EMBEDDED HANDLE THEORY, CONCORDANCE AND ISOTOPY

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The purpose of this paper is to announce a new method for dealing with concordances of one manifold in another and some results obtained by using it. The basic idea is to regard the concordance as consisting of a series of "steps," each of which has the form of an attached handle (see Section 2 for precise definitions). Then one proceeds to operate on these embedded handles exactly as in Smale's proof of his h-cobordism theorem. The operations are realized by an ambient isotopy of the whole concordance. If one can succeed in realizing all the moves then one can "cancel off" all the handles and so replace the concordance by an isotopy. The method works equally well in the PL and Diff categories; however, for simplicity we will work from now on in the PL category only.

Roughly speaking, there are no obstructions to realizing all handle moves in codims ≥ 2 except for "canceling complementary handles" and no obstruction to canceling handles in codims ≥ 3. Thus we immediately recover Hudson's results [3]. In codim 2 we can measure the obstruction to canceling a pair of complementary handles as an element of the integral group ring of the fundamental group of the complement of the concordance at that level. However since a Whitney-type process is involved, we need to assume that the ambient dimension is at least five. We thus obtain a new theorem of concordance implies isotopy for codimension 2 embeddings with simply connected complements. Applying our results to codim 0 concordances, using a handle induction argument, we recover (a version of) Cerf's theorem [2].

Precise statements of the results on concordance and isotopy are given in Section 1. Definitions of embedded handles and statements of results on realizing handle moves are given in Section 2 together with sketches of proofs. In Section 3 we sketch the derivation of the results of Section 1 from those of Section 2.

Full details will be given elsewhere and I also hope to be able to give a better description of the obstruction in codim 2.

1. Concordance and Isotopy

Let Q^q be a PL manifold (of dim q) with boundary ∂Q (neither assumed to be compact) and X a compact polyhedron. A *concordance* of X in Q is an embedding $F: X \times I \to Q \times I$ such that $F^{-1}(Q \times \{i\}) = X \times \{i\}$ for $i = 0, 1$. An *isotopy* is a level-preserving concordance; i.e., $F^{-1}(Q \times \{t\}) = X \times \{t\}$ for all $t \in I$. An *ambient isotopy* is the restriction of an isotopy of Q (cf. [4] and [12]).

A concordance F *respects the boundary* if there is a subpolyhedron $X_0 \subset X$ such that $F^{-1}(\partial Q \times I) = X_0 \times I$ (i.e., F restricts to a concordance of X_0 in ∂Q.)

F is *locally unknotted* (in the sense of Akin [1]) if the ambient intrinsic dimension of $F(X \times I)$ is constant on "flowlines." More precisely, extend F to $F_+: X \times R \to Q \times R$ by defining $F_+|X \times (-\infty, 0]$ and $F_+|X \times [1, \infty)$ to be products and then we require that the ambient intrinsic dimension of $F_+(X \times R)$ at (x, t) is independent of t.

Notice that F might be locally knotted in the sense of Lickorish [6] and that in codims ≥ 3 all embeddings are locally unknotted by Lickorish's main theorem [6].

We are interested in whether F can be moved to an isotopy, that is, whether there is an ambient isotopy of $Q \times I$ carrying F to $F_0 \times \mathrm{id}$, where $F_i: X \to Q$, $i = 0, 1$, is the embedding determined by $F|X \times \{i\}$. If it can be so moved, then F_0 and F_1 are isotopic embeddings and we have "concordance implies isotopy." It is clear that we will have to assume F is locally unknotted and respects the boundary. This we always do from now on.

Theorem 1. *Suppose $F: X \times I \to Q \times I$ is a concordance which is trivial on the boundary* (i.e., $F|X_0 \times I = (F_0|X_0) \times \mathrm{id}$) *and that X collapses to $X_1 \supset X_0$ with $X_1 - X_0$ of dimension $\leq q - 3$. Then there is an ambient isotopy of $Q \times I$ carrying F to $F_0 \times$ id and keeping $Q \times \{0\} \cup \partial Q \times I$ fixed.*

Theorem 2. *Same hypothesis and conclusions as Theorem* 1, *except that*

(a) *$X_1 - X_0$ has dimension $\leq q - 2$;*

(b) *$q \geq 5$;*

(c) $\pi_1(Q - F_0(X)) = \pi_1(Q - F_1(X)) = \pi_1(Q \times I - F(X \times I)) = 0$.

Theorem 1 is a slightly improved version of Hudson's main result [3], while Theorem 2 is basically new. There are well-known examples to show that condition (c) is necessary (slice knots, etc.) but I have no idea whether condition (b) is necessary.

Corollary (Improved Cerf). *Let* $F : M \times I \to M \times I$ *be a concordance of the closed manifold* M *in itself and assume that* $\dim M \geq 5$, $\pi_1(M) = 0$. *Then* F *is ambient isotopic to* $F_0 \times \mathrm{id}$.

Proof. Apply Theorem 2 with $X = M$-{disk}. This moves F to F' with $F'|X$ trivial. Finally straighten F' on the disk by the Alexander trick.

A similar dodge gives a theorem for bounded manifolds. There are theorems for polyhedra in polyhedra deduced by induction over intrinsic strata using Stone's results [10] to straighten neighborhoods.

Notice that our proof of the Cerf theorem uses a PL device (the Alexander trick); in the Diff case our methods only prove the result modulo the theorem for a disk.

2. Embedded Handles

In this section X will be a manifold M^n and $X_0 = \partial M$ (in fact in Section 3 all we will need is the case M is a disk). A point $(x, t) \in Q \times I$ is a *regular point* of $F : M \times I \to Q \times I$ if either (1) $(x, t) \notin \mathrm{im}\, F$ or (2) there is a neighborhood J of t in I, a manifold M' and a level-preserving embedding $G : M' \times J \to Q \times J$ onto a neighborhood of (x, t) in $F(M \times I)$. In other words, near (x, t), $F(M \times I)$ is (setwise) the track of an isotopy.

A value $t \in I$ is a *regular value* of F if (x, t) is a regular point for all $x \in Q$. One easily proves

Lemma 1. *If* t *is a regular value of* F, *then there is a manifold* $M(t)$, *a neighborhood* J *of* t *in* I *and an isotopy* $G : M(t) \times J \to Q \times J$ *such that* $\mathrm{im}\, G = \mathrm{im}\, F \cap Q \times J$. That is, *in a neighborhood of* t, F *is* (*setwise*) *the track of an isotopy.*

We say t is a critical value if it is not a regular value. We next define a *standard critical value* of *index* p, $0 \leq p \leq n + 1$ (the analogue of a p-saddle): Suppose there are manifolds $M_-(t)$ and $M_+(t)$, neighborhoods J_-, J_+ of t in $[0, t]$, $[t, 1]$, respectively (it is implicit that $t \in \mathrm{int}\, I$) and isotopies $G_\varepsilon : M_\varepsilon \times J_\varepsilon \to Q \times J_\varepsilon$, $\varepsilon = +, -$. Suppose further that $\mathrm{im}(F) \cap Q \times (J_+ \cup J_-) = \mathrm{im}\, G_+ \cup \mathrm{im}\, G_- \cup h^p$, where $h^p \subset Q \times \{t\}$ and that as triples we have a PL isomorphism,

$$(h^p, h^p \cap \mathrm{im}\, G_-, M^p \cap \mathrm{im}\, G_+) \cong (I^p \times I^r, \partial I^p \times I^r, I^p \times \partial I^r),$$

where $I^p = [-1, +1]^p \subset R^p$ is the standard p-disk and $p + r = n + 1$. Suppose finally that in the level t, $\mathrm{im}\, G_+ \cap \mathrm{im}\, G \;\; = \mathrm{cl}(\mathrm{im}\, G_\varepsilon - h)$ for $\varepsilon = +, -$ (this last condition can actually be deduced from the fact that $\mathrm{im}\, F$ is a manifold).

We think of h^p as a p-handle attached to M_- in $Q \times \{t\}$ and defining a surgery to M_+. See Figure 1 for a picture of a standard critical level.

A regular or standard critical level is *locally unknotted* if the isotopies are locally unknotted in the sense of Hudson–Zeeman [4] and the embedding of h^p in $Q \times \{t\}$ is locally flat. From now on all such levels are assumed to be locally unknotted.

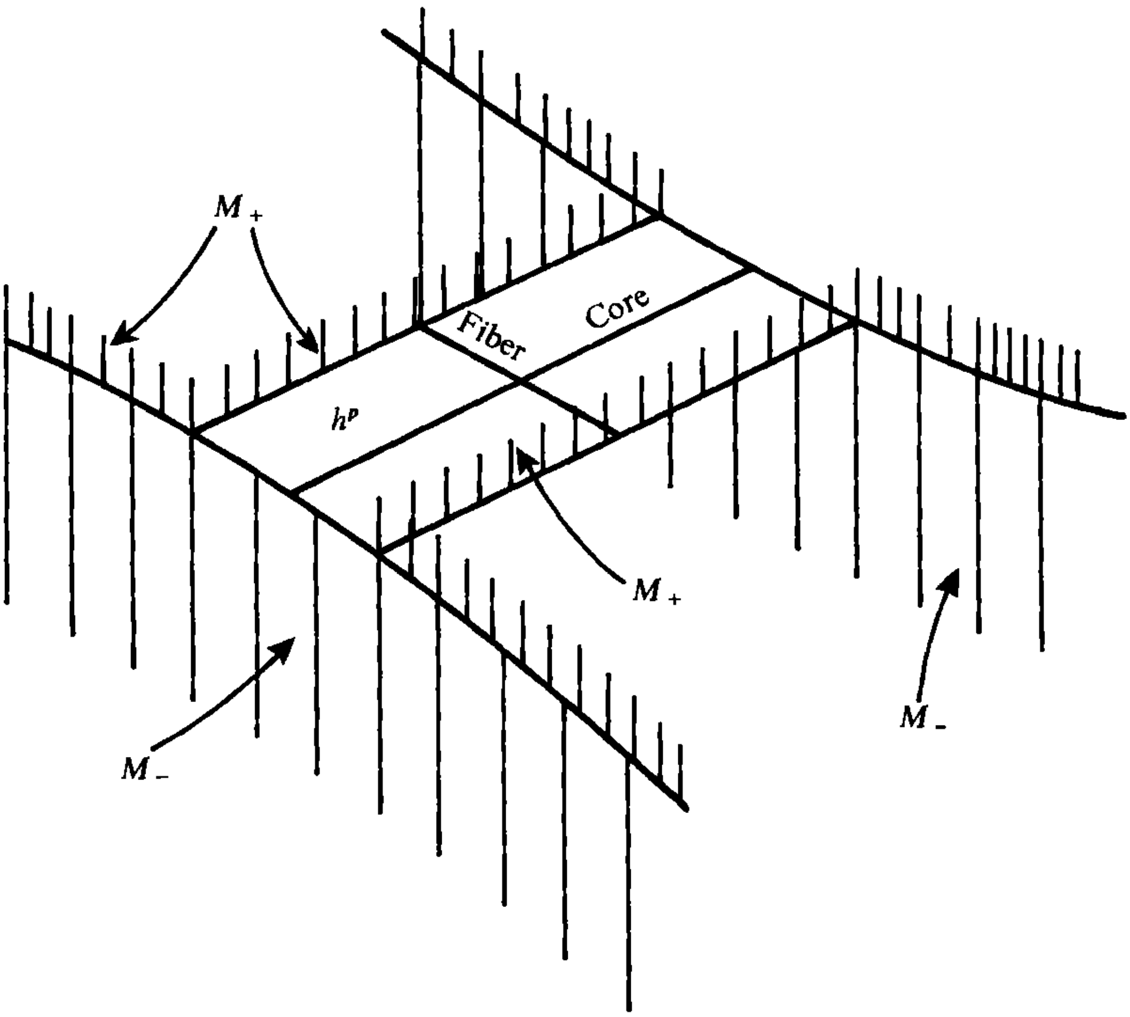

Figure 1. Standard critical level.

Lemma 2. *Suppose $F : M \times I \to Q \times I$ is a locally flat concordance which is trivial on the boundary. Then there is an ε-isotopy of $Q \times I$* mod $Q \times \{0\} \cup \partial Q \times I$ *carrying F to a concordance with only a finite number of critical levels each of standard type.*

Lemma 2 follows from the methods of Kosinski [5]; see [7] for a direct proof. We call the result of Lemma 2 a "handle decomposition" of the concordance F. Our aim is to echo the proof of the h-cobordism theorem through handle decompositions of F. We first observe that by an ambient isotopy of F we can straighten all the isotopies in the handle decomposition so that F has the form of vertical "walls" with a finite number of "bridges"

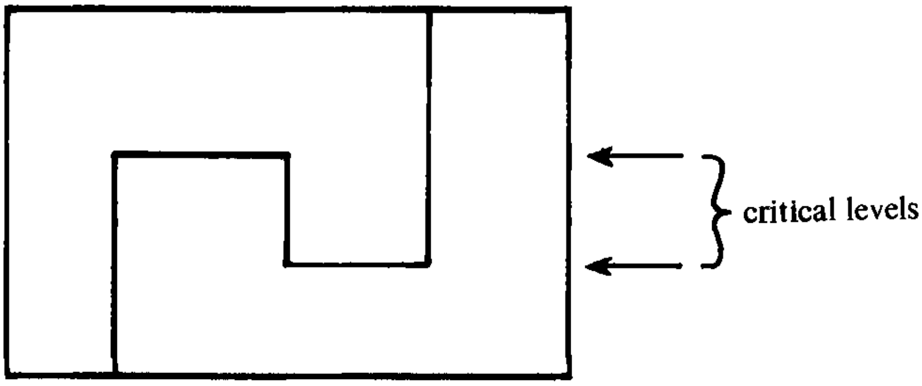

Figure 2. Handle decomposition of a concordance.

(see Figure 2). Observe also that the handle decomposition of F gives a relative decomposition of $M \times I \bmod M \times \{0\}$ by projecting handles vertically downwards (see [8, 9, and 11] for treatments of PL handle theory). Finally, notice that the handles have a preferred ordering given by the values of their levels.

Lemma 3 (Reordering Handles). *Suppose $q - n \geq 2$ and suppose given a handle decomposition of F with handles h^p, $h^{p'}$ consecutive with $p' \leq p$. Then we can find another decomposition with the same number of handles of each index, the same ordering except that $h^{p'}$ precedes h^p.*

Sketch of Proof. The idea is to isotope the embedding of $h^{p'}$ in its level keeping $M_-(h^{p'})$ setwise fixed. This can be realized by an ambient isotopy of $Q \times I$ which affects im F setwise only above the level of $h^{p'}$. First, by general position we can assume that the core of $h^{p'}$ misses the fiber of h^p on projection into Q. Then, by trimming $h^{p'}$ down to its core, that $h^{p'}$ misses the fiber of h^p, finally slide $h^{p'}$ away from the fiber of h^p and obtain h^p and $h^{p'}$ disjoint on projection on Q. Now there is no obstruction to sliding $h^{p'}$ down past h^p, i.e., to reordering.

Lemma 4 (Adding Handles). *The isotopy which "swings" one p-handle over another can be realized.* (There are no codimension requirements.)

Lemma 4 is easy—one simply applies the covering isotopy theorem (Hudson–Zeeman [4]).

Lemma 5 (Canceling Complementary Handles). *Suppose h^p and h^{p+1} are complementary handles in a decomposition of F. Then there is a decomposition of F with the same number of handles of each index in the same order, except that h^p and h^{p+1} are missing. Provided either*

(a) *$q - n \geq 3$, or*

(b) *$q - n = 2$, $q \geq 5$, $p \geq 1$, and an obstruction $\theta \in Z(\pi)$ vanishes where $\pi = \pi_1(Q - M_+(h^p)) = \pi_1(Q - M_-(h^{p+1}))$.*

Sketch of Proof. Case (a). The idea is to make the projections of h^{p+1} and h^p maximally disjoint. Then slide the two handles into the same level. This level now consists of a trivial step (a disk glued onto M_- by a face) and the critical level can be removed by isotoping the face across to the complementary one (Figure 3).

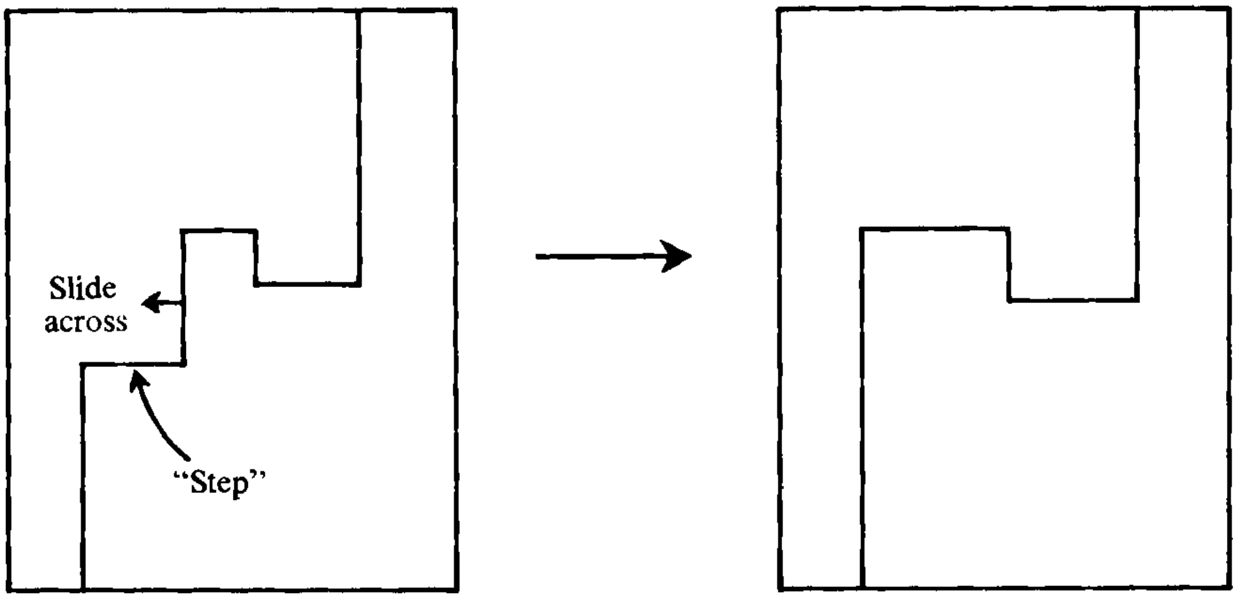

Figure 3. Removing a step.

Since we have one extra dimension to spare, by general position as in Lemma 2, we may assume that the core of h^{p+1} projects disjoint as far as possible from the fiber of h^p (i.e., they intersect in the one point p on their boundaries which occurs in the definition of complementary handles; cf. [8]). Then by a little care we get them disjoint near p. Then the argument is as in Lemma 2.

Case (b). The same ideas as case (a) apply. The general position however only yields the fiber of h^p and the core of h^{p+1} intersecting in a finite number of points. Use a Whitney-type process and the extra hypotheses to remove them and then the argument of case (a) applies.

3. Sketches of Proofs of Theorems 1 and 2

The proofs of Theorems 1 and 2 are similar and we proceed simultaneously until further notice.

Let N be a regular neighborhood of $F(X \times I)$ in $Q \times I$ which meets the subsets $Q \times \{i\}$, $i = 0, 1$, and $\partial Q \times I$ regularly. By Akin's product neighborhood theorem [1], the product structure on $X \times I$ extends to N. This remark enables us to assume that $X = q$-manifold M and $X_0 = M_0$ is a $(q-1)$-dimensional submanifold of ∂M. Moreover, the hypotheses imply that M has a handle decomposition mod M_0 with handles of indices $\leq q-3$ only in the case of Theorem 1, or $q-2$ in Theorem 2. That is, $M \cong M_0 \times$

$I \cup h_1 \cup \cdots \cup h_t$, say. We now straighten F by induction. First on $M_0 \times I$ by a similar argument to Lemma 7 below. Then straighten F on each h_i by Lemmas 6 and 7.

Lemma 6. *Theorems* 1 *and* 2 *are true for* $X = I^p$, $X_0 = \partial I^p$.

Lemma 7. *Suppose* $X = I^p \times I^r$, $p + r = q$, $X_0 = \partial I^p \times I^r$ *and* $F_0|X_0 \cup I^p \times \{0\}$ *is trivial. Then* F *can be straightened keeping* $Q \times \{0\} \cup \partial Q \times I \cup F(I^p \times \{0\})$ *fixed.*

Proof of Lemma 6. We apply the embedded handle theory of Section 2 to straighten F setwise, finally use the Alexander trick to straighten F pointwise. Following the proof of the h-cobordism theorem gives the result if $p \geq 5$. For $p < 5$ we push all the handles into two adjacent indices (use Wall's new result† [11] for $p = 3$) and then push them all into the same level by a similar argument to the proof of Lemma 5. Now F has only one critical level which can be removed by a similar argument to Lemma 5 (Figure 3).

In the case of $q - p = 2$ we have omitted to show how to cancel O-handles [we assumed $p \geq 1$ in case (b) of Lemma 5]. For this one uses the obvious presentation of $\pi_1(Q \times I - F(I^p \times I))$ given by the handle decomposition. Since this group is trivial we can trivialize this presentation by Tietze moves. One then shows how to realize the Tietze moves by embedded handle moves.

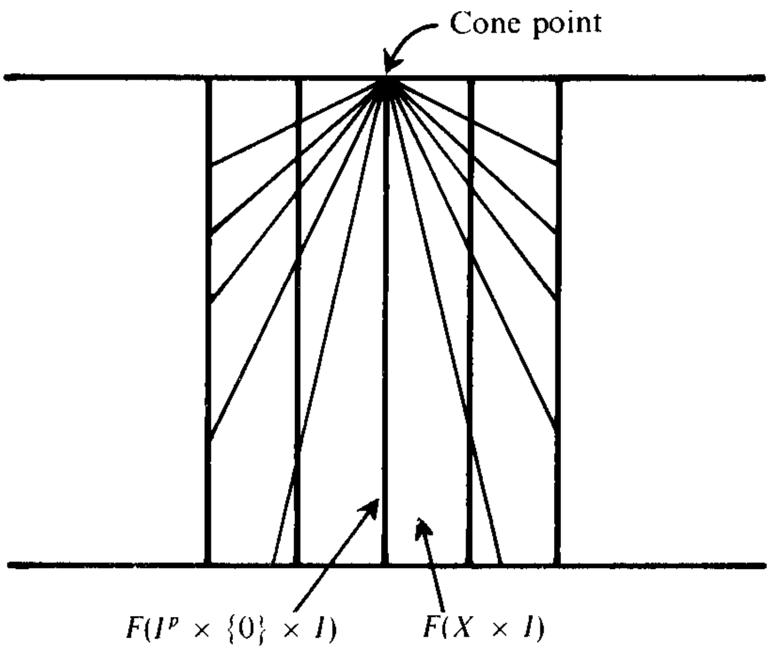

Figure 4

† There appears to be a gap in Wall's proof. However, we do not need the full strength of his asserted result since we know the h-cobordism is trivial. we can therefore use a 1-parameter PL approximation theorem, proved in a similar way to [5]. to know that the handle decomposition can be trivialized by handle moves.

Proof of Lemma 7.† We can assume that F preserves X setwise by the relative regular neighborhood theorem $G : Q \times I \to Q \times I$ to be $F \circ (F_0^{-1} \times \text{id})$ on $X \times I$ and the identity outside a collar neighborhood of $F(X \times I)$. This is easy since F is concordant to the identity. Now G is the identity outside a disk and can be isotoped to the identity by the Alexander trick with the cone point chosen in $F_1(I^p \times \{0\})$. This carries F to the trivial concordance (see Figure 4).

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† This proof appeared in Haefliger's unpublished *Lissage des immersions, II*.