Chain Complexes and Assembly

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Introduction

The topological applications of algebraic \(K\)- and \(L\)-theory involve a chain level procedure which assembles an \(R[\pi]\)-module chain complex from a local system of \(R\)-module chain complexes over a space \(X\), with \(R\) a commutative ring and \(\pi\) the group of covering translations of a regular covering \(\tilde{X}\) of \(X\). In this paper we investigate the assembly of chain complexes from the categorical point of view, replacing \(X\) by a \(\Delta\)-set.

The term assembly originated with the work of Quinn on the Sullivan-Wall surgery exact sequence, interpreting the passage from the bordism of normal maps to the surgery obstruction groups as a geometric \(L\)-theory assembly map from a generalized homology theory, piecing together a non-simply-connected surgery obstruction from a local system of simply-connected normal maps over a non-simply-connected space. The algebraic \(L\)-theory assembly maps of Ranicki [14] can be constructed by extending the methods of this paper to chain complexes with Poincaré duality. The fibre of the \(L\)-theory assembly map has homotopy groups the manifold surgery structure sets. The fibre of the algebraic \(K\)-theory assembly map has homotopy groups the higher Whitehead groups, which are closely related to the pseudo-isotopy of manifolds and the Waldhausen algebraic \(K\)-theory of spaces. Assembly also features in controlled topology: local systems of chain complexes can be viewed as combinatorial analogues of the geometric algebraic objects of Quinn [11]. We make no attempt here to relate the chain complex assembly to the assembly maps in algebraic \(K\)- and \(L\)-theory, or to controlled topology, being only concerned with the passage in chain homotopy theory from local systems of \(R\)-module chain complexes over \(X\) to \(R[\pi]\)-module chain complexes afforded by assembly.

A local system \(C = \{C(x) | x \in X\}\) is a collection of \(R\)-module chain complexes \(C(x)\) indexed by the simplexes \(x \in X\) of a \(\Delta\)-set \(X\), together with \(R\)-module chain maps \(\partial_i: C(x) \to C(\partial_i x)\) satisfying the usual simplicial relations. (Readers unfamiliar with \(\Delta\)-sets can work instead with ordered simplicial complexes.) A local system \(C\) can be viewed as a contravariant functor to the category of \(R\)-module chain complexes and chain maps from the category also denoted by \(X\), with objects the simplexes of \(X\) and morphisms the face inclusions. The assembly of \(C\) is defined
in §1 to be an $R[\pi]$-module chain complex $C[\tilde{X}]$, which can be viewed as the chain homotopy colimit of the functor defined by the composite of $C$ and the covering projection $\tilde{X} \to X$.

An $R[\pi]$-module chain complex $C$ is finite if it is a bounded complex of $f.g.$ free $R[\pi]$-modules. A local system of $R$-module chain complexes $C = \{C(x) \mid x \in X\}$ is finite if each $C(x)$ is a finite $R$-module chain complex and $C(x) = 0$ for all but a finite number of $x \in X$. The assembly of a finite local system $C$ is a finite chain complex $C[\tilde{X}]$. If a finite $R[\pi]$-module chain complex $D$ is chain equivalent to an assembly $C[\tilde{X}]$ of a finite local system $C$ of $R$-module chain complexes we shall say that "$D$ can be fragmented over $X$", following the terminology of Anderson and Munkholm [2].

We shall be particularly concerned with the case of the universal cover $\tilde{X}$ of a connected $\Delta$-set $X$, with $\pi = \pi_1(X)$ the fundamental group.

**Problem.** Fix $R$, $X$, $\tilde{X}$. Which finite $R[\pi_1(X)]$-module chain complexes $D$ can be fragmented over $X$?

The (chain) homotopy theory developed in §1–§5 provides a criterion for answering the question, namely that $D$ be chain equivalent to a finite $f.g.$ free $R[\pi]$-module chain complex which is a comodule over the coalgebra defined by the cellular chain complex $\text{cl}(\tilde{X}; R)$ with respect to the Alexander-Whitney diagonal chain map. This criterion is not effective, in that it cannot be used to formulate an algebraic obstruction to fragmenting $D$ over $X$. Nevertheless, in certain cases we can decide whether fragmentation is possible or not.

The single most important property of chain assembly is that geometric chain complexes can be fragmented:

**Example 0.1.** If $f: Y \to X$ is a map between $\Delta$-sets with $Y$ finite then the cellular $R[\pi]$-module chain complex $\text{cl}(\tilde{Y}; R)$ of the pullback $\tilde{Y} = f^*\tilde{X}$ can be fragmented over $X$. For a simplicial map $f: Y \to X$ of simplicial complexes with barycentric subdivision $f': Y' \to X'$ and $Y$ finite the $R[\pi]$-module chain complex $\text{cl}(\tilde{Y}; R)$ is chain equivalent to the assembly of the local system of $R$-module chain complexes $\{C(x) \mid x \in X\}$, with $\tilde{x} \in X'$ the barycentre of $x \in X$ and $C(x) = \text{cl}(f'^{-1}(\tilde{x}); R)$.

In §1 we define the category $\mathcal{C}^f(R, X)$ of finite local systems $C = \{C(x) \mid x \in X\}$ of $R$-module chain complexes over $X$ and maps. In §2 we define a convenient full subcategory $\mathcal{C}^f(R, X) \subset \mathcal{C}^f(R, X)$ of local systems $C$ such that the structure maps $\partial_i; C(x) \to C(\partial_i x)$ $(0 \leq i \leq |x|, x \in X)$ are split injections in each dimension. The inclusion $\mathcal{C}^f(R, X) \subset \mathcal{C}^f(R, X)$ induces an equivalence of homotopy categories (2.14). We want to view local systems as equivalent if they become homotopy equivalent after assembly. In §3 we develop the theories of triangulated categories and localization necessary for making this precise. In §4 we define a triangulated category $\mathcal{K}^f(R, X)$ with the objects of $\mathcal{C}^f(R, X)$, such that a morphism $f: C \to D$ is an isomorphism if and only if it assembles with respect to the universal cover $\tilde{X}$ to a
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chain equivalence \( f[\tilde{X}] : C[\tilde{X}] \to D[\tilde{X}] \) of finite \( R[\pi_1(X)] \)-module chain complexes. Assembly defines a functor

\[
h\mathcal{E}^f(R, X) \to H\mathcal{E}^f(R[\pi_1(X)]); C \to C[\tilde{X}]
\]

to the homotopy category \( H\mathcal{E}^f(R[\pi_1(X)]) \) of finite \( R[\pi_1(X)] \)-module chain complexes. We investigate the extent to which the assembly functor is an equivalence of triangulated categories. This is done in §4 by comparing the graded endomorphism ring of a generator of \( h\mathcal{E}^f(R, X) \) with the graded endomorphism ring \( R[\pi_1(X)] \) of the generator \( R[\pi_1(X)] \) of \( H\mathcal{E}^f(R[\pi_1(X)]) \). Our main result is:

**Theorem 4.2.** The triangulated category \( h\mathcal{E}^f(R, X) \) is generated by the single object \( \Gamma \) with \( \Gamma[*] = R \) for a base vertex \( \ast \in X^{(0)} \) and \( \Gamma[x] = 0 \) for \( x \neq \ast \). The graded endomorphism ring of \( \Gamma \) is isomorphic to the Pontrjagin ring of the pointed loop space \( \Omega X \)

\[
\text{End}_*(\Gamma) \cong H_*(\Omega X; R).
\]

The assembly of \( \Gamma \) is the 0-dimensional \( R[\pi_1(X)] \)-module chain complex \( R[\pi_1(X)] \). The assembly functor \( h\mathcal{E}^f(R, X) \to H\mathcal{E}^f(R[\pi_1(X)]) \) corresponds to the augmentation map

\[
\text{End}_*(\Gamma) \cong H_*(\Omega X; R) \to \text{End}_*(R[\pi_1(X)]) \cong H_0(\Omega X; R) = R[\pi_1(X)].
\]

Dwyer and Kan [6] have obtained a similar result.

The most direct application of 4.2 is in the case \( X = K(\pi, 1) \):

**Example 0.2.** For an Eilenberg–MacLane space \( X = K(\pi, 1) \) the augmentation map \( H_*(\Omega X; R) \to R[\pi] \) is an isomorphism, so that the assembly functor \( h\mathcal{E}^f(R, X) \to H\mathcal{E}^f(R[\pi]) \) with respect to the universal cover \( \tilde{X} = E\pi \) is an equivalence of categories. Thus every finite \( R[\pi] \)-module chain complex can be fragmented over \( K(\pi, 1) \).

2-dimensional chain complexes can be fragmented. This can be deduced from 4.2, since every \( A \)-set \( X \) has the 2-skeleton of an Eilenberg–MacLane space \( K(\pi_1(X), 1) \), or else may be obtained geometrically using 0.1:

**Example 0.3.** A 2-dimensional finite \( R[\pi] \)-module chain complex

\[
C : C_2 \xrightarrow{d} C_1 \xrightarrow{d} C_0
\]

can be fragmented over any \( A \)-set \( X \), since by the Hurewicz theorem \( C \) can be realized by attaching a finite number of cells to \( X \) in dimensions \( k, k + 1, k + 2 \) for \( k \geq 2 \), to obtain a \( CW \) complex \( Y \) containing \( X \) as a retract and such that \( cl(Y, X)_r = C_{r-k} \) (\( r \geq 0 \)).
In §5 (which is independent of §§3, 4) we investigate the extra algebraic structure of chain complexes which can be fragmented. The cellular $R$-module chain complex $\cl(\tilde{X}; R)$ is a coalgebra via the Alexander-Whitney diagonal chain map $\Delta: \cl(\tilde{X}; R) \to \cl(\tilde{X}; R) \otimes_R \cl(\tilde{X}; R)$.

**Theorem 5.1.** A finite $R[\pi_1(X)]$-module chain complex $D$ can be fragmented over $X$ if and only if it is homotopy equivalent to a finite $R[\pi_1(X)]$-module chain complex $B$ which admits a coaction $\mathcal{V}: B \to \cl(\tilde{X}; R) \otimes_R B$ of $\cl(\tilde{X}; R)$ compatible with the $R[\pi_1(X)]$-actions, with $\tilde{X}$ the universal cover of $X$.

The following result holds for any covering $\tilde{X}$ of $X$, with $\pi$ the group of covering translations.

**Corollary 0.4.** If a finite $R[\pi]$-module chain complex $C$ can be fragmented over $X$ then the $R[\pi]$-module chain map

$$e = \text{augmentation} \otimes \text{id}: \cl(\tilde{X}; R) \otimes_R C \to R \otimes_R C = C$$

is a chain homotopy split surjection, with a splitting $f: C \to \cl(\tilde{X}; R) \otimes_R C$ such that $ef \simeq \text{id}: C \to C$.

In general it is not possible to fragment chain complexes of dimensions $\geq 3$. In §6 we shall use 0.4 to prove:

**Example 0.5.** The 3-dimensional finite $\mathbb{Z}[\mathbb{Z}_2]$-module chain complex $\cl(S^3)$ of the universal cover $S^3 = \tilde{\mathbb{R}P}^3$ of $\mathbb{R}P^3$ cannot be fragmented over $\mathbb{R}P^2$.

We shall use 0.5 in §6 to construct an example of a finite chain complex over an amalgamated free product of group rings (in the non-injective case) which does not have a corresponding Mayer–Vietoris decomposition.

### §1. Assembly

We refer to Rourke and Sanderson [15] for the general theory of $A$-sets.

Let $A$ be the category with objects the sets $[n] = \{0, 1, 2, \ldots, n\}$ ($n \geq 0$) and morphisms $[m] \to [n]$ the injective-order preserving functions. In particular, for $0 \leq i \leq n$ there are defined morphisms

$$\partial_i: [n - 1] \to [n]; j \mapsto \begin{cases} j & \text{if } j \leq i \\ j + 1 & \text{if } j > i \end{cases}.$$ 

A $A$-set $X$ is a contravariant functor

$$X: A \to \{\text{sets and functions}\}; [n] \to X^{(n)}.$$
Equivalently, a $\Delta$-set $X$ is a simplicial set without digeneracy operators, that is a collection of sets $X^{(n)} (n \geq 0)$ and maps $\tilde{\partial}_i: X^{(n)} \to X^{(n-1)} (0 \leq i \leq n)$ satisfying the usual relations of face maps. We regard $X$ as the category with objects the elements $x \in X^{(n)} (n \geq 0)$, writing $|x| = n$, calling them the $n$-simplices of $X$. The morphisms $e: y \to x$ in $X$ are defined by morphisms $e: [|y|] \to [|x|]$ in $\Delta$ such that 

$$e^*(x) = y \in X^{(0)}.$$ 

In particular, for each $x \in X^{(n)}$ and $0 \leq i \leq n$ there is defined a morphism $\tilde{\partial}_i: \partial_i x \to x$ in $X$.

**Example 1.1.** An ordered simplicial complex $X$ determines a $\Delta$-set (also denoted $X$) with the same simplices. The category $X$ has one object for each simplex $x \in X$ and one morphism $x \to y$ for each face inclusion $x \subseteq y$.

Given a ring $R$ and a $\Delta$-set $X$ let $\mathcal{A}(X; R)$ be the simplicial $R$-module chain complex, with $\mathcal{A}(X; R)$ the free $R$-module generated by the $n$-simplices $x \in X^{(n)}$ and differential

$$d: \mathcal{A}(X; R)_n \to \mathcal{A}(X; R)_{n-1}; x \to \sum_{i=0}^n (-1)^i \tilde{\partial}_i x.$$ 

**Definition 1.2.** i) Let $\mathcal{C}(R)$ be the category of free $R$-module chain complexes $C$ which are bounded below, such that for some $n \in \mathbb{Z} C_r = 0$ for $r < n$. A morphism in $\mathcal{C}(R)$ is a chain map $f: C \to D$.

ii) Let $\mathcal{C}^f(R) \subset \mathcal{C}(R)$ be the full subcategory of the finite complexes $C$, such that each $C_r$ is a free $R$-module and $C_r = 0$ for all but a finite number of $r \in \mathbb{Z}$.

A $\Delta$-set $X$ determines a covariant functor

$$\Delta(\cdot; R): X \to \mathcal{C}(R); x \to \mathcal{A}(x; R)$$

with $\mathcal{A}(x; R) = \mathcal{A}(\Delta^n; R)$ defined for $x \in X^{(n)}$ to be the simplicial $R$-module chain complex of the standard $n$-simplex $\Delta^n$. The generators of $\Delta(x; R)$ are the morphisms $f: z \to x$ in $X$. A morphism $e: x \to y$ in $X$ induces an $R$-module chain map

$$e^*: \Delta(x; R) \to \Delta(y; R); f \mapsto ef.$$ 

As in the introduction we assume that there is given a regular covering $\tilde{X}$ of the $\Delta$-set $X$ with group of covering translations $\pi$ and covering projection $p: \tilde{X} \to X$, and that $R$ is a commutative ring.

**Definition 1.3.** i) A local system over $[R, X]$ is a contravariant functor $\mathcal{C}: X \to \mathcal{C}(R); x \to C[x]$.

ii) The local system $C$ is finite if each $C[x]$ is a finite $R$-module chain complex and $C[x] = 0$ for all but finitely many $x \in X$.

iii) The local system $C$ is bounded below if there exists $n \in \mathbb{Z}$ with $C[x], = 0$ for all $x \in X$ and $r < n$. 


A local system $C$ over $[R, X]$ is a collection \( \{ C[x] \mid x \in X \} \) of free $R$-module chain complexes, together with structure maps $\partial_i : C[x] \to C[\partial_i x] \ (0 \leq i \leq |x|)$ satisfying the usual relations of face maps.

**Definition 1.4.** The assembly of a local system $C$ over $[R, X]$ is the $R[\pi]$-module chain complex
\[
C[\tilde{X}] = \left( \bigoplus_{y \in \tilde{Y}} \Delta(y; R) \otimes_R C[y] \right) / \sim
\]
with $\sim$ the equivalence relation generated by
\[
a \otimes f^* b \sim f_* a \otimes b \text{ for a morphism } f : y \to z \text{ in } \tilde{X},
\]
with $a \in \Delta(y; R)$ and $b \in C(z)$. \hfill \square

**Proposition 1.5.** The assembly $C[\tilde{X}]$ of a local system $C$ over $[R, X]$ is a free $R[\pi]$-module chain complex $C[\tilde{X}]$. If $C$ is finite (resp. bounded below) then the chain complex $C[\tilde{X}]$ is finite (resp. bounded below).

**Proof.** For each $x \in X$ choose a lift $\tilde{x} \in \tilde{X}$ and define $R[\pi]$-module isomorphisms
\[
\bigoplus_{x \in \tilde{X}} R[\pi] \otimes_R C[x]_{-|x|} \to C[\tilde{X}], \quad g \otimes a \to \Delta(g\tilde{x}) \otimes a \quad (g \in \pi)
\]
with $\Delta(g\tilde{x}) = 1 \in \Delta(\Delta g\tilde{x}; R)|_{|x|} = R$. \hfill \square

Let $f : Y \to X$ be a map of $\Delta$-sets, and let $\tilde{Y} = f^* \tilde{X}$ be the pullback of $\tilde{X}$ to a cover of $Y$. In 1.9 below we show that $f$ determines a local system $C$ over $[R, X]$ such that the assembly $C[\tilde{X}]$ is chain equivalent to the simplicial $R[\pi]$-module chain complex $\Delta(\tilde{Y}; R)$. $C$ is defined using the following $\Delta$-set analogue of the duals in the barycentric subdivision of a simplicial complex.

**Definition 1.6.** The barycentric subdivision of a $\Delta$-set $X$ is the $\Delta$-set $X'$ defined by the nerve of the category $X$, with one $n$-simplex for each string $x_0 \to x_1 \to \ldots \to x_n$ of composable non-identity morphisms in $X$. \hfill \square

The “under” category $x/X$ of a simplex $x \in X$ is the category with objects the morphisms $x \to y$ in $X$. A morphism $y \to z$ in $x/X$ is a commutative diagram in $X$
\[
\begin{array}{ccc}
x & \rightarrow & y \\
\downarrow & & \downarrow \\
z
\end{array}
\]

**Definition 1.7.** The dual $x^+$ of a simplex $x \in X$ is the nerve of the category $x/X$, which we regard as a $\Delta$-set. \hfill \square
The category \( x/X \) has an initial object, so that the nerve \( x^\perp \) is contractible. The rule \( x \rightarrow x^\perp \) is contravariant, i.e. every morphism \( x \rightarrow y \) in \( X \) induces a map of \( \Delta \)-sets \( y^\perp \rightarrow x^\perp \). For each \( x \) there is an evident forgetful map \( x^\perp \rightarrow X' \) (from the nerve of \( x/X \) to the nerve of \( X \)). We can write

\[
X' = \lim_{x \in X} x^\perp
\]

where the direct limit is taken over the opposite category of \( X \).

**Example 1.8.** For the \( \Delta \)-set \( X \) of a simplicial complex (as in 1.1) the barycentric subdivision \( X' \) is the \( \Delta \)-set of the barycentric subdivision of the simplicial complex. For a simplex \( x \in X \) the category \( x/X \) has one object for each simplex \( y \) such that \( x \leq y \). The dual \( x^\perp \) is the \( \Delta \)-set of the dual cell of \( x \), the simplicial complex with vertices the barycentres \( \hat{y} \) of the simplexes \( y \in X \) such that \( x \leq y \).

Let \( f: Y \rightarrow X \) be a map of \( \Delta \)-sets, and let \( f': Y' \rightarrow X' \) be its barycentric subdivision. For any simplex \( x \) in \( X \) define a \( \Delta \)-set \( f/x \) to fit into a strict pullback square of \( \Delta \)-sets

\[
\begin{array}{ccc}
  f/x & \rightarrow & Y' \\
  \downarrow & & \downarrow f \\
  x^\perp & \rightarrow & X'
\end{array}
\]

**Proposition 1.9.** The simplicial \( R[\pi] \)-module chain complex \( \Delta(\tilde{Y}; R) \) is chain equivalent to the assembly \( C[\tilde{X}] \) of the local system over \( [R, X] \) defined by

\[
C: X \rightarrow \mathcal{C}(R); x \mapsto C[x] = \Delta(f/x; R).
\]

**Proof.** Define a chain equivalence

\[
C[\tilde{X}] = \left( \bigoplus_{x \in \tilde{X}} \Delta(\tilde{x}; R) \otimes_R C[p\tilde{x}] \right) \sim \lim_{x \in \tilde{X}} C[p\tilde{x}] = \Delta(\tilde{Y}; R);
\]

\[
a \otimes b \rightarrow e(a) \otimes b
\]

with \( e: \Delta(\tilde{x}; R) \rightarrow R \) the augmentation chain maps.

An interesting phenomenon arises when \( f: Y \rightarrow X \) is a Kan fibration of \( \Delta \)-sets. In this case the \( \Delta \)-sets \( f/x \) will all be homotopy equivalent to the fibre of \( f \), since \( x^\perp \) is contractible for any \( x \) in \( X \). Therefore \( C[x] \) will have the same chain homotopy type for all \( x \) in \( X \). Such a local system \( C \) over \([R, X]\) is called homogeneous; we shall return to this notion in §4.

**Definition 1.10.** A map of local systems \( f: C \rightarrow D \) over \([R, X]\) is a natural transformation of contravariant functors \( X \rightarrow \mathcal{C}(R) \), as defined by a collection of
$R$-module chain maps $\{f[x] : C[x] \to D[x] | x \in X\}$ which commute with the structure maps.

\[\square\]

**Definition 1.11.** Let $\mathcal{C}[R, X]$ be the category of local systems over $[R, X]$ which are bounded below. Let $\mathcal{C}'[R, X] \subset \mathcal{C}[R, X]$ be the full subcategory of finite local systems.

\[\square\]

Assembly defines functors

$\mathcal{C}[R, X] \to \mathcal{C}(R[x]); C \mapsto C[\widetilde{x}]$,

$\mathcal{C}'[R, X] \to \mathcal{C}'(R[x]); C \mapsto C[\widetilde{x}]$.

There is an evident notion of homotopy $g : f \simeq f' : C \to D$ for maps of local systems over $[R, X]$, given by a collection of $R$-module chain homotopies $\{g[x] : f[x] \simeq f'[x] : C[x] \to D[x] | x \in X\}$ which commute with the structure maps.

**Definition 1.12.**

i) A map of local systems $f : C \to D$ over $[R, X]$ is a strong equivalence if it is a homotopy equivalence, that is if there exists a homotopy inverse $f^{-1} : D \to C$.

ii) A map of local systems $f : C \to D$ over $[R, X]$ is a weak equivalence if each $R$-module chain map $f[x] : C[x] \to D[x] (x \in X)$ is a chain equivalence.

\[\square\]

A strong equivalence is a weak equivalence, but the converse does not hold in general. In §2 we shall define a full subcategory $\mathcal{C}(R, X) \subset \mathcal{C}[R, X]$ of local systems for which weak equivalences are strong equivalences. An object of $\mathcal{C}(R, X)$ is a local system $C$ over $[R, X]$ such that the structure maps $\partial_i : C[x] \to C[\partial_i x]$ ($0 \leq i \leq |x|$) are compatibly split injections. A generalized mapping cylinder construction will be used in 2.14 to show that every local system over $[R, X]$ is weakly equivalent to an object in $\mathcal{C}(R, X)$.

**Example 1.13.** Let $X = d^1$ be the standard 1-simplex, with vertices 0, 1. Let $C, M$ be the finite local systems over $[Z, X]$ defined by

$C[0] = C[1] = 0$, $C[01]_0 = M[01]_0 = \mathbb{Z}$,

$d_{M[0]} = d_{M[1]} = 1 : M[0], C[1] = \mathbb{Z} \to M[0] = M[1] = \mathbb{Z}$,

$C[01]_i = M[01]_i = M[0]_i = M[1]_i = 0$ ($i \neq 0$),

$\partial_i = 1 : M[01]_0 = \mathbb{Z} \to M[i]_0 = \mathbb{Z}$ ($i = 0, 1$),

so that $M[i]$ is the mapping cylinder of $\partial_i : C[01] \to C[i]$. $M$ is an object of $\mathcal{C}(Z, X) \subset \mathcal{C}(Z, X)$ weakly equivalent to $C$. The map $f : M \to C$ of local systems over $[Z, X]$ defined by

$f[01] = 1 : M[01]_0 = \mathbb{Z} \to C[01]_0 = \mathbb{Z}$
is a weak equivalence which is not a strong equivalence, since there do not exist
chain homotopy inverses \( \{ f[x]^{-1}; C[x] \to M[x], x \in X \} \) such that
\[
\partial_i f[01]^{-1} = f[i]^{-1} \partial_i C[01] \to M[i] \quad (i = 0, 1).
\]

A map \( f: C \to D \) of local systems over \([R, X] \) induces a chain map \( f[\vec{X}]: C[\vec{X}] \to D[\vec{X}] \) of the assembly \( R[\pi] \)-module chain complexes.

**Proposition 1.14.** A weak equivalence \( f: C \to D \) of local systems over \([R, X] \) induces a chain equivalence of the assembly \( R[\pi] \)-module chain complexes \( f[\vec{X}]: C[\vec{X}] \to D[\vec{X}] \).

**Proof.** The filtration of \( X \) by skeletons induces filtrations of \( C[\vec{X}] \) and \( D[\vec{X}] \), and \( f_* \) is a chain equivalence on the filtration quotients.

\( \Box \)

\section{Equivalences}

Let \( R \) be a ring, so that \( \mathcal{C}(R) \) is defined as in 1.2.

**Definition 2.1.** The homotopy category \( H\mathcal{C}(R) \) is the category with the objects of \( \mathcal{C}(R) \) (i.e. free \( R \)-module chain complexes which are bounded from below) and morphisms the chain homotopy classes of chain maps. Also, let \( H\mathcal{C}^f(R) \) be the full subcategory of \( H\mathcal{C}(R) \) consisting of the finite complexes.

\( \Box \)

The isomorphisms in \( H\mathcal{C}(R) \) are the chain equivalences. \( H\mathcal{C}(R) \) is equivalent to the category of fractions of \( \mathcal{C}(R) \) obtained by inverting all the chain equivalences. Similarly for \( H\mathcal{C}^f(R) \) and \( \mathcal{C}^f(R) \).

**Definition 2.2.** Let \( H\mathcal{C}[R, X] \) (resp. \( H\mathcal{C}^f[R, X] \)) be the category of fractions of \( \mathcal{C}[R, X] \) (resp. \( \mathcal{C}^f[R, X] \)) obtained by inverting all the weak equivalences.

\( \Box \)

The forgetful functor \( e: \mathcal{C}[R, X] \to H\mathcal{C}[R, X] \) sends weak equivalences to isomorphisms. It is characterized by the universal property that any other functor from \( \mathcal{C}[R, X] \) which sends weak equivalences to isomorphisms factors uniquely through \( e \). Objects \( C, D \) in \( \mathcal{C}[R, X] \) are isomorphic in \( H\mathcal{C}[R, X] \) if and only if there exists a sequence of objects \( C(0) = C, C(1), C(2), \ldots, C(k) = D \) and weak equivalences \( C = C(0) \to C(1) \leftarrow \ldots \to C(k) = D \). In view of 1.13 \( H\mathcal{C}[R, X] \) is not equivalent to the category of local systems over \([R, X] \) and homotopy classes of maps. We shall now use the theory of modules over categories due to \( \text{Lück} \) [10] to define a full subcategory \( \mathcal{C}(R, X) \subset \mathcal{C}[R, X] \) such that the category of fractions of \( \mathcal{C}(R, X) \) by \( \text{invertible} \) weak equivalences is equivalent to the category of objects in \( \mathcal{C}(R, X) \) and homotopy classes of maps.
Fix a small category $J$. (Later we shall take $J = X^{op}$, the opposite of the
category of a $A$-set $X$.)

**Definition 2.3.** ([10]) An $R[J]$-module is a covariant functor

$$M: J \to \text{M}(R) = \{R\text{-modules}\}; j \mapsto M[j].$$

$M$ is weakly projective if $M[j]$ is projective for each object $j$ in $J$. $M$ is weakly free if $M[j]$ is free for each $j$.

In practice these properties are not as useful as the more categorical:

**Definition 2.4.** ([10]) i) An $R[J]$-module $M$ is strongly projective if for each surjective $R[J]$-module morphism $p: N \to M$ there exists an $R[J]$-module morphism $s: M \to N$ such that $ps: M \to M$ is the identity. (Surjectivity of $p$ means that $p[j]: N[j] \to M[j]$ is surjective for each $j$ in $J$.)

ii) A base for an $R[J]$-module $M$ is a collection $\{b(j) | j \in J\}$ of subsets $b(j) \subseteq M[j]$ such that for each object $j \in J$ the evident $R$-module morphism

$$\bigoplus_{f:i \to j} R\langle b(i) \rangle \to M[j]$$

is an isomorphism, with the direct sum ranging over all morphisms $f: i \to j$ in $J$ and $R\langle b(i) \rangle$ the free $R$-module generated by the set $b(i)$. $M$ is strongly free if there exists a base.

A strongly free $R[J]$-module $M$ is strongly projective, since any natural transformation $M \to N$ is determined by its values on the base elements $b(j)$ $(j \in J)$. Every strongly projective $R[J]$-module is a direct summand of a strongly free one.

**Definition 2.5.** A chain complex of $R[J]$-modules $C$ is strongly projective (resp. strongly free) if the $R[J]$-modules $C_r$ $(r \in \mathbb{Z})$ are strongly projective (resp. strongly free).

A chain complex $C$ of $R[J]$-modules can also be regarded as a covariant functor

$$C: J \to \{R\text{-module chain complexes}\}; j \mapsto C[j]$$

with $C[j]_r = C_r[j]$ for any $j$ in $J$, etc.

There are evident notions of chain map and chain homotopy for $R[J]$-module chain complexes.

**Definition 2.6.** i) A chain map of $R[J]$-module chain complexes $f: C \to D$ is a strong equivalence if it is a homotopy equivalence, that is if there exists a homotopy inverse $f^{-1}: D \to C$. 
ii) An $R[J]$-module chain map $f: C \to D$ is a weak equivalence if $f[J]$: $C[j] \to D[j]$ is a homotopy equivalence for each $j$ in $J$.

Again, a strong equivalence is a weak equivalence, but in general the converse does not hold.

From now on we shall only be concerned with $R[J]$-module chain complexes $C$ which are bounded from below, i.e. such that there exists $n \in \mathbb{Z}$ with $C_r = 0$ for all $r < n$.

**Proposition 2.7.** A weak equivalence $f: C \to D$ of strongly projective $R[J]$-module chain complexes is a strong equivalence.

**Proof.** By Proposition 2.5 of Ranicki [13] the chain map $f: C \to D$ is a strong equivalence if and only if the mapping cone $E$ of $f: C \to D$ is contractible as an $R[J]$-module chain complex. We assume therefore that $E$ is weakly contractible, strongly projective and bounded from below, and we have to show that it is strongly contractible. Let $r$ be the least integer such that $E_r \neq 0$; then the differential $E_{r+1} \to E_r$ is surjective, so it has a splitting since $E_r$ is strongly projective. This shows that $E$ splits into two pieces $E'$ and $E''$, where $E''$ is strongly contractible and concentrated in dimensions $r + 1$ and $r$, and where $E'$ is zero in dimensions $\leq r$. Further, $E'$ is weakly contractible and strongly projective, etc. It follows that $E$ splits into a direct sum of strongly contractible pieces.

**Lemma 2.8.** A weakly projective (resp. weakly free) $R[J]$-module $M$ admits a natural resolution by a chain complex of strongly projective (resp. strongly free) $R[J]$-modules

$$E(M): \ldots \to E_2(M) \to E_1(M) \to E_0(M)$$

such that

$$H_i(E(M)) = \begin{cases} M & \text{if } i = 0 \\ 0 & \text{if } i \geq 1 \end{cases}.$$

**Proof.** For each object $j$ in $J$, let $I^j$ be the free $R[J]$-module with base $\{b(i) | i \in I\}$ given by

$$b(i) = \begin{cases} 1 & \text{if } i = j \\ \emptyset & \text{if } i \neq j \end{cases}.$$

If $M$ is a weakly projective $R[J]$-module then the $R[J]$-module

$$T(M) = \bigoplus_{j \in J} M[j] \otimes_R I^j$$

is strongly projective. There is an obvious surjection $T(M) \to M$ whose kernel is again weakly projective (resp. weakly free). Therefore the following recursive
definition will do:

\[ E_0(M) = T(M), \quad E_1(M) = T(\ker(E_0(M) \to M)), \]
\[ E_{r+1}(M) = T(\ker(E_r(M) \to E_{r-1}(M))) \quad \text{for } r > 0. \]

\[ \square \]

**Proposition 2.9.** For any chain complex C of weakly projective (resp. weakly free) \( R[J] \)-modules there exists a chain complex \( \hat{C} \) of strongly projective (resp. strongly free) \( R[J] \)-modules, and an \( R[J] \)-module chain map \( \phi: \hat{C} \to C \) which is a weak equivalence. The construction is natural in C.

**Proof.** The rule \( E: M \to E(M) \) of 2.8 is a functor; it can therefore be applied to each chain module \( C_r \) of \( C \), and to the differentials in \( C \). The result is a "chain complex of chain complexes", or a double chain complex

\[ \ldots \to E(C_{r+1}) \to E(C_r) \to E(C_{r-1}) \to \ldots \]

whose "horizontal" differentials \( d_h: E(C_r) \to E(C_{r-1}) \) are obtained by applying \( E \) to the differentials in \( C \). (There are also the "vertical" differentials \( d_v: E_j(C_r) \to E_{j-1}(C_r), \) since each \( E(C_r) \) is a chain complex.) This double chain complex gives rise to an ordinary chain complex \( \hat{C} \) graded over \( \mathbb{Z} \), namely

\[ \hat{C}_r = \bigoplus_{p+q=r} E_p(C_q). \]

The differential \( \hat{C}_r \to \hat{C}_{r-1} \) restricted to \( E_{r-p}(C_p) \subseteq \hat{C}_r \) is equal to

\[ (d_v(-p) d_h) : E_{r-p}(C_p) \to E_{r-p-1}(C_p) \oplus E_{r-p-1}(C_{p-1}) \subseteq \hat{C}_{r-1}. \]

There is an obvious chain map \( \phi: \hat{C} \to C \) which on \( E_{r-p}(C_p) \subseteq \hat{C}_r \) is zero if \( r > p \), and agrees with the augmentation \( E_{r-p}(C_p) \to C_p \) if \( r = p \). It is not difficult to see that \( \phi \) is a weak equivalence. \[ \square \]

Suppose now that \( X \) is a \( \Delta \)-set. As in §1 regard \( X \) as a category, and let \( X^{op} \) be the opposite category. From now on \( J = X^{op} \).

The category \( \mathcal{C}(R, X) \) of 1.11 has objects the local systems of \( R \)-module chain complexes over \( X \), which are the weakly free \( R[X^{op}] \)-module chain complexes bounded from below.

**Definition 2.10.** Let \( \mathcal{C}(R, X) \) be the full subcategory of \( \mathcal{C}(R, X) \) with objects the local systems \( C \) over \([R, X]\) such that each \( C_r, (r \in \mathbb{Z}) \) is a strongly free \( R[X^{op}] \)-module.

**Example 2.11.** The local system \( C \) over \([R, X]\) determined by a map of \( \Delta \)-sets \( f: Y \to X \) in 1.9 by

\[ C[x] = \Delta(f(x); R) \quad (x \in X) \]
is a local system in \( \mathcal{C}(R, X) \), with \( C[x] \), the strongly free \( R[X^\text{op}] \)-module with one base element for each \( r \)-simplex \( (y_0 \to y_1 \to \ldots \to y_r, x \to f(y_0)) \) of \( f/x \) such that the morphism \( x \to f(y_0) \) is the identity.

\[ \square \]

**Example 2.12.** Let \( X \) be a simplicial complex (cf. 1.1), and let \( C \) be an \( R \)-module chain complex such that each \( C_r (r \in \mathbb{Z}) \) is expressed as a direct sum

\[ C_r = \bigoplus_{x \in X} C_r(x) \]

of based free \( R \)-modules, and such that

\[ d(C(x)) \subseteq \bigoplus_{y \geq x} C(y)_{r-1} \quad (x \in X, r \in \mathbb{Z}). \]

Then \( C \) determines a local system \( D \) in \( \mathcal{C}(R, X) \) with

\[ D[x] = \bigoplus_{y \geq x} C(y)_r \quad (r \in \mathbb{Z}). \]

In fact, every local system in \( \mathcal{C}(R, X) \) arises in this way. The bases for the free \( R \)-modules \( C(x), (x \in X) \) define a base for the strongly free \( R[X^\text{op}] \)-module \( D[x] \).

Associating to each base element the barycentre \( x \in X' \) of \( x \in X \) displays \( C \) as a chain complex of geometric groups (or rather \( R \)-modules) in the sense of Connell and Hollingsworth [3] and Quinn [11].

\[ \square \]

**Definition 2.13.** Let \( \mathcal{C}^f(R, X) \subset \mathcal{C}(R, X) \) be the full subcategory with objects the finite local systems \( C \) such that \( C_r[x] \) is a \( f.d. \) free \( R \)-module for all \( x \in X \) and \( r \in \mathbb{Z} \), and equal to zero for all but finitely many \( x \) and \( r \), so that

\[ \mathcal{C}^f(R, X) = \mathcal{C}^f[R, X] \cap \mathcal{C}(R, X). \]

Let \( H\mathcal{C}(R, X) \) and \( H\mathcal{C}^f(R, X) \) be the categories of fractions obtained from \( \mathcal{C}(R, X) \) and \( \mathcal{C}^f(R, X) \) by formally inverting all weak equivalences (in the sense of 2.6 ii)).

\[ \square \]

By 2.7 a morphism \( C \to D \) in the category of fractions \( H\mathcal{C}(R, X) \) is the same as a chain homotopy class of morphisms \( C \to D \) in \( \mathcal{C}(R, X) \).

**Proposition 2.14.** The inclusion \( \mathcal{C}(R, X) \subset \mathcal{C}[R, X] \) induces equivalences of categories \( H\mathcal{C}(R, X) \to H\mathcal{C}[R, X], H\mathcal{C}^f(R, X) \to H\mathcal{C}^f[R, X] \).

**Proof.** Taking \( J = X^\text{op} \) in 2.8, we have a functor from \( \mathcal{C}[R, X] \) to \( \mathcal{C}(R, X) \) which sends an object \( C \) to \( \hat{C} \), and we have a natural chain map \( \phi: \hat{C} \to C \) which is a weak equivalence for any \( C \in \mathcal{C}[R, X] \). This proves immediately that the functor \( H\mathcal{C}(R, X) \to H\mathcal{C}[R, X] \) is an equivalence of categories. The rest of the proof consists in checking that if \( C \) has certain finiteness properties, then \( \hat{C} \) also has these properties. Since \( \hat{C} \) is built from strongly projective resolutions \( E(C_r) \) of \( C_r \) for \( r \in \mathbb{Z} \), it is sufficient to check that each \( E(C_r) \) has the required finiteness properties.
This uses the following observation: if $C_r[x] = 0$ for all simplices of dimension $> s$, say, then $E_0(C_r)[x] = 0$ for all simplices of dimension $> s$, and $E_1(C_r)[x] = 0$ for all simplices of dimension $> s - 1$, and $E_2(C_r)[x] = 0$ for all simplices of dimension $> s - 2$, etc. In particular, $E_{s+1}(C_r)[x] = 0$ for all simplices $x$, so that $E(C_r)$ does have the required finiteness properties if $C$ has them.

\[\]

**Proposition 2.15.** The assembly of a local system $C$ in $\mathcal{C}(R, X)$ is such that there is defined a natural chain equivalence of free $R[\pi]$-module chain complexes

\[ u: C(\tilde{X}) \rightarrow \lim_{y \in \tilde{X}} C[py], \]

taking the direct limit over the category $\tilde{X}^{op}$.

**Proof.** $C(\tilde{X})$ is the free $R[\pi]$-module with one base element for each element of the $R[j]$-module base of $C_r$. Define $u$ in such a way that for each simplex $y$ in $\tilde{X}$ the diagram

\[
\begin{array}{ccc}
A(y) \otimes C[py] & \xrightarrow{\text{augmentation} \otimes \text{id}} & Z \otimes C[py] \\
\text{obvious} & & \text{obvious} \\
C(\tilde{X}) & \xrightarrow{u} & C(\tilde{X})
\end{array}
\]

is commutative. Clearly $u$ is onto, so it is sufficient to prove that $\ker(u)$ is contractible. Write

\[\ker(u) = \bigoplus_{y \in \tilde{X}} (\ker(A(y) \rightarrow Z) \otimes C[py])/\sim,\]

with the usual relations. Note that $\ker(A(y) \rightarrow Z)$ is contractible for all $y$. The filtration of $\tilde{X}$ by skeletons induces a filtration of $\ker(u)$, and it is easy to check by hand that the filtration quotients are contractible. Therefore $\ker(u)$ is contractible.

\[\]

§3. Triangulated Categories

A triangulated category $\mathbb{A}$ is an additive category equipped with an automorphism, the *suspension* $\Sigma: \mathbb{A} \rightarrow \mathbb{A}$, and with a distinguished class of diagrams of the form

\[B \rightarrow C \rightarrow D \rightarrow \Sigma B\]

satisfying certain axioms. See Hartshorne [9] and Verdier [16] for details. In a distinguished triangle $B \rightarrow C \rightarrow D \rightarrow \Sigma B$, we call $D$ the cofibre of the morphism $B \rightarrow C$; it is determined up to non-unique isomorphism by $B \rightarrow C$.

**Example 3.1.** The categories $\mathcal{H}(R)$, $\mathcal{H}(R, X)$ and $\mathcal{H}^J(R, X)$ are triangulated, with the suspension defined by dimension shift. The algebraic mapping cone $C(f)$ of
a chain map \( f: D \to D' \) determines a distinguished triangle \( D \to D' \to C(f) \to \Sigma D, \)
and up to isomorphism all distinguished triangles in one of these categories have
this form (by definition).

We need to recall two notions from Verdier [16], full triangulated subcategory
(sous-catégorie triangulée) and fat subcategory (sous-catégorie epaisse). A full
triangulated subcategory \( \mathcal{N} \) of \( \mathcal{A} \) is a full subcategory having the property that \( D \)
belongs to \( \mathcal{N} \) for any distinguished triangle \( B \to C \to D \to \Sigma B \) in \( \mathcal{A} \) such that \( B \) and
\( C \) belong to \( \mathcal{N} \). (Verdier merely requires that \( D \) be isomorphic to an object in \( \mathcal{N} \), but
the difference is slight.)

Suppose now that \( \mathcal{A} \) is equivalent to a small category. Given a full triangulated
subcategory \( \mathcal{N} \subset \mathcal{A} \) we can formally invert all the morphisms in \( \mathcal{A} \) whose cofibre
belongs to \( \mathcal{N} \). The resulting category of fractions is denoted \( \mathcal{A}/\mathcal{N} \). It has an explicit
description which is familiar to topologists, e.g. from chapter 14 of Part III of
Adams [1]. We repeat it briefly.

Let \( C \) be an object of \( \mathcal{A} \). Form the category \( I(C) \) whose objects are morphisms
\( e: C \to K \) in \( \mathcal{A} \) whose cofibre belongs to \( \mathcal{N} \). A morphism from \( e: C \to K \) to \( e': C \to K' \)
is an arrow \( K \to K' \) in \( \mathcal{A} \) making the appropriate diagram commutative. The
category \( I(C) \) is directed; that is, any two objects \( e, e' \) in \( I(C) \) appear in a diagram
\( e \to e' \to e' \) and for any two morphisms \( m_1, m_2: f \to g \) in \( I(C) \) there exists another
morphism \( n: g \to h \) such that \( nm_1 = nm_2 \). This follows easily from the octahedron
axiom which holds in any triangulated category. (See also Beilinson–Bernstein–Deligne [3].)

Given objects \( B, C \) in \( \mathcal{A} \) let \([B, C]\) denote the set of morphisms \( B \to C \) in \( \mathcal{A} \). Use
the directed structure of \( I(C) \) to define

\[
[[B, C]] = \lim_{e: C \to K} [B, K]
\]

where the direct limit is taken over all objects \( e \) in \( I(C) \). There is a composition law

\[
[[B, C]] \times [[C, D]] \to [[B, D]]
\]

as follows. Represent elements in \([B, C] \) and \([C, D] \) by diagrams

\[
\begin{array}{ccc}
K & \overset{s}{\to} & B \\
\downarrow e & & \downarrow t \\
C & \overset{t}{\to} & C \\
\downarrow f & & \downarrow f \\
D & & D
\end{array}
\]

such that the cofibres of \( e \) and \( f \) belong to \( \mathcal{N} \). Let \( M \) be the cofibre in \( \mathcal{A} \) of \( e \) \( t \):
\( C \to K \oplus L \). The commutative diagram

\[
\begin{array}{ccc}
K & \overset{s}{\to} & M \\
\downarrow e & & \downarrow e \\
C & \overset{t}{\to} & C \\
\downarrow f & & \downarrow f \\
D & \overset{t}{\to} & D
\end{array}
\]
represents an element of \([[[B, D]]]\) after deletion of \(C, e\) and \(t\). Summarizing, we have a new category with the same objects as \(A\) but with the morphisms from \(B\) to \(C\) given by \([[B, C]]\) rather than \([B, C]\). The universal property of the category of fractions \(A/\mathbb{N}\) gives an isomorphism from \(A/\mathbb{N}\) to the new category, which we use as an identification. \(A/\mathbb{N}\) is a triangulated category, in which a triangle is distinguished if and only if there exists an isomorphic triangle in \(A/\mathbb{N}\) which is induced from a distinguished triangle in \(A\).

Let \(\mathbb{N}\) be the full subcategory of \(A\) consisting of all objects which are isomorphic to 0 in \(A/\mathbb{N}\). Clearly \(\mathbb{N} \subset \mathbb{N}\) and \(\mathbb{N}\) is a full triangulated subcategory of \(A\) such that \(A/\mathbb{N}\) maps to \(A/\mathbb{N}\) by an isomorphism. But we need not have \(\mathbb{N} = \mathbb{N}\) — cf. example 3.4 below.

**Proposition 3.2.** If \(\mathbb{N} \subset A\) is a full triangulated subcategory then \(\mathbb{N}\) is the full subcategory consisting of all objects in \(A\) which are direct summands of objects in \(\mathbb{N}\).

**Proof.** For an object \(B\) in \(\mathbb{N}\) the identity morphism \(1: B \to B\) and the zero morphism \(0: B \to B\) agree in \(A/\mathbb{N}\), so that there exists a commutative diagram

\[
\begin{array}{ccc}
K & \xrightarrow{e} & 0
\
\downarrow & & \\
B & \xrightarrow{1} & B
\end{array}
\]

in \(A\) such that the cofibre of \(e\) belongs to \(\mathbb{N}\). But \(e \cdot 1 = 0\) means \(e = 0\), which means that the cofibre of \(e\) is isomorphic to \(\Sigma B \oplus K\) in \(A\). If this cofibre belongs to \(\mathbb{N}\) then \(\Sigma B\) is a direct summand of an object in \(\mathbb{N}\), and the same will hold for \(B\).

**Definition 3.3.** A full triangulated subcategory \(\mathbb{N}\) is **fat** if \(N = \mathbb{N}\).

From Proposition 3.2 we get the following criterion for a subcategory to be fat, which is perhaps less mysterious than that of Verdier [16].

**Proposition 3.4.** A full triangulated subcategory \(\mathbb{N} \subset A\) is fat if and only if for any object \(C\) in \(\mathbb{N}\) with a splitting \(C \cong B \oplus D\) in \(A\) the objects \(B\) and \(D\) belong to \(\mathbb{N}\).

**Example 3.5.** Let \(A = \mathcal{H} \mathcal{O}(\mathbb{Z})\), the homotopy category of finite \(\mathbb{Z}\)-module chain complexes. Let \(\mathbb{N} \subset A\) be the full triangulated subcategory consisting of the chain complexes whose Euler characteristic is zero. There exist objects in \(A\) which are not in \(\mathbb{N}\), so that \(\mathbb{N} \neq A\). Every object \(B\) in \(A\) is a direct summand of the object \(B \oplus \Sigma B\) in \(\mathbb{N}\), so that \(\mathbb{N} = A\).

**§4. The Loop Space**

Throughout this section we assume that \(X\) is a connected pointed \(\Delta\)-set, and that \(p: \tilde{X} \to X\) is the universal covering with group of covering translations \(\pi = \pi_1(X)\).
Let $R$ be a commutative ring, as before. Theorem 4.2 states our main result, relating a quotient of the triangulated category $\mathcal{C}^f(R, X)$ of finite local systems in $\mathcal{C}(R, X)$ to the Pontryagin ring $H_\bullet(\Omega X; R)$.

**Definition 4.1.** i) Let $h\mathcal{C}^f(R, X)$ be the triangulated category of finite local systems in $\mathcal{C}(R, X)$ defined by

$$h\mathcal{C}^f(R, X) = H\mathcal{C}^f(R, X)/\mathbb{N},$$

with $\mathbb{N} \subseteq H\mathcal{C}^f(R, X)$ the full triangulated subcategory consisting of all the finite local systems $C$ in $\mathcal{C}(R, X)$ such that the assembly $C[\bar{X}]$ is a contractible finite $R[\pi]$-module chain complex.

ii) For any finite local systems $B, C$ in $\mathcal{C}(R, X)$ and $n \in \mathbb{Z}$ let

$$[B, C]_n = \text{Hom}_{H\mathcal{C}^f(R, X)}(\Sigma^n B, C),$$

$$[[B, C]]_n = \text{Hom}_{h\mathcal{C}^f(R, X)}(\Sigma^n B, C).$$

It follows from 3.4 that $\mathbb{N}$ is a fat subcategory. A finite local system $C$ in $\mathcal{C}(R, X)$ is isomorphic to 0 in $h\mathcal{C}^f(R, X)$ if and only if the assembly $C[\bar{X}]$ is contractible. For any finite local system $C$ in $\mathcal{C}(R, X)$ we can replace the assembly $C[\bar{X}]$ by the chain equivalent complex $C(\bar{X})$ of 2.15, obtaining a functor of triangulated categories

$$h\mathcal{C}^f(R, X) \rightarrow H\mathcal{C}^f(R[\pi]); C \rightarrow C(\bar{X})$$

which we also call assembly. Note that $H\mathcal{C}^f(R[\pi])$ is generated as a triangulated category by the finite $R[\pi]$-module chain complex $R[\pi]$ concentrated in dimension 0.

**Theorem 4.2.** i) The triangulated category $h\mathcal{C}^f[R, X]$ is generated by the finite local system $\Gamma$ in $\mathcal{C}(R, X)$ with

$$\Gamma[x] = \begin{cases} R & \text{if } x = \ast \in X^{(0)} \\ 0 & \text{otherwise} \end{cases},$$

which assembles to $\Gamma(\bar{X}) = R[\pi]$.

ii) There is an identification of graded ring morphisms

assembly = augmentation: $[[\Gamma, \Gamma]]_0 = H_\bullet(\Omega X; R)$

$$\rightarrow [R[\pi], R[\pi]]_0 = H_0(\Omega X; R) = R[\pi].$$

The proof of 4.2 occupies the remainder of §4.

**Definition 4.3.** For each simplex $x \in X$ let $\Gamma^x$ be the finite local system in $\mathcal{C}(R, X)$ concentrated in degree 0, with $(\Gamma^*)_0$ the strongly free $R[X^{op}]$-module with base

$$b(y) = \begin{cases} \{1\} & \text{if } y = x \\ \emptyset & \text{if } y \neq x \end{cases}.$$

For the base vertex $\ast \in X$ this is the local system $\Gamma = \Gamma^\ast$ of 4.2.
The proof of 4.2 requires an effective method for describing the graded abelian group $[[B, C]]_\ast$ for finite local systems in $\mathcal{C}(R, X)$. The idea is to construct for each finite local system $C$ in $\mathcal{C}(R, X)$ a kind of “Kan extension” of $C$: an infinite local system $V^\infty C$ in $\mathcal{C}(R, X)$ which contains $C$ and such that the inclusion $C(\tilde{X}) \subset V^\infty C(\tilde{X})$ is a chain equivalence of $R[\pi]$-module chain complexes. By construction $V^\infty C$ is “homogeneous”, meaning that the $R$-module chain complexes $V^\infty C[x]$ have the same homotopy type for all simplices $x$ in $X$. One finds that

$$[[B, C]]_\ast = [B, V^\infty C]_\ast.$$

For $B = C = \Gamma$ this gives $[[\Gamma, \Gamma]]_\ast = H_\ast(\Omega X; R)$.

**Proposition 4.4.** The finite local system $\Gamma$ generates the triangulated category $h\mathcal{C}^f(R, X)$.

**Proof.** It is clear that the objects $\Gamma^x$ generate the triangulated category $H\mathcal{C}^f(R, X)$, in the following sense: the smallest full triangulated subcategory of $H\mathcal{C}^f(R, X)$ containing all the $\Gamma^x$ is $H\mathcal{C}^f(R, X)$ itself.

Suppose now that $x$ and $y$ are simplices in $X$, and $j: x \to y$ is a morphism in $X$. There is a unique map of local systems in $\mathcal{C}(R, X)$

$$\gamma^j: \Gamma^x \to \Gamma^y$$

which sends the generator of $\Gamma^x$ to $j^\ast$ (generator of $\Gamma^y$). The algebraic mapping cone $E^j$ of $\gamma^j$ is such that the assembly $E^j(\tilde{X})$ is contractible. Therefore $\Gamma^x$ and $\Gamma^y$ are isomorphic in $h\mathcal{C}^f(R, X)$.

If $x$ and $y$ are arbitrary simplices in $X$ then we can always find a sequence of morphisms $x \to x_1 \leftarrow x_2 \to \ldots \leftarrow y$ in $X$, since $X$ is connected. Therefore $\Gamma^x$ and $\Gamma^y$ are isomorphic in $h\mathcal{C}^f(R, X)$ for arbitrary $x$ and $y$.

The assembly functor

$$h\mathcal{C}^f(R, X) \to H\mathcal{C}^f(R[\pi]); C \to C(\tilde{X})$$

sends the generator $\Gamma$ to $\Gamma(\tilde{X}) = R[\pi]$, which generates the chain complex category $H\mathcal{C}^f(R[\pi])$. The assembly induces a morphism of graded rings $\alpha: [[\Gamma, \Gamma]]_\ast \to [R[\pi], R[\pi]]_\ast = R[\pi]$. Repeated application of the five lemma shows that the assembly functor $h\mathcal{C}^f(R, X) \to H\mathcal{C}^f(R[\pi])$ is an equivalence of categories if and only if $\alpha$ is an isomorphism.

**Definition 4.5.** A local system $C$ in $\mathcal{C}(R, X)$ is homogeneous if for each morphism $j: x \to y$ in $X$ the chain map $j^\ast: C[y] \to C[x]$ is a chain equivalence of $R$-module chain complexes.

**Lemma 4.6.** A local system $C$ in $\mathcal{C}(R, X)$ is homogeneous if and only if $[E^j C]_\ast = 0$ for all morphisms $j: x \to y$ in $X$.

**Proof.** A morphism $j: x \to y$ in $X$ determines a morphism $\gamma^j: \Gamma^x \to \Gamma^y$ in $\mathcal{C}(R, X)$ with mapping cone $E^j$. For any local system $C$ in $\mathcal{C}(R, X)$ the homotopy
classes of maps \( \Sigma^* E \rightarrow C \) correspond to elements of \( H_{\ast-1}(j^*: C[y] \rightarrow C[x]) \). So \([E, C] = 0\) implies that \( j^*: C[y] \rightarrow C[x] \) is a homology equivalence, and therefore a homotopy equivalence since \( C[y] \) and \( C[x] \) are free and bounded from below.

\[ \square \]

**Lemma 4.7.** A homogenous finite local system \( C \) in \( \mathcal{C}(R, X) \) with contractible assembly \( C(X) \) is contractible.

**Proof.** We use the result of 2.15 stating that \( C(X) \cong C(X) \). The filtration of \( X \) by skeletons induces a filtration of \( C[X] \), which is obvious from the definition 1.4. The filtration of \( C[X] \) gives rise to a spectral sequence converging to \( H_{\ast}(C[X]) \), whose \( E^2 \)-term can be analysed because \( C \) is homogeneous: it is

\[ E^2_{p,q} \cong H_p(X; H_q(C[x])) \]

where \( x \) can be any fixed simplex in \( X \), preferably the base vertex. (There is an obvious analogy here with the Leray–Serre spectral sequence of a fibre over \( X \).) Suppose if possible that \( C \) is not contractible. By 2.7 this amounts to saying that \( C(x) \) is not contractible for some \( x \) (and then for all \( x \) because \( C \) is homogeneous). Then \( H_{\ast}(C[x]) \neq 0 \) since \( C[x] \) is free over \( R \) and bounded from below. Let \( s \) be the least integer such that \( H_s(C[x]) \neq 0 \). A look at the differentials in the spectral sequence shows that

\[ E_{p,q}^2 = H_{p}(X; H_{q}(C[x])) \cong H_{q}(C[x]) \]

maps injectively to the \( E^\infty \)-term \( H_{q}(C[X]) \). Therefore \( C[X] \cong C(X) \) is not contractible, which is a contradiction.

\[ \square \]

**Lemma 4.8.** Let \( C \) be a finite local system in \( \mathcal{C}(R, X) \). The following conditions on a homogenous local system \( D \) in \( \mathcal{C}(R, X) \) together with a map \( e: C \rightarrow D \) are equivalent:

i) Every morphism \( g: C \rightarrow E \) in \( H\mathcal{C}(R, X) \) with \( E \) homogeneous has a unique factorization

\[ \begin{array}{ccc} & & g \\ & \searrow & \downarrow \\
C & \rightarrow & E \\
\searrow & & \\
& \nwarrow & \\
& D \end{array} \]

in \( H\mathcal{C}(R, X) \).

ii) the chain map \( C(X) \rightarrow D(X) \) induced by \( e \) is a chain equivalence of \( R[\pi] \)-module chain complexes.

**Proof.** Assume without loss of generality that \( C_r = 0 \) for \( r < 0 \). We first construct a specific homogenous local system \( D \) in \( \mathcal{C}(R, X) \) with a chain map \( e: C \rightarrow D \) satisfying i) and ii), with \( D = V^\infty C \) the direct limit of a sequence of local systems in \( \mathcal{C}(R, X) \) and inclusions

\[ C \subset VC \subset V^2C \subset V^3C \subset \ldots \]
obtained inductively as follows. Suppose that \( n > 0 \) and that \( V^{n-1}C \) has already been constructed, with \( V^0C = C \). Form

\[
\bigoplus_f D_f,
\]

the direct sum being taken over all the morphisms \( f: D_f \to V^{n-1}C \) in \( \mathcal{G}(R, X) \) where \( D_f \) has the form \( \Sigma^kE^j \) for some morphism \( f: x \to y \) in \( X \) and some integer \( k \geq -1 \). (See 4.6.) Let \( V^nC = V(V^{n-1}C) \) be the algebraic mapping cone of the tautological chain map

\[
\bigoplus_f D_f \to V^{n-1}C
\]

which agrees with \( f \) on the summand \( D_f \). It follows from 4.6 that the union

\[
V^nC = UV^nC
\]

is homogeneous: any chain map \( \Sigma^kE^j \to V^nC \) will factor through \( V^nC \) for sufficiently large \( n \), and will then be nullhomotopic in \( V^{n+1}C \) by construction, if \( k \geq -1 \). (If \( k < -1 \), then any chain map \( \Sigma^kE^j \) is zero because the chain complex \( V^nC \) is zero in dimensions \( < 0 \), whereas \( \Sigma^kE^j \) is zero in dimensions \( \geq 0 \).) Further, it is clear from the construction that the inclusions of \( R[\pi] \)-module chain complexes

\[
C(\vec{x}) \subset V^1C(\vec{x}) \subset V^2C(\vec{x}) \subset V^3C(\vec{x}) \subset \ldots
\]

are all homotopy equivalences, so that the inclusion of \( C(\vec{x}) \) in \( V^nC(\vec{x}) \) is also a homotopy equivalence.

Next we have to show that the inclusion \( e: C \to V^nC \) has the universal property i). Suppose then that \( F \) is a homogeneous local system in \( \mathcal{G}(R, X) \). We want to show that the restriction \( e^*: [V^nC, F] \to [C, F] \) is an isomorphism, where square brackets denote morphism sets in \( H\mathcal{G}(R, X) \). Suppose given a map \( g: C \to F \) of local systems in \( \mathcal{G}(R, X) \). Suppose moreover, for induction purposes, that we have already found an extension of \( g \) to a map from \( V^{n-1}C \) to \( F \), for some \( n > 0 \). Recall that \( V^nC \) was constructed from \( V^{n-1}C \) by attaching cones on objects of the form \( \Sigma^kE^j \), using attaching maps \( f: \Sigma^kE^j \to V^{n-1}C \). By 4.6, these attaching maps become nullhomotopic upon composing with the map from \( V^{n-1}C \) to \( F \), because \( F \) is homogeneous. Therefore the map \( V^{n-1}C \to F \) can be extended to a map \( V^nC \to F \), and the extension will be essentially unique. Letting \( n \) tend to infinity, one finds that \( e^*: [V^nC, F] \to [C, F] \) is indeed an isomorphism.

This shows that the inclusion \( e: C \to V^nC \) satisfies the conditions i), ii). It also shows that i) implies ii), because up to suitable isomorphism in \( H\mathcal{G}(R, X) \) there can only be one morphism \( C \to D \) having the universal property, and that particular one also satisfies condition ii).

Suppose finally that \( g: C \to D \) is a morphism to a homogeneous local system \( D \) in \( \mathcal{G}(R, X) \) such that condition ii) is satisfied, i.e. that \( g \) assembles to an \( R[\pi] \)-module chain equivalence \( C(\vec{x}) \to D(\vec{x}) \). We can use the universal property of \( e: C \to V^nC \) to construct a map of local systems \( u: V^nC \to D \) such that \( ug = g \). Then \( u \) will induce a chain equivalence \( V^nC(\vec{x}) \simeq D(\vec{x}) \) of \( R[\pi] \)-module chain complexes; from this information we ought to deduce that \( u \) is an isomorphism in \( H\mathcal{G}(R, X) \). The mapping cone \( K = C(u) \) is a homogeneous local system in \( \mathcal{G}(R, X) \).
with contractible assembly $K(X)$. By 4.7 $K$ is contractible, i.e. isomorphic to zero in 
$H\mathcal{C}(R, X)$.

\[
\square
\]

**Definition 4.9.** A homogenous envelope of a finite local system $C$ in $\mathcal{C}(R, X)$ is a homogenous local system $D$ in $\mathcal{C}(R, X)$ together with a map $e: C \rightarrow D$ satisfying the equivalent conditions of 4.8.

\[
\square
\]

**Proposition 4.10.** Every finite local system $C$ in $\mathcal{C}(R, X)$ has a homogenous envelope.

**Proof.** Immediate from the proof of 4.8.

\[
\square
\]

**Proposition 4.11.** For any two finite local systems $B, C$ in $\mathcal{C}(R, X)$ there is a natural isomorphism of graded abelian groups

\[
[[B, C]]_* \cong [B, V^\infty C]_*
\]

where $V^\infty C$ is the homogenous envelope of $C$.

**Proof.** To define a morphism from $[[B, C]]_*$ to $[B, V^\infty C]$, represent an element in $[[B, C]]_*$ by a diagram

\[
\begin{array}{c}
\Sigma^* B \\
\downarrow f \\
C
\end{array}
\]

in $H\mathcal{C}(R, X)$, where the cofibre $E$ of $g: C \rightarrow D$ has contractible assembly $E(X)$. By the characterization 4.8 ii) of homogeneous envelopes, this means that the homogeneous envelope of $E$ is zero. Since any map from $E$ to a homogeneous local system in $\mathcal{C}(R, X)$ such as $V^\infty C$ factors through the homogeneous envelope of $E$, it follows that $[E, V^\infty C]_* = 0$. Therefore $g^*: [D, V^\infty C]_* \rightarrow [C, V^\infty C]_*$ is an isomorphism, and in particular the inclusion $C \rightarrow V^\infty C$ factors through a map $j: D \rightarrow V^\infty C$. The composition $j^* \Sigma^* B \rightarrow V^\infty C$ is the required element in $[B, V^\infty C]_*$.

A morphism in the opposite direction, from $[B, V^\infty C]_*$ to $[[B, C]]_*$, is obtained as follows. We constructed $V^\infty C$ from $C$ by successively attaching cones on objects of the form $\Sigma^* E^!$. Therefore any map $f: \Sigma^* B \rightarrow V^\infty C$ has image contained in some finitely generated subcomplex $D \subseteq V^\infty C$ such that $C \subseteq D$ and $D$ is obtained from $C$ by successively attaching only finitely many cones on objects of the form $\Sigma^* E^!$. The diagram

\[
\begin{array}{c}
\Sigma^* B \\
\downarrow f \\
C
\end{array}
\]
then represents a morphism in \([[[B, C]]_\ast, \ast]\), since the inclusion \(C \to D\) induces a chain equivalence \(C(\hat{X}) \simeq D(\hat{X})\) of \(R[\pi]\)-module chain complexes.

We leave it to the reader to check that the two morphisms are inverse to one another.

\[\square\]

**Proposition 4.12.** The graded endomorphism ring \([[[\Gamma, \Gamma]]_\ast, \ast]\) of the generator \(\Gamma\) of \(h\Omega^f(R, X)\) is isomorphic to the Pontrjagin ring \(H_\ast(\Omega X; R)\).

**Proof.** Let \(f : \hat{P} \to X\) the path fibration, a Kan fibration of \(\Delta\)-sets with contractible total space \(P\). Assume also that \(P\) is pointed and that \(f\) preserves base points. Let \(E\) be the local system in \(\Omega^f(R, X)\) associated to \(P\) by the method of 1.9. The same method associates to the base point of \(P\) a subcomplex of \(E\) which we can identify with \(\Gamma\). Then \(E\) is homogeneous and the inclusion of \(\Gamma\) in \(E\) induces a homotopy equivalence \(\Gamma(\hat{X}) \simeq E(\hat{X})\). Therefore \(E\) is "the" homogeneous envelope of \(\Gamma\) and can be identified with \(\Omega^\infty \Gamma\). It follows from 4.11 that the endomorphism ring of \(\Gamma\) in \(h\Omega^f(R, X)\) is

\[
[[[\Gamma, \Gamma]]_\ast, \ast] \cong [[\Gamma, \Omega^\infty \Gamma]]_\ast \cong [[\Gamma, E]]_\ast \cong H_\ast(E[x]),
\]

where \(x\) is the base vertex of \(X\). By 1.9 there is defined a chain equivalence

\[E[x] \simeq \Delta(\Omega X; R)\]

It remains to be seen that this isomorphism of graded groups is multiplicative. This is clear from looking at the inverse isomorphism, as follows. The topological monoid \(\Omega X\) acts on the total space \(P\) of \(f : \hat{P} \to X\), by an action preserving the fibres. It follows that the Pontrjagin algebra \(\Delta(\Omega X; R)\) acts on the local system \(E\). At the level of homology this gives a ring morphism from the Pontrjagin ring \(H_\ast(\Omega X; R)\) to the ring \([E, E]]_\ast\) of endomorphisms of \(E\). But

\[[E, E]]_\ast \cong [[\Gamma, E]]_\ast \cong [[[\Gamma, \Gamma]]_\ast, \ast]
\]

since \(E\) is the homogeneous envelope of \(\Gamma\) and has the universal property with regard to maps from \(\Gamma\) to arbitrary homogeneous objects, such as \(E\) itself. The identification \([E, E]]_\ast \cong [[[\Gamma, \Gamma]]_\ast, \ast]\) is clearly also a ring isomorphism.

\[\square\]

**Example 4.13.** As an illustration of the proof of 4.12 we show how an element \(g\) in \(\pi \subset R[\pi] \cong H_0(\Omega X; R)\) corresponds to an endomorphism \(\Gamma \to \Gamma\) in \(h\Omega^f(X, X)\). Represent \(g\) by a map \(y : \text{Int} \to X\), where \(\text{Int}\) is the unit interval (suitably subdivided and made into a \(\Delta\)-set), and \(\gamma\) is a map of \(\Delta\)-sets sending the two endpoints to the base point in \(X\). Let \(i, j : \{\ast\} \to \text{Int}\) be the maps sending \(\{\ast\}\) to the endpoints 0 and 1, respectively. By the method of 1.9 the diagram

\[
\begin{array}{ccc}
\{\ast\} & \xrightarrow{i} & \text{Int} \\
\downarrow \gamma & \leftarrow & \downarrow j \\
X & & \{\ast\}
\end{array}
\]
of $\mathcal{A}$-sets over $X$ determines a diagram

$$
\Gamma \overset{i}{\rightarrow} \mathcal{C} \overset{j}{\rightarrow} \Gamma
$$

of finite local systems in $\mathcal{C}(R, X)$. Both $i$ and $j$ represent isomorphisms in $k\mathcal{C}(R, X)$, so that $i^{-1}j$ is an automorphism of $\Gamma$ in $k\mathcal{C}(R, X)$.

\section{Comodule Structures}

In this section we find it convenient to work in one of the categories $\mathcal{C}[R, X]$ or $k\mathcal{C}(R, X)$ of local systems over $[R, X]$. As in §4 $\tilde{X}$ is the universal cover of $X$. We wish to prove:

**Theorem 5.1.** A finite $R[\pi_1(X)]$-module chain complex $C$ can be fragmented over $X$ if and only if it is homotopy equivalent to a finite $R[\pi_1(X)]$-module chain complex $B$ which admits a coaction $V: B \rightarrow \Delta(\tilde{X}; R) \otimes R B$ of $\Delta(\tilde{X}; R)$ which is compatible with the actions of $R[\pi]$. 

By analogy with 4.5:

**Definition 5.2.** A local system $C$ over $[R, X]$ is homogeneous if for every morphism $j: x \rightarrow y$ in $X$ the induced chain map $j^*: C[y] \rightarrow C[x]$ is a chain equivalence.

The simplicial chain complex $\Delta(\tilde{X}; R)$ is an associative coalgebra, with comultiplication given by the Alexander–Whitney diagonal chain approximation

$$
AW: \Delta(\tilde{X}; R) \rightarrow \Delta(\tilde{X}; R) \otimes R \Delta(\tilde{X}; R).
$$

See Eilenberg and Zilber [8], Eilenberg and Moore [7]. The comultiplication is compatible with $R[\pi]$-actions if we use the diagonal $R[\pi]$-action on $\Delta(\tilde{X}; R) \otimes R \Delta(\tilde{X}; R)$.

**Proposition 5.3.** The assembly $D[\tilde{X}]$ of a local system $D$ over $[R, X]$ is a left comodule over $\Delta(\tilde{X}; R)$, i.e. admits a coaction.

**Proof.** The coaction

$$
V: D[\tilde{X}] \rightarrow \Delta(\tilde{X}; R) \otimes R D[\tilde{X}]
$$

is defined in such a way that for any simplex $y$ in $\tilde{X}$ with image $py$ in $X$, the following diagram commutes:

$$
\begin{array}{c}
\Delta(y) \otimes R D[py] \overset{AW \otimes id}{\rightarrow} (\Delta(y) \otimes R \Delta(y)) \otimes R D[py] \\
\downarrow \\
(\Delta(\tilde{X}; R) \otimes R \Delta(y)) \otimes R D[py] \\
\downarrow \\
\Delta(\tilde{X}; R) \otimes R (\Delta(y) \otimes R D[py]) \\
\downarrow \\
D[\tilde{X}] \overset{V}{\rightarrow} \Delta(\tilde{X}; R) \otimes R D[\tilde{X}].
\end{array}
$$
(Here $\Delta(y)$ is the simplicial chain complex of the simplex $y.$) Again, $V$ is compatible with $R[\pi]$-actions if we use the diagonal action on $\Delta(\tilde{X}; R) \otimes_R D[\tilde{X}]$.

We shall now obtain a partial converse to 5.3.

**Definition 5.4.** Let $D(X)$ be the category whose objects are chain complexes of free $R[\pi]$-modules, bounded from below and equipped with a coaction $\nabla: D \to \Delta(\tilde{X}; R) \otimes_R D$ which is compatible with the $R[\pi]$-actions. The morphisms in $D(X)$ are chain maps preserving the comodule structure (and commuting with $R[\pi]$-actions).

**Proposition 5.5.** For any comodule $D$ in $D(X)$ there exists another comodule $D'$ in $D(X)$ and a homogeneous local system $C$ over $[R, X]$, as well as comodule maps

$$C[\tilde{X}] \to D' \leftarrow D$$

whose underlying $R[\pi]$-module chain maps are chain equivalences.

**Proof.** We begin with some notation. If $E$ and $E'$ are comodules in $D(X)$ let $\text{Hom}_R(E, E')$ be the chain complex of comodule maps from $E$ to $E'$. In other words, $\text{Hom}_R(E, E')$ is the kernel of the chain map

$$\text{Hom}_{R[\pi]}(E, E') \to \text{Hom}_R(\Delta(\tilde{X}; R) \otimes_R E',)$$

$$f \mapsto (\text{id} \otimes f) - V'f$$

where $V: E \to \Delta(\tilde{X}; R) \otimes_R E$ and $V': E' \to \Delta(\tilde{X}; R) \otimes_R E'$ are the coactions.

For a simplex $x$ in $X$ let $\tilde{x}$ be the collection of all simplices $y$ in $\tilde{X}$ such that $p(y) = x$; denote by $\Delta(\tilde{x})$ the simplicial chain complex of their (disjoint) union, so that $\Delta(\tilde{x})$ is isomorphic to $\Delta(\tilde{X}; R) \otimes_R [\pi]$, where the isomorphism is only well determined up to translation by elements of $\pi$. The evident chain map $\Delta(\tilde{x}) \to \Delta(\tilde{X}; R)$ is a map between coalgebras (using Alexander-Whitney diagonals); in particular, $\Delta(\tilde{x})$ is a comodule over $\Delta(\tilde{X}; R)$ and belongs to $D(X)$.

Given a comodule $D$ let $V_D$ be the contravariant functor from the category $X$ to the category of $R$-module chain complexes defined by

$$V_D(x) = \text{Hom}_R(\Delta(\tilde{x}), D)$$

for any simplex $x$ in $X$. Without further assumptions on $D$ not much can be said about $V_D$; for example the chain complexes $V_D(x)$ need not be free or even projective over $R$. This does not prevent us from defining the assembly $V_D[\tilde{X}]$ by the usual formula (see 1.4); we can also endow it with a coaction $V: V_D[\tilde{X}] \to \Delta(\tilde{X}; R) \otimes_R V_D[\tilde{X}]$ using the formula in 5.3. There is an evident chain map $\mu_D: V_D[\tilde{X}] \to D$ which is compatible with the coactions; namely, if $x$ is a simplex in $X$, then the composition $\Delta(\tilde{x}) \otimes_R V_D(x) \to V_D[\tilde{X}] \to D$ agrees with the evaluation chain map $\Delta(\tilde{x}) \otimes_R \text{Hom}_R(\Delta(\tilde{x}), D) \to D$. We now seek conditions on $D$ which ensure that $V_D$ is a local system over $[R, X]$ and that $\mu_D$ is a weak equivalence. Suppose for example that $D$ is an extended comodule. This means that $D$ is
isomorphic in $\mathcal{D}(X)$ to an object of the form $A(\tilde{X}; R) \otimes_R B$, where $B$ is a chain complex of free $R[\pi]$-modules (bounded from below) and the coaction is given by

$$AW \otimes \text{id}_B: A(\tilde{X}; R) \otimes_R B \to A(\tilde{X}; R) \otimes_R A(\tilde{X}; R) \otimes_R B.$$ 

In this case $\text{Hom}_{\mathcal{D}(X)}(E, D) \cong \text{Hom}_{\mathcal{D}(X)}(E, B)$ for any other comodule $E$ in $\mathcal{D}(X)$. This shows in particular that $V_D$ is a local system over $[R, X]$, and it is also obvious that $\mu_D$ is a chain equivalence of the underlying chain complexes.

More generally, we will say that $D$ is solvable if it admits a descending filtration by $R[\pi]$-module chain subcomplexes

$$\ldots \subset \text{Fil}_2 D \subset \text{Fil}_1 D \subset \text{Fil}_0 D = D$$

such that $\text{Fil}_r D$ is a sub-comodule of $D$, and such that the quotient $\text{Fil}_r D/\text{Fil}_{r+1} D$ belongs to $\mathcal{D}(X)$ and is extended in the sense above (for all $r \geq 0$). The filtration is also required to be convergent, which means that for fixed $k \in \mathbb{Z}$ there are only finitely many $r \geq 0$ such that $\text{Fil}_r D$ is nonzero in dimensions $< k$. If in this situation $E$ is any finite dimensional object in $\mathcal{D}(X)$, then the filtration of $D$ determines a (convergent) filtration of $\text{Hom}_{\mathcal{D}(X)}(E, D)$, with quotients of the form $\text{Hom}_{\mathcal{D}(X)}(E, B(r))$.

Here $B(r)$ is a chain complex of free $R[\pi]$-modules such that $\text{Fil}_r D/\text{Fil}_{r+1} D$ is isomorphic to the extended comodule $A(\tilde{X}; R) \otimes_R B(r)$. It follows that $V_D$ is a local system over $[R, X]$ if $D$ is solvable, and that $\mu_D: V_D[\tilde{X}] \to D$ is a chain equivalence of the underlying chain complexes. (The proof for $\mu_D$ is by induction on the number of nontrivial quotients $\text{Fil}_r D/\text{Fil}_{r+1} D$; if there are infinitely many, the induction still works because of the convergence assumption.) Therefore 5.5 is a consequence of Lemma 5.6 below.

\[ \square \]

**Lemma 5.6.** Any comodule $D$ in $\mathcal{D}(X)$ admits a morphism to a solvable comodule, say $f: D \to D'$, whose underlying chain map is a chain equivalence.

**Proof.** (This is implicit in Eilenberg–Moore [7], especially in chap. 6 of [7].) Fix some $k$ in $\mathbb{Z}$ such that $D$ is zero in dimensions $\leq k$. Suppose that we have already constructed a short exact sequence

$$0 \to D \overset{i(r)}{\to} E^{(r)} \overset{e^{(r)}}{\to} F^{(r)} \to 0$$

in $\mathcal{D}(X)$ such that $H_j(E^{(r)}) = 0$ for $j \leq k + r + 1$, and such that $E^{(r)}$ is solvable with a specific filtration

$$\text{Fil}_{r+1} E^{(r)} = 0 \subset \text{Fil}_r E^{(r)} \subset \ldots \subset \text{Fil}_1 E^{(r)} \subset \text{Fil}_0 E^{(r)} = E^{(r)}$$

with $\text{Fil}_i E^{(r)}/\text{Fil}_{i+1} E^{(r)}$ extended for $0 \leq i \leq r$. Choose a chain equivalence $\omega(r): E^{(r)} \to G^{(r)}$ of free $R[\pi]$-module chain complexes, such that $G^{(r)}$ is zero in dimensions $\leq k + r + 1$. Let $E^{(r+1)}$ be the desuspension of the mapping cone of the composite comodule map

$$E^{(r)} \overset{\nu}{\to} A(\tilde{X}; R) \otimes_R E^{(r)} \overset{\text{id} \otimes \omega^{(r)} \cdot p^{(r)}}{\to} A(\tilde{X}; R) \otimes_R G^{(r)}.$$
(Note that the coaction $V$ is also a comodule map from $E^{(r)}$ to the extended comodule $\Delta(X; R) \otimes_R E^{(0)}$; note also that the mapping cone of a comodule map is again a comodule.) There is an obvious projection $E^{(r+1)} \to E^{(0)}$ and an obvious lift of $i(r); D \to E^{(r)}$ to $i(r+1); D \to E^{(r+1)}$. The cokernel $E^{(r+1)}$ of $i(r+1)$ will be such that $H_j(F^{(r+1)}) = 0$ if $j \leq k + r + 2$, and by construction the kernel of $E^{(0)} \to E^{(0)}$ is a sub-comodule which is extended (it is equal to $\Delta(X; R) \otimes_R \Sigma^{-1} G^{(0)}$). So if we take $D'$ to be the inverse limit of the $E^{(r)}$, and if we take $f; D \to D'$ to be the inverse limit of the $i(r)$, then $D'$ is solvable and $f$ is a chain equivalence of the underlying chain complexes. The induction process starts with $E^{(0)} = \Delta(X; R) \otimes_R D$ and $i(0) = V; D \to \Delta(X; R) \otimes_R D$.

Suppose that in the situation of 5.5 the underlying chain complex of $D$ has the usual finiteness properties (belongs to $\Phi_f(R[\pi])$). It is claimed in 5.1 that the chain complex $D$ is chain equivalent to the assembly of a finite local system over $[R, X]$, but 5.5 will normally yield an infinite local system over $[R, X]$ with assembly chain equivalent to $D$. This explains the need for the following lemma.

**Lemma 5.7.** If a homogeneous local system $C$ over $[R, X]$ is such that the assembly $C[X]$ is chain equivalent to a finite chain complex then there exists a finite local system $B$ over $[R, X]$ and a natural map $B \to C$ such that the induced chain map $B[X] \to C[X]$ is a chain equivalence.

**Proof.** The proof is by induction on the minimum length of a chain complex in $\Phi_f(R[\pi])$ chain equivalent to the assembly $C[X]$. For the induction step, note that the first nontrivial homology group of $C[X]$, say $H_0(C[X])$, is isomorphic to the first nontrivial homology group $H_0(C[y])$ of $C[y]$, for any simple $y$ in $X$ (preferably the base vertex). This follows from a spectral sequence argument, using the fact that $X$ is simply connected. Further, $H_0(C[y])$ agrees with the group of homotopy classes of natural chain maps from $\Sigma I^r$ to $C$ (in the notation of §4); it also agrees with the group of homotopy classes of natural chain maps from the homogeneous envelope of $\Sigma I^r$ to $C$. (See §4.) We can therefore find a natural map $e; C \to C$ of local systems over $[R, X]$ such that $C'$ is the homogeneous envelope of a direct sum of finitely many copies of $\Sigma I^r$, and such that the homomorphism from $H_0(C[x])$ to $H_0(C[x])$ induced by $e$ is a surjection.

By induction, the cofibre $C''$ of $e$, which is a homogeneous local system over $[R, X]$, admits a finite approximation $B'' \to C''$ as in 5.7. We can think of $C$ as the desuspension of the mapping cone of the connecting chain map $C'' \to \Sigma C'$; both $C''$ and $C'$ have finite approximations, and moreover $C'$ is a direct limit of finite local systems over $[R, X]$. It follows that the connecting chain map $C'' \to \Sigma C'$ has a finite approximation. Therefore $C$ has a finite approximation.

This completes the proof of 5.1.
§6. Applications

As in the introduction let \( \tilde{X} \) be a regular covering of a \( \Delta \)-set \( X \) with group of covering translations \( \pi \).

**Definition 6.1.** A finite \( R[\pi] \)-module chain complex \( D \) can be fragmented over \( X \) if it is chain equivalent to the assembly \( C[\tilde{X}] \) of a finite local system \( C = \{ C[x] | x \in X \} \) of \( R \)-module chain complexes.

By Corollary 0.4 a finite \( R[\pi] \)-module chain complex \( C \) can be fragmented over \( X \) only if the augmentation chain map \( e: \Delta(\tilde{X}; R) \otimes_R C \to C \) is a split surjection up to chain homotopy. This is a non-trivial condition which will now be used to construct an example of a chain complex which cannot be fragmented, from which we derive some applications.

**Proposition 6.2.** Let \( R = \mathbb{Z}, \pi = \mathbb{Z}_2, C = \Delta(S^3) \), with \( S^3 = \tilde{\mathbb{R}P}^3 \) the double cover of \( \mathbb{R}P^3 \). The \( \mathbb{Z}[\mathbb{Z}_2] \)-module chain complex \( C \) cannot be fragmented over \( X = \mathbb{R}P^3 \).

**Proof.** Consider the commutative diagram of spaces

\[
\begin{array}{ccc}
S^2 \times_{\mathbb{Z}_2} S^3 & \longrightarrow & \mathbb{R}P^3 \\
\downarrow & & \downarrow \\
S^2 \times_{\mathbb{Z}_2} S^\infty & \longrightarrow & \mathbb{R}P^\infty.
\end{array}
\]

The inclusion \( \mathbb{R}P^3 \to \mathbb{R}P^\infty \) induces the projection

\[ H_3(\mathbb{R}P^3) = \mathbb{Z} \to H_3(\mathbb{R}P^\infty) = \mathbb{Z}_2, \]

whereas the projection \( S^2 \times_{\mathbb{Z}_2} S^\infty \cong \mathbb{R}P^2 \to S^\infty/\mathbb{Z}_2 = \mathbb{R}P^\infty \) induces the zero map

\[ H_3(\mathbb{R}P^2) = 0 \to H_3(\mathbb{R}P^\infty) = \mathbb{Z}_2. \]

Thus \( H_3(S^2 \times_{\mathbb{Z}_2} S^3) \to H_3(\mathbb{R}P^3) \) is not onto, the augmentation chain map \( e: \Delta(S^2) \otimes \Delta(S^3) \to \Delta(S^3) \) is not a \( \mathbb{Z}[\mathbb{Z}_2] \)-module chain homotopy split surjection, and \( \Delta(S^3) \) does not fragment over \( \mathbb{R}P^2 \) by 0.4. (In fact, \( e \) induces

\[ e_* = 2: H_3(S^2 \times_{\mathbb{Z}_2} S^3) = \mathbb{Z} \to H_3(\mathbb{R}P^3) = \mathbb{Z}_2. \]

\[ \square \]

**Remark 6.3.** Example 6.2 was originally detected by the “pseudo-chain complexes” of Ranicki [12, p. 784].

A group morphism \( f: H \to G \) induces a functor

\[ f_*: (\text{left } \mathbb{Z}[H]\text{-modules}) \to (\text{left } \mathbb{Z}[G]\text{-modules}); \]

\[ P \to f_* P = \mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} P. \]

Given a union of connected \( \Delta \)-sets \( X = X_1 \cup \gamma X_2 \) write the fundamental groups as

\[ \pi_1(X) = G, \pi_1(X_\gamma) = G_\gamma (p = 1, 2), \pi_1(Y) = H, \]
let $i_p \colon H \to G_p$, $j_p \colon G_p \to G$ be the group morphisms induced by the inclusions $Y \to X_p$, $X_p \to X$ ($p = 1, 2$), and let

$$k = j_1 i_1 = j_2 i_2 : H \to G.$$ 

By the Van Kampen theorem $G = G_1 \ast_H G_2$ is an amalgamated free product defining a pushout square of groups

$$
\begin{array}{ccc}
H & \xrightarrow{i_1} & G_1 \\
\downarrow{j_1} & & \downarrow{j_1} \\
G_2 & \xrightarrow{j_2} & G
\end{array}
$$

**Definition 6.4.** A finite $R[G]$-module chain complex $C$ splits over $\Phi$ if it fits into a chain homotopy pushout square

$$
\begin{array}{ccc}
k_1 D & \xrightarrow{j_{11} f_1} & j_{11} C_1 \\
\downarrow{j_{12}} & & \downarrow{j_1} \\
 j_{21} C_2 & \xrightarrow{j_2} & C
\end{array}
$$

with $C_p$ a finite $R[G_p]$-module chain complex, $D$ a finite $R[H]$-module chain complex, and $f_p : i_p^* D \to C_p$ ($p = 1, 2$) $R[G_p]$-module chain maps.

Equivalently, $C$ is chain equivalent to the algebraic mapping cone of an $R[G]$-module chain map

$$(j_{11} f_1 j_{21} f_2) : k_1 D \to j_{11} C_1 \oplus j_{21} C_2,$$

defining a Mayer–Vietoris presentation

$$0 \to k_1 D \to j_{11} C_1 \oplus j_{21} C_2 \to C \to 0$$

up to chain homotopy. If $C$ fragments over $X$ then it splits over $\Phi$, with $C_1$, $C_2$, $D$ the assemblies of the restrictions of $C$ to the subcomplexes $X_1$, $X_2$, $Y \subset X$. (Here we interpret the word assembly as in 1.4.) The morphism $k : H \to G$ is an injection if and only if both the morphisms $i_p : H \to G_p$ ($p = 1, 2$) are injections. Waldhausen [17] proved that every f.g. free $R[G]$-module chain complex splits over $\Phi$ in the injective case. (See Quinn [11] for an account from the point of view of geometric algebra.) The example of Proposition 6.2 will be used to show that there is no such splitting in the non-injective case. We use the result of Corollary 0.2 that chain complexes can be fragmented over Eilenberg–MacLane spaces, and take $X_1$, $X_2$, $Y$ to be such. If $C$ splits over $\Phi$ then $C_1$, $C_2$, $D$ can be fragmented over $X_1$, $X_2$, $Y$ compatibly using the full strength of the equivalence of Example 0.2, and hence $C$ can be fragmented over $X$. Since $X$ need not be an Eilenberg–MacLane space this imposes a non-trivial necessary condition for $C$ to split.
Proposition 6.5. Let $R = \mathbb{Z}$, $X = X_1 \cup_\partial X_2 = \mathbb{R}P^2$, $X_1$ is the Möbius band $\tilde{M}$, $X_2 = D^2$, $Y = S^1 = \partial \tilde{M} = \partial D^2$, so that the pushout square of fundamental groups is

$$
\begin{array}{ccc}
Z & \xrightarrow{2} & Z \\
\downarrow & & \downarrow \\
\{1\} & \xrightarrow{\phi} & Z_2 \\
\end{array}
$$

The $\mathbb{Z} \langle \mathbb{Z}_2 \rangle$-module chain complex $C = \Delta(S^3)$ cannot be fragmented over $X$ (by 6.2), and hence does not split over $\Phi$.


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References


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