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On the universal covering group of the real symplectic group

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ABSTRACT

A model for the universal covering group of the symplectic group as a Lie group, and some calculations based on the model, as well as defining a similar model for the Lagrangian Grassmannian and relating our construction to the Maslov Index.

1. Introduction

The real symplectic group in $2n$ dimensions, $\mathrm{Sp}(2n, \mathbb{R})$, has a number of non-trivial central extensions such as the metaplectic double covering $\mathrm{Mp}(2n, \mathbb{R})$, an extension by $\mathbb{Z}_2$, the circle extension $\mathrm{Mp}^c(2n, \mathbb{R})$ as the automorphism group of the Heisenberg group, and the universal covering group $\widetilde{\mathrm{Sp}}(2n, \mathbb{R})$ an extension by $\mathbb{Z}$. These extensions do not have faithful finite dimensional representations, so there are no nice models for them as Lie groups of matrices, and this can make calculations with them cumbersome. Paths in $\mathrm{Sp}(2n, \mathbb{R})$ are not easily lifted to an extension unless they lie in some subgroup whose inverse image in the extended group is easy to describe.

In [1], an explicit model for the circle extension of the symplectic group $\mathrm{Sp}(2n, \mathbb{R})$ (the Mp^c group) was used to facilitate the definition of a theory of symplectic spinors on any symplectic manifold and to extend the Kostant theory of metaplectic half-forms to this case. The group manifold can be described explicitly as a hypersurface in $\mathrm{Sp}(2n, \mathbb{R}) \times \mathbb{C}^*$ and the group multiplication given by a single global formula. In [2], these ideas have been used to define Dirac operators on any symplectic manifold and extend the theory of Habermann [3] to this case.

The method used can be adapted to other central extensions. We do this here for the universal covering group $\widetilde{\mathrm{Sp}}(2n, \mathbb{R})$. We show how to make models for the universal covering manifold using suitable maps to the circle and to write the group multiplication in terms of an associated cocycle. We then show how to construct a particularly nice explicit circle map using the ideas from [1] and obtain the formula for its cocycle.

As an application we look at the universal covering group of $\mathrm{Sp}(2, \mathbb{R}) = \mathrm{SL}(2, \mathbb{R})$ and the inverse image of $\mathrm{SL}(2, \mathbb{Z})$ in it. It is a theorem of Milnor that this extension of $\mathrm{SL}(2, \mathbb{Z})$ is isomorphic to the braid group on three strands. We use our explicit formulae for the multiplication to write down generators and show that they satisfy the braid relation.

Finally, we show how the same methods can be used to construct the universal covering manifold of the Lagrangian Grassmannian and the Maslov Index.

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2. Basic construction

In this section, we show how a smooth circle function on a connected Lie group G with $\pi_1(G, e) = \mathbb{Z}$ can be used to construct a manifold underlying the covering group.

Definition 1. Let G be a connected Lie group with $\pi_1(G, e) = \mathbb{Z}$. A *circle function* on G is a smooth map $\varphi: G \to S^1$ which induces an isomorphism of the fundamental groups. It is said to be *normalised* if it satisfies

(i) $\varphi(e) = 1$;
(ii) $\varphi(g^{-1}) = \varphi(g)^{-1}$.

We shall regard S^1 as the unit circle in the complex plane, and φ as a complex valued function.

Lemma 2. *If φ is a circle function then there is a unique smooth map $\widehat{\varphi}: G \to S^1$ with*

$$\widehat{\varphi}(g)^2 = \varphi(g)/\varphi(g^{-1}), \qquad \widehat{\varphi}(e) = 1,$$

and $\widehat{\varphi}$ is a normalised circle function.

Proof. By the connectedness of G there can be at most one such map $\widehat{\varphi}$.

I claim that the map $\Psi: G \to S^1$ given by $\Psi(g) = \varphi(g)\varphi(g^{-1})$ is trivial on $\pi_1(G, e)$. This follows since if γ is a loop at e in G then $\Psi_*[\gamma] = [\Psi \circ \gamma] = [\varphi \circ \gamma] + [\varphi \circ \gamma^{-1}] = \varphi_*[\gamma] + \varphi_*[\gamma^{-1}] = \varphi_*([\gamma] - [\gamma]) = 0$ since the multiplication in the fundamental group of a topological group is induced by pointwise multiplication of loops. Moreover, $\Psi(e) = \varphi(e)^2$ thus Ψ has a unique smooth square root ψ with $\psi(e) = \varphi(e)$. We set $\widehat{\varphi}(g) = \varphi(g)\psi(g)^{-1}$. Property (i) is immediate in view of the normalisation of ψ. Further, $\widehat{\varphi}(g)^2\widehat{\varphi}(g^{-1})^2 = \varphi(g)/\varphi(g^{-1})\varphi(g^{-1})/\varphi(g) = 1$. Thus $\widehat{\varphi}(g)\widehat{\varphi}(g^{-1}) = \pm 1$ and (ii) follows since G is connected and $\widehat{\varphi}(e)\widehat{\varphi}(e^{-1}) = 1$. Since Ψ_* is trivial on π_1 so is ψ_* and hence $\widehat{\varphi}_*$ is an isomorphism. □

Lemma 3. *If φ is a normalised circle function for G, then there is a unique smooth function $\eta : G \times G \to \mathbb{R}$ such that*

$$\varphi(g_1 g_2) = \varphi(g_1)\varphi(g_2)\, e^{i\,\eta(g_1, g_2)}$$

with $\eta(e, e) = 0$. Further

(i) *$\eta(g, e) = \eta(e, g) = \eta(g, g^{-1}) = 0$, for all $g \in G$;*
(ii) *$\eta(g_1, g_2) + \eta(g_1 g_2, g_3) = \eta(g_1, g_2 g_3) + \eta(g_2, g_3)$ for all $g_1, g_2, g_3 \in G$.*

Proof. The existence of η is a consequence of the following. If $\varphi : G \to H$ is a continuous map (not necessarily a homomorphism) of connected topological groups with $\varphi(e_G) = e_H$ and $\Phi : G \times G \to H$ is defined by $\Phi(g_1, g_2) = \varphi(g_1 g_2)\varphi(g_1)^{-1}\varphi(g_2)^{-1}$, then $\Phi_* : \pi_1(G \times G, (e_G, e_G)) \to \pi_1(H, e_H)$ is trivial. From this it follows that Φ has a smooth logarithm which is unique when its value is specified at one point since G, and hence $G \times G$, is connected.

(i) and (ii) follow by continuity arguments again based on the connectivity of G. For example, $\exp i\eta(g, e) = \varphi(ge)\varphi(g)^{-1}\varphi(e)^{-1} = 1$ so $\eta(g, e) \in 2\pi\mathbb{Z}$ and thus must be constant. It vanishes for $g = e$. □

Definition 4. If φ is a normalised circle function for the Lie group G set

$$\widetilde{G} = \left\{(g, c) \in G \times \mathbb{R} \mid \varphi(g) = e^{ic}\right\}.$$

It is easy to see that 1 is a regular value of the map $G \times \mathbb{R} \ni (g, c) \mapsto \varphi(g)e^{-ic} \in S^1$ and hence that $\widetilde{G}$ is a smooth manifold of the same dimension as G.

Proposition 5. *Let $\widetilde{G}$ be defined as above and set*

$$(g_1, c_1) \cdot (g_2, c_2) = (g_1 g_2, c_1 + c_2 + \eta(g_1, g_2))$$

then $\widetilde{G}$ is a Lie group with identity $(e_G, 0)$ and inverse $(g, c)^{-1} = (g^{-1}, -c)$. If $\sigma : \widetilde{G} \to G$ is given by $\sigma(g, c) = g$ then σ is a surjective homomorphism with kernel $\{(1, 2\pi\, k) \mid k \in \mathbb{Z}\}$.

Proof. Associativity and the group properties follow from Lemma 3. σ is obviously smooth and clearly a homomorphism. □

It follows from this proposition that σ is a homomorphism of Lie groups with discrete kernel, and so induces an isomorphism of Lie algebras.

Theorem 6. *$\sigma : \widetilde{G} \to G$ is the universal covering group of G.*

Proof. Let $\gamma\colon [0,1] \to G$ be a loop in G, set $z(t) = \varphi(\gamma(t))$ and let $k \in \mathbb{Z}$ be the winding number of the origin of $z(t)$ as a contour in $\mathbb{C}$. This means that we can lift $z(t) = \mathrm{e}^{ic(t)}$ with $c(0) = 0$ and then $c(1) = 2\pi k$. Since φ induces an isomorphism on the fundamental group, k is also the homotopy class of γ. If (γ, c) is a loop in $\widetilde{G}$ at $(1,0)$ then $\varphi(\gamma(t)) = \mathrm{e}^{ic(t)}$ with $c(1) = c(0) = 0$. It follows that $k = 0$ and hence that γ is contractible in G. So there is a curve of loops δ_s in G with $\delta_0 = \gamma$ and $\delta_1 \equiv 1$. Then $\varphi(\delta_s(t))$ lifts to $\mathbb{R}$ so we can write $\varphi(\delta_s(t)) = \mathrm{e}^{ic_s(t)}$ with $c_s(0) = c_s(1) = 0$ for all s and $c_1(t) \equiv 0$. (δ_s, c_s) is then a homotopy of (γ, c) to a constant loop in $\widetilde{G}$. Since (γ, c) was arbitrary, it follows that $\widetilde{G}$ is simply connected and hence that it is the universal covering group of G. □

Remark 7. We have a converse to this result. If $\widetilde{G} \to G$ is the universal covering group of a Lie group G and $\pi_1(G, e_G) = \mathbb{Z}$, then $\widetilde{G}$ is an extension of G by $\mathbb{Z}$. Take a good cover U_α of G and choose sections $s_\alpha\colon U_\alpha \to \widetilde{G}$. Then on overlaps we have $s_\beta = s_\alpha . n_{\alpha\beta}$. Viewed as real valued functions, the 1-cocycle $n_{\alpha\beta}$ is a coboundary, so there are smooth functions f_α on U_α with $f_\beta - f_\alpha = n_{\alpha\beta}$. If we set $\varphi = \exp 2\pi i f_\alpha$ then φ is globally defined and smooth on G. Moreover, if we form the cover as in Definition 4, then it is left as an exercise to show that $(g, c) \mapsto s_\alpha(g).(c - f_\alpha(g))$, when $g \in U_\alpha$, gives a globally defined isomorphism of Lie groups.

Example 8. As a first example we take $G = \mathrm{U}(n)$ then $\varphi(g) = \det g$ is a normalised circle function. In this case $\eta \equiv 0$ and $\widetilde{\mathrm{U}(n)}$ is a subgroup of $\mathrm{U}(n) \times \mathbb{R}$.

Lemma 9. *$\widetilde{\mathrm{U}(n)}$ is isomorphic to* $\mathrm{SU}(n) \times \mathbb{R}$.

Proof. Define a map

$$\mu : \mathrm{SU}(n) \times \mathbb{R} \to \widetilde{\mathrm{U}(n)} : (k, c) \mapsto (\mathrm{e}^{ic}k, nc),$$

which is well defined since $\det(\mathrm{e}^{ic}k) = \mathrm{e}^{nic}$. Moreover μ is a homomorphism since φ is a homomorphism and $\eta \equiv 0$ giving

$$\begin{aligned}\mu(k_1, c_1)\mu(k_2, c_2) &= (\mathrm{e}^{ic_1}k_1, nc_1)(\mathrm{e}^{ic_2}k_2, nc_2)\\ &= (\mathrm{e}^{i(c_1+c_2)}k_1k_2, nc_1 + nc_2)\\ &= \mu(k_1k_2, c_1 + c_2).\end{aligned}$$

The map μ is onto, for if $k \in \mathrm{U}(n)$ with $\det k = \mathrm{e}^{ic}$ then $\det(\mathrm{e}^{-ic/n}k) = 1$. So the image of $(\mathrm{e}^{-ic/n}k, c/n) \in \mathrm{SU}(n) \times \mathbb{R}$ under μ is $(k, c) \in \widetilde{\mathrm{U}(n)}$. The kernel of μ is

$$\left\{(k, c) \in \mathrm{SU}(n) \times \mathbb{R} \mid \mathrm{e}^{ic}k = 1,\ nc = 0\right\} = \{(1, 0)\},$$

which proves the lemma. □

Example 10. The symplectic group is semisimple, so there are no homomorphisms $\mathrm{Sp}(2n, \mathbb{R}) \to S^1$. However, $U(n)$ is a maximal compact subgroup and so the same fundamental group. Suppose we have a smooth map $\varphi : \mathrm{Sp}(2n, \mathbb{R}) \to S^1$ such that $\varphi(g^{-1}) = \varphi(g)^{-1}$, $\varphi(1) = 1$, and whose restriction to $U(n)$ is det, then φ is a normalised circle function for $\mathrm{Sp}(2n, \mathbb{R})$ and so such a map φ will induce an isomorphism on the fundamental group.

In fact we could take any retraction $\rho\colon \mathrm{Sp}(2n, \mathbb{R}) \to \mathrm{U}(n)$ and set $\varphi = \det \circ \rho$ to get a circle function and then normalise it. But the result in this generality might not be all that easy to work with.

In order to have a useable model for $\widetilde{\mathrm{Sp}}(2n, \mathbb{R})$ it is necessary to have a nice formula for the multiplication, and this depends on being able to compute the corresponding cocycle η. We shall see in what follows that there is a good choice of φ which leads to an explicit formula for η. After defining this we shall make some computations with our formulae, to show that they are indeed 'useable'.

3. Calculation of a symplectic circle function φ

We move to a slightly more abstract setting, taking a general symplectic vector space as our starting point. Let (V, Ω) be a $2n$-dimensional real symplectic vector space. Fix $J \in \mathrm{End}(V)$ such that $J^2 = -1$, $\Omega(Ju, Jv) = \Omega(u, v)$ for every $u, v \in V$ and $\Omega(v, Jv) > 0$ if $v \neq 0$. Let

$$\mathrm{Sp}(V, \Omega) = \{g \in \mathrm{End}(V) \mid \Omega(gv, gw) = \Omega(v, w),\ \forall v, w \in V\};$$
$$\mathrm{Gl}(V, J) = \{g \in \mathrm{Gl}(V) \mid gJ = Jg\};$$
$$\mathrm{U}(V, \Omega, J) = \{g \in \mathrm{Sp}(V, \Omega) \mid gJ = Jg\} = \mathrm{Sp}(V, \Omega) \cap \mathrm{Gl}(V, J).$$

View V as a complex vector space using J, then $\mathrm{Sp}(V, \Omega)$ is isomorphic to $\mathrm{Sp}(2n, \mathbb{R})$, $\mathrm{U}(V, \Omega, J)$ to $\mathrm{U}(n)$ and $\mathrm{Gl}(V, J)$ to $\mathrm{Gl}(n, \mathbb{C})$.

Given $g \in \mathrm{Sp}(V, \Omega)$ it can be written *uniquely* as a sum of a J-linear and J-antilinear part, namely,

$$g = C_g + D_g,$$

where $C_g = \frac{1}{2}(g - JgJ)$ and $D_g = \frac{1}{2}(g + JgJ)$.

Lemma 11. *$C_g \in \mathrm{Gl}(V, J)$. If $g \in \mathrm{U}(V, \Omega, J)$, then $C_g = g$.*

Proof. Suppose that $C_g v = 0$. Then

$$0 = 4\Omega(C_g v, JC_g v) = 2\Omega(v, Jv) + \Omega(gv, Jgv) + \Omega(gJv, JgJv).$$

But each term in the right hand side of the above equation is positive definite and so $v = 0$. □

The definition of the circle function φ is now easy, namely,

$$\varphi(g) = \frac{\det C_g}{|\det C_g|}.$$

4. Calculation of the cocycle η

We now find a formula for the cocycle η whose existence is guaranteed by Lemma 3. First we make some preliminary calculations.

Put $Z_g = C_g^{-1} D_g$. Then $g = C_g(1 + Z_g)$. Z_g is $\mathbb{C}$-antilinear. We have $g^{-1} = C_{g^{-1}}(1 + Z_{g^{-1}})$. Equating $\mathbb{C}$-linear and $\mathbb{C}$-antilinear parts in

$$1 = g^{-1}g = C_{g^{-1}}(1 + Z_{g^{-1}})C_g(1 + Z_g)$$

gives

$$1 = C_{g^{-1}}(C_g + Z_{g^{-1}}C_g Z_g) \quad \text{and} \quad 0 = C_{g^{-1}}(Z_{g^{-1}}C_g + C_g Z_g).$$

Hence

$$Z_{g^{-1}} = -C_g Z_g C_g^{-1} \quad \text{and} \quad 1 = C_{g^{-1}} C_g (1 - Z_g^2),$$

since $C_{g^{-1}}$ is invertible. Thus $1 - Z_g^2$ is invertible with $(1 - Z_g^2)^{-1} = C_{g^{-1}} C_g$. We can also decompose a product

$$C_{g_1 g_2}(1 + Z_{g_1 g_2}) = C_{g_1}(1 + Z_{g_1}) C_{g_2}(1 + Z_{g_2})$$

as

$$C_{g_1 g_2} = C_{g_1}(C_{g_2} + Z_{g_1} C_{g_2} Z_{g_2}) = C_{g_1}(1 - Z_{g_1} Z_{g_2^{-1}}) C_{g_2}$$

and

$$\begin{aligned} Z_{g_1 g_2} &= C_{g_2}^{-1}(1 - Z_{g_1} Z_{g_2^{-1}})^{-1}(Z_{g_1} C_{g_2} + C_{g_2} Z_{g_2}) \\ &= C_{g_2}^{-1}(1 - Z_{g_1} Z_{g_2^{-1}})^{-1}(Z_{g_1} - Z_{g_2^{-1}}) C_{g_2}. \end{aligned}$$

To define the cocycle η we need to see that the function

$$(g_1, g_2) \mapsto \frac{\det(1 - Z_{g_1} Z_{g_2^{-1}})}{|\det(1 - Z_{g_1} Z_{g_2^{-1}})|}$$

has a smooth logarithm. To see this, we determine that the set where Z_g lives is essentially the Siegel domain.

Introduce a Hermitian structure on (V, J) as a $\mathbb{C}$ vector space by defining

$$\langle v, w\rangle = \Omega(v, Jw) - i\,\Omega(v, w).$$

Then

$$\langle Jv, w\rangle = i\,\langle v, w\rangle = -\langle v, Jw\rangle \quad \text{and} \quad \langle v, v\rangle = \Omega(v, Jv) > 0, \quad \text{for } v \neq 0.$$

We calculate

$$\begin{aligned} 2\,\langle C_g^* v, w\rangle &= 2\,\langle v, C_g w\rangle \\ &= \Omega(v, Jgw) + \Omega(v, gJw) - i\,\Omega(v, gw) + i\,\Omega(v, JgJw) \\ &= -\Omega(g^{-1}Jv, w) + \Omega(g^{-1}v, Jw) - i\,\Omega(g^{-1}v, w) - i\,\Omega(g^{-1}Jv, Jw) \\ &= \langle g^{-1}v, w\rangle - i\,\langle g^{-1}Jv, w\rangle = \langle g^{-1}v, w\rangle - \langle Jg^{-1}Jv, w\rangle \\ &= 2\,\langle C_{g^{-1}} v, w\rangle. \end{aligned}$$

Therefore

$$C_g^* = C_{g^{-1}},$$

which implies

$$\varphi(g^{-1}) = \frac{\det C_{g^{-1}}}{|\det C_{g^{-1}}|} = \frac{\det C_g^*}{|\det C_g^*|} = \overline{\varphi(g)}.$$

Moreover,

$$1 - Z_g^2 = (C_{g^{-1}} C_g)^{-1} = (C_g^* C_g)^{-1},$$

which is positive definite. Since Z_g is antilinear and $\langle v, w\rangle$ is antilinear in w, the function $(v, w) \mapsto \langle v, Z_g w\rangle$ is complex bilinear. We claim that it is symmetric.

Lemma 12. *For every $v, w \in V$, we have $\langle v, Z_g w\rangle = \langle w, Z_g v\rangle$.*

Proof. Now

$$\Omega(D_g v, w) = \Omega(v, D_{g^{-1}} w) = -\Omega(D_{g^{-1}} w, v);$$

and $\Omega(D_g, Jw) = -\Omega(D_{g^{-1}} w, Jv)$. Hence $\langle D_g v, w\rangle = -\langle D_{g^{-1}} w, v\rangle$ and so

$$\langle Z_g v, C_g^* w\rangle = \langle C_g Z_g v, w\rangle = -\langle C_{g^{-1}} Z_{g^{-1}} w, v\rangle = \langle Z_g C_g^* w, v\rangle. \quad \square$$

So Z_g has the three properties:

(i) Z_g is $\mathbb{C}$-antilinear;
(ii) $(v, w) \mapsto \langle v, Z_g w\rangle$ is symmetric;
(iii) $1 - Z_g^2$ is self-adjoint and is positive definite.

Let $\mathbb{B}(V, \Omega, J)$ be the *Siegel domain* consisting of $Z \in \mathrm{End}_{\mathbb{R}}(V)$ such that

$$ZJ = -JZ, \qquad \langle v, Zw\rangle = \langle w, Zv\rangle, \quad \text{and} \quad 1 - Z^2 \quad \text{is positive definite.}$$

Then we have a map

$$\mathrm{Sp}(V, \Omega) \to \mathrm{Gl}(V, J) \times \mathbb{B}(V, \Omega, J) : g \mapsto (C_g, Z_g),$$

whose image is the submanifold $\left\{(C, Z) \mid 1 - Z^2 = (C^* C)^{-1}\right\}$.

Proposition 13. *If $Z_1, Z_2 \in \mathbb{B}(V, \Omega, J)$, then $1 - Z_1 Z_2$ is invertible and its real part is positive definite.*

Proof. Suppose that $v \neq 0$. Then

$$\begin{aligned}
\langle\left[(1 - Z_1Z_2) + (1 - Z_1Z_2)^*\right] v, v\rangle &= \langle(1 - Z_1Z_2)v, v\rangle + \langle v, (1 - Z_1Z_2)v\rangle \\
&= \langle(1 - Z_1^2)v, v\rangle + \langle(1 - Z_2^2)v, v\rangle + \langle Z_1^2 v, v\rangle \\
&\quad + \langle Z_2^2 v, v\rangle - \langle Z_1Z_2 v, v\rangle - \langle v, Z_1Z_2 v\rangle \\
&= \langle(1 - Z_1^2)v, v\rangle + \langle(1 - Z_2^2)v, v\rangle + \langle Z_1 v, Z_1 v\rangle \\
&\quad + \langle Z_2 v, Z_2 v\rangle - \langle Z_1 v, Z_2 v\rangle - \langle Z_2 v, Z_1 v\rangle \\
&= \langle(1 - Z_1^2)v, v\rangle + \langle(1 - Z_2^2)v, v\rangle + \|(Z_1 - Z_2)v\|^2.
\end{aligned}$$

The first two terms in the last equality are positive, while the last term is nonnegative. Consequently, the kernel of $1 - Z_1Z_2$ is $\{0\}$ and its real part is positive definite, which proves the proposition. $\square$

Thus $1 - Z_1Z_2 \in \mathrm{Gl}(V, J)$ and it lies in the contractible subset

$$\left\{g \in \mathrm{Gl}(V, J) \mid g + g^* \text{ is positive definite}\right\}.$$

It follows that there is a smooth logarithm

$$a : \mathbb{B}(V, \Omega, J) \times \mathbb{B}(V, \Omega, J) \to \mathfrak{gl}(V, J)$$

such that $1 - Z_1Z_2 = \mathrm{e}^{a(Z_1, Z_2)}$. We can define the cocycle η by

$$\eta(g_1, g_2) = \mathrm{Im}\,\mathrm{Tr}\,(a(Z_{g_1}, Z_{g_2^{-1}})).$$

Therefore

$$\mathrm{e}^{i\,\eta(g_1, g_2)} = \frac{\det(1 - Z_{g_1} Z_{g_2^{-1}})}{|\det(1 - Z_{g_1} Z_{g_2^{-1}})|},$$

which implies

$$\varphi(g_1 g_2) = \varphi(g_1)\varphi(g_2)\,\mathrm{e}^{i\,\eta(g_1, g_2)}.$$

5. The relationship with angle functions

In [4], Milnor is interested in lifting $\varphi\circ\sigma : \widetilde{G}\to S^1$ to a map $\theta : \widetilde{G}\to\mathbb{R}$, called an *angle function*, so that $\varphi\circ\sigma = e^{i\theta}$. In our case, this is easy: set $\theta(g,c)=c$. Then

$$\theta((g_1,c_1)\cdot(g_2,c_2)) - \theta(g_1,c_1) - \theta(g_2,c_2) = c_1+c_2+\eta(g_1,g_2) - c_1 - c_2 = \eta(g_1,g_2).$$

Lemma 3 in [4] becomes the $n=1$ case of the following.

Lemma 14. $|\eta(g_1,g_2)| < n\pi/2.$

Proof. The above inequality states that $|\arg(\det(1-Z_1Z_2))| < n\pi/2$ if we choose the branch of arg which is 0 at 1. We know that $1-Z_1Z_2 = X+iY$ with X positive definite. So $1-Z_1Z_2 = X^{\frac12}(1+iX^{-\frac12}YX^{-\frac12})X^{\frac12}$ which implies

$$\det(1-Z_1Z_2) = \det X\ \det(1+i\widetilde{Y}),$$

where $\widetilde{Y} = X^{-\frac12}YX^{-\frac12}$ is self-adjoint. Therefore

$$\arg(\det(1-Z_1Z_2)) = \arg(\det X\ \det(1+i\widetilde{Y})) = \arg\prod_{j=1}^{n}(1+i\widetilde{y}_j),$$

where $\widetilde{y}_j$ are the eigenvalues of $\widetilde{Y}$. Since $\arg(1+iy)\in(-\pi/2,\pi/2)$, we obtain the desired inequality. □

It can be seen that by taking $1-Z_1^2$ and $1-Z_2^2$ positive definite that η takes values in the whole interval $(-n\pi/2, n\pi/2)$ so the bound in Lemma 14 is sharp.

6. Special case: $\widetilde{\mathbf{Sp}}(2,\mathbb{R})$

In this section, we work out the above theory in detail for the universal covering group of $\mathrm{Sp}(2,\mathbb{R}) = \mathrm{SL}(2,\mathbb{R})$.

Let V be $\mathbb{R}^2$ with standard basis $\{e_1,e_2\}$. With respect to this basis the symplectic form Ω has matrix $J=\begin{pmatrix}0&1\\-1&0\end{pmatrix}$. Suppose that P is a 2×2 real matrix which is J-linear, that is, $PJ=JP$. Then $P=\begin{pmatrix}a&b\\-b&a\end{pmatrix}$, which we identify with the complex number $a+ib$. Let $\Sigma=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$, which represents complex conjugation on (V,J) thought of as a complex vector space. A J-antilinear map Z has a matrix of the form $\begin{pmatrix}\alpha&\beta\\\beta&-\alpha\end{pmatrix}$, which can be identified with $(\alpha-\beta J)\Sigma$. In other words, Z can be identified with the $\mathbb{C}$-antilinear map $\widehat{Z}:\mathbb{C}\to\mathbb{C} : z\mapsto(\alpha-i\beta)\bar{z}$.

When $g=\begin{pmatrix}a&b\\c&d\end{pmatrix}\in\mathfrak{gl}(2,\mathbb{R})$, a calculation shows that

$$C_g = \frac12\begin{pmatrix}a+d & b-c\\ c-b & a+d\end{pmatrix}$$

and as a real matrix

$$Z_g = \frac{1}{(a+d)^2+(b-c)^2}\begin{pmatrix}a^2+c^2-b^2-d^2 & 2(ab+cd)\\ 2(ab+cd) & -(a^2+c^2-b^2-d^2)\end{pmatrix}.$$

Using the above identification, this gives

$$\widehat{Z}_g z = \alpha_g\bar{z} = \left(\frac{(a^2+c^2-b^2-d^2)-2i(ad+bc)}{(a+d)^2+(b-c)^2}\right)\bar{z}.$$

For $g_1,g_2\in\mathrm{Sp}(2,\mathbb{R})$ the value of the cocycle $\eta(g_1,g_2)$ defined by

$$e^{i\eta(g_1,g_2)} = \frac{\det(1-Z_{g_1}Z_{g_2^{-1}})}{|\det(1-Z_{g_1}Z_{g_2^{-1}})|} = \frac{1-\alpha_{g_1}\overline{\alpha}_{g_2^{-1}}}{|1-\alpha_{g_1}\overline{\alpha}_{g_2^{-1}}|} \tag{1}$$

where $\eta(g_1,g_2)$ is continuous and lies in the open interval $(-\pi/2,\pi/2)$.

Example 15. To show how useable these formulae are we calculate with the generators of the integer matrices $\mathrm{Sp}(2,\mathbb{Z})$. This subgroup is generated by

$$s_1=\begin{pmatrix}1&1\\0&1\end{pmatrix}\quad\text{and}\quad s_2=\begin{pmatrix}1&0\\-1&1\end{pmatrix},$$

subject to the relations

$$s_1 s_2 s_1 = s_2 s_1 s_2 \quad \text{and} \quad (s_1 s_2 s_1)^4 = 1.$$

Using the formula for α_g, we obtain the table

	g	α_g
s_1	$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$	$-\frac{1}{5}(1+2\,i)$
s_2	$\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$	$\frac{1}{5}(1+2\,i)$
s_1^{-1}	$\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$	$\frac{1}{5}(-1+2\,i)$
s_2^{-1}	$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$	$\frac{1}{5}(1-2\,i)$
$s_1 s_2$	$\begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$	$\frac{1}{5}(-1+2\,i)$
$(s_1 s_2)^{-1}$	$\begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$	$\frac{1}{5}(1+2\,i)$

Next we compute the values of the cocycle η at (s_1, s_2), $(s_1 s_2, s_1)$, and $(s_2, s_1 s_2)$. From (1) and the above table we obtain

$$e^{i\,\eta(s_1,s_2)} = \frac{1 - \frac{1}{5}\,(-1-2\,i)\frac{1}{5}(1+2\,i)}{\left|1 - \frac{1}{5}\,(-1-2\,i)\frac{1}{5}(1+2\,i)\right|} = \frac{22+4\,i}{10\sqrt{5}} = \frac{11+2\,i}{5\sqrt{5}}.$$

Let $\theta_0 = \tan^{-1}\frac{1}{2}$. Then

$$e^{3\theta_0\,i} = \left(\frac{2+i}{\sqrt{5}}\right)^3 = \frac{2+11\,i}{5\sqrt{5}} = i\,e^{-\eta(s_1,s_2)\,i},$$

which implies

$$\eta(s_1, s_2) = \pi/2 - 3\theta_0.$$

Note that $\eta(s_1, s_2)$ lies in $(-\pi/2, \pi/2)$. Now

$$e^{i\,\eta(s_1 s_2,s_2)} = \frac{1 - \frac{1}{5}\,(-1+2\,i)\frac{1}{5}(1+2\,i)}{\left|1 - \frac{1}{5}\,(-1+2\,i)\frac{1}{5}(1+2\,i)\right|} = 1.$$

Consequently, $\eta(s_1 s_2, s_2) = 0$. Similarly,

$$e^{i\,\eta(s_2,s_1 s_2)} = \frac{1 - \frac{1}{5}\,(1+2\,i)\frac{1}{5}(1-2\,i)}{\left|1 - \frac{1}{5}\,(1+2\,i)\frac{1}{5}(1-2\,i)\right|} = 1$$

and thus $\eta(s_2, s_1 s_2) = 0$.

7. The braid group B_3 lies in $\widetilde{\mathrm{Sp}}(2, \mathbb{R})$

In this section, we show that the three strand braid group B_3, given by the presentation

$$\langle \sigma_1, \sigma_2 : \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 \rangle, \tag{2}$$

is a discrete subgroup of the universal covering group $\widetilde{\mathrm{Sp}}(2, \mathbb{R})$ of $\mathrm{Sp}(2, \mathbb{R})$.

First, we look for elements $\widetilde{\sigma}_j = (s_j, c_j)$, $j = 1, 2$ of $\widetilde{\mathrm{Sp}}(2, \mathbb{R})$ which satisfy the braid relation (2) and thus lie in the image $\widetilde{B}_3$ of B_3 under the map

$$\tau : B_3 \to \widetilde{\mathrm{Sp}}(2, \mathbb{R}) : \sigma_j \mapsto \widetilde{\sigma}_j, \quad \text{for } j = 1, 2.$$

Write

$$\widetilde{\sigma}_1 = (s_1, \theta_0 + 2\pi\, k_1) \quad \text{and} \quad \widetilde{\sigma}_2 = (s_2, \theta_0 + 2\pi\, k_2),$$

for some $k_1, k_2 \in \mathbb{Z}$. Then

$$\begin{aligned}\widetilde{\sigma}_1 \cdot \widetilde{\sigma}_2 &= (s_1 s_2, 2\theta_0 + 2\pi\, k_1 + 2\pi\, k_2 + \eta(s_1, s_2))\\ &= (s_1 s_2, 2\theta_0 + 2\pi\, k_1 + 2\pi\, k_2 + \pi/2 - 3\theta_0)\\ &= (s_1 s_2, -\theta_0 + 2\pi\, k_1 + 2\pi\, k_2 + \pi/2).\end{aligned}$$

Consequently,

$$\begin{aligned}(\widetilde{\sigma}_1 \cdot \widetilde{\sigma}_2) \cdot \widetilde{\sigma}_1 &= (s_1 s_2, -\theta_0 + 2\pi\, k_1 + 2\pi\, k_2 + \pi/2) \cdot (s_1, \theta_0 + 2\pi\, k_1)\\ &= (s_1 s_2 s_1, 4\pi\, k_1 + 2\pi\, k_2 + \pi/2 + \eta(s_1 s_2, s_1))\\ &= (s_1 s_2 s_1, 4\pi\, k_1 + 2\pi\, k_2 + \pi/2)\end{aligned}$$

and

$$\begin{aligned}\widetilde{\sigma}_2 \cdot (\widetilde{\sigma}_1 \cdot \widetilde{\sigma}_2) &= (s_2, \theta_0 + 2\pi\, k_2) \cdot (s_1 s_2, -\theta_0 + 2\pi\, k_1 + 2\pi\, k_2 + \pi/2)\\ &= (s_2 s_1 s_2, 2\pi\, k_1 + 4\pi\, k_2 + \pi/2 + \eta(s_2, s_1 s_2))\\ &= (s_2 s_1 s_2, 2\pi\, k_1 + 4\pi\, k_2 + \pi/2).\end{aligned}$$

Therefore the braid relation

$$(\widetilde{\sigma}_1 \cdot \widetilde{\sigma}_2) \cdot \widetilde{\sigma}_1 = \widetilde{\sigma}_1 \cdot (\widetilde{\sigma}_2 \cdot \widetilde{\sigma}_1)$$

is satisfied if and only if $k_1 = k_2 = k$. Hence

$$\widetilde{\sigma}_1 \cdot \widetilde{\sigma}_2 \cdot \widetilde{\sigma}_1 = \widetilde{\sigma}_1 \cdot \widetilde{\sigma}_2 \cdot \widetilde{\sigma}_1 = (J, 6\pi\, k + \pi/2),$$

for some $k \in \mathbb{Z}$. Moreover,

$$\begin{aligned}(\widetilde{\sigma}_1 \cdot \widetilde{\sigma}_2)^3 &= (\widetilde{\sigma}_1 \cdot \widetilde{\sigma}_2 \cdot \widetilde{\sigma}_1) \cdot (\widetilde{\sigma}_2 \cdot \widetilde{\sigma}_1 \cdot \widetilde{\sigma}_2)\\ &= (J^2, 12\pi\, k + \pi) = (-1, 12\pi\, k + \pi).\end{aligned}$$

Setting $k = 0$, we have shown that

$$\widetilde{\sigma}_1 \cdot \widetilde{\sigma}_2 \cdot \widetilde{\sigma}_1 = \widetilde{\sigma}_1 \cdot \widetilde{\sigma}_2 \cdot \widetilde{\sigma}_1 = (J, \pi/2)$$

and

$$(\widetilde{\sigma}_1 \cdot \widetilde{\sigma}_2)^3 = (-1, \pi).$$

We now show that $(\widetilde{\sigma}_1 \cdot \widetilde{\sigma}_2)^3$ generates the centre of $\widetilde{\mathrm{Sp}}(2, \mathbb{R})$. Suppose that

$$(g_1, a_1) \cdot (g_2, a_2) = (g_2, a_2) \cdot (g_1, a_1)$$

for every $(g_2, a_2) \in \widetilde{\mathrm{Sp}}(2, \mathbb{R})$. Then

$$(g_1 g_2, a_1 + a_2 + \eta(g_1, g_2)) = (g_2 g_1, a_2 + a_1 + \eta(g_2, g_1))$$

for every $g_2 \in \mathrm{Sp}(2, \mathbb{R})$. Hence g_1 lies in the centre $Z(\mathrm{Sp}(2, \mathbb{R}))$ of $\mathrm{Sp}(2, \mathbb{R})$; so in particular, $g_1 \in \mathrm{U}(1)$. Consequently, $\eta(g_1, g_2) = \eta(g_2, g_1) = 0$ and $\mathrm{e}^{i a_1} = \varphi(g_1) = \det g_1 = \pm 1$. In other words, (g_1, a_1) is either $(-1, 2n\pi + 1)$ or $(1, 2n\pi)$ for some $n \in \mathbb{Z}$. But $(-1, \pi)^{2n} = (1, 2n\pi)$. Thus $Z(\widetilde{\mathrm{Sp}}(2, \mathbb{R}))$ is generated by $(-1, \pi)$.

Proposition 16. $\widetilde{B}_3 = \widetilde{\mathrm{Sp}}(2, \mathbb{Z})$.

Proof. Let $\widetilde{\sigma} \in \widetilde{\mathrm{Sp}}(2, \mathbb{Z}) \subseteq \widetilde{\mathrm{Sp}}(2, \mathbb{R})$. Then $\widetilde{\sigma} = (g, c)$, where $g \in \mathrm{Sp}(2, \mathbb{Z})$. Now g is a word $w(s_1, s_2)$ in s_1 and s_2 which is a product of s_j s and s_k^{-1} s. Consider

$$w(\widetilde{\sigma}_1, \widetilde{\sigma}_2) = (w(s_1, s_2), c') = (g, c').$$

Because $\widetilde{\sigma}_j \in \widetilde{\mathrm{Sp}}(2, \mathbb{R})$, we have $\varphi(g) = \mathrm{e}^{ic} = \mathrm{e}^{ic'}$, which implies that $c = c' + 2\pi\,\ell$ for some $\ell \in \mathbb{Z}$. Hence $(1, 2\pi\,\ell) = (\widetilde{\sigma}_1 \cdot \widetilde{\sigma}_2)^{6\ell}$. But

$$\begin{aligned}\widetilde{\sigma} = (g, c) &= (g, c') \cdot (1, 2\pi\,\ell), \quad \text{because } \eta(1, g) = 0\\ &= w(\widetilde{\sigma}_1, \widetilde{\sigma}_2) \cdot (\widetilde{\sigma}_1 \cdot \widetilde{\sigma}_2)^{6\ell},\end{aligned}$$

which lies in $\widetilde{B}_3$. Thus $\widetilde{\mathrm{Sp}}(2, \mathbb{Z}) \subseteq \widetilde{B}_3$, which proves the proposition, since $\widetilde{\sigma}_1$ and $\widetilde{\sigma}_2$ generate $\widetilde{\mathrm{Sp}}(2, \mathbb{Z})$. □

We now prove the following lemma.

Lemma 17. *B_3 is isomorphic to $\widetilde{B}_3$.*

Proof. Consider

$$B_3 \xrightarrow{\tau} \widetilde{B}_3 \xrightarrow{\sigma} \widetilde{\mathrm{Sp}}(2, \mathbb{R}) : \sigma_j \mapsto \widetilde{\sigma}_j, \quad \text{for } j = 1, 2.$$

Let $v(\sigma_1, \sigma_2)$ be a word in $\ker \tau$. Then $v(s_1, s_2) = 1$. But $s_1s_2s_1(s_2s_1s_2)^{-1}$ and $(s_2s_1s_2)^4$ generate the set of relations in $\mathrm{Sp}(2, \mathbb{Z})$. Hence

$$v(s_1, s_2) = u(s_1s_2s_1(s_2s_1s_2)^{-1}, (s_2s_1s_2)^4)$$

for some word u. Therefore

$$\begin{aligned} v(\sigma_1, \sigma_2) &= u((\sigma_1\sigma_2\sigma_1)(\sigma_2\sigma_1\sigma_2)^{-1}, (\sigma_1\sigma_2\sigma_2)^4) \\ &= u(1, (\sigma_1\sigma_2\sigma_1)^4) = (\sigma_1\sigma_2\sigma_1)^{4m}. \end{aligned}$$

But

$$(\sigma_1\sigma_2\sigma_1)^2 = (\sigma_1\sigma_2\sigma_1)(\sigma_2\sigma_1\sigma_2) = (\sigma_1\sigma_2)^3.$$

So

$$(\sigma_1\sigma_2\sigma_1)^{4m} = (\sigma_1\sigma_2)^{6m}.$$

However,

$$(1, 0) = \tau(v(\sigma_1, \sigma_2)) = \tau((\sigma_1\sigma_2)^{6m}) = (\widetilde{\sigma}_1\widetilde{\sigma}_2)^{6m} = ((-1, \pi))^{2m} = (1, 2m\pi).$$

Consequently, $m = 0$, which implies $v(\sigma_1, \sigma_2) = 1$. Thus the surjective homomorphism τ is injective and hence is an isomorphism. This completes the proof of the lemma. □

We have thus obtained the following result of Milnor [5].

Theorem 18. *The inverse image of* $\mathrm{Sp}(2, \mathbb{Z})$ *in the simply connected covering group of* $\mathrm{Sp}(2, \mathbb{R})$ *is isomorphic to the braid group on three strands.*

8. The Lagrangian Grassmannian

Let (V, Ω, J) be as above and let $V^{\mathbb{C}} = V + iV$ be the complexification of V. Let $v = v_1 + iv_2 \mapsto \overline{v} = v_1 - iv_2$ be complex conjugation and for a subset W of $V^{\mathbb{C}}$ let $\overline{W} = \{\overline{v} \mid v \in W\}$. A complex subspace $W \subset V^{\mathbb{C}}$ is said to be *real* if $W = U + iU$ for a subspace $U \subset V$ and this is the case if and only if $W = \overline{W}$.

Extend Ω, $\mathrm{Sp}(V, \Omega)$, J, etc. to be $\mathbb{C}$-linear (or bilinear) on $V^{\mathbb{C}}$.

Definition 19. A subspace F of $V^{\mathbb{C}}$ is (complex) *Lagrangian* if

(i) $\Omega(v, w) = 0$ for every $v, w \in F$;
(ii) $\dim_{\mathbb{C}} F = \frac{1}{2} \dim_{\mathbb{R}} V$.

A Lagrangian subspace F is *positive* if $i\,\Omega(v, \overline{v}) \geq 0$ for every $v \in F$ and *strictly positive* if $i\,\Omega(v, \overline{v}) > 0$ for every $v \neq 0 \in F$.

A Lagrangian subspace is real if $F = W + iW$, where $W = F \cap V$ is a real Lagrangian subspace of V; F is a *complex structure* if $F \cap \overline{F} = \{0\}$. $F + \overline{F}$ and $F \cap \overline{F}$ are always real subspaces. Real Lagrangian subspaces and complex structures are the two extremes where $F \cap \overline{F}$ is as large or as small as possible.

The first result is the following.

Proposition 20. *If F and G are positive Lagrangian subspaces of $V^{\mathbb{C}}$, then $F \cap \overline{G}$ is real.*

Proof. In fact we show that $F \cap \overline{G} \subseteq F \cap \overline{F}$, which implies $F \cap \overline{G} = F \cap \overline{F} \cap G \cap \overline{G}$ and hence the result.

If $v \in F \cap \overline{G}$ then $\overline{v} \in G$, so $i\,\Omega(\overline{v}, \overline{\overline{v}}) \geq 0$. Consequently, $-i\,\Omega(v, \overline{v}) \geq 0$. But $i\,\Omega(v, \overline{v}) \geq 0$ since $v \in F$. Hence $\Omega(\overline{v}, \overline{v}) = 0$. But $i\,\Omega(v, \overline{w})$ is a positive definite form on F. So by Cauchy–Schwartz, its kernel is all vectors with $\Omega(v, \overline{v}) = 0$. But its kernel is $\overline{F}$, since F is Lagrangian. Thus $v \in F$ and $\Omega(v, \overline{v}) = 0$, which implies that $v \in F \cap \overline{F}$. Hence $F \cap \overline{G} \subseteq F \cap \overline{F}$. □

Corollary 21. *If F is positive and G is strictly positive, then $F \cap \overline{G} = \{0\}$.*

Proof. If G is strictly positive, then $G \cap \overline{G} = \{0\}$. So $F \cap \overline{G} \subseteq G \cap \overline{G} = \{0\}$. Hence $F \cap \overline{G} = \{0\}$. □

Lemma 22. $F_0 = \{(1 + iJ)v \mid v \in V\}$ *is a strictly positive Lagrangian subspace of* $V^{\mathbb{C}}$.

Proof. The lemma follows from

$$i\,\Omega((1 + iJ)v, (1 - iJ)v) = 2\Omega(v, Jv) > 0,$$

provided that $v \neq 0$. □

Corollary 23. *If F is positive, then $F \cap \overline{F_0} = \{0\}$.*

Since $V^{\mathbb{C}} = F_0 \oplus \overline{F}_0$ is a direct sum, it follows that the map $F \to F_0$ given by $v \mapsto \frac{1}{2}(1+iJ)v$ is onto. Because $\dim F = \dim F_0$, this map is a $\mathbb{C}$-linear isomorphism. The map $V \to F_0$ defined by $v \mapsto \frac{1}{2}(1+iJ)v$ is an $\mathbb{R}$-linear isomorphism. Hence, given $v \in V$ there is a unique $v_1 + i\,v_2 \in F$ so that

$$\frac{1}{2}(1+iJ)(v_1+i\,v_2) = \frac{1}{2}(1+iJ)v.$$

In other words, $v = v_1 - Jv_2$ for unique $v_1 + i\,v_2 \in F$. Define $Z_F(v) = v_1 + Jv_2$.

Proposition 24. *If F is a positive Lagrangian subspace of $V^{\mathbb{C}}$, then $v \mapsto Z_F v$ is a J-antilinear map on V, $(v, w) \to \langle v, Z_F w\rangle$ is $\mathbb{C}$-symmetric, and $1 - Z_F^2$ is positive. If F is strictly positive, then $1 - Z_F^2$ is positive definite. The correspondence $F \mapsto Z_F$ is a bijection from the set of strictly positive Lagrangian subspaces of $V^{\mathbb{C}}$ onto the Siegel domain $\mathbb{B}(V, \Omega, J)$.*

Proof. If $v = v_1 - Jv_2$ with $v_1 + i\,v_2 \in F$, then $Jv = Jv_1 + v_2$ and $v_2 - i\,v_1 = -i(v_1 + i\,v_2) \in F$. So

$$Z_F J v = -Jv_1 + v_2 = -J(v_1 - Jv_2) = -JZ_F v.$$

Consequently, Z_F is J antilinear.

Let $v = v_1 - Jv_2$, $w = w_1 - Jw_2$ where $v_1 + i\,v_2, w_1 + i\,w_2 \in F$. Then

$$\begin{aligned}
\langle v, Z_F w\rangle - \langle w, Z_F v\rangle &= \langle v, w_1 + Jw_2\rangle - \langle w, v_1 + Jv_2\rangle \\
&= \Omega(v_1 - Jv_2, J(w_1 + Jw_2)) - i\,\Omega(v_1 - Jv_2, w_1 + Jw_2) \\
&\quad - \Omega(w_1 - Jw_2, J(v_1 + Jv_2)) + i\,\Omega(w_1 - Jw_2, v_1 + Jv_2) \\
&= -\Omega(v_1, w_2) - \Omega(v_2, w_1) - \Omega(v_2, w_1) - \Omega(v_1, w_2) \\
&\quad + i[-\Omega(v_1, w_1) + \Omega(v_2, w_2) - \Omega(v_1, w_1) + \Omega(v_2, w_2)] \\
&= -2i\,\Omega(v_1 - i\,v_2, w_1 - i\,w_2) = 0,
\end{aligned}$$

since $\overline{F}$ is isotropic. Now

$$\langle Z_F^2 v, v\rangle = \langle Z_F v, Z_F v\rangle = \Omega(v_1 + Jv_2, J(v_1 + Jv_2))$$

provided that $v = v_1 - Jv_2$ with $v_1 + i\,v_2 \in F$. Then we also have

$$\langle v, v\rangle = \Omega(v_1 - Jv_2, J(v_1 - Jv_2)).$$

Consequently,

$$\langle (1 - Z_F)^2 v, v\rangle = 4\,\Omega(v_1, v_2).$$

But

$$i\,\Omega(v_1 + i\,v_2, v_1 - i\,v_2) = 2\,\Omega(v_1, v_2).$$

Thus F being positive implies that $1 - Z_F^2$ is positive. F strictly positive implies that $1 - Z_F^2$ is positive definite.

To see that we get a bijection between strictly positive Lagrangian subspaces and $\mathbb{B}(V, \Omega, J)$, we define the inverse map. For $Z \in \mathbb{B}(V, \Omega, J)$, set

$$F = \left\{ \frac{1}{2}(Z+1)v - \frac{1}{2}iJ(Z-1)v \mid v \in V \right\}.$$

Since $Z + 1$ and $Z - 1$ are invertible, the map $v \mapsto \frac{1}{2}(Z+1)v - \frac{1}{2}iJ(Z-1)v$ has real rank $\dim V$. Hence $\dim_{\mathbb{R}} F = \dim V$. But in fact F is a $\mathbb{C}$-subspace. To see this, we find that

$$\begin{aligned}
i((Z+1)v - iJ(Z-1)v) &= J(Z-1)v + i(Z+1)v \\
&= -(Z+1)Jv - iJ(-Z+1)Jv \\
&= (Z+1)(-Jv) - iJ(Z-1)(-Jv) \in F.
\end{aligned}$$

Thus F has dimension $\frac{1}{2}\dim V$ as a complex space. An easy calculation shows

$$\Omega((Z+1)v - iJ(Z-1)v, (Z+1)w - iJ(Z-1)w) = 2i(\langle Zv, w\rangle - \langle Zw, v\rangle).$$

So F is isotropic is equivalent to $\langle Zv, w\rangle = \langle Zw, v\rangle$. Positivity is similar. This completes the proof of the proposition. □

Under the correspondence defined in Proposition 24, we see that F is real implies that $v_1, v_2 \in F$ if $v_1 + i\,v_2 \in F$. So

$$\langle (1 - Z_F^2)v, v\rangle = 4\,\Omega(v_1, v_2) = 0$$

and hence $Z_F^2 = 1$. Thus the real Lagrangian Grassmannian in our picture is the set of J-antilinear maps Z of V into itself with $Z^2 = 1$ and $\langle Zv, w\rangle = \langle Zw, v\rangle$.

How does the real symplectic group $\mathrm{Sp}(V, \Omega)$ act on $\Lambda(V, \Omega)$? On Lagrangian subspaces of V it acts by $F \to g \cdot F$ for $g \in \mathrm{Sp}(V, \Omega)$. What is its action on Z? In other words, give a formula for $Z_{g\cdot F}$ in terms of g and Z_F. To answer this last question, let $v = v_1 - Jv_2$ with $v_1 + i\,v_2 \in F$. Then $gv_1 + igv_2 \in g \cdot F$. So $Z_{g\cdot F}(gv_1 - Jgv_2) = gv_1 + Jgv_2$ and $Z_F(v_1 - Jv_2) = v_1 + Jv_2$. Now

$$\begin{aligned} gv_1 - Jgv_2 &= C_g(1 + Z_g)v_1 - JC_g(1 + Z_g)v_2 \\ &= C_g(1 + Z_g)v_1 - C_gJv_2 + C_gZ_gJv_2 \\ &= C_g(v_1 - Jv_2) + C_gZ_g(v_1 + Jv_2) \\ &= C_gv + C_gZ_gZ_Fv = C_g(1 + Z_gZ_F)v \end{aligned}$$

and

$$\begin{aligned} gv_1 + Jgv_2 &= C_g(1 + Z_g)v_1 + JC_g(1 + Z_g)v_2 \\ &= C_g(v_1 + Jv_2) + C_gZ_g(v_1 - Jv_2) \\ &= C_gZ_Fv + C_gZ_gv = C_g(Z_F + Z_g)v. \end{aligned}$$

Hence

$$Z_{g\cdot F}C_g(1 + Z_gZ_F)v = C_g(Z_F + Z_g)v.$$

Since $Z_g \in \mathbb{B}(V, \Omega, J)$, it follows that $1 + Z_gZ_F$ is invertible for every positive Lagrangian subspace F. Thus

$$Z_{g\cdot F} = C_g(Z_F + Z_g)(1 + Z_gZ_F)^{-1}C_g^{-1}. \tag{3}$$

Eq. (3) is a *uniform* formula for the action of $\mathrm{Sp}(V, \Omega)$ on Lagrangian subspaces of V in terms of our parameters.

Formula (3) shows that

$$g \cdot Z = C_g(Z + Z_g)(1 + Z_gZ)^{-1}C_g^{-1}$$

is a smooth action of $\mathrm{Sp}(V, \Omega)$ on the set of antilinear maps Z of V into itself with the properties that $\langle v, Zw\rangle = \langle w, Zv\rangle$ for every $v, w \in V$ and $1 - Z^2$ is positive definite.

9. The universal covering space of the Lagrangian Grassmannian

In this section, we define a circle function on the Lagrangian Grassmannian and use it to make a construction (4) of its universal covering space $\widetilde{\Lambda(V, \Omega)}$ by analogy with the universal covering group $\widetilde{\mathrm{Sp}}(V, \Omega)$ of the symplectic group. We show that our constructions are compatible by giving an explicit action (5) of $\widetilde{\mathrm{Sp}}(V, \Omega)$ on $\widetilde{\Lambda(V, \Omega)}$. Finally, we show how to lift paths to the universal covers.

Let $\Lambda(V, \Omega)$ be the real Lagrangian Grassmannian. If $\lambda \in \Lambda(V, \Omega)$, then λ and $J\lambda$ are perpendicular with respect to the Euclidean inner product $\mathrm{Re}\,\langle\ ,\ \rangle$ on V. In particular, $V = \lambda \oplus J\lambda$ is a direct sum. Define Z_λ by $Z_\lambda v = v_1 - Jv_2$ if $v = v_1 + Jv_2$ with $v_i \in \lambda$. In this special case, we have $Z_\lambda^2 = 1$ and as before $(v, w) \mapsto \langle v, Z_\lambda w\rangle$ is a $\mathbb{C}$-symmetric bilinear form. Moreover, $\lambda = \{v \in V \mid Z_\lambda v = v\}$.

To define a circle function $\psi : \Lambda(V, \Omega) \to S^1$ for $\Lambda(V, \Omega)$, we pick a base point $\lambda_0 \in \Lambda(V, \Omega)$ and consider the function $\lambda \mapsto Z_\lambda Z_{\lambda_0}$.

Proposition 25. $Z_\lambda Z_{\lambda_0} \in \mathrm{U}(V, \Omega, J)$.

Proof. $Z_\lambda Z_{\lambda_0}$ is J-linear. Moreover,

$$\begin{aligned} \langle Z_\lambda Z_{\lambda_0}u, Z_\lambda Z_{\lambda_0}v\rangle &= \langle Z_{\lambda_0}v, Z_\lambda^2 Z_{\lambda_0}u\rangle \\ &= \langle Z_{\lambda_0}v, Z_{\lambda_0}u\rangle = \langle u, Z_{\lambda_0}^2 v\rangle = \langle u, v\rangle. \quad \square \end{aligned}$$

Therefore we can define ψ by $\lambda \mapsto \det_{\mathbb{C}}(Z_\lambda Z_{\lambda_0})$.

Lemma 26. *If U is a $\mathbb{C}$-vector space and $A, B : U \to U$ are antilinear maps, then*

$$\det_{\mathbb{C}}(AB) = \overline{\det_{\mathbb{C}}(BA)}.$$

Proof. Fix a basis to identify U with $\mathbb{C}^n$. Then A becomes $z \mapsto A_1\bar{z}$ and B becomes $z \mapsto B_1\bar{z}$ with A_1, B_1 being $n \times n$ matrices with complex entries. So $ABz = A_1\bar{B}_1 z$ and $BAz = B_1\bar{A}_1 z$. Thus

$$\det_{\mathbb{C}}(AB) = \det_{\mathbb{C}}(A_1\bar{B}_1) = \det_{\mathbb{C}} A_1\, \overline{\det_{\mathbb{C}} B_1}$$

and

$$\det_{\mathbb{C}}(BA) = \det_{\mathbb{C}}(B_1\bar{A}_1) = \det_{\mathbb{C}} B_1\, \overline{\det_{\mathbb{C}} A_1}. \quad \square$$

The above formula allows us to compute $\psi(g \cdot \lambda)$ using (3) as follows:

$$Z_{g\cdot\lambda}Z_{\lambda_0} = C_g(Z_\lambda + Z_g)(1 + Z_gZ_\lambda)^{-1}C_g^{-1}Z_{\lambda_0}.$$

Consequently,

$$\begin{aligned}
\det_{\mathbb{C}}(Z_{g\cdot\lambda}Z_{\lambda_0}) &= \det_{\mathbb{C}}\Big(\underbrace{C_g(Z_\lambda + Z_g)}_{A}\,\underbrace{(1 + Z_gZ_\lambda)C_g^{-1}Z_{\lambda_0}}_{B}\Big)\\
&= \overline{\det_{\mathbb{C}}((1 + Z_gZ_\lambda)^{-1}C_g^{-1}Z_{\lambda_0}C_g(Z_\lambda + Z_g))}\\
&= \overline{\det_{\mathbb{C}}(1 + Z_gZ_\lambda)^{-1}}\,\overline{\det_{\mathbb{C}} C_g^{-1}}\,\overline{\det_{\mathbb{C}}(\underbrace{Z_{\lambda_0}}_{A}\ \underbrace{C_g(1 + Z_gZ_\lambda)Z_\lambda}_{B})}\\
&= \overline{\det_{\mathbb{C}}(1 + Z_gZ_\lambda)^{-1}}\ \overline{\det_{\mathbb{C}} C_g^{-1}}\ \det_{\mathbb{C}}(C_g(1 + Z_gZ_\lambda)Z_\lambda Z_{\lambda_0})\\
&= \overline{\det_{\mathbb{C}}(1 + Z_gZ_\lambda)^{-1}}\,\overline{\det_{\mathbb{C}} C_g^{-1}} \cdot \det_{\mathbb{C}} C_g\ \det_{\mathbb{C}}(1 + Z_gZ_\lambda)\ \det_{\mathbb{C}}(Z_\lambda Z_{\lambda_0}).
\end{aligned}$$

Hence

$$\psi(g \cdot \lambda) = \psi(\lambda)\varphi(g)^2\, e^{i\,\nu(g,\lambda)},$$

where

$$\varphi(g) = \frac{\det C_g}{|\det C_g|} \quad \text{and} \quad e^{i\,\nu(g,\lambda)} = \left(\frac{\det_{\mathbb{C}}(1 + Z_gZ_\lambda)}{|\det_{\mathbb{C}}(1 + Z_gZ_\lambda)|}\right)^2.$$

The smooth function ν is uniquely determined if we require that $\nu(e, \lambda_0) = 0$. This then implies that $\nu(g, \lambda) = 0$ for every $g \in \mathrm{U}(V, \Omega, J)$.

Theorem 27. *There is a smooth map $\psi : \Lambda(V, \Omega) \to S^1$ inducing an isomorphism of $\pi_1(\Lambda(V, \Omega))$ with $\pi_1(S^1) = \mathbb{Z}$ and satisfying*

$$\psi(g \cdot \lambda) = \psi(\lambda)(\det_{\mathbb{C}} g)^2,$$

if $g \in \mathrm{U}(V, \Omega, J)$. There is a smooth map $\nu : \mathrm{Sp}(V, \Omega) \times \Lambda(V, \Omega) \to \mathbb{R}$ such that

$$\psi(g \cdot \lambda) = \psi(\lambda)\, \varphi(g)^2\, e^{i\,\nu(g,\lambda)}.$$

Is the map ψ unique? Obviously ψ can take any value at a given point, but then it is determined. For if ψ_1 and ψ_2 satisfy $\psi_i(g \cdot \lambda) = \psi_i(\lambda)(\det g)^2$ for every $g \in U(V, \Omega, J)$, then ψ_1/ψ_2 is invariant. Since $\mathrm{U}(V, \Omega, J)$ acts transitively on $\Lambda(V, \Omega)$, it follows that ψ_1/ψ_2 is constant. Thus the map ψ is *independent* of the choice of circle function φ. Can any choice of circle function do? Yes, for if φ_0 and ν_0 satisfy

$$\psi(g \cdot \lambda) = \psi(\lambda)\, \varphi_0(\lambda)^2\, e^{i\,\nu_0(g,\lambda)}$$

then φ/φ_0 is trivial on $\mathrm{U}(V, \Omega, J)$. So $\varphi(g) = \varphi_0(g)\, e^{i\,\tau(g)}$ for some smooth map $\tau : \mathrm{Sp}(V, \Omega) \to \mathbb{R}$ with $\tau(g) = 0$ for every $g \in \mathrm{U}(V, \Omega, J)$. Hence

$$\psi(g \cdot \lambda) = \psi(\lambda)\, \varphi(g)^2\, e^{i\,\nu(g,\lambda)},$$

where $\nu(g, \lambda) = 2\tau(g) + \nu_0(g, \lambda)$.

Define the universal covering space of $\Lambda(V, \Omega)$ by

$$\widetilde{\Lambda(V, \Omega)} = \left\{(\lambda, c) \in \Lambda(V, \Omega) \times \mathbb{R} \mid \psi(\lambda) = e^{ic}\right\} \tag{4}$$

and the covering projection $\sigma : \widetilde{\Lambda(V, \Omega)} \to \Lambda(V, \Omega) : (\lambda, c) \mapsto \lambda$. Also define a map $\rho : \widetilde{\Lambda(V, \Omega)} \to \mathbb{R} : (\lambda, c) \mapsto c$. Then

$$e^{i\,\rho(\lambda,c)} = e^{ic} = \psi(\lambda) = \psi(\sigma(\lambda, c)).$$

Consequently, $e^{i\rho} = \psi \circ \sigma$. Define an action of $\widetilde{\mathrm{Sp}}(V, \Omega)$ on $\widetilde{\Lambda(V, \Omega)}$ by

$$(g, a) \cdot (\lambda, c) = (g \cdot \lambda, c + 2a + \nu(g, \lambda)). \tag{5}$$

The above action is smooth. Moreover, $\widetilde{\mathrm{Sp}}(V, \Omega)$ acts transitively on $\widetilde{\Lambda(V, \Omega)}$. Combining the above definitions we obtain

$$\rho(\widetilde{g} \cdot \widetilde{\lambda}) = \rho(\widetilde{\lambda}) + 2\,\theta(\widetilde{g}) + \nu(\sigma(\widetilde{g}), \sigma(\widetilde{\lambda})). \tag{6}$$

Using (6), we now derive a formula for the cocycle η in terms of the function ν. Applying (6) twice gives

$$\begin{aligned}\rho((\widetilde{g_1} \cdot \widetilde{g_2}) \cdot \widetilde{\lambda}) &= \rho(\widetilde{g_2} \cdot \widetilde{\lambda}) + 2\,\theta(\widetilde{g_1}) + \nu(\sigma(\widetilde{g_1}), \sigma(\widetilde{g_2} \cdot \lambda)) \\ &= \rho(\widetilde{\lambda}) + 2\,\theta(\widetilde{g_2}) + \nu(\sigma(\widetilde{g_2}), \sigma(\widetilde{\lambda})) + 2\,\theta(\widetilde{g_1}) + \nu(\sigma(\widetilde{g_1}), \sigma(\widetilde{g_2} \cdot \widetilde{\lambda})).\end{aligned}$$

But

$$\rho((\widetilde{g_1}\widetilde{g_2}) \cdot \widetilde{\lambda}) = \rho(\widetilde{\lambda}) + 2\,\theta(\widetilde{g_1}\widetilde{g_2}) + \nu(\sigma(\widetilde{g_1}\widetilde{g_2}), \sigma(\widetilde{\lambda})).$$

Consequently,

$$\begin{aligned}2\,\eta(g_1, g_2) &= 2(\theta(\widetilde{g_1}\widetilde{g_2}) - \theta(\widetilde{g_1}) - \theta(\widetilde{g_2})) \\ &= \nu(\sigma(\widetilde{g_2}), \sigma(\widetilde{\lambda})) + \nu(\sigma(\widetilde{g_1}), \sigma(\widetilde{g_2} \cdot \widetilde{\lambda})) - \nu(\sigma(\widetilde{g_1}\widetilde{g_2}), \sigma(\widetilde{\lambda})) \\ &= \nu(g_2, \lambda) + \nu(g_1, g_2 \cdot \lambda) - \nu(g_1 \cdot g_2, \lambda).\end{aligned}$$

Given a continuous closed curve $[0, 1] \to \Lambda(V, \Omega) : t \mapsto \lambda_t$ with $\lambda_0 = \lambda_1$, we lift it to a continuous curve $[0, 1] \to \widetilde{\Lambda(V, \Omega)} : t \mapsto \widetilde{\lambda}_t = (\lambda_t, c_t)$ with c_t a continuous real-valued function of t satisfying $e^{ic_t} = \psi(\lambda_t)$ for all $t \in [0, 1]$. Then $e^{ic_1} = \psi(\lambda_1) = \psi(\lambda_0) = e^{i\lambda_0}$ and so $(c_1 - c_0)/2\pi$ is an integer. If (λ_t, d_t) is a second lift then $(c_t - d_t)/2\pi$ is integer valued and continuous; hence constant. Thus $(d_1 - d_0)/2\pi = (c_1 - c_0)/2\pi$ and so the integer is independent of which lift is chosen.

Definition 28. If λ_t is a continuous closed curve in the real Lagrangian Grassmannian $\Lambda(V, \Omega)$ and (λ_t, c_t) is a continuous lift into $\widetilde{\Lambda(V, \Omega)}$ then the integer $(c_1 - c_0)/2\pi$ constructed as above is called the *Maslov index* of the curve λ_t.

Lift λ_t to a curve $g_t \in \mathrm{Sp}(V, \Omega)$ with $\lambda_t = g_t \cdot \lambda_0$ and set $g_0 = 1$. Then we can lift g_t to a curve $\widetilde{g}_t$ in $\widetilde{\mathrm{Sp}}(V, \Omega)$ with $\widetilde{g}_0 = (1, 0)$. We have $\widetilde{g}_t = (g_t, a_t)$. Moreover,

$$\widetilde{g}_t \cdot \widetilde{\lambda}_0 = (g_t, a_t) \cdot (\lambda_0, 0) = (g_t \cdot \lambda_0, 2a_t + \nu(g_t, \lambda_0)) = (\lambda_t, 2a_t + \nu(g_t, \lambda_0)).$$

Therefore $c_1 - c_0 = 2a_1 + \nu(g_1, \lambda_0)$. In other words,

$$c_1 - c_0 = 2\theta(\widetilde{g}_1) + \nu(g_1, \lambda_0). \tag{7}$$

10. The graph map

The graph of a symplectic linear map of a symplectic vector space is a Lagrangian subspace of the symplectic double so we get a map of the real symplectic group into a Lagrangian Grassmannian. In this section, we show how our circle function for the symplectic group is related to the circle function in the Lagrangian Grassmannian of the symplectic double.

If $g \in \mathrm{Sp}(V, \Omega)$, let

$$\Gamma_g = \{(v, gv) \mid v \in V\} \subseteq V \oplus V$$

be its graph. Then Γ_g is Lagrangian in $(V \oplus V, \Omega \oplus (-\Omega))$. Hence we get a map

$$\Gamma : \mathrm{Sp}(V, \Omega) \to \Lambda(V \oplus V, \Omega \oplus (-\Omega)) : g \mapsto \Gamma_g,$$

called the *graph map*. We will use $\Gamma_{\mathrm{id}} = \Delta_V$, the diagonal, as a base point in $\Lambda(V \oplus V, \Omega \oplus (-\Omega))$.

Given J in (V, Ω) we obtain J_1 in $(V \oplus V, \Omega \oplus (-\Omega))$ defined by $\begin{pmatrix} J & 0 \\ 0 & -J \end{pmatrix}$. We then have the circle functions

$$\varphi : \mathrm{Sp}(V, \Omega) \to S^1 : g \mapsto \frac{\det C_g}{|\det C_g|}$$

and

$$\Psi : \Lambda(V \oplus V, \Omega \oplus (-\Omega)) \to S^1 : \Gamma_g \mapsto \det(Z_{\Gamma_g} Z_{\Gamma_{\mathrm{id}}}).$$

To aid computations we observe that $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ is J_1-linear if and only if A and D are J-linear and B and C are J-antilinear. We have the following special cases of Lemma 26.

Lemma 29.

$$\det\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} = \det A\,\overline{\det D},$$

$$\det\begin{pmatrix} 1 & B \\ 0 & 1 \end{pmatrix} = \det\begin{pmatrix} 1 & 0 \\ C & 1 \end{pmatrix} = 1.$$

Proposition 30.

$$Z_{\Gamma_g} = \begin{pmatrix} -Z_g & C_g^{-1} \\ (C_g^*)^{-1} & C_g Z_g C_g^{-1} \end{pmatrix} \quad \textit{and} \quad Z_{\Gamma_{\mathrm{id}}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Proof. Z_{Γ_g} is defined by

$$Z_{\Gamma_g}\begin{pmatrix} v \\ w \end{pmatrix} = \begin{pmatrix} v_1 \\ w_1 \end{pmatrix} - J_1\begin{pmatrix} v_2 \\ w_2 \end{pmatrix}$$

if

$$\begin{pmatrix} v \\ w \end{pmatrix} = \begin{pmatrix} v_1 \\ w_1 \end{pmatrix} + J_1\begin{pmatrix} v_2 \\ w_2 \end{pmatrix}$$

with $(v_i, w_i)^T \in \Gamma_g$. When $w_i = g v_i$, this means that

$$v = v_1 + J v_2$$
$$w = g v_1 - J g v_2 = C_g(1 + Z_g)v_1 - C_g(1 - Z_g)J v_2.$$

Hence

$$C_g^{-1} w = v_1 - J v_2 + Z_g(1 - Z_g)J v_2.$$

In other words,

$$v_1 - J v_2 = C_g^{-1} w - Z_g v \quad \text{and} \quad v_1 + J v_2 = v.$$

But

$$\begin{aligned} Z_{\Gamma_g}\begin{pmatrix} v \\ w \end{pmatrix} &= \begin{pmatrix} v_1 - J v_2 \\ g v_1 + J g v_2 \end{pmatrix} = \begin{pmatrix} v_1 - J v_2 \\ C_g[(v_1 + J v_2) + Z_g(v_1 - J v_2)] \end{pmatrix} \\ &= \begin{pmatrix} C_g^{-1} w - Z_g v \\ C_g[v + Z_g(C_g^{-1} w - Z_g v)] \end{pmatrix} = \begin{pmatrix} -Z_g v + C_g^{-1} w \\ C_g(1 - Z_g^2)v + C_g Z_g C_g^{-1} w \end{pmatrix}. \end{aligned}$$

The desired formula for Z_{Γ_g} follows because $1 - Z_g^2 = (C_g^* C_g)^{-1}$. Substituting $Z_{\mathrm{id}} = 0$ and $C_{\mathrm{id}} = 1$ into the preceding formula gives the second. □

Thus

$$\begin{aligned} Z_{\Gamma_g} Z_{\Gamma_{\mathrm{id}}} &= \begin{pmatrix} C_g^{-1} & -Z_g \\ C_g Z_g C_g^{-1} & (C_g^*)^{-1} \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & (C_g^*)^{-1} \end{pmatrix}\begin{pmatrix} 1 & -Z_g \\ 0 & 1 \end{pmatrix}\begin{pmatrix} (1 - Z_g^2)^{-1} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ Z_g & 1 \end{pmatrix}\begin{pmatrix} C_g & 0 \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

Consequently,

$$\begin{aligned} \det(Z_{\Gamma_g} Z_{\Gamma_{\mathrm{id}}}) &= \overline{\det(C_g^*)^{-1}} \times 1 \times \det(C_g^* C_g) \times 1 \times \det C_g^{-1} \\ &= \frac{\det \overline{C_g}}{\det C_g} = \varphi(g)^{-2}. \end{aligned}$$

Hence, with our normalisation we have proved the following theorem.

Theorem 31. *For every $g \in \mathrm{Sp}(V, \Omega)$ the circle functions are related by*

$$\Psi(\Gamma_g) = \varphi(g)^{-2}.$$

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