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Constructing manifolds by homotopy equivalences
I. An obstruction to constructing PL-manifolds from homology manifolds


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CONSTRUCTING MANIFOLDS
BY HOMOTOPY EQUIVALENCES I.
AN OBSTRUCTION
TO CONSTRUCTING PL-MANIFOLDS
FROM HOMOLOGY MANIFOLDS
by Hajime SATO

0. Introduction.

A homology manifold can be given a canonical cell complex structure, where each cell is a contractible homology manifold. In this paper, given a homology manifold $M$, we aim at constructing a PL-manifold with a cell complex structure, where each cell is an acyclic PL-manifold, which is cellularly equivalent to the canonical cell complex structure of $M$. We obtain a theorem that, if the dimension $n$ of $M$ is greater than 4 and if the boundary $\partial M$ is a PL-manifold or empty, there is a unique obstruction element in $H_{n-4}(M; \mathcal{H}^3)$, where $\mathcal{H}^3$ is the group of 3-dimensional PL-homology spheres modulo those which are the boundary of an acyclic PL-manifold. If the manifold is compact, the constructed PL-manifold is simple homotopy equivalent to $M$.

I have heard that similar results have been obtained independently and previously by M. Cohen and D. Sullivan, refer [1] and [9].

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1. Definition of homology manifold with boundary.$^{(1)}$

Let $K$ be a locally finite simplicial complex and let $\sigma$ be a simplex of $K$. We define the subcomplexes of $K$ as follows.

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\[ \text{Sr}(\sigma, K) = \text{St}(\sigma) = \{ \tau \in K, \exists \alpha > \tau, \alpha > \sigma \} \]
\[ \partial \text{St}(\sigma, K) = \partial \text{St}(\sigma) = \{ \tau \in \text{St}(\sigma), \tau \uparrow \sigma \} \]
\[ \text{Lk}(\sigma, K) = \text{Lk}(\sigma) = \{ \tau \in \text{St}(\sigma), \tau \cap \sigma = \emptyset \} \]

We write by \( K', K'' \), the first and the second barycentric subdivisions of \( K \).

Let \( M \) be a locally finite full simplicial complex of dimension \( n \). We say that \( M \) is a homology manifold of dimension \( n \) if the following equivalent condition holds:

\text{LEMMA 1. — The followings are equivalent:}

i) for any simplex \( \sigma \) of dimension \( p \),
\[ \widetilde{H}_i(\text{Lk}(\sigma, M)) = \widetilde{H}_i(S^{n-p-1}) \quad \text{or} \quad 0. \]

ii) for any simplex \( \sigma \) of dimension \( p \),
\[ \widetilde{H}_i(\text{St}(\sigma, M)/\partial \text{St}(\sigma, M)) = \widetilde{H}_i(S^n) \quad \text{or} \quad 0. \]

iii) for any point \( x \) of \( |M| \), where \( |M| \) denotes the underlying topological space of \( M \),
\[ \widetilde{H}_i(|M|, |M|-x) = \widetilde{H}_i(S^n) \quad \text{or} \quad 0. \]

The definition is invariant by the PL-homeomorphism in the category of simplicial complexes.

\text{LEMMA 2. — For any p-simplex \( \sigma \) of \( M \), \( \text{Lk}(\sigma, M) \) is a compact \((n - p - 1)\)-dimensional homology manifold.}

\text{Proof. — It is compact because \( M \) is locally finite. Let \( \tau \) be a q-simplex of \( \text{Lk}(\sigma, M) \). We have}
\[ \text{Lk}(\tau, \text{Lk}(\sigma, M)) = \text{Lk}(\tau \sigma, M). \]
Hence \( \widetilde{H}_i(\text{Lk}(\tau, \text{Lk}(\sigma, M))) = \widetilde{H}_i(S^{n-p-q-1}) \) or 0, which completes the proof.

Let us define the subset \( \partial M \) of \( M \) by
\[ \partial M = \{ \sigma \in M | \widetilde{H}_i(\sigma, M) = 0 \} \]
We call it as the boundary of \( M \). If \( \partial M = \emptyset \), the manifold is classical
and the following Poincaré duality is well known (see for example [7: (7,4)]).

**Lemma 3.** — Let $M$ be an orientable compact $n$-dimensional homology manifold without boundary. Let $A_1 \supset A_2$ be subcomplexes of $M$. Then we have the isomorphism

$$H^i(A_1, A_2) = H_{n-i}(|M| - |A_2|, |M| - |A_1|).$$

Using this we will prove the followings. By lemma 2, for $p$-simplex $\sigma$, $Lk(\sigma, M)$ is a homology manifold and we can define $\partial Lk(\sigma, M)$.

**Lemma 4.** — If $\partial Lk(\sigma, M) \neq \emptyset$, $Lk(\sigma, M)$ is acyclic and $\partial Lk(\sigma, M)$ is an $(n - p - 2)$-dimensional homology manifold such that

$$\tilde{H}_i(\partial Lk(\sigma, M)) = \tilde{H}_i(S^{n-p-2}).$$

**Proposition 5.** — If $\partial M \neq \emptyset$, $\partial M$ is a subcomplex and is an $(n - 1)$-dimensional homology manifold without boundary.

We prove that lemma 4 for $n = k$ implies proposition 5 for $n = k$ and proposition 5 for $n \leq k$ implies lemma 4 for $n = k + 1$. Since lemma 4 holds for $n = 1$, we can continue by induction.

Lemma $4_{n-k} \Rightarrow$ Proposition $5_{n-k}$. Let $\sigma$ be a $p$-simplex of $\partial M$ and let $\sigma_0 < \sigma$. Then we can write $\sigma = \sigma_0 \sigma_1$. We have

$$\tilde{H}_*(Lk(\sigma_1, Lk(\sigma_0, M))) = \tilde{H}_*(Lk(\sigma, M)) = 0,$$

which shows that $\sigma_1 \in \partial Lk(\sigma_0, M)$ and so $\partial Lk(\sigma_0, M) \neq \emptyset$. By the lemma 4, $Lk(\sigma_0, M)$ is acyclic and it follows that $\sigma_0 \in \partial M$. Hence $\partial M$ is a well-defined subcomplex of $M$. A $q$-simplex $\tau$ of $Lk(\sigma, M)$ is in $Lk(\sigma, \partial M)$ if and only if $\tilde{H}_i(Lk(\tau \sigma, M)) = 0$. Since

$$Lk(\tau \sigma, M) = Lk(\tau, Lk(\sigma, M)),$$

it is equivalent to that $\tau$ belongs to $\partial Lk(\sigma, M)$. Hence the complex $Lk(\sigma, \partial M)$ coincides with $\partial Lk(\sigma, M)$. By lemma $4_k$, we have

$$\tilde{H}_i(\partial Lk(\sigma, M)) = \tilde{H}_i(S^{k-p-2}),$$

which shows that $\partial M$ is a $(k - 1)$-dimensional homology manifold without boundary.

Proposition $5_{n \leq k} \Rightarrow$ Lemma $4_{n=k+1}$. Let $M$ be a homology manifold of dimension $k + 1$. Let $\sigma$ be a $p$-simplex of $M$. By lemma 2,
$Lk(a, M)$ is a homology manifold of dimension $k - p$. By proposition 5 for $n = k - p$, $\partial Lk(a, M)$ is a $(k - p - 1)$-dimensional homology manifold without boundary if it is not empty. Let $2Lk(a, M)$ be the double of $Lk(a, M)$, i.e.,

$$2Lk(a, M) = Lk(a, M) \cup _{\partial Lk(a, M)} Lk(a, M).$$

Let $\tau$ be a $q$-simplex of $2Lk(a, M)$. If $\tau$ is not a simplex of $\partial Lk(a, M)$, clearly,

$$\widetilde{H}_i(Lk(\tau, 2Lk(a, M))) = \widetilde{H}_i(Lk(\tau, Lk(a, M))) = \widetilde{H}_i(S^{k-p-q-1}).$$

If $\tau$ is a simplex of $\partial Lk(a, M)$, we have

$$Lk(\tau, 2Lk(a, M)) = Lk(\tau, Lk(a, M)) \cup _{Lk(\tau, \partial Lk(a, M))} Lk(\tau, Lk(a, M)).$$

By definition $\widetilde{H}_i(Lk(\tau, Lk(a, M))) = 0$ and by the proposition 5 for $n = k - p - 1$, we have

$$\widetilde{H}_i(Lk(\tau, \partial Lk(a, M))) = \widetilde{H}_i(S^{k-p-q-2}).$$

Hence in any case $\widetilde{H}_i(Lk(\tau, 2Lk(a, M))) = \widetilde{H}_i(S^{k-p-q-1})$, which shows that $2Lk(a, M)$ is a $(k - p)$-dimensional homology manifold without boundary. Applying lemma 3, we have

$$H^i(Lk(a, M), \partial Lk(a, M)) = H_{k-p-i}(|Lk(a, M)| - |\partial Lk(a, M)|).$$

Notice that for any homology manifold $M$, $H_i(|M| - |\partial M|) = H_i(M)$. Hence $H^i(Lk(a, M), \partial Lk(a, M)) = H_{k-p-i}(S^{k-p})$ or $H_{k-p-i}(pt.)$. But if it is isomorphic to $H_{k-p-i}(S^{k-p})$, we have

$$H^0(Lk(a, M), \partial Lk(a, M)) = \mathbb{Z},$$

which contradicts to the definition that $\widetilde{H}_0(Lk(a, M)) = 0$. Hence $Lk(a, M)$ is acyclic and consequently $\widetilde{H}_i(\partial Lk(a, M)) = \widetilde{H}_i(S^{k-p-1})$, which completes the proof.

2. Cell decomposition of a homology manifold.

We mean by a homology cell (resp. pseudo homology cell) of dimension $n$ or homology $n$-cell (resp. pseudo homology $n$-cell) a
compact contractible (resp. acyclic) homology manifold of dimension \( n \) with a boundary, the boundary being a homology sphere but not necessarily simply connected. A (pseudo) homology cell complex is a complex \( K \) with a locally finite family of (pseudo) homology cells \( C = \{ C_\alpha \} \), such that:

i) \( K = \bigcup C_\alpha \)

ii) \( C_\alpha, C_\beta \in C \) implies \( \partial C_\alpha, C_\alpha \cap C_\beta \) are unions of cells in \( C \)

iii) If \( \alpha \neq \beta \), then \( \text{Int } C_\alpha \cap \text{Int } C_\beta = \emptyset \).

If a homology manifold \( M \) has a (pseudo) homology cell complex structure, we call it a (pseudo) cellular decomposition of \( M \). Two (pseudo) homology cell complexes \( K = \bigcup C_\alpha \), \( K' = \bigcup C'_\alpha \) are isomorphic if there exists a bijection \( k : C \to C' \) such that both \( k \) and \( k^{-1} \) are incidence preserving. In such a case we say that they are cellularly equivalent.

Now we have the following:

**Proposition.** — If two finite homology cell complexes \( K, K' \) are cellularly equivalent, then they are simple homotopy equivalent.

We can define a simplicial map \( f : K \to K' \) inductively by the dimension of the cells. Hence it is sufficient to prove the following lemma.

**Lemma 2.** — Let \( A^i_j (j = 1, 2, \ldots, r) \) be subcomplex of simplicial complexes \( B^i \) for \( i = 1, 2 \) respectively such that \( B^i = \bigcup_j A^i_j \), and let \( f : B^1 \to B^2 \) be a simplicial map. For any subset \( s \) of \( \{ 1, 2, \ldots, r \} \), let \( A^i_s = \bigcap_{j \in s} A^i_j \) and let \( f_s \) be the restriction of \( f \) on \( A_s \). If \( f_s \) is a mapping from \( A^1_s \) to \( A^2_s \) which is a simple homotopy equivalence for any \( s \), then \( f \) itself is a simple homotopy equivalence.

**Proof.** — First suppose that \( r = 2 \). We have the exact sequence

\[
0 \to C_* (A^1_1) \to C_* (B^1) \to C_* (A^1_1 \cap A^2_1) \to 0
\]

of the chain complexes. Let \( g : A^1_2 / (A^1_1 \cap A^1_2) \to A^2_2 / (A^2_1 \cap A^2_2) \) be the map induced by \( f \) and let us denote by \( w( ) \) the Whitehead torsion. Then by theorem 10 of [8], we have
\[ w(f) = w(f_{(1)}) + w(g). \]

Remark here that \( f \) and \( g \) can easily be seen to be homotopy equivalences. Further we have the exact sequence

\[ 0 \to C_\ast(A_1^i \cap A_2^j) \to C_\ast(A_2^j) \to C_\ast(A_2^j/(A_1^i \cap A_2^j)) \to 0 \]

which shows that

\[ w(f_{(2)}) = w(f_{(1,2)}) + w(g). \]

Since \( w(f_{(1)}) = w(f_{(2)}) = w(f_{(1,2)}) = 0 \), we have \( w(f) = 0 \). If \( r \geq 3 \), we can repeat this argument, which shows that \( f \) is a simple homotopy equivalence for any \( r \).

Now let \( \sigma \) be a simplex of a locally finite simplicial complex \( K \). We denote by \( b_\sigma \in K' \) its barycenter. We define dualcomplex \( D(\sigma) \) and its subcomplex \( \delta D(\sigma) \) which are subcomplexes of \( K' \) by

\[ D(\sigma) = D(\sigma, K) = \{ b_{\sigma_0} \ldots b_{\sigma_r} | \sigma < \sigma_0 < \ldots < \sigma_r \in K \} \]

\[ \delta D(\sigma) = \delta D(\sigma, K) = \{ b_{\sigma_0} \ldots b_{\sigma_r} | \sigma \neq \sigma_0 < \ldots < \sigma_r \in K \} \]

The followings are easy to see.

i) if \( \sigma < \sigma' \Rightarrow D(\sigma) \supset D(\sigma') \)

ii) \( D(\sigma) = b_\sigma * \delta D(\sigma) \)

iii) \( \delta D(\sigma) = \bigcup_\tau D(\tau) \) where \( \tau > \sigma \) and \( \tau \neq \sigma \)

iv) \( \delta D(\sigma) \) is isomorphic to \( Lk(\sigma, K)' \).

Let \( M \) be a homology manifold. For each simplex

\[ \sigma = b_{\sigma_0} b_{\sigma_1} \ldots b_{\sigma_r} \]

of \( M' \), where \( \sigma_0^{n_0} < \sigma_1^{n_1} < \ldots < \sigma_r^{n_r} \) are a set of simplexes of \( M \), we have the dual cell \( D(\sigma, M') \). It is a compact homology manifold by lemma 2 of § 1. Further we have

\[ \delta D(\sigma, M') \cong Lk(\sigma, M) \]
\[ \cong Lk(\sigma, \sigma_r) * Lk(\sigma_r, M) \]
\[ \cong S^{n_r-r-1} * Lk(\sigma_r, M) \]
\[ \cong Lk(\sigma_r, M) \times D^{n_r-r} \cup (Lk(\sigma_r, M) \ast (pt.)) \times S^{n_r-r-1}. \]
where $\cong$ denotes that both sides are PL-homeomorphic and let
\[ d_\sigma : \delta D(\sigma, M') \to \text{Lk}(\sigma, M) \times D^{n-r} \cup (\text{Lk}(\sigma, M) \ast (pt.)) \times S^{n-r-1} \]
be the PL-homeomorphism, which we call the trivialization of $\delta D(\sigma, M')$. If $\sigma$ is not in $\partial M$, $\delta D(\sigma, M')$ is a homology manifold whose homology groups are isomorphic to those of $S^{n-1}$, boundary being empty. If $\sigma \in \partial M$, $\delta D(\sigma, M')$ is an acyclic homology manifold with the boundary $\text{Lk}(\sigma, \partial M'')$ which is PL-homeomorphic to $\partial \text{Lk}(\sigma, M) \times D^{n-r} \cup (\partial \text{Lk}(\sigma, M) \ast (pt.)) \times S^{n-r-1}$. The union $\text{St}(\sigma, \partial M'') \cup \delta (\sigma, M') = \delta D(\sigma, M')$ is a homology manifold without boundary whose homology groups are isomorphic to those of $S^{n-1}$. Hence in any case $D(\sigma, M')$ is a homology cell. The union $\cup D(\sigma, M')$, $\sigma$ moving all simplexes of $M'$, gives the cellular decomposition of $M$, which we call the canonical one.

We define the handle $M_i$ of index $i$ by the disjoint union
\[ M_i = \cup D(b_{\sigma n-i}) \]
where $\sigma$ changes all $(n-i)$-simplexes of $M$. We have $\delta D(b_\sigma) = \cup D(\tau)$, where $\sigma < \tau \in M'$, and it gives a cellular decomposition of $M_i$. We can devide the boundary as $\delta D(b_\sigma) = \text{LD}(b_\sigma) \cup \text{HD}(b_\sigma)$, which consists of unions of celles attached to the handles of lower indexes and higher indexes. We define them as
\[
\begin{align*}
\text{LD}(b_\sigma) &= \delta D(b_\sigma) \cap \left( \bigcup_{i < i} M_i \right) \\
\text{HD}(b_\sigma) &= \delta D(b_\sigma) \cap \left( \bigcup_{i > i} M_i \right).
\end{align*}
\]
Let $\tau = b_{\tau_0} b_{\tau_1} \ldots b_{\tau_r} \neq \sigma$ be a simplex of $M'$, where
\[ \tau_0^{m_0} < \tau_1^{m_1} \ldots < \tau_r^{m_r} \in M. \]
Then $D(\tau) \in \text{LD}(b_\sigma)$ if and only if $\tau_r > \sigma$ and $D(\tau) \in \text{HD}(b_\sigma)$ if and only if $\tau_0 < \sigma$. It is easy to see that
\[
\begin{align*}
\text{LD}(b_\sigma) &\cong \text{Lk}(\sigma, M) \times D^{n-i} \\
\text{HD}(b_\sigma) &\cong (\text{Lk}(\sigma, M) \ast (pt.)) \times S^{n-i-1}.
\end{align*}
\]
and these isomorphism together give the trivialization $d_{b_\sigma}$ of $\delta D(b_\sigma)$. 

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where $\cong$ denotes that both sides are PL-homeomorphic and let
\[ d_\sigma : \delta D(\sigma, M') \to \text{Lk}(\sigma, M) \times D^{n-r} \cup (\text{Lk}(\sigma, M) \ast (pt.)) \times S^{n-r-1} \]
be the PL-homeomorphism, which we call the trivialization of $\delta D(\sigma, M')$. If $\sigma$ is not in $\partial M$, $\delta D(\sigma, M')$ is a homology manifold whose homology groups are isomorphic to those of $S^{n-1}$, boundary being empty. If $\sigma \in \partial M$, $\delta D(\sigma, M')$ is an acyclic homology manifold with the boundary $\text{Lk}(\sigma, \partial M'')$ which is PL-homeomorphic to $\partial \text{Lk}(\sigma, M) \times D^{n-r} \cup (\partial \text{Lk}(\sigma, M) \ast (pt.)) \times S^{n-r-1}$. The union $\text{St}(\sigma, \partial M'') \cup \delta (\sigma, M') = \delta D(\sigma, M')$ is a homology manifold without boundary whose homology groups are isomorphic to those of $S^{n-1}$. Hence in any case $D(\sigma, M')$ is a homology cell. The union $\cup D(\sigma, M')$, $\sigma$ moving all simplexes of $M'$, gives the cellular decomposition of $M$, which we call the canonical one.

We define the handle $M_i$ of index $i$ by the disjoint union
\[ M_i = \cup D(b_{\sigma n-i}) \]
where $\sigma$ changes all $(n-i)$-simplexes of $M$. We have $\delta D(b_\sigma) = \cup D(\tau)$, where $\sigma < \tau \in M'$, and it gives a cellular decomposition of $M_i$. We can devide the boundary as $\delta D(b_\sigma) = \text{LD}(b_\sigma) \cup \text{HD}(b_\sigma)$, which consists of unions of celles attached to the handles of lower indexes and higher indexes. We define them as
\[
\begin{align*}
\text{LD}(b_\sigma) &= \delta D(b_\sigma) \cap \left( \bigcup_{i < i} M_i \right) \\
\text{HD}(b_\sigma) &= \delta D(b_\sigma) \cap \left( \bigcup_{i > i} M_i \right).
\end{align*}
\]
Let $\tau = b_{\tau_0} b_{\tau_1} \ldots b_{\tau_r} \neq \sigma$ be a simplex of $M'$, where
\[ \tau_0^{m_0} < \tau_1^{m_1} \ldots < \tau_r^{m_r} \in M. \]
Then $D(\tau) \in \text{LD}(b_\sigma)$ if and only if $\tau_r > \sigma$ and $D(\tau) \in \text{HD}(b_\sigma)$ if and only if $\tau_0 < \sigma$. It is easy to see that
\[
\begin{align*}
\text{LD}(b_\sigma) &\cong \text{Lk}(\sigma, M) \times D^{n-i} \\
\text{HD}(b_\sigma) &\cong (\text{Lk}(\sigma, M) \ast (pt.)) \times S^{n-i-1}.
\end{align*}
\]
and these isomorphism together give the trivialization $d_{b_\sigma}$ of $\delta D(b_\sigma)$. 

Let $\Delta^{n-i}$ be the standard $(n-i)$-simplex and let
\[ \partial \Delta^{n-i} = S^{n-i-1} = \bigcup_\alpha C_\alpha \]
be the cell decomposition defined as above, which we call the standard decomposition of $S^{n-i-1}$. The decomposition
\[ \text{HD}(b_\sigma) = \bigcup D(\tau) \]
is equal to the standard product decomposition
\[ \{ \text{Lk}(\sigma, M) \ast (pt.) \} \times \left( \bigcup_\alpha C_\alpha \right) . \]
All the cells of $\text{HD}(b_\sigma)$ which is not contained in $\text{LD}(b_\sigma) \cap \text{HD}(b_\sigma)$ is written as
\[ (\text{Lk}(\sigma, M) \ast (pt.)) \times C_\alpha . \]
Finally we define $M_{(i)}$ the subcomplex of $M$ composed of handles whose indexes are inferior or equal to $i$, that is,
\[ M_{(i)} = \bigcup_{j \leq i} M_j \subset M . \]
Then we have
\[ M_{(i)} = M_{(i-1)} \cup M_i \]
attached on $\bigcup_\sigma \text{LD}(b_\sigma)$, $\sigma$ being $(n-i)$-simplexes.

3. PL-homology spheres.

We call an $n$-dimensional homology manifold whose homology groups are isomorphic to those of $S^n$ a homology $n$-sphere or homology sphere of dimension $n$. If it is a PL-manifold, it is called a PL-homology $n$-sphere.

If dimension is smaller than 3, a homology sphere is the natural sphere. And so any 3-dimensional homology manifold is a PL-manifold. In order to study higher dimensional cases we define the group $\mathcal{H}^3$.

Let $X^3$ be the set of oriented 3-dimensional PL-homology spheres. Note that any homology sphere is orientable. We say that $H_1^3 \in X^3$ is equivalent to $H_2^3 \in X^3$ if $H_1^3 \# (-H_2^3)$ is the boundary of an acyclic PL-manifold, where $\#$ denotes the connected sum and
- $H_3^2$ is $H_3^2$ with the orientation inversed. Let $\mathcal{H}^3 = X^3/\sim$ be the set of equivalence classes. By the connected sum operation, $\mathcal{H}^3$ is an abelian group. Let $G$ be the binary dodecahedral group. The quotient space $S^3/G$ is a PL-homology sphere whose class in $\mathcal{H}^3$ is non trivial.

On the contrary, for higher dimensions the following is known [2] [6] [4].

**Proposition 1** (Hsiang-Hsiang, Tamura, Kervaire). — Any PL-homology sphere is the boundary of a contractible PL-manifold, if the dimension is greater than 3.

We will prove the followings, where $x$ is a point in $S^1$, $i \geq 1$.

**Proposition 2.** — Let $H^3 \in X^3$, then $H^3 \times S^1$ is the boundary of a PL-manifold $K^3$ such that $H_\ast(K) \cong H_\ast(S^1)$ and the inclusion

$$j : S^1 \hookrightarrow \{x\} \times S^1 \hookrightarrow H^3 \times S^1 \hookrightarrow K$$

induce an isomorphism of the fundamental groups.

**Proposition 3.** — Let $H^3 \in X^3$ and let $i \geq 2$. Then $H^3 \times S^i$ is the boundary of a PL-manifold $K^{4+i}$ such that the inclusion

$$j : S^i \hookrightarrow \{x\} \times S^i \hookrightarrow H^3 \times S^i \hookrightarrow K$$

induces a homotopy equivalence.

*Proof of Proposition 2.* — Since any orientable closed 3-dimensional PL-manifold is a boundary of a 4-dimensional parallelizable PL-manifold (See by example [3]), we have a parallelizable PL-manifold $L^4$ such that $\partial L = H$. By doing surgery we can assume that $\pi_1(L) = 0$. By the Poincaré duality theorem, $H_2(L)$ is free abelian. Let $p : L \times S^1 \rightarrow S^1$ be the projection. Then it induces an isomorphism of the fundamental groups. Remark that if we have a manifold $K$ with boundary $H^3 \times S^1$ such that $H_2(K) \cong 0$ and the inclusion

$$j : S^1 \hookrightarrow K$$

induces the isomorphism of the fundamental groups, then, by the Poincaré duality, we have $H_i(K) = 0$ for $i \geq 2$. Hence it is sufficient to kill $H_2(L \times S^1)$. Since $H_2(L)$ is free, so is $H_2(L \times S^1)$. We can follow the method of lemma 5.7 of Kervaire-Milnor [5]. Since $\pi_1(L) = 0$, the Hurewicz map of $L$, $\pi_2(L) \rightarrow H_2(L)$, is isomorphic,
and so is the Hurewicz map of $L \times S^1$

$$h : \pi_2(L \times S^1) \to H_2(L \times S^1).$$

Hence we can represent any element of $H_2(L \times S^1)$ by an embedded sphere. In our case the boundary $\partial(L \times S^1)$ is $H^3 \times S^1$ and it does not satisfy the hypothesis of that lemma. But since we have

$$H_2(\partial(L \times S^1)) = 0,$$

the result is the same.

\textit{Proof of Proposition 3.} — Let $K^5$ be the 5-dimensional PL-manifold of proposition 2. Attach $K$ with $H^3 \times D^2$ by the identity map on $H^3 \times S^1$. The constructed manifold $W^5$ is a simply connected PL-homology sphere, and by the generalized Poincaré conjecture, it is the natural sphere $S^5$. It shows that we can embed $H^3$ in $S^5$ with a trivial normal bundle. By composing with the natural embedding $S^5 \hookrightarrow S^{4+i}$, we have an embedding of $H^3$ in $S^{4+i}$ with the trivial normal bundle. The manifold $N$ which is the complement of the open regular neighbourhood of $H^3$ in $S^{4+i}$ has $H^3 \times S^i$ as the boundary and the inclusion $j : S^i \hookrightarrow N$ induces an isomorphism of homology groups, hence homotopy equivalence, which completes the proof.

4. An obstruction to constructing PL-manifold.

Let $M$ be a homology manifold of dimension greater than 4. We assume that the boundary $\partial M$ is a PL-manifold if it is not empty. As in § 2, it has the handle decomposition

$$M = M_{(n)} = \bigcup_{0 \leq i \leq n} M_i$$

which has also the canonical homology cell complex structure. We want to construct a PL-manifold with a pseudo homology cell complex structure which is cellularly equivalent to $M$. Since $M_{(3)}$ is a PL-manifold, a problem first arises when we attach handles of index 4.

Let $\sigma$ be an $(n - 4)$-simplex in the interior of $M$. Then $\text{Lk}(\sigma, M)$ is a 3-dimensional PL-homology sphere. Connecting $\sigma$ by a path from
a fixed base point of $M$, we can give the orientation for the neighbourhood of $\sigma$, and hence for $Lk(\sigma, M)$.

Let $Lk(\sigma, M)$ be the class in the group $\mathcal{C}^3$. To each $(n - 4)$-simplex $\sigma$ of $M$, we define a function $\lambda(M) : \{(n - 4)\text{-simplex}\} \rightarrow \mathcal{C}^3$ by

$$\lambda(M)(\sigma) = \begin{cases} \{Lk(\sigma, M)\} & \text{if } \sigma \in \text{Int. } M \\ 0 & \text{otherwise.} \end{cases}$$

Then $\lambda(M)$ is an element of the chain group $C_{n-4}(M, \mathcal{C}^3)$. The coefficient may be twisted if the manifold is not orientable.

**Lemma 1.** $\lambda(M)$ is a cycle.

**Proof.** Let $\mu$ be an $(n - 5)$-simplex. In the homology 4-sphere $Lk(\mu)$, the complex $\bigcup Lk(\sigma_i) \ast (x_i)$, where $x_i$ denotes the barycenter of the 1-simplex $b_\mu b_{\sigma_i}$ and the sum extends to all the $(n - 4)$-simplexes such that $\sigma_i > \mu$, is a subcomplex whose complement in $Lk(\mu)$ is a PL-manifold. So the connected-summed PL-manifold $\sum Lk(\sigma_i)$ bounds an acyclic PL-manifold.

Hence $\lambda(M)$ represents an element $\{\lambda(M)\}$ of $H_{n-4}(M, \mathcal{C}^3)$. Now we have the theorem:

**Theorem.** Let $M^n$ be a homology manifold with the dimension $n > 4$. Assume that $\partial M$ is a PL-manifold if $\partial M \neq \emptyset$. If the obstruction class

$$\{\lambda(M)\} \in H_{n-4}(M, \mathcal{C}^3)$$

is zero, then there exists a PL-manifold $N$ with a pseudo homology cell decomposition which is cellularly equivalent to $M$.

**Proof.** Since $\{\lambda(M)\} = 0$, there exists a correspondance

$$g : \{(n - 3)\text{-simplex}\} \rightarrow \mathcal{C}^3$$

such that

$$\sum_{\tau_i > \sigma} g(\tau_i) = \{Lk(\sigma, M)\} \in \mathcal{C}^3.$$

We will inductively construct PL-manifolds $N_p$ and $N_{(p)} = \bigcup_{q \leq p} N_q$ with a pseudo homology cell decomposition $N_p = \bigcup E_\alpha$ where all
pseudo cells are PL-manifolds such that $N_{(p)}$ is cellularly equivalent to $M_{(p)}$.

(a) $p \leq 2$. In this case, the manifolds $N_p$, $N_{(p)}$ and their cells are just equal to $M_p$, $M_{(p)}$ and their cells. That is, for any $j$-simplex $\sigma$, $j \geq n - 2$, we define the PL-manifolds as

$$E(\sigma) = D(\sigma)$$

$$N_p = \bigcup \{E(\sigma) | \dim \sigma = n - p\} = \bigcup \{D(\sigma) | \dim \sigma = n - p\} = M_p$$

For any simplex $\mu \in M'$ such that $\mu > b_\sigma$, we put

$$E(\mu) = D(\mu).$$

Hence $\partial E(\sigma) = \partial D(\sigma) = \bigcup D(\mu) = \bigcup E(\mu)$, and $N_{(p)} = M_{(p)}$.

(b) $p = 3$. Let $\tau_i$ be an $(n - 3)$-simplex. Let $H_i^3$ be the 3-dimensional PL-homology sphere which represents $g(\tau_i)$ and let $K_i$ be the PL-manifold whose boundary is $H_i^3 \times S^{n-4}$ such that the inclusion $j: S^{n-4} \hookrightarrow K_i$ induces the isomorphisms of the fundamental groups and the homology groups, whose existence is shown by propositions 2 and 3 of § 3. Let $D^3 \subset H_i^3$ be a disc. Then $D^3 \times S^{n-4} \subset \partial K_i$. We have the PL-homeomorphism $\partial D(b_{\tau_i}) = S^2 \times D^{n-3} \cup D^3 \times S^{n-4}$. We define the PL-manifolds $E(b_{\tau_i})$ and $N_3$ by

$$E(b_{\tau_i}) = D(b_{\tau_i}) \cup D^3 \times S^{n-4} K_i$$

$$N_3 = \bigcup_i E(b_{\tau_i})$$

where $D(b_{\tau_i})$ is attached to $K_i$ by the identity map on $D^3 \times S^{n-4}$. It is easy to see that $E(b_{\tau_i})$ is a homology cell. We will give the pseudo cell decomposition for $\partial E(b_{\tau_i})$. First we divide $\partial E(b_{\tau_i})$ as the union $\partial E(b_{\tau_i}) = LE(b_{\tau_i}) \cup HE(b_{\tau_i})$, where

$$LE(b_{\tau_i}) = \partial D(b_{\tau_i}) - D^3 \times D^{n-3}$$

$$HE(b_{\tau_i}) = \partial K_i - D^3 \times S^{n-4} = (H_i^3 - D^3) \times S^{n-4}.$$

Since $LE(b_{\tau_i}) = LD(b_{\tau_i})$, we give the cell decomposition by that of $LD(b_{\tau_i})$. We give the pseudo cell decomposition in the interior of $HE(b_{\tau_i})$ as
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\[(H_i^3 - D^3) \times S^{n-4} = (H_i^3 - D^3) \times \left( \bigcup \alpha C_\alpha \right) = \bigcup \alpha (H_i^3 - D^3) \times C_\alpha, \]

where \(S^{n-4} = \bigcup C_\alpha\) is the standard decomposition. These decompositions of \(LE(b_{\tau_i})\) and \(HE(b_{\tau_i})\) fit together on their intersection and give the decomposition of \(\partial E(b_{\tau_i})\), which is clearly cellular equivalent to that of \(\partial D(b_{\tau_i})\). For each simplex \(\mu > b_{\tau_i}, \mu \in M'\), we denote by \(E(\mu)\) the pseudo cell of \(\partial E(b_{\tau_i})\) which corresponds by the equivalence to \(D(\mu) \in \partial D(b_{\tau_i})\). We have \(\partial E(b_{\tau_i}) = \bigcup E(\mu)\). We define \(N_{(3)}\) by

\[N_{(3)} = N_{(2)} \cup N_3\]

attached by the identity on \(LE(b_{\tau_i})\). \(N_{(3)}\) is cellurally equivalent to \(M_{(3)}\).

(c) \(p = 4\). Let \(\sigma\) be a \((n - 4)\)-simplex. Let \(\bigcup E(\mu) \subset \partial N_{(3)}\) be the union of pseudo cells such that \(b_\sigma < \mu \in M', \mu \neq b_\sigma\). Then by the definition, it is PL-homeomorphic to the PL-manifold

\[(Lk(\sigma) \# \Sigma (-H_i^3)) \times D^{n-4}\]

where \(H_i^3\) represents \(g(\tau_i)\) and the sum extends to all \(\tau_i > \sigma\).

Since \(\{Lk(\sigma)\} = \Sigma g(\tau_i)\) in \(\mathcal{G}^3\), the PL-homology 3-sphere

\[H_\sigma^3 = Lk(\sigma) \# \Sigma (-H_i^3)\]

is the boundary of an acyclic PL-manifold \(W_\sigma^4\). The union

\[W_\sigma^4 \times S^{n-5} \cup H_\sigma^3 \times D^{n-4}\]

is a PL-homology \((n - 1)\)-sphere. By the proposition 1 of § 3, it is the boundary of a contractible PL-manifold \(Y_\sigma\). We define the PL-manifolds \(E(b_\sigma)\) and \(N_4\) as

\[E(b_\sigma) = Y_\sigma\]

\[N_4 = \bigcup E(b_\sigma)\, .\]

Further we define \(LE(b_\sigma)\) and \(HE(b_\sigma)\) by

\[LE(b_\sigma) = H_\sigma^3 \times D^{n-4}\]

\[HE(b_\sigma) = W_\sigma^4 \times S^{n-5}\, .\]

The pseudo cellular decomposition for \(LE(b_\sigma)\) is already defined and we give for \(HE(b_\sigma)\) by the product with the standard decomposition.
They give a pseudo cellular decomposition of
\[ \partial E(b_\sigma) = \text{LE}(b_\sigma) \cup \text{HE}(b_\sigma), \]
which is cellularly equivalent to that of \( \partial D(b_\sigma) \). For each simplex \( \mu > b_\sigma, \mu \in M' \), we define \( E(\mu) \) by the pseudo cell which corresponds to \( D(\mu) \) by this equivalence. We define \( N_{(4)} \) by \( N_{(3)} \cup N_4 \) attached by the identity of \( \text{LE}(b_\sigma) \), which is cellularly equivalent to \( M_{(4)} \).

(d) \( p \geq 5 \). Let \( \sigma \) be a \( j \)-simplex \( j \leq n - 5 \). Let \( \cup E(\mu) \subset \partial N_{(n-j-1)} \) be the union of pseudo cells such that \( \mu > b_\sigma, \mu \neq b_\sigma \). Then by our definition, it is a PL-manifold
\[ H^p_{a} \times D^{n-p} \]
where \( H^p_{a} \) is a PL-homology \((p-1)\)-sphere, where \( p = n - j \). By the proposition 1 of § 3, \( H^p_{a} \) is the boundary of a contractible PL-manifold \( W^p_{a} \). We define \( E(b_\sigma) \) by
\[ E(b_\sigma) = W^p_{a} \times D^{n-p}. \]
The other definitions are just similar to the case when \( p = 4 \).

Continuing this process, we obtain a PL-manifold \( N = N_{(n)} \) which is cellularly equivalent to \( M = M_{(n)} \).

Q.E.D.

5. Simple homotopy equivalence.

By the theorem of § 4, for the same \( M \), if the obstruction class is 0, we can construct a PL-manifold \( N \). In this section, we prove the following.

**Theorem. —** If \( M \) is compact, the constructed manifold \( N \) is simple homotopy equivalent to \( M \).

Let \( M^{(k)} \) denote the \( k \)-skelton of \( M \). Let \( L \) be a subcomplex of \( M^{(k)} \), we define the PL-submanifold \( N^{(L)} \) of \( N \) by
\[ N^{(L)} = \cup \{ E(b_\sigma) | \sigma \in L \}. \]
We put
\[ N^{(k)} = N^{(M^{(k)})} = \cup \{ E(b_\sigma) | \sigma \in M^{(k)} \}. \]

By the induction of \( k \), we prove the stronger
**Lemma 1.** There exists a simple homotopy equivalence
\[ f : M^{(k)} \to N^{(k)} \]
such that, for any \((k + 1)\)-simplex \(\mu\), \(f(\partial \mu) \subseteq N(\partial \mu)\) and
\[ f|\partial \mu : \partial \mu \to N(\partial \mu) \]
is a simple homotopy equivalence.

**Proof.** If \(k = 0\), it holds obviously. Now we will prove the lemma for \(k + 1\) assuming the lemma for \(k\). Let \(\mu\) be a \((k + 1)\)-simplex. Since the collar of \(\partial \mu\) is PL-homeomorphic to \(S^k \times I\), we can write
\[ \mu = S^k \times I \cup S^k \ast (b_\mu) \]
where \(S^k_0 = S^k \times \{0\} = \partial \mu\) and \(S^k_1 = S^k \times \{1\} = S^k \times I \cap S^k \ast (b_\mu)\).

Recall that \(N(M^{(k)} \cup \mu) = N^{(k)} \cup E(b_\mu)\)
\[ N^{(k)} \cap E(b_\mu) = N(\partial \mu) \cap E(b_\mu) = HE(b_\mu) = W^{n-k-1}_\mu \times S^k \]
where \(W^{n-k-1}_\mu\) is an acyclic (or contractible) PL-manifold. Let \(x\) be a point in the interior of \(W_\mu\) and let \(d : S^k \to W_\mu \times S^k\) be the embedding defined by \(d(S^k) = \{x\} \times S^k\). We define a map
\[ \tilde{f} : S^k_0 \cup S^k_1 \to N^{(k)} \]
by
\[ \tilde{f} | S^k_0 = f, \quad \tilde{f} | S^k_1 = d. \]
Since \(\tilde{f} | \partial M\) gives a simple homotopy equivalence \(\partial \mu \to N(\partial \mu)\), \(N(\partial \mu)\) is homotopy equivalent to \(S^k\), and so \(\tilde{f} | S^k_0\) and \(\tilde{f} | S^k_1\) are homotopic. Hence we can extend \(\tilde{f}\) on \(S^k \times I\). Further since \(E(b_\mu)\) is contractible, we can extend \(\tilde{f}\) to a map from \(\mu = S^k \times I \cup S^k \ast (b_\mu)\) to \(N(M^{(k)} \cup \mu)\).

By the definition, \(f\) and \(\tilde{f}\) coincide on \(\partial \mu\), and so we have a map
\[ g = f \cup \tilde{f} : M^{(k)} \cup \mu \to N(M^{(k)} \cup \mu). \]

Repeating this for all \((k + 1)\)-simplexes of \(M\), we obtain a map \(g : M^{(k+1)} \to N^{(k+1)}\). We have the exact sequences of chain groups,
\[ 0 \to C_\ast(M^{(k)}) \to C_\ast(M^{(k+1)}) \to \Sigma C_\ast(\mu/\partial \mu) \to 0, \]
\[ 0 \to C_\ast(N^{(k)}) \to C_\ast(N^{(k+1)}) \to \Sigma C_\ast(E(b_\mu)/HE(b_\mu)) \to 0, \]
where we regard them as \(\mathbb{Z}_\pi_1(M^{(k+1)}) = \mathbb{Z}_\pi_1(N^{(k+1)})\)-modules.
The map $g$ induces $f_*$ on the first elements and $id_*$ on the third elements. Since they are chain equivalences with trivial Whitehead torsion, so is $g_*$ by [8]. Hence $g$ is a simple homotopy equivalence. It is easy to see that, for any $(k + 2)$-simplex $\tau$, $g$ induce a simple homotopy equivalence
\[ g|_{\partial \tau} : \partial \tau \to N(\partial \tau). \]
Q.E.D.

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