A NEW PROOF THAT $\Omega_3$ IS ZERO

COLIN ROURKE

It is a fact that any closed orientable 3-manifold can be changed into $S^3$ by a finite number of elementary surgeries on embedded circles (which implies that $\Omega_3$, the 3-dimensional oriented cobordism group, is zero). Existing proofs of this fact either use a significant amount of algebraic topology (Thom [2]) or a lengthy calculation involving curves on a surface (Lickorish [1]). In this note, I shall give a short elementary proof which avoids both algebraic topology and calculation.

Suppose that $S$ is an orientable surface of genus $n$, then $x = (x_1, x_2, \ldots, x_n)$ is said to be a complete system of curves on $S$ provided that each $x_i$ is a simple closed curve, the curves $\{x_i\}$ are pairwise disjoint and the union $\bigcup_i x_i$ does not separate $S$.

A Heegaard diagram $S(x, y)$ is an orientable surface $S$ with two complete systems $x, y$. The diagram determines a closed orientable 3-manifold $M(x, y)$ obtained by attaching thickened 2-discs to $S \times I$: along the $x_i$ on $S \times \{0\}$ and along the $y_i$ on $S \times \{1\}$, and then filling in the resulting $S^2$-boundaries with 3-balls. The resulting 3-manifold $M$ has a specific handle presentation with one 0-handle, one 3-handle, $n$ 1-handles and $n$ 2-handles; the curves $x, y$ are drawn on a level surface between the 1-handles and the 2-handles—the $x$ being the $b$-spheres for the 1-handles and the $y$ being the $a$-spheres for the 2-handles. Any handle presentation with one 0-handle and one 3-handle can be regarded as a Heegaard diagram in this way and it therefore follows from elementary results in handle theory that any orientable 3-manifold is given by some Heegaard diagram. Notice that if one of the $x$ curves meets one of the $y$ curves transversally in a single point then the corresponding handles are complementary and can be cancelled; therefore $M$ has a Heegaard diagram of lower genus. (In fact the reduced diagram can be obtained explicitly by cutting out a neighbourhood in $S$ of the two transverse curves, filling in the resulting circle boundary with a disc and completing any other curves, cut in the process, across the disc.)

I need the following observation.

**Lemma 1.** Suppose that $S(x, y)$ is a Heegaard diagram and that $z$ is a third complete system of curves on $S$. Let $\chi(M, z)$ denote the result of performing surgery on $M = M(x, y)$ using the curves $z$ (with framings given by parallel curves in the surface $S$). Then $\chi(M, z)$ is homeomorphic to the connected sum $M(x, z) \# M(y, z)$.

**Proof.** Assume that the surgeries are performed at level $\{\frac{1}{2}\}$ in $S \times I$. Surgery on $z_t \times \{\frac{1}{2}\}$ is performed as follows. First, remove a neighbourhood of $z_t \times \{\frac{1}{2}\}$: this has the same effect as cutting $S \times I$ open at $S \times \{\frac{1}{2}\}$ near to $z_t$, and results in a new boundary torus $(\alpha_t \cup \beta_t) \times S^1$ when $\alpha_t \times S^1$ and $\beta_t \times S^1$ are annuli with common boundary in the two copies of $S \times \{\frac{1}{2}\}$, see Figure 1.

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Secondly, glue back a solid torus, namely \((\alpha_i \cup \beta_i) \times D^2\), where \(\partial D^2\) is identified with \(S^1\).

Now let \(M_1, M_2\) be the results of cutting \(M\) completely along \(S \times \{\frac{1}{2}\}\) (where \(M_1\) contains the lower half \(S \times [0, \frac{1}{2}]\)). Let

\[
M_1^+ = M_1 \cup \alpha_i \times D^2, \quad M_2^+ = M_2 \cup \beta_i \times D^2.
\]

Then \(M_1^+, M_2^+\) are homeomorphic to \(M(x, z), M(y, z)\) respectively, with a 3-ball removed from each, and \(\chi(M, z)\) is the union of \(M_1^+, M_2^+\) along their common 2-sphere boundary.

I also need the following easy lemma.

**Lemma 2.** Suppose that \(x, y\) are two non-separating curves on a surface \(S\) which meet transversally in a finite number of points. Let \(|x \cap y|\) denote the number of points of intersection.

(a) If \(|x \cap y| = 0\) (that is, \(x \cap y = \emptyset\)) then there is a third non-separating curve \(z\) which meets each of \(x\) and \(y\) transversally in a single point.

(b) If \(|x \cap y| > 1\) there is a third non-separating curve \(z\) such that \(|x \cap z| < |x \cap y|\) and \(|y \cap z| < |x \cap y|\).

**Proof.** (a) Cut \(S\) along \(x\) and glue in discs \(D_1, D'_1\) to get a new surface \(S'\).

**Subcase (a), in which \(y\) separates \(S'\).** In this case \(D_1, D'_1\) must lie on opposite sides of \(y\) (or else \(y\) would separate \(S\)). Join corresponding points of \(D_1\) and \(D'_1\) by a simple path \(\alpha\) crossing \(y\) once. Then \(\alpha\) gives the required curve in \(S\) (Figure 2).

**Proof.** (b) Cut \(S\) along \(y\) and glue in discs \(D_2, D'_2\) to get a new surface \(S'\).

**Subcase (b), in which \(x\) separates \(S'\).** In this case \(D_2, D'_2\) must lie on opposite sides of \(x\) (or else \(x\) would separate \(S\)). Join corresponding points of \(D_2\) and \(D'_2\) by a simple path \(\beta\) crossing \(x\) once. Then \(\beta\) gives the required curve in \(S\) (Figure 2).
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Subcase (a), in which $y$ does not separate $S'$. Cut $S'$ along $y$ and glue in discs $D_x, D'_x$ then join $D_1$ to $D_2$ by a simple arc $\alpha_1$ and join the corresponding points of $D_1', D_2'$ by another arc $\alpha_2$ not meeting $\alpha_1$. Then $\alpha_1 \cup \alpha_2$ gives the required curve in $S$ (Figure 3).

(b) By choosing two points of $x \cap y$ which are adjacent in $x$ we can find an arc $\alpha$ in $x$ which meets $y$ only at its end points $A, B$. Let $\beta, \gamma$ be the two arcs of $y$ joining $A$ to $B$. Since $y$ does not separate $S$, one of $\alpha \cup \beta, \alpha \cup \gamma$ does not separate $S$. Suppose, without loss of generality, that $\alpha \cup \beta$ does not separate $S$. Shift $\alpha$ off itself, starting by pushing in the $\beta$-direction at $A$. Complete by an arc close to $\beta$ to get a simple closed curve which meets $x$ in at least one fewer point and $y$ in at most one point (Figure 4(a) or (b)).

![Figure 4](image_url)

**Theorem.** Any closed orientable 3-manifold can be reduced to $S^3$ by a finite number of surgeries on embedded curves.

**Proof.** Let $M$ be a closed orientable 3-manifold and, without loss of generality, assume that $M = M(x, y)$. Associate to the diagram $S(x, y)$ two integers, namely

$$n = \text{genus}(S) \text{ and } r = \min_{i,j} |x_i \cap y_j|.$$ 

The theorem is proved by double induction on $n$ and $r$. We assume, inductively, that the theorem is true for smaller $n$ or for the same $n$ and smaller $r$. The induction starts with $n = 0$, when $M$ is already $S^3$ and there is nothing to prove. The induction step is as follows.

*Case 1, in which $r > 1$.** Without loss of generality, we assume that $r = |x_1 \cap y_1|$. By Lemma 2(b), choose a curve $z_1$ such that $|x_1 \cap z_1| < r$ and $|y_1 \cap z_1| < r$. Extend $z_1$ to a complete system $z$ and apply Lemma 1:

$$\chi(M, z) \simeq M(x, z) \neq M(y, z).$$
Each of the 3-manifolds on the right of the equation has diagram with the same $n$ but smaller $r$. By induction, each can be reduced to $S^3$ by surgery and thus, by performing all three sets of surgeries, $M$ can be reduced to $S^3$ by surgery.

Case 2, in which $r = 0$. In this case we use Lemma 2(a) to find $z_1$ meeting $x_1, y_1$ transversally in one point, and complete to $z$ as before. Then $M(x, z), M(y, z)$ each has a diagram containing a transverse pair of curves meeting in one point, and therefore, as remarked earlier, each has another diagram of smaller genus. Hence, by induction, each can be reduced to $S^3$ by surgery and it follows, as in case 1, that $M$ can be reduced to $S^3$ by surgery.

Case 3, in which $r = 1$. In this case the diagram for $M$ contains a transverse pair of curves and therefore $M$ has a diagram of lower genus. Hence, by induction, $M$ can be reduced to $S^3$ by surgery.

References


Mathematics Institute
University of Warwick
Coventry CV4 7AL