THE PROJECTIVE CLASS GROUP TRANSFER INDUCED BY AN $S^1$-BUNDLE

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Introduction

This note gives an explicit algebraic description of the geometric transfer map induced in the (reduced) projective class groups by an $S^1$-bundle $\xymatrix{ S^1 \ar[r] & E \ar[r] & B }$

$$
p_*^\ast : \tilde K_0(\mathbb{Z}[\rho]) \ar[r] & \tilde K_0(\mathbb{Z}[\pi])
$$

with $\pi = \pi_1(E), \rho = \pi_1(B)$. This is the transfer map (1.4) of the preceding paper, Munkholm and Pedersen [4], to which we refer for terminology and background material. In particular, $t \in \pi$ is the canonical generator of the cyclic group $\ker(p_*:\pi \to \rho)$ represented by the inclusion $S^1 \to E$ of a fibre, $\phi: \mathbb{Z}[\pi] \to \mathbb{Z}[\pi]/(t-1) = \mathbb{Z}[\rho]; r \mapsto \tilde{r}$ is the projection of fundamental group rings induced by $p_*:\pi \to \rho$, and $\mathbb{Z}[\pi] \to \mathbb{Z}[\pi]; r \mapsto r^t$ is a ring automorphism determined by the orientation class $w_1(p) \in H^1(B; \mathbb{Z}_2)$ such that $(t-1)r = r^t(t-1)$. In the orientable case $w_1(p) = 0$, $t \in \pi$ is central and $r^t = r$.

Our main results are:

Proposition 2.1 The projection of rings $\phi: \mathbb{Z}[\pi] \to \mathbb{Z}[\rho]$ gives rise to an algebraic transfer map in the projective class groups

$$
\phi^\ast : K_0(\mathbb{Z}[\rho]) \to K_0(\mathbb{Z}[\pi]); \{ \text{im}(\tilde{x}) \} \mapsto \{ \text{im}(x') \} \in \mathbb{Z}[\pi]^n.
$$

Here $\tilde{x} \in M_{n}(\mathbb{Z}[\rho])$ is a projection (i.e. an $n \times n$ matrix $\tilde{x}$ with entries in $\mathbb{Z}[\rho]$ such that $\tilde{x}^2 = \tilde{x}$) and $X^t \in M_{2n}(\mathbb{Z}[\pi])$ is the projection defined by

$$
X^t = \begin{pmatrix} X & Y \\ (t-1) & 1-X^t \end{pmatrix} \in M_{2n}(\mathbb{Z}[\pi])
$$

for any $X, Y \in M_{n}(\mathbb{Z}[\pi])$ such that $\phi(X) = \tilde{x}, X(1-X) = Y(t-1), XY = YX^t$.

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Proposition 4.1 The algebraic and geometric transfer maps in the reduced projective class groups coincide, that is if $B,E$ are finitely dominated CW complexes

$$\delta^1_0 = p^*_{\overline{K}_0} : \overline{K}_0(\mathbb{Z}[\pi]) \longrightarrow \overline{K}_0(\mathbb{Z}[\pi]) ;$$

$$[B] \longrightarrow \delta^1_0([B]) = p^*_{\overline{K}_0}([B]) = [E]$$

with $[B],[E]$ the Wall finiteness obstructions.

[1]

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References
§1. Rings with pseudostructure

Let \( R \) be an associative ring with 1. We shall be using the following conventions regarding matrices and morphisms over \( R \).

Given (left) \( R \)-modules \( M, N \) let \( \text{Hom}_R(M, N) \) denote the additive group of \( R \)-module morphisms
\[
    f : M \longrightarrow N ; \quad x \longmapsto f(x)
\]

For \( m, n \geq 1 \) let \( M_{m,n}(R) \) be the additive group of \( m \times n \) matrices \( X = (x_{ij}) \) (\( 1 \leq i \leq m, 1 \leq j \leq n \)) with entries \( x_{ij} \in R \), and use the isomorphism of abelian groups
\[
    M_{m,n}(R) \overset{\sim}{\longrightarrow} \text{Hom}_R(R^m, R^n) ; \quad X = (x_{ij}) \longmapsto (f : (r_1, r_2, \ldots, r_m) \longmapsto \sum_{i=1}^{m} r_i x_{i1}, \sum_{i=1}^{m} r_i x_{i2}, \ldots, \sum_{i=1}^{m} r_i x_{in})
\]
to identify
\[
    M_{m,n}(R) = \text{Hom}_R(R^m, R^n)
\]

If the \( R \)-module morphisms \( f \in \text{Hom}_R(R^m, R^n) \), \( g \in \text{Hom}_R(R^n, R^p) \) have matrices \( X = (x_{ij}) \in M_{m,n}(R) \), \( Y = (y_{jk}) \in M_{n,p}(R) \) the composite \( R \)-module morphism
\[
    g f : R^m \longrightarrow R^n \longrightarrow R^p ; \quad r \longmapsto g(f(r))
\]
has the product matrix
\[
    XY = (\sum_{j=1}^{n} x_{ij} y_{jk}) \in M_{m,p}(R)
\]

The \( n \times n \) matrix ring \( M_n(R) = M_{n,n}(R) \) is thus identified with the endomorphism ring \( \text{Hom}_R(R^n, R^n) \) of the f.g. free \( R \)-module \( R^n \)
of rank \( n \), as usual.

A projection over \( R \) is a matrix \( X \in M_n(R) \) such that
\[
    X(1-X) = 0 \in M_n(R)
\]
so that \( \text{im}(X) \subseteq R^n \) is a f.g. projective \( R \)-module with
\[
    \text{im}(X) \oplus \text{im}(1-X) = R^n
\]
and \( \text{im}(1-X) \) is a f.g. projective inverse of \( \text{im}(X) \). Let
\[
    P_n(R) = \{ X \in M_n(R) | X(1-X) = 0 \} \subseteq M_n(R)
\]
denote the subset of \( M_n(R) \) consisting of projections. Every f.g. projective \( R \)-module \( P \) is isomorphic to \( \text{im}(X) \) for some \( X \in P_n(R) \).
A pseudostructure $\phi = (\alpha, t)$ on the ring $R$ consists of an automorphism
\[ \alpha : R \rightarrow R ; \quad r \mapsto r^t \]
and an element $t \in R$ such that
\[ t^t = t , \quad (t-1)r = r^t(t-1) . \]
Let $\phi$ also denote the projection onto the quotient of $R$ by the two-sided principal ideal $(t-1)R$.
\[ \phi : R \rightarrow \tilde{R} = R/(t-1) ; \quad r \mapsto \tilde{r} . \]
An $S^1$-bundle $S^1 \rightarrow E \rightarrow P \rightarrow B$ with $p_* = \phi : \pi_1(E) = \pi \rightarrow \pi_1(B) = \emptyset$ determines a pseudostructure $\phi = (\alpha, t)$ on $R = \mathbb{Z}[\pi]$ with $\tilde{R} = \mathbb{Z}[\phi]$ (cf. Munkholm and Pedersen [3,4]).
Let then $(R, \phi)$ be a ring $R$ with pseudostructure $\phi = (\alpha, t)$.
A pseudoprojection over $(R, \phi)$ is a pair of matrices over $R$
\[ (X, Y) \in M_n(R) \times M_n(R) \]
such that
\[ X(1-X) = Y(t-1) , \quad XY = YX^t \in M_n(R) , \]
where $X^t = \alpha(X) = (x_{ij}^t) \in M_n(R)$. The pseudoprojection $(X, Y)$ gives rise to a projection over $\tilde{R}$
\[ \tilde{X} \in P_n(\tilde{R}) \]
with $\tilde{X} = \phi(X) = (\tilde{x}_{ij}) \in M_n(\tilde{R})$, and also to a projection over $R$
\[ X^t = \left( \begin{array}{cc} X & Y \\ t-1 & 1-X^t \end{array} \right) \in P_{2n}(R) . \]
Let
\[ P_n(R, \phi) = \{(X, Y) \in M_n(R) \times M_n(R) | X(1-X) = Y(t-1) , \ XY = YX^t \} \]
denote the subset of $M_n(R) \times M_n(R)$ consisting of the pseudoprojections over $(R, \phi)$.

Proposition 1.1 Every projection $\tilde{X} \in P_n(\tilde{R})$ over $\tilde{R}$ lifts to a pseudoprojection $(X, Y) \in P_n(R, \phi)$ (non-uniquely), with $\phi(X) = \tilde{X}$.
Proof: Every matrix $\tilde{X} \in M_n(\tilde{R})$ lifts to some $X \in M_n(R)$, with any two such lifts $X_1, X_2$ differing by
\[ X_1 - X_2 = W(t-1) \in M_n(R) \]
for some $W \in M_n(R)$. Thus if $X \notin M_n(R)$ is a lift of a projection $\tilde{X} \in P_n(R)$ there exists $W \in M_n(R)$ such that
\[ X(1-X) = W(t-1) \in M_n(R) . \]
Define the matrix
\[ Z = \begin{pmatrix} X & W \\ t-1 & 1-Xt \end{pmatrix} \in \text{GL}_{2n}(\mathbb{R}) . \]

Now
\[ Z(1-Z) = \begin{pmatrix} 0 & WX^t - XW \\ 0 & 0 \end{pmatrix} \in \text{GL}_{2n}(\mathbb{R}) , \]
so that \((Z(1-Z))^2 = 0\) and
\[ Z^2 + (1-Z)^2 = 1 - 2Z(1-Z) \in \text{GL}_{2n}(\mathbb{R}) \]
is invertible, with inverse
\[ (Z^2 + (1-Z)^2)^{-1} = 1 + 2Z(1-Z) \in \text{GL}_{2n}(\mathbb{R}) , \]
so that there is defined a projection
\[ X^! = (Z^2 + (1-Z)^2)^{-1}Z^2 \in \text{P}_{2n}(\mathbb{R}) . \]

(The principal ideal \((Z(1-Z))\) of the matrix ring \(M_{2n}(\mathbb{R})\) is nilpotent, and \(X^! \in \text{P}_{2n}(\mathbb{R})\)\(\subset \text{M}_{2n}(\mathbb{R})\) is an idempotent (= projection) lifting the idempotent \([Z] \in M_{2n}(\mathbb{R})/(Z(1-Z))\) - cf. Bass [0,III.2.10], Swan [9,5.17]). Substituting the relation \(Z^4 = 2z^3 - 2z^2\) we have
\[ X^! = (1 + 2Z(1-Z))Z^2 \]
\[ = (1 + 2Z)Z^2 - 2(2Z^3 - Z^2) \]
\[ = 3Z^2 - 2Z^3 \in \text{P}_{2n}(\mathbb{R}) , \]
with
\[ X^! - z = (2Z-1)Z(1-Z) \]
\[ = \begin{pmatrix} 2X-1 & 2W \\ 2t-2 & 1-2X^t \end{pmatrix} \begin{pmatrix} 0 & WX^t - XW \\ 0 & 0 \end{pmatrix} \]
\[ = \begin{pmatrix} 0 & (2X - 1)(WX^t - XW) \\ 0 & 0 \end{pmatrix} \in \text{M}_{2n}(\mathbb{R}) . \]

Defining
\[ Y = W + (2X - 1)(WX^t - XW) \in \text{M}_{n}(\mathbb{R}) , \]
we have
\[ X^! = \begin{pmatrix} X & Y \\ t-1 & 1-Xt \end{pmatrix} \in \text{P}_{2n}(\mathbb{R}) \]
with \(\phi(X) = X, X(1-X) = Y(t-1)\), \(XY = WX^t\). The projection \(\bar{X} \in \text{P}_{n}(\bar{R})\) has been lifted to a pseudoprojection \((X,Y) \in \text{P}_{n}(\bar{R},\text{P})\).
Given an $R$-module $\bar{M}$ let $\phi^1\bar{M}$ be the $R$-module with the same additive group as $\bar{M}$ and
\[ R \times \phi^1\bar{M} \longrightarrow \phi^1\bar{M}; \quad (r, x) \longmapsto r \cdot x. \]
An $R$-module morphism $f \in \text{Hom}_R(\bar{M}, \bar{N})$ also defines an $R$-module morphism
\[ \phi^1 f : \phi^1\bar{M} \longrightarrow \phi^1\bar{N}; \quad \bar{x} \longmapsto \bar{f}(\bar{x}). \]
Given a pseudoprojection $(X, Y) \in P_n(R, \phi)$ define the f.g. projective $R$-module $\bar{P} = \text{im}(\bar{X})$, and define the associated pseudoresolution of the restricted $R$-module $\phi^1\bar{P}$ to be the 1-dimensional f.g. projective $R$-module chain complex $C^1$ with
\[ d_{C^1} = \begin{bmatrix} 1 - X \\ l - t \end{bmatrix}; \quad C^1_0 = \text{coker}(X^1 = \begin{bmatrix} x \\ y \end{bmatrix}); \quad R^n \otimes R^n \longrightarrow R^n \otimes R^n \]
\[ = C^1_0 = R^n. \]
The homology $R$-modules of $C^1$ are given by
\[ H_0(C^1) = \text{coker}(\begin{bmatrix} 1 - X \\ l - t \end{bmatrix}; \quad R^n \otimes R^n \longrightarrow R^n) = \phi^1\bar{P}, \]
\[ H_1(C^1) = \ker((t - 1 - X^t) : R^n \longrightarrow R^n \otimes R^n), \]
and in many respects $C^1$ is like a f.g. projective $R$-module resolution of $\phi^1\bar{P}$. However, $C^1$ is a genuine resolution of $\phi^1\bar{P}$ (with $H_1(C^1) = 0$) if and only if $t - 1 \in R$ is a non-zero-divisor.

By Proposition 1.1 there exists a pseudoresolution $C^1$ of $\phi^1\bar{P}$ for any f.g. projective $R$-module $\bar{P}$. As for uniqueness, we have:

**Proposition 1.2** Given pseudoprojections $(X, Y) \in P_n(R, \phi)$, $(X', Y') \in P_n(R, \phi)$ and a morphism of f.g. projective $R$-modules
\[ \bar{f} : \bar{P} = \text{im}(\bar{X}) \longrightarrow \bar{P}' = \text{im}(\bar{X}') \]
there is defined an $R$-module chain map of the associated pseudoresolutions
\[ f^1 : C^1 \longrightarrow C'^1 \]
uniquely up to chain homotopy, such that
\[ (f^1)_* = \phi^1 f : H_0(C^1) = \phi^1\bar{P} \longrightarrow H_0(C'^1) = \phi^1\bar{P}'. \]
The construction of $f^1$ is functorial up to chain homotopy, with
\[ 1^1 = 1, \quad (f'f)^1 = f'^1f^1 \]
up to chain homotopy. In particular, if $\bar{f} \in \text{Hom}_R(\bar{P}, \bar{P}')$ is an isomorphism then $f^1 : C^1 \longrightarrow C'^1$ is a chain equivalence.
Proof: Let $F \in M_{n,n'}(R)$ be the matrix of the composite $R$-module morphism

$$P : \tilde{R}^n \xrightarrow{\text{projection}} \text{im}(\tilde{X}) = \widetilde{F} \xrightarrow{\tilde{f}} \text{im}(\tilde{X}') = \tilde{R}^{n'}.$$  

Choose a lift $F \in M_{n,n'}(R)$ of $\tilde{F}$ and define

$$P' = \begin{pmatrix} XFX' & XFY' - YFX'_X, t \\ 0 & X^t_F X, t \end{pmatrix} \in M_{2n,2n'}(R)$$

such that

$$X^tP' = F'X^t \in M_{2n,2n'}(R).$$

The $R$-module chain map $f : C \to C'$ is defined by

$$f : C \xrightarrow{[F]} C' \xrightarrow{XFX'} C'^{\prime}.$$ 

If $F_1, F_2 \in M_{n,n'}(R)$ are two different lifts of $\tilde{F}$ there exists

$$G \in M_{n,n'}(R)$$

such that

$$F_1 - F_2 = G(t-1) \in M_{n,n'}(R),$$

and the $R$-module morphism

$$g : [0 \ \hat{X}G] : C_0 = R^n \to C_1 = \text{coker}(X^t)$$

defines a chain homotopy

$$g : f_1 = f_2 : C \to C'$$

between the corresponding $R$-module chain maps $f_1, f_2 : C \to C'$.

If $(X,Y) = (X',Y') \in P_n(R,\phi)$ and $\tilde{F} = \tilde{F} = \text{im}(\tilde{X}) \to \tilde{R} = \text{im}(\tilde{X})$

then $P = X \in M_{n}(R)$ is a lift of the composite $R$-module morphism

$$\tilde{P} : \tilde{X} : \tilde{R}^n \xrightarrow{\text{projection}} \tilde{R} \xrightarrow{\text{inclusion}} \tilde{R}^{n'},$$

so that

$$P = \begin{pmatrix} X^3 & 0 \\ 0 & (X^t)^3 \end{pmatrix} \in M_{2n}(R)$$

and the $R$-module morphism

$$h = [1 + X + X^2 \ 0] : C_0 = R^n \to C_1 = \text{coker}(X^t)$$

defines a chain homotopy
Given pseudoprojective resolutions $(X,Y) \in P_n(R,\phi)$, $(X',Y') \in P_{n'}(R,\phi)$, $(X'',Y'') \in P_{n''}(R,\phi)$ and $R$-module morphisms

\[ \bar{f} : \bar{P} = \text{im}(\bar{X}) \rightarrow \bar{P}' = \text{im}(\bar{X}') \rightarrow \bar{P}'' = \text{im}(\bar{X}'') \]

let

\[ \bar{f}'' = \bar{f}' \bar{f} : \bar{P} \rightarrow \bar{P}' \rightarrow \bar{P}'' \]

be the composite $R$-module morphism. If $P \in M_{n,n}(R)$ and $P' \in M_{n',n''}(R)$ are lifts of the composite $R$-module morphisms

\[ \bar{P} : R^n \rightarrow P \rightarrow P' \rightarrow P'' \rightarrow R^{n''} \\
\bar{P}' : R^{n'} \rightarrow P' \rightarrow P'' \rightarrow R^{n''} \]

then the product

\[ P'' = \text{im}(P') \in M_{n,n''}(R) \]

is a lift of the composite $R$-module morphism

\[ \bar{P}'' : R^n \rightarrow P \rightarrow P'' \rightarrow R^{n''} \]

such that

\[ f'' = f'f'' \in M_{2n,2n''}(R) \]

and so

\[ f'' = f'f'' : \bar{C} \xrightarrow{f'} \bar{C}' \xrightarrow{f''} \bar{C}''. \]

§2. The projective class transfer

Proposition 2.1 Given a ring $R$ with pseudostructure $\phi = (a,t)$ there is defined an algebraic transfer map in the projective class groups

\[ \hat{\phi}^I : K_0(R) \rightarrow K_0(R) : [\bar{P}] \mapsto [\text{im}(\bar{X}^I)] - [R^n], \]

sending a f.g. projective $R$-module $\bar{P} = \text{im}(\bar{X}) (\bar{X} \in P_n(R))$ to the projective class $[C^I] = [\text{im}(X^I)] - [R^n] \in K_0(R) ((X,Y) \in P_n(R,\phi))$ of any pseudoresolution $C^I$ of $\phi^I$. If $P$ is a (stably) f.g. free $R$-module then $\hat{\phi}^I([\bar{P}]) = 0 \in K_0(R)$, so that there is also defined an algebraic transfer map in the reduced projective class groups

\[ \tilde{\hat{\phi}}^I : \tilde{K}_0(R) \rightarrow \tilde{K}_0(R) : [\bar{P}] \mapsto [\text{im}(\bar{X}^I)]. \]
Prove: Given a f.g. projective $\tilde{R}$-module $\tilde{P}$ use Proposition 1.1 to lift a projection $\tilde{X} \in P_n(\tilde{R})$ such that $\tilde{P} = \text{im}(\tilde{X})$ to a pseudoprojection $(X, Y) \in P_n(R, \psi)$, and let $\text{C}^1 : \text{im}(X^1) \rightarrow R^n$ be the corresponding pseudoresolution of $\psi^1 \tilde{P}$. Up to $R$-module isomorphism

$$\text{im}(X^1) \otimes \text{coker}(X^1) = \text{im}(X^1) \otimes \text{im}(1-X^1) = R^n,$$

so that

$$[\text{C}^1] = [R^n] - [\text{coker}(X^1)] = [\text{im}(X^1)] - [R^n] = \phi^1_0([\tilde{P}]) \in \text{K}_0(R).$$

An element of $\text{K}_0(R)$ is the formal difference $[\tilde{P}] - [\tilde{P}']$, for some f.g. projective $R$-modules $\tilde{P} = \text{im}(\tilde{X})$, $\tilde{P}' = \text{im}(\tilde{X}')$. Now $[\tilde{P}] - [\tilde{P}'] = 0 \in \text{K}_0(R)$ if and only if there exists an $R$-module isomorphism $\tilde{f} : \tilde{P} \oplus \tilde{P}' \rightarrow \tilde{P} \oplus \tilde{P}'$ for some f.g. projective $R$-module $\tilde{Q}$, in which case Proposition 1.2 gives a chain equivalence $f^1 : \text{C}^1 \rightarrow \text{C}'^1$ of the corresponding pseudoresolutions of $\psi^1 \tilde{P}, \psi^1 \tilde{P}'$.

As the projective class of a chain complex is a chain homotopy invariant it follows that

$$\phi^1_0([\tilde{P}] - [\tilde{P}']) = [\text{C}^1] - [\text{C}'^1] = 0 \in \text{K}_0(R),$$

and so $\phi^1_0 : \text{K}_0(R) \rightarrow \text{K}_0(R)$ is well-defined.

For $P = R^n$ take $\tilde{X} = 1 \in P_n(\tilde{R})$, $(X, Y) = (1, 0) \in P_n(R, \psi)$, so that the projection

$$X^1 = \begin{pmatrix} 1 & 0 \\ t-1 & 0 \end{pmatrix} : R^n \oplus R^n \rightarrow R^n \oplus R^n$$

has $\text{im}(X^1) \cong R^n$ and so

$$\phi^1_0([R^n]) = [R^n] - [R^n] = 0 \in \text{K}_0(R).$$

Thus $\tilde{R}^1 : \text{K}_0(R) \rightarrow \text{K}_0(R)$ is also well-defined.

[1]

The original algebraic description in terms of matrices of the Whitehead group $S^1$-bundle transfer map

$$F^*_{WH} = \tilde{\phi}^1_1 : \text{Wh}(\varphi) \rightarrow \text{Wh}(\Pi)$$

due to Munkholm and Pedersen [3] was reformulated by Ranicki [6, §7.8] in terms of the theory of pseudo chain complexes. We shall now recall this theory, and show how it applies to the projective class group $S^1$-bundle transfer.
Given an R-module $M$ let $M^t$ denote the R-module with the same additive group and

$$R \times M^t \longrightarrow M^t ; (r,x) \longmapsto r^{-t}x,$$

where $a^{-1} : R \longrightarrow R; r \longmapsto r^t$ is the inverse of the ring automorphism $a : R \longrightarrow R; r \longmapsto r^t$ in the pseudostructure $\phi = (a, t)$. An R-module morphism $f \in \text{Hom}_R(M, N)$ also defines an R-module morphism

$$f^t : M^t \longrightarrow N^t ; x \longmapsto f(x),$$

such that

$$f(t-1) = (t-1)f^t : M^t \longrightarrow N$$

with $t-1 \in \text{Hom}_R(M^t, M)$ defined by

$$t-1 : M^t \longrightarrow M ; x \longmapsto tx - x.$$

For $M = R^n$ use the R-module isomorphism

$$M^t \sim R^n ; (r_1, r_2, \ldots, r_n) \longmapsto (r_1^t, r_2^t, \ldots, r_n^t)$$

to identify $M^t = R^n$, so that $t-1 \in \text{Hom}_R(M^t, M)$ has matrix $t-1 \in M_n(R)$. If $f \in \text{Hom}_R(M^t, M)$ has matrix $X = (x_{ij}) \in M_{n \times n}(R)$ then

$$f^t \in \text{Hom}_R(R^n, M^t) = \text{Hom}_R(R^n, R^n)$$

has matrix $X^t = (x^t_{ij}) \in M_{n \times n}(R)$.

A pseudo chain complex over $(R, t)$ $\zeta = (C, d, e)$ consists of a collection of R-modules $\{C_r| r \geq 0\}$ and two collections of R-module morphisms $\{d \in \text{Hom}_R(C_r, C_{r-1})| r \geq 1\}, \{e \in \text{Hom}_R(C_r, C_{r-2})| r \geq 2\}$ such that

$$d^2 = (t-1)e : C_r \longrightarrow C_{r-2}, \quad d^t e = ed : C_r \longrightarrow C_{r-3}.$$

Note that $\zeta$ determines an R-module chain complex $\bar{C}$ with

$$d_{\bar{C}} = 1 \otimes d : \bar{C}_r = \oplus_{i} C_i \longrightarrow \bar{C}_{r-1} = \oplus_{i} C_{i-1} ; a \otimes x \longmapsto a \otimes d(x),$$

and an R-module chain complex $C^t$ with

$$d_{C^t} = \left( \begin{array}{cc} d & (-)^t e \\ (-)^t (t-1) & d^t \end{array} \right)$$

$$: C^t_r = C_r \otimes C^t_{r-1} \longrightarrow C^t_{r-1} = C_{r-1} \otimes C^t_{r-2} ;$$

$$(x, y) \longmapsto (d(x) + (-)^t (t-1)(y), (-)^t e(x) + d^t(y)).$$

Proposition 7.8.8 of Ranicki [6] associates to an $S^1$-bundle of CW complexes $S^1 \longrightarrow E \rightarrow B$ with $p_* \otimes \pi_1(E) = \pi_1(B) = p$ a pseudo chain complex $\zeta(p) = (C, d, e)$ over $(\Z[\pi], \phi)$ with $C_r$ $(r \geq 0)$ the f.g. free $\Z[\pi]$-module of rank the number of $r$-cells in $B$, such that the cellular chain complexes of the universal covers $\tilde{B}, \tilde{E}$ of $B, E$ are given by
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$C(B) = \overline{C}$, $C(E) = C^1$.

If $B$ is finitely dominated then so is $E$, and the Wall finiteness obstructions are given by the reduced projective classes

$[B] = [C(B)] = [C] \in K_0(\mathbb{Z}[\rho])$,
$[E] = [C(E)] = [C^1] \in K_0(\mathbb{Z}[\eta])$.

The geometric transfer map $p^*_K : \tilde{K}_0(\mathbb{Z}[\rho]) \to \tilde{K}_0(\mathbb{Z}[\eta])$ is defined by

$p^*_K([B]) = [E] \in \tilde{K}_0(\mathbb{Z}[\eta])$,

so that it will follow from the identification $p^*_K = \tilde{\phi}_O^1$ in §4 below that

$[C^1] = [C(E)] = [E] \\
= p^*_K([B]) = \tilde{\phi}_O^1([C]) \in \tilde{K}_0(\mathbb{Z}[\eta])$.

In Ranicki [8] it will be shown algebraically that for any finitely dominated pseudo chain complex $\mathcal{P} = (C,d,e)$ over a ring with pseudostructure $(R,\phi)$ the algebraic transfer map $\phi_O^1 : K_0(R) \to K_0(R)$ sends the projective class $[C] \in K_0(R)$ to

$\phi_O^1([C]) = [C^1] \in K_0(R)$

(which will give an alternative proof of $p^*_K = \tilde{\phi}_O^1$ on setting $R = \mathbb{Z}[\eta]$, $\mathcal{P} = \mathcal{P}(\rho)$). At any rate, for any pseudoprojection $(X,Y) \in P_n(R,\phi)$ there is defined a finitely dominated pseudo chain complex $\mathcal{P} = (C,d,e)$ over $(R,\phi)$ with

\[
d = \begin{cases} 1 - X : C_{2i+1} = R^n & \to \ C_{2i} = R^n \\ X : C_{2i+2} = R^n & \to \ C_{2i+1} = R^n \ (i \geq 0) \\ e = Y : C_{j} = R^n & \to \ C_{j-2} = R^n \ (j \geq 2) \end{cases}
\]

for which

$[C] = [\text{im}(X)] \in K_0(R)$,

$[C^1] = [\text{im}(X^1)] \to [R^n] = \phi_O^1([C]) \in K_0(R)$.

Note that $C^1$ is an infinite f.g. free $R$-module chain complex which is chain equivalent to the f.g. projective pseudoresolution $C^1$ of $\phi^1(\text{im}(X))$ associated to $(X,Y) \in P_n(R,\phi)$ in §1 above.

In the case when $t - 1 \in R$ is a non-zero-divisor (which for a group ring $R = \mathbb{Z}[\eta]$ is equivalent to $t \in \eta$ being of infinite order) $\phi^1_R$ is an $R$-module of homological dimension 1, with a f.g. free $R$-module resolution

$0 \to R \xrightarrow{(t-1)} R \xrightarrow{\phi} \phi^1_R \to 0$.
If $\tilde{P}$ is a f.g. projective $R$-module then $\phi^!\tilde{P}$ is therefore an $R$-module of homological dimension 1, with a f.g. projective resolution

$$
\begin{array}{cccc}
O & \longrightarrow & P_1 & \longrightarrow & P_0 & \longrightarrow & \phi^!\tilde{P} & \longrightarrow & O
\end{array}
$$

The classical transfer map in the projective class groups is defined by

$$
\phi^! : K_0(\tilde{R}) \longrightarrow K_0(R) ; [\tilde{P}] \longmapsto [P_0] - [P_1]
$$

and this definition extends by the Bass-Quillen resolution theorem to transfer maps in the higher $K$-groups

$$
\phi^ ! : K_m(\tilde{R}) \longrightarrow K_m(R) \quad (m \geq 1).
$$

(More generally, the classical methods give transfer maps $\phi^! : K_s(\tilde{R}) \longrightarrow K_s(R)$ for any morphism of rings $\phi : R \longrightarrow \tilde{R}$ such that $\phi^! R$ is an $R$-module of finite homological dimension).

**Proposition 2.2** If $(R, \phi)$ is a ring with pseudostructure such that $t-1 \in R$ is a non-zero-divisor the projective class group transfer map $\phi^!_O$ defined above agrees with the classical transfer map

$$
\phi^!_O = \phi^! : K_0(\tilde{R}) \longrightarrow K_0(R).
$$

**Proof:** In this case the pseudoresolution $C^!$ of $\phi^!(\text{im}(X))$

associated to a pseudoprojection $(X,Y) \in P^*_n(R,\phi)$ in §1 above is a 1-dimensional f.g. projective $R$-module resolution of $\phi^!(\text{im}(\tilde{X}))$

$$
\begin{array}{cccc}
O & \longrightarrow & \text{coker}(X^!) & \longrightarrow & R^0 & \longrightarrow & \phi^!(\text{im}(\tilde{X})) & \longrightarrow & O
\end{array}
$$

so that

$$
\phi^!_O([\text{im}(\tilde{X})]) = [C^!] = \phi^!(\text{im}(\tilde{X})) \in K_0(R).
$$

For a group ring $R = \mathbb{Z}[\pi]$ the identification

$$
\phi^!_O = \phi^! : \tilde{K}_0(\mathbb{Z}[\pi]) \longrightarrow \tilde{K}_0(\mathbb{Z}[\pi])
$$

given by Proposition 2.2 may also be obtained by combining the identifications $\phi^! = p^* \phi^!_O$ of §4 and $p^*_R = \phi^!$ of Munkholm and Pedersen [2].

In Proposition 3.2 below the algebraic $S^1$-bundle transfer map $\phi^! : K_1(R) \longrightarrow K_1(R)$ of Munkholm and Pedersen [3] in the case when $t-1 \in R$ is a non-zero-divisor will be similarly identified with the classical transfer map $\phi^! : K_1(\tilde{R}) \longrightarrow K_1(\tilde{R})$. It would be interesting to know if the definitions of $\phi^!_O$ and $\phi^!_1$ extend to algebraic transfer maps in the higher $K$-groups

$$
\phi^!_m : K_m(\tilde{R}) \longrightarrow K_m(R) \quad (m \geq 2)
$$
in the case when \( t-1 \in R \) is a zero divisor, so that \( \phi_1^R \) is an \( R \)-module of infinite homological dimension and the classical methods fail.

§3. The Whitehead torsion transfer

The Whitehead torsion transfer map of Munkholm and Pedersen \([3]\) was defined for any ring with pseudostructure \((R, \Phi)\) to be

\[
\phi_1^I : K_1(\overline{R}) \longrightarrow K_1(R) ; \tau(\overline{X}) \longmapsto \tau\left( \begin{array}{cc}
X & -Z \\
-t & Yt
\end{array} \right)
\]

with \( X \in M_n(R) \) a lift of \( \overline{X} \in GL_n(\overline{R}) \) and \( Y, Z \in M_n(R) \) such that

\[
XY = 1 - Z(t-1) \in M_n(R).
\]

In Ranicki \([6, \S 7.8]\), \( \phi_1^I(\tau(\overline{X})) \in K_1(R) \) was interpreted as the torsion \( \tau(C^I) \) of the acyclic \( R \)-module chain complex

\[
C^I : \mathbb{R}^n \longrightarrow R^n \oplus R^n \longrightarrow \mathbb{R}^n
\]

associated to the pseudo chain complex \( \mathcal{E} = (C, d, e) \) with

\[
d = X : C_1 = \mathbb{R}^n \longrightarrow C_0 = \mathbb{R}^n , \quad C_r = 0 \ (r \geq 2) , \quad e = 0,
\]

for which

\[
\tau(C) = \tau(\overline{X}; \mathbb{R}^n \longrightarrow \mathbb{R}^n) \in K_1(R).
\]

(The identification \( \phi_1^I(\tau(\overline{X})) = \tau(C^I) \in K_1(R) \) is immediate from the observation that

\[
\left( \begin{array}{c}
-X \\
Yt
\end{array} \right) : R^n \oplus R^n \longrightarrow \mathbb{R}^n
\]

is a splitting map for \( (1-t)x^t) : R^n \longrightarrow R^n \oplus R^n \). It will be shown in Ranicki \([8]\) that for any finite pseudo chain complex \( \mathcal{E} = (C, d, e) \) over \((R, \Phi)\) with each \( C_r \ (r \geq 0) \) a based f.g. free \( R \)-module with \( \overline{C} \) (and hence \( C^I \)) acyclic

\[
\phi_1^I(\tau(C)) = \tau(C^I) \in K_1(R).
\]

We shall now interpret \( \phi_1^I \) in terms of the pseudoresolution construction \((X, Y) \longrightarrow C^I \) of §1.

**Proposition 3.1** The Whitehead torsion transfer map

\[
\phi_1^I : K_1(\overline{R}) \longrightarrow K_1(R)
\]

sends the torsion \( \tau(\overline{f}) \in K_1(R) \) of an automorphism \( \overline{f} \in \text{Hom}_R(\overline{P}, \overline{P}) \) of a f.g. projective \( R \)-module \( \overline{P} \) to the torsion

\[
\phi_1^I(\tau(\overline{f})) = \tau(f^I) \in K_1(R)
\]

of the induced self chain equivalence \( f^I : C^I \longrightarrow C^I \), with \( C^I \) the
pseudoresolution of \( \phi^1 \) associated to any pseudoprojection 
\((X,Y) \in P_n(R,\phi)\) with \( \overline{P} = \text{im}(\overline{X})\).

**Proof:** Stabilizing \( \overline{f} \) by \( 1 \in \text{Hom}_R(\text{im}(1-\overline{X}), \text{im}(1-\overline{X})) \) it may be
assumed that \( \overline{P} = \overline{R}^n \) is a f.g. free \( \overline{R} \)-module, and
\((X,Y) = (1,0) \in P_n(R,\phi)\), so that \( C^i : R^n \xrightarrow{1-t} R^n \).

If \( \overline{f} \in \text{Aut}_R(\overline{R}^n,\overline{R}^n) \) has matrix \( \overline{X} \in \text{GL}_n(\overline{R}) \) then

\[
\begin{array}{c}
C^i : R^n \xrightarrow{1-t} R^n \\
\downarrow \overline{f}^t \\
\downarrow C^i \\
R^n \xrightarrow{1-t} R^n \\
\end{array}
\]

for any lift \( X \in \text{M}_n(R) \) of \( \overline{X} \), so that

\[
\tau(\overline{f}^t) = \tau(C(\overline{f}^t)) : R^n \xrightarrow{(1-t \ X^t)} R^n \otimes_{\overline{R}} R^n \xrightarrow{\text{t}-1} R^n \\
= \phi^1(\tau(\overline{X})) = \phi^1(\tau(\overline{f})) \in K_1(R).
\]

By analogy with Proposition 2.2:

**Proposition 3.2** If \( \text{t}^{-1} \in R \) is a non-zero-divisor the Whitehead torsion transfer map \( \phi^1 \) agrees with the classical transfer map 
\( \phi^1 = \phi^1 : K_1(\overline{R}) \longrightarrow K_1(R) \).

**Proof:** Given an automorphism \( \overline{f} \in \text{Aut}_R(\overline{R}^n,\overline{R}^n) \) note that the self
chain equivalence \( f^i : C^i \longrightarrow C^i \) defined in the proof of
Proposition 3.1 is a resolution of the automorphism 
\( \overline{f} \in \text{Aut}_R(\overline{R}^n,\overline{R}^n) \), so that

\[
\phi^1(\tau(\overline{f})) = \tau(f^i) = \phi^1(\tau(\overline{f})) \in K_1(R).
\]

For a group ring \( R = \mathbb{Z}[\pi] \) the identification
\( \overline{\phi}^1 = \phi^1 : \text{Wh}(\rho) \longrightarrow \text{Wh}(\pi) \) given by Proposition 3.2 may also be
obtained by combining the identifications \( \phi^1 = \phi^1 \) of Munkholm and
Pedersen [3] and \( \phi^* = \phi^1 \) of Munkholm [1].

In §4 we shall make use of the following relation between
the projective class group transfer \( \phi^1 : K^0(\overline{R}) \longrightarrow K^0(R) \) for a
ring with pseudostructure \((R,\phi)\), the Whitehead torsion transfer 
\((\phi \times 1)^1 : K_1(\overline{R}[z,z^{-1}]) \longrightarrow K_1(R[z,z^{-1}])\) for the polynomial
extension ring with pseudostructure \((R[z,z^{-1}],\phi \times 1)\) and the
canonical Bass-Heller-Swan injections.
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\[ h_R : K_0(R) \rightarrow K_1(R[z,z^{-1}]) ; \quad [P] \mapsto \tau(z;[z,z^{-1}] \sim P[z,z^{-1}]) \]

and \( h_R : K_0(R) \rightarrow K_1(R[z,z^{-1}]) \) defined similarly.

**Proposition 3.3** There is defined a commutative diagram

\[
\begin{array}{ccc}
K_0(R) & \xrightarrow{\phi_O} & K_0(R) \\
\downarrow h_R & & \downarrow h_R \\
K_1(R[z,z^{-1}]) & \xrightarrow{(\phi \times 1)_1} & K_1(R[z,z^{-1}]) \\
\end{array}
\]

**Proof:** Given a f.g. projective \( \mathbb{R} \)-module \( \mathbb{P} \) let \( (X,Y) \in P_n(R,\phi) \) be a pseudoprojection such that \( \mathbb{P} = \text{im}(X) \), and let \( \mathbb{C}^l \) be the corresponding pseudoresolution of \( \phi \mathbb{P} \). Now

\[
(\phi \times 1)_1^l h_R^l([\mathbb{P}]) = (\phi \times 1)_1^l (\tau(z;\mathbb{P}[z,z^{-1}] \sim \mathbb{P}[z,z^{-1}] ))
\]

\[
= \tau(z;\mathbb{C}^l[z,z^{-1}] \sim \mathbb{C}^l[z,z^{-1}]) \quad \text{(by Proposition 3.1)}
\]

\[
= h_R^l([\mathbb{C}^l]) = h_R^l \phi_O^l([\mathbb{P}]) \in K_1(R[z,z^{-1}]),
\]

so that \( (\phi \times 1)_1^l h_R^l = h_R^l \phi_O^l \).

[1]

54. The algebraic and geometric transfer maps coincide

Let \( S^1 \rightarrow E \xrightarrow{\mathbb{P}} B \) be an \( S^1 \)-bundle with \( p_* = \phi : \pi_1(E) = \pi \rightarrow \pi_1(B) = 0 \), and let \( (R = \mathbb{Z}[\pi],\phi) \) be the corresponding ring with pseudostructure.

**Proposition 4.1** The algebraic and geometric transfer maps in the reduced projective class groups coincide, that is

\[ z^l_O = p_\mathbb{P}^*_O : \bar{K}_0(\mathbb{Z}[\phi]) \rightarrow \bar{K}_0(\mathbb{Z}[\pi]). \]

**Proof:** We offer two proofs, in fact.

1) Given a pseudoprojection \( (X,Y) \in P_n(\mathbb{Z}[\pi],\phi) \) and a number \( m \gg 2 \) the proof of Theorem F of Wall [10] gives an \( S^1 \)-bundle of CW pairs

\[
S^1 \rightarrow (E,F) \xrightarrow{(p,q)} (B,K)
\]

with \( K \) finite and \( B \) finitely dominated, such that \( \pi_1(B) = \pi_1(K) = 0 \) and such that the relative pseudo chain complex \( \mathbb{C}(p,q) = (C,d,e) \) is given by

\[
C_r = \begin{cases} 
\mathbb{Z}[\pi]^n & \text{if } r \gg 2m \\
0 & \text{if } r \ll 2m-1
\end{cases}
\]
\( \begin{align*}
&\begin{aligned}
  d &= \left\{ \begin{array}{c}
  1-X : C_{2i+1} \longrightarrow C_{2i} & (i \geq m) \\
  X : C_{2i+2} \longrightarrow C_{2i+1}
  \end{array} \right.
  \vspace{1em}
  e = Y : C_{r} \longrightarrow C_{r-2} & (r \geq 2m+2).
\end{aligned}
\end{align*} \)

The finiteness obstruction of \( B \) (= the reduced projective class of \( C(B) = C \)) is given by

\[ [B] = [C] = [\text{im}(X)] \in \check{K}_0(\mathbb{Z}[\rho]), \]

and that of \( E \) by

\[ [E] = [C^1] = [\text{im}(X^1)] \in \check{K}_0(\mathbb{Z}[\pi]), \]

so that

\[ p_{K_O}^\ast ([B]) = [E] = [\text{im}(X^1)] \]

\[ = \tilde{c}_{O}^1([\text{im}(X)])) = \check{c}_{O}^1([B]) \in \check{K}_0(\mathbb{Z}[\pi]). \]

ii) Consider the commutative diagram preceding Corollary 2.3 of Munkholm and Pedersen [4]

\[
\begin{array}{ccc}
\check{K}_0(\mathbb{Z}[\pi]) & \xleftarrow{h_{\pi}} & \text{Wh}(\pi \times \mathbb{Z}) \\
\downarrow{p_{K_O}^\ast} & & \downarrow{(p \times 1)_{\text{Wh}}} \\
\check{K}_0(\mathbb{Z}[\rho]) & \xrightarrow{h_{\rho}} & \text{Wh}(\rho \times \mathbb{Z}) \\
\end{array}
\]

in which \( h_{\pi} \) (resp. \( h_{\rho} \)) is the canonical Bass-Heller-Swan surjection (resp. injection). From Proposition 3.3 we have

\[ (p \times 1)_{\text{Wh}} h_{\rho} = h_{\pi} \tilde{c}_{O}^1, \]

so that

\[ \begin{align*}
  p_{K_O}^\ast &= h_{\pi}^{-1}(p \times 1)_{\text{Wh}} h_{\rho} \\
  &= h_{\pi} h_{\rho}^{-1} \tilde{c}_{O}^1 : \check{K}_0(\mathbb{Z}[\rho]) \longrightarrow \check{K}_0(\mathbb{Z}[\pi]).
\end{align*} \]

§5. The relative transfer exact sequence

A ring morphism \( \phi : R \longrightarrow S \) induces morphisms in the algebraic K-groups

\[ \begin{align*}
  \Phi : K_0(R) & \longrightarrow K_0(S) : [F] \longrightarrow [\phi_1 F], \quad \Phi_1 F = S \otimes_R F \\
  \Phi_1 : K_1(R) & \longrightarrow K_1(S) : \tau(X) \longrightarrow \tau(\phi(X)), \quad X \in \text{GL}_n(R)
\end{align*} \]

which are related by a change of rings exact sequence

\[ \begin{array}{c}
K_1(R) \xrightarrow{\Phi_1} K_1(S) \xrightarrow{j} K_1(\phi_1) \xrightarrow{\partial} K_0(R) \xrightarrow{\Phi_1} K_0(S)
\end{array} \]
with $K_1(\phi_1)$ the relative $K$-group of stable isomorphism classes of pairs $(P,f)$ consisting of a f.g. projective $R$-module $P$ and an $S$-module isomorphism $f : \phi_1 P \cong S^n$, with $(R^n,1) = 0 \in K_1(\phi_1)$ and

$$j : K_1(S) \longrightarrow K_1(\phi_1) : \tau(z) \longmapsto (R^n,z), \quad z \in \text{GL}_n(S)$$

$$\tilde{\phi} : K_1(\phi_1) \longrightarrow K_0(R) ; (P,f) \longmapsto \{ P \} - \{ R^n \}.$$  

We shall now obtain an analogous exact sequence for the transfer maps

$$K_1(\tilde{R}) \longrightarrow K_1(R) \longrightarrow K_1(\phi_1) \longrightarrow K_0(\tilde{R}) \longrightarrow K_0(R),$$

relating the projective class group transfer $\phi_1^j$ of $S^2$ to the Whitehead torsion transfer $\phi_1^j$ of $S^3$.

A base $(S,T)$ for a pseudoprojection $(X,Y) \in P_n(R,\phi)$ is a pair of matrices

$$S = \begin{pmatrix} S_1 \\ S_2 \end{pmatrix} \in M_{2n,m}(R), \quad T = \begin{pmatrix} T_1 \\ T_2 \end{pmatrix} \in M_{m,2n}(R)$$

with $S_1,S_2 \in M_{n,m}(R), T_1,T_2 \in M_{m,n}(R)$ such that

$$ST = X^t \in M_{2n}(R), \quad TS = 1 \in M_{m}(R).$$

The factorization of $R$-module morphisms

$$X^t = \begin{pmatrix} X & Y \\ t-1 & 1-x^t \end{pmatrix} : R^n \oplus R^n \longrightarrow R^n \longrightarrow R^n \oplus R^n$$

shows that a base $(S,T)$ of $(X,Y)$ determines a base (in the usual sense) of the f.g. projective $R$-module $\text{im}(X^t) \subseteq R^n \oplus R^n$ consisting of $m$ elements. Conversely, if $\text{im}(X^t)$ is a f.g. free $R$-module of rank $m$ then a choice of base for $\text{im}(X^t)$ determines a factorization

$$X^t : R^n \oplus R^n \longrightarrow S \longrightarrow R^n \longrightarrow R^n \oplus R^n$$

with $S$ onto and $T$ one-one; it follows from the identity

$$S(TS-1)T = ST(ST-1)$$

$$= X^t(X^t-1) = 0 \in M_{2n}(R)$$

that $TS = 1 \in M_{n}(R)$, and so $(S,T)$ defines a base of $(X,Y)$. There is thus a natural one-one correspondence between the bases $(S,T)$ of the pseudoprojection $(X,Y)$ and the bases of the f.g. projective $R$-module $\text{im}(X^t)$, if any such exist. In dealing with bases of pseudoprojections we shall assume that $(R,\phi)$ satisfies the
following two conditions:
i) f.g. free $R$-modules have a well-defined rank,

\[ \alpha^2 : R \xrightarrow{\sim} \bar{R} ; \bar{r} \mapsto (\bar{t}^t)^t \]
is an inner automorphism of $\bar{R}$, in which case $m = n$ for any pseudoprojection base $(S, T)$: by i)

$[\bar{R}] \in K_0(\bar{R})$ generates an infinite cyclic subgroup of $K_0(\bar{R})$, and by

\[ \alpha_{\bar{R}} : K_0(\bar{R}) \longrightarrow K_0(\bar{R}) ; [\bar{P}] \longmapsto [\bar{P}^t] \]
is an involution of $K_0(\bar{R})$ fixing $[\bar{R}]$, so that if $(S, T) \in M_{2m, m}^n(R) \times M_{m, 2n}^m(R)$ is a base for

the pseudoprojection $(X, Y) \in P_n(R, \phi)$ the f.g. projective $R$-module

$R^n = \text{im}(1-X)$ is such that up to $R$-module isomorphism

\[ \bar{R}^n = \phi_1(\text{im}(X^t)) = \text{im}(\bar{X}) \oplus \bar{P}^t , \quad \bar{R}^n = \text{im}(\bar{X}) \oplus \bar{P}^t , \]
and it is clear from the action of $\alpha_{\bar{R}}$ on the identity

\[ [\bar{P}] - [\bar{P}^t] = [\bar{R}^n] - [\bar{R}^n] \in K_0(\bar{R}) \]

that $m = n$. In particular, the conditions i) and ii) are satisfied

by the group rings with pseudostructure $(R = \mathbb{Z}[\pi], \phi)$ arising in

topology.

A based pseudoprojection $(X, Y, S, T)$ is a pseudoprojection

$(X, Y) \in P_n(R, \phi)$ together with a base $(S, T) \in M_{2n, m}^n(R) \times M_{m, 2n}^m(R)$. Given such an object define the associated based pseudoresolution of the $R$-module $\phi_1(\text{im}(X))$ to be the 1-dimensional based f.g. free $R$-module chain complex

\[
\begin{array}{ccc}
S_2 & \longrightarrow & R^n \\
\downarrow \& \downarrow \& \\
D^1 : R^n & \longrightarrow & R^n \\
\end{array}
\]

which is chain equivalent to the projective pseudoresolution $C^1$ of

$\phi^1(\text{im}(X))$ associated to $(X, Y)$ in §1. Explicitly, a chain equivalence $C^1 \sim \longrightarrow D^1$ is defined by

\[
\begin{array}{ccc}
\begin{bmatrix}
1-X \\
1-t
\end{bmatrix} & \longrightarrow & R^n \\
\downarrow \phi_1(\text{coker}(X^t)) \downarrow \& \downarrow \phi_1(S) \downarrow \\
\begin{bmatrix}
Y \\
-X^t
\end{bmatrix} & \longrightarrow & XS_1 + YS_2 \\
\downarrow \downarrow \downarrow \& \downarrow \downarrow \downarrow \\
D^1 : R^n & \longrightarrow & R^n \\
\end{array}
\]

(This is the composite $C^1 \longrightarrow B^1 \longrightarrow D^1$ of the chain equivalence

\[
\begin{array}{ccc}
\begin{bmatrix}
1-X \\
1-t
\end{bmatrix} & \longrightarrow & R^n \\
\downarrow \phi_1(\text{coker}(X^t)) \downarrow \& \downarrow \phi_1(S) \downarrow \\
\begin{bmatrix}
Y \\
-X^t
\end{bmatrix} & \longrightarrow & [X, Y] \\
\downarrow \downarrow \downarrow \& \downarrow \downarrow \downarrow \\
B^1 : R^n & \longrightarrow & \text{im}(X^t) \\
\end{array}
\]

(defined for any pseudoprojection $(X, Y)$) and the chain isomorphism
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\[ B^1 : \mathbb{R}^n \xrightarrow{[t-1 \cdot X^t]} \text{im}(X^1) \]
\[ \downarrow 1 \quad \downarrow \quad \downarrow \]
\[ D^1 : \mathbb{R}^n \xrightarrow{S_2} \mathbb{R}^n \]
\[ S_1 \]

A morphism of based pseudoprojections over $(R, \Phi)$

\[ f : (X, Y, S, T) \xrightarrow{\sim} (X', Y', S', T') \]

is just a morphism of the associated f.g. projective $\bar{R}$-modules

\[ \bar{f} : \text{im}(\bar{X}) \xrightarrow{\sim} \text{im}(\bar{X}') \]

Replacing the projective pseudo-resolutions $C^1, C'^1$ in the construction of Proposition 1.2 by the chain equivalent based pseudo-resolutions $D^1, D'^1$ there is obtained an $R$-module chain map

\[ f^1 : D^1 \xrightarrow{\sim} D'^1 \]

inducing the $R$-module morphism

\[ (f^1)_* = \phi^1 \bar{f} : H^0(D^1) = \phi^1(\text{im}(\bar{X})) \xrightarrow{\sim} H^0(D'^1) = \phi^1(\text{im}(\bar{X}')) \]

uniquely up to chain homotopy. More precisely, $f^1$ is defined by

\[ f^1 : \mathbb{R}^n \xrightarrow{S_2} \mathbb{R}^n \]
\[ \downarrow XFX' \quad \downarrow TF' \quad \downarrow \]
\[ D^1 : \mathbb{R}^n \xrightarrow{S_2} \mathbb{R}^n \]

with $F \in \text{M}_{n \times n}(R)$ the matrix of any $R$-module morphism $F \in \text{Hom}_R(R^n, R^{n'})$ lifting the composite $R$-module morphism

\[ \bar{F} : \mathbb{R}^n \xrightarrow{\text{projection}} \text{im}(\bar{X}) \xrightarrow{\bar{f}} \text{im}(\bar{X}') \xrightarrow{\text{injection}} R^n \]

and

\[ F^1 = \begin{pmatrix} XFX' & XFY' - YFP_X^t X^t \cr 0 & X^t F_X^t X^t \end{pmatrix} \in \text{M}_{2n \times 2n}(R) \]

as before.

An isomorphism of based pseudoprojections is a morphism

\[ f : (X, Y, S, T) \xrightarrow{\sim} (X', Y', S', T') \]

which is defined by an $\bar{R}$-module isomorphism $\bar{f} \in \text{Hom}_R(\text{im}(\bar{X}), \text{im}(\bar{X}'))$,

in which case $f^1 : D^1 \xrightarrow{\sim} D'^1$ is a chain equivalence of based
R-module chain complexes and the torsion of f is defined by
\[ \tau(f) = \tau(f^1 : D^1 \to D^0) \in K_1(R). \]

In general, the torsion is an invariant of f but not of \( f \).
However, if f is an automorphism (i.e. \((X,Y,S,T) = (X',Y',S',T')\))
then the torsion \( \tau(f : \text{im}(X) \to \text{im}(\overline{X})) \in K_1(R) \)
is defined, and Proposition 3.1 shows that
\[ \tau(f) = \tau(f^1) = \phi_1^1(\tau(\overline{f})) \in K_1(R). \]
An isomorphism \( f : (X,Y,S,T) \to (X',Y',S',T') \) is simple if
\[ \tau(f) = 0 \in K_1(R). \]

Define the relative transfer group \( K_1(\phi^1) \) to be the abelian
group with one generator for each simple isomorphism class of
based pseudoprojections \((X,Y,S,T) \) over \((R,\phi)\), with relations
\[(X,Y,S,T) + (X',Y',S',T') = (X \phi X', Y \phi Y', S \phi S', T \phi T') \in K_1(\phi^1). \]

**Proposition 5.1** The relative transfer group \( K_1(\phi^1) \) fits into an
exact sequence
\[ K_1(R) \xrightarrow{\phi_1^1} K_1(R) \xrightarrow{j} K_1(\phi^1) \xrightarrow{\delta} K_0(\overline{R}) \xrightarrow{\delta_0} K_0(R) \]
with
\[ j : K_1(R) \xrightarrow{} K_1(\phi^1); \]
\[ \tau(Z) \xrightarrow{} (O, O, \begin{pmatrix} O \\ O \end{pmatrix}, Z^{-1}(t=1) Z^{-1}) \ (Z \in GL_n(R)) \]
\[ \delta : K_1(\phi^1) \xrightarrow{} K_0(\overline{R}) ; (X,Y,S,T) \xrightarrow{} [\text{im}(X)] \]

**Proof:** If \( \overline{P}, \overline{Q} \) are f.g. projective \( \overline{R} \)-modules such that
\[ [\overline{P}] - [\overline{Q}] \in \ker(\phi_1^1 : K_0(\overline{R}) \to K_0(R)) \]
let \( -\overline{Q} \) be a f.g. projective inverse for \( \overline{Q} \), so that \( \overline{Q} \phi \overline{Q} = \overline{R}^m \) is a
f.g. free \( \overline{R} \)-module, and let \((X,Y) \in P_n(R,\phi) \) be a pseudoprojection
such that \( \overline{P} \phi \overline{Q} = \text{im}(X) \). Now
\[ [\text{im}(X^1)] - [\overline{R}^n] = \phi_1^1([\text{im}(X)]) \]
\[ = \phi_1^1([P] - [Q] + [(\overline{R}^m)] = 0 \in K_0(\overline{R}), \]
so that \( \text{im}(X^1) \) is a stably f.g. free \( \overline{R} \)-module. Stabilizing \( \overline{P}, \overline{Q} \)
if necessary it may be assumed that \( \text{im}(X^1) \) is an unstably f.g.
free \( \overline{R} \)-module. Choosing a base \((S,T) \) for \((X,Y) \) there is obtained
an element \((X,Y,S,T) - (1,0,\begin{pmatrix} 1 \\ t \end{pmatrix},1 O) \in K_1(\phi^1) \) \( (1 \in GL_m(R)) \)
such that
\[ [\bar{P}] - [\bar{Q}] = [\Phi - \bar{Q}] - [\Phi^R] = [\text{im}(\bar{X})] - [\Phi^R] = 3((X,Y,S,T) - (1,0,\begin{pmatrix} 1 \\ \ell - 1 \end{pmatrix},(1,0))) \in \text{im}(3:K_1(\Phi^1) \to K_0(\bar{R})) \]

verifying exactness at \( K_0(\bar{R}) \).

If \((X,Y,S,T),(X',Y',S',T')\) are based pseudoprojections such that
\[ (X',Y',S',T') - (X,Y,S,T) \in \ker(3:K_1(\Phi^1) \to K_0(\bar{R})) \]
there exists a (stable) isomorphism
\[ f : (X,Y,S,T) \isom (X',Y',S',T') \]
The torsion \( \tau(f) \in K_1(\bar{R}) \) is such that
\[ (X',Y',S',T') - (X,Y,S,T) = j(\tau(f)) \in \text{im}(j:K_1(\bar{R}) \to K_1(\Phi^1)) \]
verifying exactness at \( K_1(\Phi^1) \).

If \( Z \in GL_n(\bar{R}) \) is such that \( \tau(Z) \in \ker(j:K_1(\bar{R}) \to K_1(\Phi^1)) \)
there exists a based pseudoprojection \((X,Y,S,T)\) with a simple isomorphism
\[ f : (X,Y,S,T) \isom (X,Y,S,T) \]
The automorphism of based pseudoprojections
\[ q : (X,Y,S,T) \isom (X,Y,S,T) \]
defined by the automorphism \( f \in \text{Hom}_R(\text{im}(\bar{X}),\text{im}(\bar{X})) \) is such that
\[ \tau(Z) = \tau(q^1) = \Phi^1_1(\tau(f)) \in \text{im}(\Phi^1_1:K_1(\bar{R}) \to K_1(\bar{R})) \]
verifying exactness at \( K_1(\bar{R}) \).

For the group ring with pseudostructure \((\bar{R} = \mathbb{Z}[\pi],\Phi)\)
associated to an \( S^1 \)-bundle \( S^1 \to E \to B \) with
\[ p_* = \Phi : \pi_1(E) = \pi \to \pi_1(B) = \rho , \quad \bar{R} = \mathbb{Z}[\rho] \] there is also defined a reduced version of the exact sequence of Proposition 5.1
\[ \text{Wh}(\rho) \xrightarrow{\Phi^1_1} \text{Wh}(\pi) \xrightarrow{\bar{g}} \text{Wh}(\Phi^1) \xrightarrow{\bar{g}} K_0(\mathbb{Z}[\rho]) \xrightarrow{\Phi^1_0} K_0(\mathbb{Z}[\pi]) \]
in the Whitehead and reduced projective class groups, with \( \text{Wh}(\Phi^1) \)
defined by
\[ \text{Wh}(\mathbf{1}^1) = K_1(\mathbf{1}^1)/j(\mathbf{1}n) + (1, D, \left( \begin{array}{c} 1 \\ t \end{array} \right), \left( 1, 0 \right)) . \]

See Ranicki [7, §7] for the geometric interpretation of this sequence.

Appendix: Connection with \( L \)-theory

We note the following connection between the algebraic \( K \)-theory \( S^1 \)-bundle transfer maps

\[ \tilde{\phi}_m^1 : \tilde{\mathbb{K}}_O(\mathbb{Z}[\rho]) \longrightarrow \tilde{\mathbb{K}}_O(\mathbb{Z}[\pi]) \quad \text{and} \quad \tilde{\phi}_m^1 : \text{Wh}(\pi) \longrightarrow \text{Wh}(\pi) \]

and the algebraic \( L \)-theory \( S^1 \)-bundle transfer maps of Munkholm and Pedersen [3, 4] and Ranicki [6, 8]

\[ \phi_L^1 : L_n^X(\rho) \longrightarrow L_{n+1}^X(\pi) \quad (m = 0 \text{ or } 1) \]

which are defined for duality-invariant subgroups \( X \subseteq \mathbb{K}_O(\mathbb{Z}[\rho]) \) \((m = 0)\) and \( X \subseteq \text{Wh}(\pi) \) \((m = 1)\). The geometric interpretation of \( \phi_L^1 \) for \( m = 1 \) in terms of finite surgery obstruction theory extends to \( m = 0 \) using the projective surgery obstruction theory of Pedersen and Ranicki [5]). The duality involutions on the algebraic \( K \)-groups are defined by

\[ \ast : \mathbb{K}_O(\mathbb{Z}[\pi]) \longrightarrow \mathbb{K}_O(\mathbb{Z}[\pi]) ; [\text{im}(X)] \longmapsto [\text{im}(X^\ast)] \]

\[ \ast : \text{Wh}(\pi) \longrightarrow \text{Wh}(\pi) ; \tau(X) \longmapsto \tau(X^\ast) \]

\[ \ast : \text{Wh}(\mathbf{1}^1) \longrightarrow \text{Wh}(\mathbf{1}^1) ; \]

\[ (X, Y, \left( \begin{array}{c} S_1 \\ T_1 \end{array} \right), \left( \begin{array}{c} T_2 \\ S_2 \end{array} \right)) \longmapsto -(1-X^\ast, -tY^\ast, \left( \begin{array}{c} -tT_1^\ast \\ T_2^\ast \end{array} \right), (-tS_2^\ast S_1^\ast)) , \]

using the group ring involution

\[ \ast : \mathbb{Z}[\pi] \longrightarrow \mathbb{Z}[\pi] ; \sum g_n g \longmapsto \sum g_n g^\ast \quad (w = \text{orientation}) \]

and the corresponding matrix ring involutions

\[ \ast : M_n(\mathbb{Z}[\pi]) \longrightarrow M_n(\mathbb{Z}[\pi]) ; X = (x_{ij}) \longmapsto X^\ast = (x_{ij}^\ast) . \]

The maps in the exact sequence of [55]

\[ \text{Wh}(\rho) \longrightarrow \text{Wh}(\pi) \longrightarrow \text{Wh}(\mathbf{1}^1) \longrightarrow \tilde{\mathbb{K}}_O(\mathbb{Z}[\rho]) \longrightarrow \tilde{\mathbb{K}}_O(\mathbb{Z}[\pi]) \]

are such that

\[ \tilde{\phi}_m^1 \ast = -\ast \tilde{\phi}_m^1 \quad (m = 0, 1) , \quad \tilde{j}^\ast = \ast \tilde{j} , \quad \tilde{\delta}^\ast = \ast \tilde{\delta} . \]
The short exact sequence of $\mathbb{Z}[\mathbb{Z}_2]$-modules

$$\cdots \to \text{ker}(\delta^1) \to \text{Wh}(\phi^1) \to \text{ker}(\delta^1) \to \text{coker}(\delta^1) \to \text{Wh}(\phi^1) \to \cdots$$

gives rise to connecting maps in the Tate $\mathbb{Z}_2$-cohomology groups

$$\delta^1 = \delta : \hat{H}^n(\mathbb{Z}_2; \text{ker}(\delta^1)) \to \hat{H}^{n+1}(\mathbb{Z}_2; \text{coker}(\delta^1))$$

which appear in a transfer map of generalized Rothenberg exact sequences

$$\cdots \to L_n^h(\rho) \to L_n^l(\rho) \to \hat{H}^n(\mathbb{Z}_2; \text{ker}(\delta^1)) \to L_n^{h-1}(\rho) \to \cdots$$

In particular, for the trivial $S^1$-bundle $E = B \times S^1$, $\pi = \rho \times \mathbb{Z}$, $\gamma_m = 0$ ($m = 0, 1$) and the exact sequence

$$\cdots \to \text{Wh}(\rho \times \mathbb{Z}) \to \text{Wh}(\phi^1) \to \text{Wh}(\rho \times \mathbb{Z}) \to \cdots$$

is split by the map

$$\bar{\phi} : \bar{\text{Wh}}_O(\mathbb{Z}[\rho]) \to \text{Wh}(\phi^1) ;$$

which is related to the duality involutions $\ast$ by

$$\bar{\phi}^\ast \ast \text{Wh}(\rho \times \mathbb{Z}) \to \text{Wh}(\rho \times \mathbb{Z})$$

with

$$h^1 = \bar{\text{Wh}}_O(\mathbb{Z}[\rho]) \to \text{Wh}(\rho \times \mathbb{Z}) ; [\text{im}(X)] \to \tau(\text{im}(X)) \mapsto (X, O, \frac{1}{X-1}, \frac{1}{(X-1)^2}) ,$$

The transfer map in this case consists of split injections

$$\cdots \to L_n^h(\rho) \to L_n^l(\rho) \to \hat{H}^n(\mathbb{Z}_2; \text{Wh}(\rho \times \mathbb{Z})) \to L_n^{h-1}(\rho) \to \cdots$$

although not the standard such injections — see Ranicki [7] for a further discussion.
BIBLIOGRAPHY

[0] H. Bass  
Algebraic K-theory,  
Benjamin (1968)

Transfer on algebraic K-theory and Whitehead  
torsion for PL fibrations,  

[2]  
and E.K. Pedersen  
On the Wall finiteness obstruction for the total  
space of certain fibrations,  
Trans. A.M.S. 261, 529 - 545 (1980)

[3]  
Whitehead transfers for $S^1$-bundles, an algebraic  
description,  

[4]  
Transfers in algebraic K- and L-theory induced  
by $S^1$-bundles,  
these proceedings

Projective surgery theory,  
Topology 19, 234 - 254 (1980)

Exact sequences in the algebraic theory of surgery,  
Mathematical Notes 26, Princeton (1981)

[7]  
Algebraic and geometric splittings of the K- and  
L-groups of polynomial extensions,  
preprint

[8]  
Splitting theorems in the algebraic theory of surgery,  
in preparation

K-theory of finite groups and orders,  
Springer Lecture Notes 149 (1970)

[10] C.T.C. Wall  
Finiteness conditions for CW complexes,  
Ann. of Maths. 81, 56 - 69 (1965)

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