6 (2) and the universal coefficient theorem (Theorem 7.4) that $E^2_{p,0} \cong H_p(B; G)$. Therefore, we have a natural map

$$\hat{\pi} : H_p(E; G) \to E^\infty_{p,0} \to E^2_{p,0} \cong H_p(B; G).$$

The Serre spectral sequence is also useful in

THEOREM 9.9. *Let (E, π, B, F) be a fiber bundle whose base space B is simply connected, and let $i : F \to E$ be an embedding of F onto some fixed fiber over a point of B. Then we can calculate the induced maps π_* and i_* from Serre's spectral sequence; that is, in Serre's spectral sequence for (E, π, B, F) we have, for all $n \geq 0$, that*

$$i_* = \hat{i} : H_n(F; G) \to H_n(E; G),$$
$$\pi_* = \hat{\pi} : H_n(E; G) \to H_n(B; G).$$

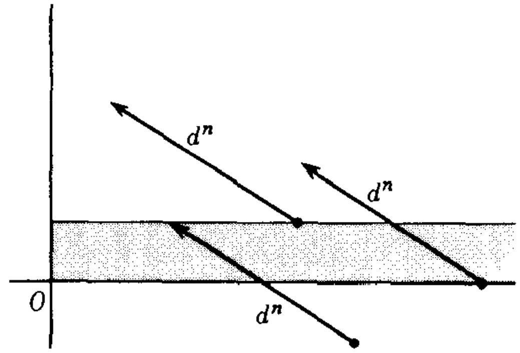


FIGURE 9.3. Homology of $P^k(\mathbb{C})$

Can we actually compute H,(E) using non-collapsing spectral sequences? In the next section we apply this theorem to compute some homology groups.

9.3. Applications of spectral sequences

It is easy to show that the complex projective space $P^k(\mathbb{C})$ is the base space of a fiber bundle whose total space is S^{2k+1} and whose fiber is S^1. Hence $P^k(\mathbb{C})$ is a $2k$-dimensional cell complex (it is in fact a manifold) with

$$H_j(P^k(\mathbb{C});\ \mathbb{Z}) = 0, \quad j < 0 \text{ and } j > 2k.$$

It is also easy to show that $P^k(\mathbb{C})$ is simply connected for $k \geq 1$. We stated earlier (Exercise 5.1) that the homology groups of the $(2k+1)$-sphere S^{2k+1} were

$$H_j(S^{2k+1};\mathbb{Z}) \cong \begin{cases} \mathbb{Z}, & j = 0, 2k+1; \\ 0, & \text{otherwise.} \end{cases}$$

We wish to compute the homology groups $H_j(P^k(\mathbb{C};\mathbb{Z})\ (j \geq 0)$ using the Serre spectral sequence of the fiber bundle $(S^{2k=1}, \pi, P^k(\mathbb{C}), S^1)$ over $\mathbb{Z}$ (cf. Figure 9.3).

Theorem 9.6 (2) says that $E^2_{p,q} \cong H_p(B;\ H_q(F;\ \mathbb{Z}))$, so that

$$E^2_{p,q} = 0,\ q < 0 \text{ or } q > 1, \quad E^2_{p,0} \cong E^2_{p,1} \cong H_p(B;\mathbb{Z}).$$

Since $E^{n+1}_{p,q} = H(E^n_{p,q}, d^n)$, we have, for all $n \geq 2$,

$$E^n_{p,q} = 0,\ q < O \quad \text{or} \quad q > 1$$

Look at the homomorphism $d^n : E^n_{p,q} \to E_{p-n,q+n-1}$. If $n > 2$, then we have $E^n_{p,q} = 0$ or $E^n_{p-n\ q+n-1} = 0$, and so

$$d^n = 0 : E^n_{p,q} \to E^n_{p-n\ q+n-1}, \quad n > 2.$$

Therefore, for all p and q we have

$$E^3_{p,q} \cong E^4_{p,q} \cong \cdots \cong E^\infty_{p,q}.$$

We have $d^2(E^2_{p,1})$ $(\subset E^2_{p-2,2}) = 0$. Furthermore, the fact $E^2_{p+2,-1} = 0$ implies that $\operatorname{im}(d^2 : E^2_{p+2,-1} \to E^2_{p,0}) = 0 \subset E^2_{p,0}$. Hence we get

$$E^\infty_{p,1} \cong E^3_{p,1} \cong E^2_{p,1} / \operatorname{im}(d^2 : E^2_{p+2,0} \to E^2_{p,1}),$$
$$E^\infty_{p,0} \cong E^3_{p,0} \cong \ker(d^2 : E^2_{p,0} \to E^2_{p-2,1}).$$

On the other hand, we have $E = S^{2k+1}$, and so from the sequence

$$F_{0,j} \subset \cdots \subset F_{j-1,1} \subset F_{j,0} = H_j(S^{2k+1}; \mathbb{Z})$$

we get

$$F_{p,q} = 0, \quad p + q \neq 0, \quad 2k + 1.$$

Recalling Theorem 9.6 (3), which says that $E^\infty_{p,q} \cong F_{p,q}/F_{p-1,q+1}$, we conclude that

$$E^\infty_{p,q}, \quad p + q \neq 0,\ 2k + 1;$$

hence it follows that

$$E^2_{p,1} = \operatorname{im}(d^2 : E^2_{p+2,0} \to E^2_{p,1}), \quad p \neq 2k,$$
$$\ker(d^2 : E^2_{p+2,0} \to E^2_{p-2,1}) = 0, \quad p \neq 2k + 1.$$

From these two equalities we conclude that if $p \neq 2k$, then

$$d^2 : E^2_{p+2,0} \cong H_{p+2}(B; \mathbb{Z}) \to E^2_{p\ 1} \cong H_p(B; \mathbb{Z})$$

is an isomorphism. We now recall that $H_{-1}(B; \mathbb{Z}) = 0$, $H_0(B; \mathbb{Z}) \cong \mathbb{Z}$ and $B = P^k(\mathbb{C})$, to obtain

$$H_j(P^k(\mathbb{C}); \mathbb{Z}) \cong \begin{cases} \mathbb{Z}, & j = 0, 2, \ldots, 2k; \\ 0, & j \text{ odd}. \end{cases}$$

We can compute the cohomology groups of $P^k(\mathbb{C})$ using the universal coefficient theorem.

9.4. Cohomology spectral sequences

We obtained Serre's homology spectral sequences from certain exact couples of fiber bundles. In a like manner we derive cohomology spectral sequences as follows. Recall that cohomology groups have cup products, which will turn out to commute with the convergence of cohomology spectral sequences. In this sense cohomology spectral sequences are even more useful than homology sequences.

In working with cohomology we habitually use subindices for derived couples $(E_n, D_n, i_n, j_n, k_n)$, where E_n and D_n are bigraded. Setting

$$F^{p,q} \equiv \ker\left(H^{p+q}(E; G) \to H^{p+q}(E^{p-1}; G)\right),$$

we get a sequence

$$H^n(E; G) = F^{0,n} \supset F^{1,n-1} \supset \cdots \supset F^{n+1,-1} = 0$$

of abelian groups.

THEOREM 9.10 (Serre's cohomology spectral sequence). *Let G be an abelian group, B a simply connected cell complex and (E, π, B, F) a fiber bundle. Then we can define a natural exact couple which has a convergent spectral sequence satisfying the following properties.*

(1) $E_n = \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} E_n^{p,q}$ *(first quadrant bigraded), where d_n is a differential $d_n(E_n^{p,q}) \subset E_n^{p+n,q-n+1}$ of bidegree $(n, -n + 1)$.*

(2) $E_2^{p,q} \cong H^p(B, H^q(F; G))$.

(3) $E_\infty^{p,q} \cong F^{p,q}/F^{p+1,q-1}$.

(4) *For every $n \geq 1$ there is a well-dejined product $E_n^{p,q} \otimes E_n^{p',q'} \to E_n^{p+p',q+q'}$*

(5) $d_n(ab) = d_n(a)b + (-1)^{p+q} a d_n(b)$, $a \in E_n^{p,q}$, $b \in E_n^{p',q'}$.

(6) *The* product

$$E_2^{p,q}\ (\cong H^p(B, H^q(F; G))) \otimes E_2^{p',q'}\left(\cong H^{p'}(B, H^{q'}))\right)$$
$$\to E_2^{p+p',q+q'}\left(\cong H^{p+p^Prime}(B, H^{q+q'}(F; G))\right)$$

equals $(-1)^{qp'}$ *times the cup product of B, whose coefficient* product *is the cup product of F.*

(7) *We derive the product for E_{n+1} from the product for E_n, and* we derive *the* product for E_∞ *from the cup product for E.*

The spectral sequence in Theorem 9.10 is the Serre *cohomology spectral sequence* of a fiber bundle (E, π, B, F) over G.

Note that the bidegree of d_n is $(n, -n+1)$, which is different from the homology bidegree $(-n, n-1)$.

We can also compute the maps of cohomology groups,

$$i^* : H^n(E; G) \to H^n(F; G),$$
$$p^* : H^n(B; G) \to H^n(E; G),$$

induced by the embedding $i : F \to E$ of the fiber F and the projection $\pi : E \to B$ of the total space E in the same way as we did in the homology case.

9.5. Cohomology groups of $P^k(\mathbb{C})$

In §9.3 we calculated the homology groups of P"(C) from the homology spectral sequence over $\mathbb{Z}$ of a certain fiber bundle, and noted that we can obtain its cohomology groups from the universal coefficient theorem. In this section, we want to take advantage of Serre's cohomology spectral sequence to compute the cohomology groups of the projective space $P^k(\mathbb{C})$ and investigate their cup-product structure.

Our situation here is just as it is with homology, and for $p \neq 2k$ the map

$$d_2 : E_2^{p,1} \cong H^p(B; \mathbb{Z}) \to E_2^{p+2,0} \cong H^{p+2}(B; \mathbb{Z})$$

is an isomorphism.

On the other hand, suppose that $a_1 \in E_2^{p,1}$ and $a_0 \in E_2^{p,0}$ correspond under isomorphisms $E_2^{p,1} \cong H^p(B; \mathbb{Z})$ and $E_2^{p,0} \cong H^p(B; \mathbb{Z})$. For this $a_0 \in E_2^{p,0} \cong H^p(B; \mathbb{Z})$, $1 \in E_2^{0,1} \cong H^0(B; \mathbb{Z}) \cong \mathbb{Z}$ satisfies

$$a_0 \cup 1 = a_1 \in E_2^{p,1} \cong H^p(B; \mathbb{Z})$$

(it may be a bit tedious, but write down three copies of the spectral sequence and consider the cup product).

Hence, by Theorem 9.10 (5) we have

$$\begin{aligned} d_2(a_1) &= d_2(a_0 \cup 1) = d_2(a_0) \cup 1 + (-1)^p a_0 \cup d_2(1) \\ &= (-1)^p a_0 \cup d_2(1), \end{aligned}$$

where $d_2(1) \in H^2(B; \mathbb{Z}) \cong \mathbb{Z}$, as well as its multiple by -1, is a generator of the group $\mathbb{Z}$. Thus we have shown the following result.

THEOREM 9.11. *The cohomology groups of* $P^k(\mathbb{C})$ *over* $\mathbb{Z}$ *are*

$$H^j(P^k(\mathbb{C});\mathbb{Z}) \cong \begin{cases} \mathbb{Z}, & j = 0, 2, \dots, 2k; \\ 0, & j \text{ odd.} \end{cases}$$

If u *generates* $H^2(P^k(\mathbb{C});\mathbb{Z})$, *then* $u^j = u \cup u \cup \dots \cup u$ generates $H^{2j}(P^k(\mathbb{C});\mathbb{Z})$.

9.6. Collapsing cohomology sequences

We see in the next proposition an example of non-trivial fiber bundles with collapsing spectral sequences. We only investigate cohomology which will be useful for us later.

By $H^{odd}(X;G)$ we mean the direct sum

$$H^{odd}(X;G) = \sum_{i=0}^{\infty} H^{2i+1}(X;G)$$

of odd-dimensional cohomology groups of X over G.

PROPOSITION 9.12. *Suppose that a fiber bundle* (E, π, B, F) *over a simply connected base space* B *satisfies*

$$H^{odd}(B;\mathbb{R}) = H^{odd}(F;\mathbb{R}) = 0.$$

Then *the spectral sequence of* (E, π, B, F) *over* $\mathbb{R}$ *collapses; that is,*

$$d_r = 0 : E_r^{p,q} \to E_r^{p+r,q-r+1},$$

for every $r \geq 2$.

PROOF. The universal coefficient theorem (Theorem 7.6) and the facts that $\mathrm{Tor}(\mathbb{Z},\mathbb{R}) = 0$ and $\mathbb{Z}\otimes\mathbb{R} \cong \mathbb{R} \cong \mathbb{R}\otimes_{\mathbb{R}}\mathbb{R}$ imply that

$$E_2^{p,q} \cong H^p(B; H^q(F;\mathbb{R})) \cong H^p(B;\mathbb{Z}) \otimes H^q(F;\mathbb{R})$$
$$\cong H^p(B;\mathbb{R}) \otimes_{\mathbb{R}} H^q(F;\mathbb{R}).$$

Therefore, if either p or q is odd then $E_2^{p,q} = 0$. Thus in this case $E_r^{p,q} = 0$ for $r \geq 2$. Moreover we get $d_r = 0 : E_r^{p,q} \to E_r^{p+r,q-r+1}$. But at least one of p, q, $p + r$, $q - r + 1$ is always odd, and so $d_n = 0$ holds whenever $r \geq 2$. ❑

We have one more proposition restricted to cohomology groups.

PROPOSITION 9.13. *Suppose that* a *fiber* bundle (E, π, B, F) *over a simply connected base space* B *has a collapsing cohomology spectral sequence.* Then *for every natural number* n *the following statements hold:*

(1) $i^* : H^n(E;G) \to H^n$ (F; G) is a *surjection;*

(2) $p^* : H^n(B;G) \to H^n(E;$ *G) is an injection;*

(3) $H^*(E;\mathbb{R}) \cong H^*(B;\mathbb{R}) \otimes_{\mathbb{R}} H^*(F;\mathbb{R})$.

PROOF. Just as in the homology case, the map i^* is equal to the composition

$$H^n(E;G) = F^{0,n} \to F^{0,n}/F^{1,n-1} \cong E_\infty^{0,n} \to E_2^{0,n} \cong H^n(F;G),$$

and the fact that $E_\infty^{0,n} \to E_2^{0,n}$ is an isomorphism implies that it is surjective.

Furthermore, p^* is equal to the composition

$$H^n(B;G) \cong H^n\left(B; H^0(F;G)\right) \cong E_2^{n,0} \to E_\infty^{n,0} \cong F^{n,0} \subset H^n(E;G),$$

but the fact that $E_2^{n,0} \to E_\infty^{n,0}$ is an isomorphism implies that it is injective.

By the universal coefficient theorem we see that

$$E_2^{p,q} \cong H^p\left(B; H^q(F;\mathbb{R})\right) \cong H^p(B;\mathbb{R}) \otimes_{\mathbb{R}} H^q(F;\mathbb{R});$$

hence (as we are working over $\mathbb{R}$), it follows that

$$H^n(E;\mathbb{R}) \cong \sum_{p+q=n} E_\infty^{p,q} \cong \sum_{p+q=n} H^p(B;\mathbb{R}) \otimes_{\mathbb{R}} H^q(F;\mathbb{R}).$$

□

9.7. Cohomology of classifying spaces

Let us calculate the cohomology groups of the classifying space

$$BU(n) = \lim_{N\to\infty} U(n+N)/U(n) \times U(N) = U(n+\infty)/U(n) \times U(\infty)$$

of n-dimensional complex vector bundles over the real coefficients $\mathbb{R}$.

There is a natural identification of T^n which is the product of n copies of U(1) $\cong S^1$ as a subgroup of *U(n); i.e.,*

$$T^n = \begin{pmatrix} * & 0 & \dots & 0 \\ 0 & * & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & * \end{pmatrix} \subset U(n).$$

We have the fiber bundle

$$U(n)/T^n \to U(n+N)/T^n \times U(N) \to U(n+N)/U(n) \times U(N),$$

and passing to the limit as N $\to \infty$ we get the fiber bundle

$$U(n)/T^n \to BT^n \equiv U(n+\infty)/T^n \times U(\infty) \\ \to BU(n) = U(n+\infty)/U(n) \times U(\infty).$$

We use this bundle to compute the cohomology of $BU(n)$.

EXAMPLE 9.14. In low dimensions we have the following:

$$U(1)/T^1 (\cong S^1/S^1) \cong \text{one point}, \quad U(2)/T^2 \cong S^2.$$

PROPOSITION 9.15. *For any natural number n,*

$$H^{odd}(U(n)/T^n : \mathbb{R}) = 0.$$

PROOF. We prove the proposition by induction on n. By the above example we know that the equality holds for n = 1, 2. Suppose that we have shown the equality up to n. We now prove the case for n + 1. Consider the fiber bundle

$$U(n)/T^n = U(n) \times T^1/T^{n+1} \to U(n+1)/T^{n+1} \to U(n+1)/U(n) \times T^1$$

whose base space $U(n+1)/U(n) \times T^1$ is homeomorphic to the complex projective space $P^n(\mathbb{C})$ through the identification $U(n+1)/U(n) = S^{2n+1}$. Hence we have

$$H^{odd}\left(U(n+1)/U(n) \times T^1; \mathbb{R}\right) = 0.$$

By the induction hypothesis we have

$$H^{odd}(U(n)/T^n; \mathbb{R}) = 0.$$

Propositions 9.12 and 9.13 imply that the cohomology spectral sequence of this fiber bundle collapses, and the fact that

$$H^*\left(U(n+1)/T^{n+1}; \mathbb{R}\right) \\ \cong H^*(U(n)/T^n; \mathbb{R}) \otimes_{\mathbb{R}} H^*\left(U(n+1)/U(n) \times T^1; \mathbb{R}\right)$$

implies that

$$H^{odd}\left(U(n+1)/T^{n+1}; \mathbb{R}\right) = 0.$$

❑

The set $\mathbb{R}[x_1, x_2, \ldots, x_n]$ of the real polynomials in n variables $x_1, x_2, \cdot, x_n$, is an abelian group which has also a ring structure.

PROPOSITION *9.16. We have the following group (actually* ring) *isomorphism:*

$$H^*(BT^n; \mathbb{R}) \cong \mathbb{R}[x_1, x_2, \cdot\ , x_n], \quad x_i \in H^2(BT^n; \mathbb{R}).$$

PROOF. Indeed,

$$\begin{aligned}\text{BT''} &= U(n+\infty)/T^n \text{ x } U(\infty)\\ &= U\left(n(1+\infty)\right)/T^1 \text{ x } U(\infty) \text{ x } \ldots \text{ x } T^1 \text{ x } U(\infty)\\ &= BT^1 \times BT^1 \times \cdots \times BT^1.\end{aligned}$$

Since $BT^1 = P^\infty(\mathbb{C}) = \lim_{n\to\infty} P^n(\mathbb{C})$, we get $H^*(BT^1;\mathbb{R}) \cong \mathbb{R}[x]$, where x corresponds to a generator of $H^2(BT^1;$ *JR).* Hence,

$$H^*(BT^n;\mathbb{R}) \cong \mathbb{R}[x] \otimes_{\mathbb{R}} \cdots \otimes_{\mathbb{R}} \mathbb{R}[x] \cong \mathbb{R}[x_1, x_2, \ldots, x_n].$$

□

PROPOSITION 9.17. *The projection* π : *BT''* $\to BU(n)$ *induces a* cohomology *homomorphism*

$$\pi^* : H^*(BU(n);\mathbb{R}) \to H^*(BT^n;\mathbb{R}) \cong \mathbb{R}[x_1, x_2, \ldots, x_n]$$

which satisfies the following:

(1) π^* *is injective;*

(2) $\operatorname{im}\pi^* = \mathbb{R}[\sigma_1, \sigma_2, \ldots, \sigma_n]$, *where* $\sigma_i = \sigma_i(x_1, x_2, \ldots, x_n)$ *is* an *elementary symmetric function of degree i.*

PROOF. We prove (1) by noting that $H^{odd}(U(n)/T^n;\mathbb{R}) = 0$ (this is the conclusion of Proposition 9.15) and examining the basic properties of cohomology spectral sequences (Theorem 9.13 (2)).

Proof of (2): The group of permutations on n letters acts on $T^n = T^1$ x . . x T^1 by interchanging T^1's, which induces self homeomorphisms of the classifying space BT^n. The set of the cohomology elements invariant under these homeomorphisms is $\mathbb{R}[\sigma_1, \sigma_2, \ldots,$ a,]. Under this action each permutation gets represented by an element of *U(n)* and thus induces the identity map of the classifying space $BU(n)$. This shows that

$$\operatorname{im}\pi^* \text{ C } \mathbb{R}[\sigma_1, \sigma_2, \ldots, \text{a,}].$$

We show the reverse inclusion by induction on n. Suppose that the inclusion holds up to n – 1, so that $\mathbb{R}[\sigma_1, \sigma_2, \ldots, \sigma_{n-1}]$ C $\operatorname{im}\pi^*$. Let $\sigma \in \mathbb{R}[\sigma_1, \sigma_2, \ldots,$ a,]. We use the cohomology spectral sequence of the fiber bundle

$$\begin{aligned}&U(n)/U(n-1) = S^{2n-1}\\ &\qquad \to BU(n-1) = U(n-1+\infty)/U(n-1) \text{ x } U(\infty)\\ &\qquad\qquad \to BU(n) = U(n+\infty)/U(n)\, x\, U(\infty)\end{aligned}$$

to construct an element a $\in H^*(BU(n);\mathbb{R})$ such that $\pi^*(a) = \sigma$ (cf. Exercise 9.2). □

EXAMPLE 9.18. For $U(2)/T^2 = S^2 \xrightarrow{\iota} BT^2 \xrightarrow{\pi} BU(2)$, *we* have

$$H^*(BT^2;\mathbb{R}) \cong \mathbb{R}[x_1, x_2],$$

$$H^*(BU(2);\mathbb{R}) \cong \mathbb{R}[\sigma_1, \sigma_2], \quad \sigma_1 = x_1 + x_2,\ \sigma_2 = x_1x_2,$$

$$H^*\left(U(2)/T^2;\mathbb{R}\right) \cong \mathbb{R}[y]/y^2, \quad i^*(y) = x_1 \quad x_2.$$

QUESTION. Show that

$$H^*(BT^2;\mathbb{R}) \cong H^*(BU(2);\mathbb{R}) \otimes_{\mathbb{R}} H^*(U(2)/T^2;\mathbb{R}).$$

We derive the following main theorem from Proposition 9.17.

THEOREM *9.19. For* $n \geq 1$,

$$H^*(BU(n);\mathbb{R}) \cong \mathbb{R}[c_1, c_2, \ldots, c_n],\ c_j \in H^{2j}(BU(n);\mathbb{R})).$$

To investigate the properties of the classifying space $BU(n)$ for complex vector bundles we often pass to BT^1 from $BT^n = BT^1 \times \cdots \times BT^1$ via the injective map

$$\pi^* : H^*(BU(n);\mathbb{R}) \cong \mathbb{R}[x_1, x_2, \cdots, x_n].$$

This technique, which has a wide variety of applications, is known as the *splitting method.*

DEFINITION 9.20 (Chern classes). An n-complex vector bundle ξ over a base space X has an expression as a pullback of the standard vector bundle γ over the classifying space $BU(n)$ by a map $f : X \to BU(n)$ (unique up to homotopies); that is, $\xi = f^*(\gamma)$. By the *j-th* Chern class of the vector bundle ξ we mean the pullback $f^*(c_j) \in H^{2j}(X;\mathbb{R})$ of the element c_j in the $2j$-th cohomology group of $BU(n)$ (by the homomorphism $f^* : H^*(BU(n);\mathbb{R}) \to H^*(X;\mathbb{R})$). It is a very important class which detects the nontriviality of a vector bundle.

In general, cohomology elements of classifying spaces are called characteristic classes. Chern classes are characteristic classes for complex vector bundles.

We can also compute the cohomology groups of the classifying space

$$BO(n) = \lim_{N\to\infty} O(n+N)/O(n) \times O(N) = O(n+\infty)/O(n) \times O(\infty)$$

of n-dimensional real vector bundles over $\mathbb{R}$ or $\mathbb{Z}_2$. The results are as follows.

Theorem 9.21. *For $n \geq 1$, we have the following:*

$$H^*(BO(n);\mathbb{R}) \cong \mathbb{R}[p_1, p_2, \ldots, p_{[n/2]}], \quad p_i \in H^{4j}(BO(n);\mathbb{R}),$$

$$H^*(BO(n);\mathbb{Z}_2) \cong \mathbb{Z}_2[w_1, w_2, \ldots, w_n], \quad w_j \in H^j(BO(n);\mathbb{Z}_2).$$

Definition 9.22. (Pontrjagin and Stiefel-Whitney classes) Let ξ be a real n-dimensional vector bundle over a base space X. We define the *j-th Pontrjagin* class of the vector bundle ξ to be the pullback of the cohomology element $p_j \in H^{4j}(BO(n);\mathbb{R})$ of the classifying space *BO(n)*, and the *j-th Stiefel–Whitney class* to be the pullback of $w_j \in H^j(BO(n);\mathbb{Z}_2)$ (p_j and w_j are as in Theorem 9.21).

Note that Pontrjagin classes and Stiefel-Whitney classes are characteristic classes of real vector bundles.

When a manifold is smooth we have the tangent bundle over it. We define characteristic classes of a manifold to be those of its tangent bundles. These characteristic classes measure global curvatures of the manifold.

Summary

9.1 For an exact couple, we construct its n-th derived couple from its (n − 1)-st derived couple, n ≥ 1 (when n = 1 think of the "0-th derived couple" to be the starting exact couple). Each derived couple is exact. These are the building blocks for a spectral sequence.

9.2 We can calculate the homology of the total space of a fiber bundle from Serre's spectral sequence.

9.3 We can actually experience the potency of Serre's spectral sequence in computing the homology of the complex projective space $P^n(\mathbb{C})$.

9.4 We can also compute the cohomology groups of the total space of a fiber bundle from Serre's cohomology spectral sequence.

9.5 We can calculate that the cohomology ring of the classifying space of the complex vector bundles is the same as the polynomial ring of the Chern classes.

Exercises

9.1 Determine the integral homology groups of an n-sphere bundle (E, π, S^2, S^2) over the base space S^2 with fiber S^2.

9.2 Consider the cohomology spectral sequence of a three-sphere bundle $U(2)/U(1) = S^3 \to BU(1) \xrightarrow{\pi} BU(2)$ over $\mathbb{R}$. Show that

$E_2^{p,3} = E_4^{p,3}$ and $E_2^{p+4,0} = E_4^{p+4,0}$. Using $H^*(BU(1);\mathbb{R}) \cong \mathbb{R}[c_1]$, $c_1 \in H^2(BU(1);\mathbb{R})$, show that $d_4 : E_2^{p,3} \to E_2^{p+4,0}$ is an injection for $p \geq 0$.

9.3 In Serre's cohomology spectral sequence of (E, π, B, S^j), show that there exists a cohomology element $\Omega \in H^{j+1}(B; \mathbb{R})$ such that the map $\Psi : H^p(B;\mathbb{R}) \to H^{p+j+1}(B;\mathbb{R})$ defined by $\Psi(u) = u \cup \Omega$, $u \in H^p(B;\mathbb{R})$, induces the following exact sequence:

$$\cdots \to H^{p+j}(E;\mathbb{R}) \to H^p(B;\mathbb{R}) \xrightarrow{\Psi} H^{p+j+1}(B;\mathbb{R}) \xrightarrow{\pi^*} H^{p+j+1}(E;\mathbb{R}) \to \cdots$$

This is the Gysin *cohomology sequence* of a sphere bundle.

A View from Current Mathematics

We will give an easy explanation of the geometric (combinatorial) representation of characteristic classes as it stands today; at present the problem is not yet solved completely.

Euler Numbers. Denote by k_j the number of j-dimensional cells of an n-dimensional simplicial complex $\mathcal{S}$. We say that the alternating sum

$$k_0 - k_1 + \cdots + (-1)^j k_j + \cdots + (-1)^n k_n$$

is the *Euler* number or the *Euler–Poincaré characteristic* of the simplicial complex $\mathcal{S}$, and denote it by χ or $\chi(\mathcal{S})$. The surface of a regular tetrahedron is a two-dimensional simplicial complex, and from $4 - 6 + 4 = 2$ it follows that $\chi(\mathcal{S}) = 2$. The surface of a cube is also a two-dimensional simplicial complex, and since $8 - 12 + 6 = 2$ we again get $\chi(\mathcal{S}) = 2$. In both cases the topological space $|\mathcal{S}|$ realized by the simplicial complex $\mathcal{S}$ is homeomorphic to the two-sphere S^2. Now do we conclude that Euler numbers are topological invariants (invariant under homeomorphisms)? The answer is YES, and we can show the even stronger result that they are homotopy invariants (invariant under homotopy equivalences), using a simple homology argument as follows.

Recall that for a simplicial complex $\mathcal{S}$ there is naturally defined a certain chain complex $\{C_q = C_q(\mathcal{S}; \mathbb{Z}), \partial\}$ over $\mathbb{Z}$, and that we have exact sequences

$$0 \to Z_q \to C_q \xrightarrow{\partial} B_{q-1} \to 0,$$
$$0 \to B_q \to Z_q \to H_q \to 0,$$

linking the groups of cycles $Z_q = Z_q(\mathcal{S}; \mathbb{Z})$, the groups of boundaries $B_q = B_q(\mathcal{S}; \mathbb{Z})$, and the q-th homology groups $H_q = H_q(\mathcal{S}; \mathbb{Z})$.

Recall also that a finitely generated abelian group G is isomorphic to a direct sum of a certain number of copies of the cyclic group $\mathbb{Z}$ and a certain number of finite cyclic groups $\mathbb{Z}_p$. We say that the number of copies of the infinite cyclic group $\mathbb{Z}$ in the direct sum for G is the *rank* of G, which we denote by rank G. From $\mathbb{Z} \otimes \mathbb{R} \cong \mathbb{R}$ and $\mathbb{Z}_p \otimes \mathbb{R} = 0$, we see that the rank of G is equal to the number of $\mathbb{R}$'s appearing in the direct sum decomposition $G \otimes \mathbb{R} \cong \mathbb{R} \oplus .. \oplus \mathbb{R}$. Hence for any exact sequence

$$0 \to A \to B \to C \to 0$$

of abelian groups we have

$$\text{rank } B = \text{rank } A + \text{rank } C.$$

For a chain complex C_q, this reads

$$\text{rank } C_q = \text{rank } \mathbb{Z}_q + \text{rank } B_{q-1},$$
$$\text{rank } \mathbb{Z}_q = \text{rank } Bq + \text{rank } Hq.$$

We multiply both sides of these two equations by $(-1)^q$ and sum from $q = 1$ to $q = n$ (= dim S) on each side to obtain the following

THEOREM 1.

$$\chi(\mathcal{S}) = \sum_{q=0}^{n} \text{rank } C_q(\mathcal{S};\mathbb{Z}) = \sum_{q=0}^{n} \text{rank } H_q(\mathcal{S};\mathbb{Z}).$$

Going back to the proof of the theorem, we see that if G is a field (for example G is $\mathbb{Z}_p$, with a prime number p), we get

$$\chi(\mathcal{S}) = \sum_{q=0}^{n} \dim\ H_q(\mathcal{S};\ G).$$

Since each $H_q\ (S; \mathbb{Z})$ is homotopyyinvariant, it follows that the Euler number is also a homotopy invariant (a topological invariant as well).

Using alternating sums in place of usual sums of simplexes, we established one form of invariance as above. In fact, contemporary mathematics owes much to this simple trick. The *Atiyah–Singer index* theorem, one of the most important theorems in modern mathematics, which states that the analytic index and the topological index are the same, is a variation of the above theorem concerning Euler numbers.

EULER SPACES. Let S be an n-dimensional simplicial complex and let $\sigma^q \in$ S be a q-simplex. Define the *link complex* of σ in S by

$$\text{Lk}(\sigma^q, S) = \{\ \tau \in \mathcal{S} |\ \tau * \sigma^q \in \mathcal{S},\ \ \tau \cap \sigma^q = \emptyset \},$$

where $\tau * \sigma^q$ is a simplex spanned by the vertices of τ and the vertices of σ^q (Fig. 1). If S is a standard triangulation of Euclidean space $\mathbb{R}^n$,

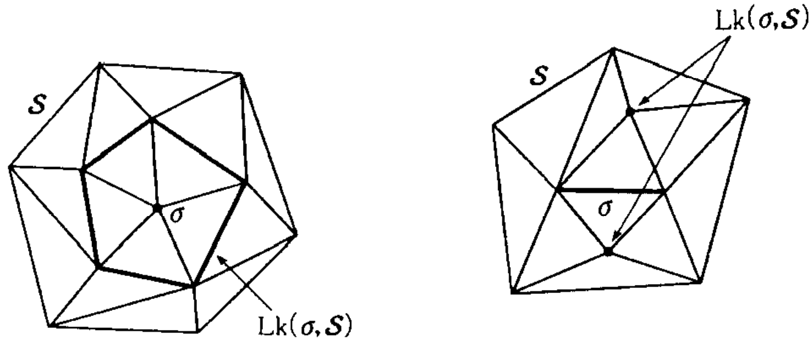


FIGURE 1. Link complex

then $\mathrm{Lk}(\sigma^q, \mathcal{S})$ is homeomorphic to S^{n-p-1}, and so $\chi(\mathrm{Lk}(\sigma^q, \mathcal{S})) = \chi(S^{n-q-1}) \equiv 0 \pmod 2$.

In general, we say that a simplicial complex S is an *Euler space* if every simplex $\sigma \in$ S satisfies

$$\chi(\mathrm{Lk}(\sigma, \mathrm{S})) \equiv 0 \pmod 2.$$

Evidently, any triangulation of a smooth manifold is an Euler space.

Homology Stiefel–Whitney Classes. Denote by $\hat{\sigma}$ the barycenter of a simplicial complex S. We define a new simplicial complex S' which is more finely divided than S but still satisfies $|\mathcal{S}'| = |\mathcal{S}|$. This is done as follows. The zero simplexes consist of the barycenters $\hat{\sigma}$ of the simplexes σ. We insist that the psimplex spanned by $\hat{\sigma}_0$, $\hat{\sigma}_1, \ldots, \hat{\sigma}_p$ is in S' only if $p+1$ simplexes $\sigma_0, \sigma_1, \ldots, \sigma_p$ satisfy $\sigma_0 \prec \sigma_1 \prec \cdots \prec \sigma_p$. We say that $\mathcal{S}'$ is the barycentric *subdivision* of S. The barycentric subdivision of a triangle divides the triangle into six subtriangles.

We define a q-chain $s_q(\mathcal{S}) \in C_q(\mathcal{S}'; \mathbb{Z})$ of the barycentric subdivision $\mathcal{S}'$ over $\mathbb{Z}_2$ by

$$s_q(\mathcal{S}) = \sum_{\sigma^q \in \mathcal{S}'} 1\langle \sigma^q \rangle, \quad 1 \in \mathbb{Z}_2.$$

THEOREM 2. *If $\mathcal{S}$ is an Euler space, then $s_q(\mathcal{S}) \in C_q(\mathcal{S}'; \mathbb{Z}_2)$ is a cycle.*

PROOF. For the q-chain $s_q(\mathcal{S}) \in C_q(\mathcal{S}'; \mathbb{Z}_2)$, the coefficient of the term $\langle \hat{\sigma}_0, \hat{\sigma}_1, \ldots, \hat{\sigma}_{q-1} \rangle$ in $\partial(s_q(\mathcal{S})) \in C_{q-1}(\mathcal{S}'; \mathbb{Z}_2)$ is given by

$$\chi(\partial\sigma_0) + \chi\left(\sum_{i=0}^{q-1} \mathrm{Lk}(\sigma_{i-1}, \partial\sigma_i)\right) + \chi(\sigma_{q-1}, \mathcal{S}) \in \mathbb{Z}_2.$$

Each of the first two terms (being the Euler number of a sphere) is zero, and the third term is zero by the assumption that S is an Euler space. □

I hope you will check this theorem by looking at some simple examples.

DEFINITION 3. For the q-cycle $s_q(\mathcal{S}) \in C_q(\mathcal{S}'; \mathbb{Z}_2)$, we also denote by $s_q(\mathcal{S})$ its homology class in $H_q(\mathcal{S}; \mathbb{Z}_2)$. We call this element the q-th *homology Stiefel–Whitney* class of the simplicial complex S.

EXAMPLE 4. The number of zero-simplexes (vertices) of the barycentric subdivision S' of a connected simplicial complex S is equal to the sum of the numbers of simplexes of each dimension. The even number of vertices constitute the boundary of a one-chain mod 2. It follows that

$$(s_0(\mathcal{S}) \bmod 2) = \chi(\mathcal{S}) \in H_0(\mathcal{S}; \mathbb{Z}_2) \cong \mathbb{Z}_2.$$

EXAMPLE 5. Triangulate the real projective plane $P^2(\mathbb{R})$ and take its barycentric subdivision. Label these simplicial complexes by $P^2(\mathbb{R})$ and T^2 respectively. Calculations according to the definition give us the following:

$$s_0\left(P^2(\mathbb{R})\right) = 1 \in H_0\left(P^2(\mathbb{R}); \mathbb{Z}_2\right) \cong \mathbb{Z}_2;$$
$$s_1\left(P^2(\mathbb{R})\right) = 1 \in H_1\left(P^2(\mathbb{R}); \mathbb{Z}_2\right) \cong \mathbb{Z}_2;$$
$$s_2\left(P^2(\mathbb{R})\right) = 1 \in H_2\left(P^2(\mathbb{R}); \mathbb{Z}_2\right) \cong \mathbb{Z}_2,$$
$$s_0(T^2) = 0 \in H_(T^2; \mathbb{Z}_2) \cong \mathbb{Z}_2;$$
$$s_1(T^2) = 0 \in H_1(T^2; \mathbb{Z}_2) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2;$$
$$s_2(T^2) = 1 \in H_2(T^2; \mathbb{Z}_2) \cong \mathbb{Z}_2.$$

We see in Figure 2 the barycentric subdivision of a triangulation of T^2 and the two-chain of T^2 whose boundary is the one-cycle $s_1(T^2) \in Z_1(T^2; \mathbb{Z}_2) \equiv \ker \partial_1$ (Definition 5.21).

Recall that we calculated the cohomology of the classifying space

$$BO(n) = \lim_{N \to \infty} O(n + N)/O(n) \times \mathrm{O(N)}$$

of the real n-vector bundles over the coefficients $\mathbb{Z}_2$ to be

$$H^*(BO(n); \mathbb{Z}_2) \cong \mathbb{Z}_2[w_1, w_2, \ldots, w_n], \quad w_1 \in H^i(BO(n); \mathbb{Z}_2)).$$

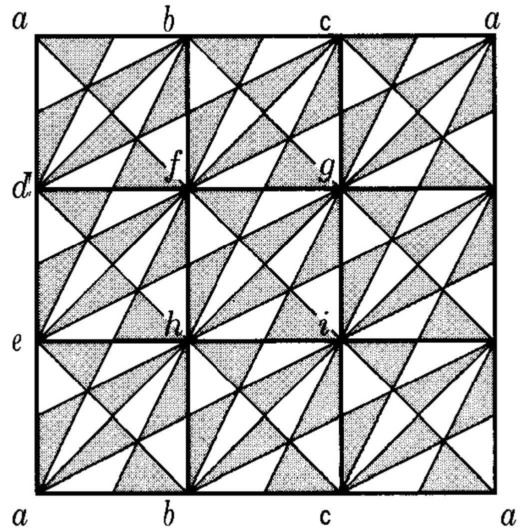


FIGURE 2. Barycentric subdivision and two-chains of the torus

Suppose that $\mathcal{S}$ is a triangulation of an n-dimensional smooth manifold (that is, $|\mathcal{S}| = M$). The pullback of w_1 by the classifying map $f : M \to BO(n)$ of the tangent bundle of M,

$$w_i(M) \equiv f^*(w_1) \in H^i(M : \mathbb{Z}_2),$$

defines a characteristic class of the manifold M called the i-th Stiefel-Whitney class, which depends only on the homotopy type of M. We mention that $w_1(M) = 0$ if and only if M is orientable.

For each manifold M there exists an isomorphism called the Poincaré duality,

$$\mu : H^i(M; \mathbb{Z}_2) \to H_{n-i}(M; \mathbb{Z}_2), \quad \mathrm{n} = \dim M.$$

THEOREM 6. *Let $\mathcal{S}$ be a triangulation of a smooth manifold M of* dimension n. *Then the q-th homology Stiefel-Whitney class $s_q(\mathcal{S}') \in H_q(\mathcal{S}'; \mathbb{Z}_2)$ is Poincard-dual to the (n q)-th Stiefel- Whitney class $w_{n-q}(M) \in H^{n-q}(M; \mathbb{Z}_2)$; that is, we have the equality*

$$\mu(w_{n-q}(M)) = s_q(\mathcal{S}').$$

Thus homology Stiefel-Whitney classes are homotopy-invariant in the category of manifolds; however, we can construct any number of Euler spaces which have the same homotopy type but have different homology Stiefel-Whitney classes. Complex analytic spaces with singularities are not manifolds, but they turn out to be Euler spaces, whose triangulations are familiar.

From the above discussion, we might conclude that the i-th homology Stiefel-Whitney class is a geometric representation of the characteristic class $w_i(M)$. In a similar manner we can also define Euler classes as the pullbacks of cohomology elements of the classifying space $BSO(n)$ by the classifying map of the oriented tangent bundles. We can regard the alternating sum of the number of simplexes as a geometrical representation of Euler classes. So far nobody has come up with any satisfactory geometric realization either for Chern classes or for Pontrjagin classes. We expect a bright future in developing these, as well as in a deeper study of Grassmannian manifolds. We have already seen some progress in this direction, and in fact we already have a rich collection of its byproducts, such as the discovery of the Chern-Simons invariant.

Appendix

This chapter offers a short list of basic definitions and results at an introductory level, which will help the reader start our book.

Sets

We will *not* give *a* rigorous definition of *sets*. A *set* is a collection of items. The items in a set are its elements or members. We often denote a set by a capital letter and list its elements by lower case letters within braces. For instance, $A = \{a, b, c\}$ represents a set consisting of the elements a, b and c. Suppose $P(x)$ is some statement about x. Then we write $\{x|\ P(x)\}$ to represent the set of all x for which $P(x)$ is valid. For instance $\mathbb{Z}_e = \{x|\ x \text{ is an even integer }\}$ says that $\mathbb{Z}_e$ is the set of the even integers; we can also write $\mathbb{Z}_e = \{\ldots. -4, -2, 0, 2, 4, \ldots\}$. If x belongs to a set A, we write $x \in A$. If x is not in A, we write $x \notin A$.

DEFINITION 1. Let X be a set. A set Y is a subset of X (we write $Y \subset X$ or $X \supset Y$) if every $x \in Y$ satisfies $x \in X$. If $X \subset Y$ and $Y \subset X$ we write $X = Y$ and say that they are the same.

If Q(x) is some statement about x, then $S = \{x \in X|\ Q(x)\}$ reads "S is the subset of X for which Q(x) is true". If a set is empty we call it the empty set (all empty sets are equal) and denote it by $\emptyset$. We say that X is finite if it contains only finitely many elements.

When A is an infinite set we often use an index *set* A to label its elements. For instance we can write $\{a_i\}_{i\in\mathbb{N}}$ instead of $\{a_1, a_2, a_3, \ldots,\}$. Here $\mathbb{N}$ is the set of the natural numbers. An indexing set is most likely to appear when one defines a set whose elements are sets (in this case we use the expression a family of *sets).*

DEFINITION 2. Let A and B be subsets of a set X (We often ignore the set X and pretend A and B are just sets). We define the union $A \cup B$, the intersection $A \cap B$ and the Cartesian product $A \times B$ of A and B by

$$A \cup B = \{ x \in X | x \in A \text{ or } x \in B \},$$
$$A \cap B = \{ x \in X | x \in A \text{ and } x \in B \},$$
$$A \times B = \{ (x, y) \,|\, x \in X, y \in B \}.$$

Two sets A and B are *disjoint* when $A \cap B = \emptyset$.

Evidently the operations $\cup$ and $\cap$ satisfy the following properties for any sets A, B and C.

(1) $A \cup A = A \cap A = A$.
(2) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$; $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.
(3) $(A \cup B) \cup C = A \cup (B \cup C)$; $(A \cap B) \cap B = A \cap (B \cap C)$ (associativity).

Because of the associativity property we can denote by $A_1 \cup A_2 \cup \cdots \cup A_n$ the union of n sets A_i. We write this more briefly as $\bigcup_i A_i$, $i = 1, 2, \ldots, n$. Similarly, $\bigcap_i A_i$ denotes the intersection of n sets A_i.

The Cartesian product $\times$ satisfies the associativity property: $A \times (B \times C) = (A \times B) \times C$. It is also distributive over the union and the intersection.

We can generalize these operations over a family of sets indexed by a set Λ.

If $B \subset A$ then the complement $A - B$ of B in A is defined by $A - B = \{ x \in A \mid x \notin B \}$. Sometimes we just write $-B$.

Definition 3. A map f from A to B, written as $f : A \to B$, is a subset f of $A \times B$ with the properties: (1) for each $x \in X$ there is $y \in Y$ such that $(x, y) \in f$; (2) if (x, y) and (x, y') are both in f, then $y = y'$.

A map $f : A \to B$ is an *injection* (we also say that f is one-to-one) if $f(x) = f(y)$ implies $x = y$. A map $f : A \to B$ is a surjection (f is onto) if for every $y \in B$ there is $x \in A$ with $f(x) = y$. If $f : A \to B$ is both injective and surjective we say that it is *a bijection*, or a bijective map.

We have the identity map $I_A : A \to A$ of A defined by $I_A(x) = x$, $x \in A$. In particular, if A is a subset of B then we have the inclusion (map) $i : A \to B$ defined by $i(x) = x$, $x \in A$. The inclusion (map) i is the identity map I_A of A if we choose to ignore the set B.

By the inverse image of a subset $B' \subset A$ by f we mean the subset $\{x \in A \mid f(x) \in B' \}$. With an abuse of notation we indicate this set by $f^{-1}(B)$.

The *composite* of maps $f : A \to B$ and $g : B \to C$ is a map $g \circ f : A \to C$ defined by $g \circ f(x) = g(f(x))$ (check that this makes sense).

If A' is a subset of A, then the composite $f \circ i : A' \to B$ of $i : A' \to A$ and $f : A \to B$ is the restriction of f to A'. We denote this restriction by $f|_{A'}$.

DEFINITION 4. We say $\sim$ is a binary relation on a set A if for any two elements a and b (in this order) of A, a is related to b by $\sim$ (we write this as $a \sim b$) or a is not related to b by $\sim$ ($a \nsim b$). A binary relation $\sim$ in A is an equivalence relation if it satisfies the following properties:

(1) $a \sim a$ for every $a \in A$ (reflexivity).
(2) $a \sim b$ implies $b \sim a$ for $a, b \in A$ (symmetry).
(3) $a \sim b$ and $b \sim c$ implies $a \sim c$ for $a, b, c \in A$ (transitivity).

Let $\sim$ be an equivalence relation on A. For an element $a \in A$ we denote by [u] the subset $\{ x \in A \quad x \sim a \}$. Then we say that [u] is the equivalence class of a with respect to $\sim$ and that a (or any element in *[u])* is a *representative* of [u]. We denote by $A/\sim$ (read "A modulo tilde") the set of all equivalence classes of A with respect to A and say that it is the *quotient set* of A (with respect to $\sim$). Each element of $A/\sim$ is some subset of A, and if $[u] \neq [b]$ then $[a] \cap [b] = \emptyset$. Moreover the union of the equivalence classes of A is equal to A. Note that the number of equivalence classes may not be finite and in that case we must consider an infinite union. But we do not go into a theoretical discussion of this nature. Let's say that we only consider the circumstance where this type of union exists. We define the projection $\pi : A \to A/\sim$ by $\pi(x) = [x]$ (verify that this is well-defined).

The set of integers modp, denoted by $\mathbb{Z}_p$, p a prime number, is the quotient set of $\mathbb{Z}$ by the equivalence relation $\sim$: $a \sim b \Leftrightarrow a - b$ is divisible by p.

Topological spaces and continuous maps

DEFINITION 5. Let X be a set. A topology in X is a family $\mathcal{U}$ of subsets of X, which we call open *sets,* satisfying the following properties:

(1) A union of elements of $\mathcal{U}$ is again an element of $\mathcal{U}$.
(2) A finite intersection of elements of $\mathcal{U}$ is again an element of $\mathcal{U}$.
(3) The empty set $\emptyset$ and X both belong to $\mathcal{U}$.

In terms of open sets we can rephrase these properties:
(1) A union of open sets is an open set.
(2) A finite intersection of open sets is an open set.
(3) $\emptyset$ and X are both open sets.

We say that X is a *topological space* whenever X has some topology $\mathcal{U}$ defined in it. We often write $(X,\mathcal{U})$ to indicate that X is a topological space with topology $\mathcal{U}$.

DEFINITION 6. In a topological space $(X,\mathcal{U})$ a subset Y c X is *closed* in X if -Y is open; that is, if -Y $\in \mathcal{U}$.

The closed sets of a topological space (X, $\mathcal{U}$) satisfy the following properties:
(1') The intersection of closed sets is a closed set.
(2') The union of finitely many closed set is a closed set.
(3') X and $\emptyset$ are both closed sets.

DEFINITION 7. Let $(X,\mathcal{U})$ be a topological space and let $x \in$ X. We say that $U \in \mathcal{U}$ is a *neighborhood* of x if x $\in U$.

Let $(X,\mathcal{U})$ be a topological space. Let Y be a subset of X. Set

$$\mathcal{U}_Y = \{ U \cap Y \mid U \in \mathcal{U} \}.$$

It is easy to check that $\mathcal{U}_Y$ is a topology in Y, making Y a topological space. We say that $\mathcal{U}_Y$ is the *relative topology* on Y with respect to $\mathcal{U}$ and that $(Y,\mathcal{U}_Y)$ is a *topological subspace* (subspace, for short) of X.

DEFINITION 8. Let X and Y be topological spaces. We say that a map $f : X \to Y$ is *continuous* if the inverse image of an arbitrary open set in Y is open in X.

Suppose X is a topological space and f : X $\to$ Y is a surjection (here Y is a set). Then

$$\text{v} = \{ \text{v c Y} \quad f^{-1}(V) \text{ is an open subset of X} \}$$

defines the identification *topology* or *quotient topology* on Y.

In particular, if $\sim$ is an equivalence relation on a topological space X, then the projection of X to $X/\sim$ is an onto map. So $X/\sim$ becomes a topological space with the quotient topology with respect to $\sim$. We then say that $X/\sim$ is the *quotient space*

THEOREM 9. *In the above setting $f : X \to Y$ is continuous. Moreover, if we give Y another topology V' for which f is continuous, then V' c V.*

In other words, the quotient topology is the strongest (largest) topology of Y such that $f : X \to Y$ is continuous.

Let (Y, $\mathcal{V}$) be a topological space. Let $f : X \to Y$ be a map from a set X to Y. Then the family $\{ f^{-1}(V) \ \ V \in \mathcal{V} \}$ of subsets of X defines a topology on X, called the *topology induced* by $f : X \to Y$. Evidently $f : (X, \mathcal{U}) \to (Y, \mathcal{V})$ is continuous. Moreover, if f : (X, U') $\to$ (Y, $\mathcal{V}$) is continuous with respect to any other topology $\mathcal{U}'$ on X then $\mathcal{U} \subset \mathcal{U}'$. In other words, the topology on X induced by $f : X \to Y$ is the *weakest* (smallest) topology on X making f continuous.

Let X_1 and X_2 be topological spaces and consider their set product $X_1 \times X_2$. Let $\pi_i : X_1 \times X_2 \to X_i$, $i = 1, 2$, be the projections. Then there is a smallest topology on X_1 x X_2 making both π_1 and π_2 continuous. This topology is called the product topology (also called the weak topology) on the *product space* X_1 x X_2 (the same name as before, but notice that this time it has become a topological space). This generalizes to the product of spaces indexed by an index set A.

DEFINITION 10. A topological space X is connected if it is not a union of two open sets.

A subset of a topological space is connected if it is connected as a subspace (with respect to the relative topology).

THEOREM 11. *A space X is connected if and* only *if the only subspaces of X that are both* open *and closed are the empty set* $\emptyset$ and X.

Groups

Suppose that a set G satisfies the following properties: To every pair a, b of elements of G there corresponds a third element $a \cdot b$, in such a way that

(1) a . (b . c) = $(a \cdot b)$ c (associativity),

(2) there exists an element ι in G such that a . $\iota = \iota a$ for every a $\in$ G (ι is the identity element of G),

(3) to every element a $\in$ G, there corresponds a unique element $\tilde{a}$ such that a . $\tilde{a} = \tilde{a}$. a = ι (every a in G has its inverse element $\tilde{a}$).

Then we say that G is a *group*.

If G satisfies $a \,.\, b = b \,.\, a$ as well, we say that the G is *abelian* or *commutative*.

Question: what is purple and commutes? Answer: an abelian grape.

EXAMPLE 12. Let $\mathbb{Z}$ be the set of the integers with the usual addition, +. Then $(\mathbb{Z}, +)$ is an abelian group (we usually say an additive group). The identity element is 0 and the inverse of a is -a, of course. From now on $\mathbb{Z}$ implies this group structure.

On the other hand, $\mathbb{Z}$ with the usual multiplication fails to satisfy property (3), and so it is not a group.

The quotient set $G/\sim$ of a group G with respect to an equivalence relation $\sim$ inherits the group structure of G: *[a] [b] = [a . b].* One must show that this operation does not depend on the choice of representatives (easy). We say that $G/\sim$ is the quotient group of G with respect to $\sim$.

A subset H of a group G is a *subgroup* of G if H is *closed* under the group operation of G ($ab \in H$ for any a, $b \in H$).

Let $(G_1, +)$ and $(G_2, +)$ be abelian groups. A map $\phi : G_1 \to G_2$ is a (group) *homomorphism* if $\phi(a+b) = \phi(a) + \phi(b)$ for all a, b in G_1 (the operation + on the left-hand side is for G_1, and that on the right-hand side is for G_2).

For a homomorphism $\phi : G_1 \to G_2$ we use the following notation:

$$\ker \phi = \{ x \in G_1 \quad \phi(x) = 0\},$$

$$\operatorname{im} \phi = \{ y \in G_2 \mid \phi(x) = y \text{ for some } x \in G_1 \}.$$

Then ker ϕ (read the *kernel* of ϕ) is a subgroup of G_1, and im ϕ (read the image of G_2 under ϕ, or simply the image ϕ) is a subgroup of G_2.

The homomorphism $\phi : G_1 \to G_2$ is a monomorphism if ker $\phi = \emptyset$ or equivalently if ϕ is one–to–one, and ϕ is an epimorphism if im $\phi = G_2$ or equivalently if ϕ is onto. If ϕ is a monomorphism and an epimorphism then it is an *isomorphism.*

Answers to Exercises

CHAPTER ONE

1.1. Show that both are homeomorphic to the letter I.
1.2. Let $f : P \to R$ be the identity map of P as the subspace of R. Let g : R $\to$ P be the map which send the subspace P of R onto P (as the identity map of P) and the leg of R to the joint of P. Then $g \circ f = id : \mathsf{P} \to \mathsf{P}$ and so it is enough to show that $f \circ g : \mathsf{R} \to \mathsf{P} \subset \mathsf{P}$ is homotopic to the identity map. We can construct a homotopy by pulling a leg continuously out of the joint.
1.3. Extend the homeomorphism between $\triangle$ and D to a homeomorphism from A onto R.

CHAPTER TWO

2.1. This quotient space is homeomorphic to the quotient space of $\triangle = \{ (x,y) \in I^2 | y \leq x \}$ in which the points on each of the three boundary segments are identified with respect to its center. The latter space is homeomorphic to S^2.
2.2. If one opens up the Mobius band along its latitudinal center line one gets a band that is homeomorphic to I x S'. By resewing the cut we get the suggested attaching map.
2.3. We get a space homeomorphic to a square by opening up the double torus along a suitable set of four loops joined at a single point. Therefore, the double torus has a cell division consisting of one O-cell, four l-cells and one 2-cell. More generally, the n-ple torus has a cell division of one O-cell, $2n$ l-cells and one 2-cell.

CHAPTER THREE

3.1. Given a map of (I", ∂I^n) into (S^k, x_0), we can choose a point $x_1 \in S^k$ and change the map by a small homotopy so that its image misses $x_1 (\neq x_0) \in S^k$. Now $S^k - x_1$ is homeomorphic to the interior of D^k. Follow the proof of $\pi_n(D^k) = 0$.
3.2. $\mathbb{Z}_2$.

3.3. The group generated by $\alpha_1, \beta_1, \alpha_2, \beta_2$ with the relation

$$\alpha_1\beta_1\alpha_1^{-1}\beta_1^{-1}\alpha_2\beta_2\alpha_2^{-1}\beta_2^{-1}.$$

CHAPTER FOUR

4.1. For $p \neq q$, $h^0(S^p \vee S^q) \cong G$, $h^p(S^p \vee S^q) \cong G$, $h^q(S^p \vee S^q) \cong G$; all others are 0. For $p = q$, $h^p(S^p \vee S^q) \cong G$, $h^p(S^p \vee S^q) \cong G \oplus G$; all others are 0.

CHAPTER FIVE

5.1.

$$H_j(S^n;\mathbb{Z}) \cong \begin{cases} \mathbb{Z}, & j = 0, n, \\ 0, & \text{otherwise}; \end{cases} \qquad H_j(D^n;\mathbb{Z}) \cong \begin{cases} \mathbb{Z}, & j = 0, \\ 0, & \text{otherwise}. \end{cases}$$

5.2. From the Mayer-Vietoris sequence for $T^2 = T_0^2 \cup_{S^1} D^2$ *we get* the homology groups of T_0^2 (the robot's glove):

$$H_0(T_0^2;\mathbb{Z}) \cong \mathbb{Z}, \quad H_1(T_0^2;\mathbb{Z}) \cong \mathbb{Z}\oplus\mathbb{Z}, \quad H_i(T_0^2;\mathbb{Z}) = 0, \quad i \geq 2.$$

Further, we use the Mayer-Vietoris sequence for

$$M_2 = T_0^2 \cup_{S^1} T_0^2$$

to obtain

$$H_0(M_2;\mathbb{Z}) \cong \mathbb{Z},\ H_1(M_2;\mathbb{Z}) \cong \mathbb{Z}^4 = \mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z},\ H_2(M_2;\mathbb{Z}) \cong \mathbb{Z}.$$

5.3. Divide M_n into one O-cell, 2n l-cells and one 2-cell. Then the boundary operators are all zero maps. Therefore, we get

$$H_0(M_n;\mathbb{Z}) \cong \mathbb{Z},\ H_1(M_n;\mathbb{Z}) \cong \mathbb{Z}^{2n} = \overbrace{\mathbb{Z}\oplus\cdots\oplus\mathbb{Z}}^{2n\ \textbf{copies}},\ H_2(M_n;\mathbb{Z}) \cong \mathbb{Z}.$$

CHAPTER SIX

6.1. $H_0(T^2;\mathbb{Z}) \cong \mathbb{Z}$, $H^1(T^2;\mathbb{Z}) \cong \mathbb{Z}^2 = \mathbb{Z}\oplus\mathbb{Z}$, $H^2(T^2;\mathbb{Z}) \cong \mathbb{Z}$.

6.2. $H^0(P^2(\mathbb{R});\mathbb{Z}) \cong \mathbb{Z}$, $H^1(P^2(\mathbb{R});\mathbb{Z}) = 0$, $H^2(P^2(\mathbb{R});\mathbb{Z}) \cong \mathbb{Z}_2$.

CHAPTER SEVEN

7.1. $H_0(P^2(\mathbb{R}) \times P^2(\mathbb{R});\mathbb{Z}) \cong \mathbb{Z}$, $H_1(P^2(\mathbb{R}) \times P^2(\mathbb{R});\mathbb{Z}) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$, $H_2(P^2(\mathbb{R}) \times P^2(\mathbb{R});\mathbb{Z}) \cong \mathbb{Z}_2$, $H_3(P^2(\mathbb{R}) \times P^2(\mathbb{R});\mathbb{Z}) \cong \mathbb{Z}_2$, $H_4(P^2(\mathbb{R}) \times P^2(\mathbb{R});\mathbb{Z}) = 0$.

7.2. $H^0(P^2(\mathbb{R});\mathbb{Z}) \cong \mathbb{Z}$, $H^1(P^2(\mathbb{R});\mathbb{Z}) = 0$, $H_2(P^2(\mathbb{R});\mathbb{Z}) \cong \mathbb{Z}_2$.

7.3. $H^0(P^2(\mathbb{R}) \times P^2(\mathbb{R});\mathbb{Z}) \cong \mathbb{Z}$, $H^1(P^2(\mathbb{R}) \times P^2(\mathbb{R});\mathbb{Z}) = 0$, $H^2(P^2(\mathbb{R}) \times P^2(\mathbb{R});\mathbb{Z}) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$, $H^3(P^2(\mathbb{R}) \times P^2(\mathbb{R});\mathbb{Z}) \cong \mathbb{Z}_2$, $H^4(P^2(\mathbb{R}) \times P^2(\mathbb{R});\mathbb{Z}) \cong \mathbb{Z}_2$.

7.4. $f^* : H^2(S^2; \mathbb{Z}) \cong \mathbb{Z} \to H^2(S^2 \vee S^4; \mathbb{Z}) \cong \mathbb{Z}$ is an isomorphism. Hence, an arbitrary element a_i of $H^2(S^2 \vee S^4; \mathbb{Z})$ is of the form $a_i = f^*(\hat{a}_i)$, $\hat{a}_i \in H^2(S^2; \mathbb{Z})$. Since $\hat{a}_1 \cup \hat{a}_2 \in H^4(S^2; \mathbb{Z}) = 0$, it follows that $a_1 \cup a_2 = f^*(\hat{a}_1) \cup f^*(\hat{a}_2) = f^*(\hat{a}_1 \cup \hat{a}_2) = 0$.

7.5. $H_0(P^2(\mathbb{R}); \mathbb{Z}_2) \cong \mathbb{Z}_2$, $H_1(P^2(\mathbb{R}); \mathbb{Z}_2) \cong \mathbb{Z}_2$, $H_2(P^2(\mathbb{R}); \mathbb{Z}_2) \cong \mathbb{Z}_2$.

7.6. $H^0(P^2(\mathbb{R}); \mathbb{Z}_2) \cong \mathbb{Z}_2$, $H^1(P^2(\mathbb{R}); \mathbb{Z}_2) \cong \mathbb{Z}_2$, $H^2(P^2(\mathbb{R}); \mathbb{Z}_2) \cong \mathbb{Z}_2$, $H^2(P^2; \mathbb{Z}_2) \cong \mathbb{Z}_2$.

CHAPTER EIGHT

8.1. The natural projection $\pi : I \times S^1 \to I$ becomes the projection of the Klein bottle onto S^1. The local triviality is obvious, and we have the fiber bundle over S^1 with the fiber S^1 whose total space is the Klein bottle.

8.2. There is a one-to-one correspondence between the real Grassmannian manifold $G^{\mathbb{R}}$ and S^1 (the real numbers $\mathbb{R}$ plus the point at infinity). Similarly, each complex line through the origin in the complex plane has the slope that corresponds to a complex number (including the point infinity). The space $\mathbb{C}$ of complex numbers plus the point ∞ is S^2.

CHAPTER NINE

9.1. The spectral sequence collapses and we have

$$H_0(E; \mathbb{Z}) \cong \mathbb{Z},\ H_2(E; \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z},\ H_4(E; \mathbb{Z}) \cong \mathbb{Z}.$$

All others are zero.

9.2. See the discussion in §9.3.

9.3. As in §9.3, we show that the sequence

$$\cdots \to H^{p+j}(E; \mathbb{R}) \to E_2^{p,j} \xrightarrow{d_{j+1}} E_2^{p+j+1,0} \xrightarrow{\pi^*} H^{p+j+1}(E; \mathbb{R}) \to \cdots$$

is exact. We can do this by setting $\Omega \equiv d_{n+1}(1)$, $1 \in H^0(B; \mathbb{R}) \cong \mathbb{R}$, to show that $\Psi(u) = (-1)^p d_{j+1}(u)$ just as in §9.5, and then by sliding the sign factor one term over.

Recommended Reading

SET THEORY AND ALGEBRA

Essentially any undergraduate text will do. Some of the classics are the following:

[1] Birkhoff, G. and Mac Lane, S., A Survey of *Modern Algebra*, A. K. Peters, 1997.

[2] Halmos, P., *Naive Set Theory,* Undergraduate texts in mathematics, Springer-Verlag New York, 1987.

[3] Lang, *S., Algebra,* Advanced Book Program, Addison-Wesley Publishing Company, 1984.

POINT SET TOPOLOGY

[4] Dugundji, J, *Topology,* William C. Brown, 1989.

[5] Kelley, J., *General Topology,* Graduate texts in mathematics; 27, Springer-Verlag, New York and Tokyo, 1975.

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[6] Fulton, W., *Algebraic Topology, A First Course,* Graduate texts in mathematics, Springer-Verlag New York, 1995.

[7] Greenberg, F. and Harper, R., *Algebraic* Topology, A *First Course*, Mathematics lecture notes series, vol. 58, Benjamin/Cummings Pub. Co., 1981.

[8] Massey, W., *A* Basic *Course in Algebraic Topology,* Graduate texts in mathematics; 127, Springer-Verlag New York, 1991.

[9] Spanier, E., *Algebraic Topology,* McGraw-Hill, New York, 1966.

[10] Whitehead, G., Elements *of Homotopy Theory,* Graduate texts in mathematics; 61, Springer-Verlag, 1978.

MANIFOLDS, LIE GROUPS, CHARACTERISTIC CLASSES

[11] Guillemin, V. and Pollack A., *Differential Topology*, Prentice Hall, 1974.

[12] Milnor, J. and Stasheff, J., *Characteristic Classes,* Annals of mathematics studies; 76, Princeton University Press, 1974.

[13] Miura, M. and Toda, H., *Topology of Lie Groups,* Translation of mathematical monographs; 91, American Mathematical Society, 1991.

[14] Steenrod, N., *The Topology of Fibre Bundles,* Princeton mathematical series; 14, 1951.

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