A GEOMETRIC FORMULATION OF SURGERY

Frank Quinn

1. Motivation

Here a theory of spaces is presented whose homotopy groups are the surgery obstruction groups of Wall [7]. These ideas form part of a thesis written at Princeton University in 1969 [3] and are similar to a treatment of the simply connected case given by Rourke in [4]. To motivate such a construction we begin with a problem such a space can be used to solve, or at least characterize.

A general question in the study of manifolds is: Given a map $f: M^m \to N^n$ of compact manifolds, when is it homotopic to some particularly nice type of map, given that some obviously necessary homotopy condition is satisfied. An example is the theorem that, if $n \geq m - 3$ and M can be Poincaré embedded in N (a homotopy condition), then f is homotopic to an embedding of M in N. The question we pose here is the complementary one, namely if $m \geq n$ when is f homotopic to some sort of fibration of M over N. If $m = n$ this is essentially the problem of deforming a homotopy equivalence to an isomorphism, which was considered by Sullivan in his thesis [6]. A partial solution for $N = S^n$ and $m \geq n + 5$ is given by Casson in [2].

The obviously necessary homotopy information for this problem is: make f into a fibration $\pi: E \to N$ with a homotopy equivalence $h: M \to E$ and $\pi h = f$; then the fiber F of π must have the homotopy type of a finite complex. In this case a spectral sequence argument shows that F must be a Poincaré duality space, and that if $V \subset N$ is a submanifold transversal to f, then $h: f^{-1}(V) \to \pi^{-1}(V)$ is a degree one map. The normal bundles of M, N, and V can be used to cover $h|f^{-1}(V)$ by a bundle map of the normal bundle of $f^{-1}(V)$, and so we actually get a surgery map. Thus if we take a triangulation of N and make f transversal to it, we get a surgery problem $h: f^{-1}(\sigma^k) \to \pi^{-1}(\sigma^k)$ over each simplex $\sigma^k \in N$. If this surgery problem is solvable over each simplex, then the solutions essentially make f into a *block fibration* with fiber $f^{-1}(\sigma^0) \simeq \pi^{-1}(\sigma^0)$, i.e., a map $f: M \to N$ transversal to a triangulation of N such that $f^{-1}(\sigma^k)$ is isomorphic with $f^{-1}(\sigma^0) \times \sigma^k$, $\sigma^k \in N$.

Now we notice that if we were to define a simplicial set of surgery problems with k-simplices surgery maps with the structure of $f^{-1}(\sigma^k) \to \pi^{-1}(\sigma^k)$, call it $\mathsf{L}_{m-n}(F)$, the transversality construction gives a map $N \to \mathsf{L}_{m-n}(F)$. If solvability correspond to null homotopy in $\mathsf{L}_{m-n}(F)$, then this map must be trivial for f to fiber. If the map is trivial, moreover, then (M, f) is homotopy equivalent and normally cobordant to a map which is a block fibration, and we are only one surgery obstruction away from a "solution" of the problem.

2. The L Spaces

Motivated, we turn to the definition and properties of the surgery spaces. First, objects with the boundary structure of a k-simplex must be defined. If Δ^k is the convex closure of $k + 1$ independent ordered vertices $(v_0,..., v_k)$ in a real linear space, then define $\partial_j\Delta^k$ = convex closure of $(\{v_0,..., v_k\} - \{v_j\})$, and for $\alpha \subseteq \{0,..., k\}$, let $\partial_\alpha\Delta^k = \bigcap_{j\in\alpha} \partial_j\Delta^k$. A *topological n-ad* is a space together with $n - 1$ subspaces denoted $(X; \partial_0X,..., \partial_{n-2}X)$, or just X. Note n denotes the number of things appearing between the parentheses. A k-simplex Δ^k has $k + 1$ faces, however, so a space with the boundary structure of Δ^k would actually have to be a $k + 2$-ad. For an n-ad X and $\alpha \subseteq \{0,..., n - 2\}$, let $\partial_\alpha X = \bigcap_{j\in\alpha} \partial_j X$. CW n-ads are defined similarly, but something more elaborate is required for manifolds and Poincaré complexes. A manifold n-ad is a topological n-ad $(M, \partial_0M,..., \partial_{n-2}M)$ which is a manifold (topological, PL, or differentiable with corners on the boundary) with ∂_jM codimension 0 submanifolds of the boundary, $\bigcup \partial_jM = \partial M$, any two faces ∂_jM and ∂_kM intersect in a codimension 0 submanifold of their boundaries, and so on. Poincaré n-ads are finite complex n-ads with the homology structure of a manifold n-ad; $(K, \bigcup_j \partial_jK)$ is a Poincaré complex, as is each face $(\partial_\alpha K, \bigcup_{j\notin\alpha} \partial_{\{j\}\cup\alpha}K)$, and the fundamental classes are related by the homology boundary map. A map of n-ads is just a face-preserving map of the total spaces.

Now to define the surgery space. Let X be a topological n-ad, $m \in Z$, and $w: \pi_1X \to Z_2$ be a homomorphism. Define $\mathsf{L}^k_m(X)$ or $\mathsf{L}^s_m(X)$ as a Δ-set (see [5], essentially a simplicial set without degeneracies) with k-simplices the set of topological surgery maps of $(k + n + 3)$-ads of dimension $m + k$, $f: M \to K$, homotopy equivalence on the $(k + 3)$rd face (simple for L^s), and with a reference map $h: K \to X$ with the orientation homomorphism $\pi_1X \to Z_2$ factoring through w and $h(\partial_{k+4+j}K) \subseteq \partial_jX$. The first $k + 2$

boundaries of this object are its boundaries as a k-simplex of the set. The $(k+3)$rd face, where it is a homotopy equivalence, is the connection with surgery, and all the rest of the faces are there to map into the faces of X.

To get a good theory, we must also assume the "vertices" are all disjoint like those of Δ^k, i.e., let $\alpha = \{0,\ldots,k\} - \{j\}$ and $\beta = \{0,\ldots,k\} - \{l\}$ for $j \neq l$, then $\partial_\alpha(f, h) \cap \partial_\beta(f, h) = \varnothing$. Comparison with chapter 9 of Wall [7] shows $\pi_j \mathsf{L}_m^s(X) = L_{j+m}(\pi_1 X, w(X))$. Where the last denotes the surgery obstruction *group*, so we have in fact geometrically realized the obstruction groups.

The L spaces are functorial on the category of topological n-ads with orientation homomorphism, natural maps being formed by composition. This functor actually takes values in the category of abelian h-spaces and homomorphisms, since a natural h-space structure is induced on $\mathsf{L}_m(X)$ by disjoint union. The empty surgery problem serves as an identity element for this structure, and will be taken as base point in what follows.

$\mathsf{L}_m(X)$ is also an infinite loop space in a natural way. The natural inclusion $\mathsf{L}_m(X) \subseteq \Omega(\mathsf{L}_{m-1}(X), \varnothing)$ obtained by considering an object in the first space as the image of a map of Δ^1 with the trivial triangulation into $\mathsf{L}_{m-1}(X)$ with empty ends, is a homotopy equivalence. An inverse is given by taking the disjoint union (over common boundaries) of the images of the simplices in a map $\Delta^1 \to \mathsf{L}_{m-1}(X)$. In fact, since objects of negative dimension are empty if $m < 0$, $\mathsf{L}_m(X)$ has only $\{\varnothing\}$ in its $|m| - 1$-skeleton. Thus for $m \leq 0$, $\mathsf{L}_{m-1}(X)$ is a classifying space $B_{\mathsf{L}_m(X)}$.

The next basic property of L is the appearance of the long exact sequence of a cofibration as the homotopy sequence of a fibration. First some notation must be established. If X is an n-ad, $\partial_j X$ is naturally an $(n-1)$-ad with faces the intersections of $\partial_j X$ with the other faces of X. Another $(n-1)$-ad $\delta_j X$ is formed by omitting $\partial_j X$ from the list of faces of X, and the natural inclusion $\partial_j X \to \delta_j X$ is a map of n-ads. Adding the empty set as a new jth face makes δ_j an n-ad again, and gives a natural map of n-ads $\delta_j X \to X$. The sequence $\partial_j X \to \delta_j X \to X$ is a cofibration sequence, and induces a sequence of maps $\mathsf{L}_n(\partial_j X) \to \mathsf{L}_n(\delta_j X) \to \mathsf{L}_n(X)$. This sequence can be continued both ways using the natural map which takes the jth boundary of an object; $\partial_j : \mathsf{L}_n(X) \to \mathsf{L}_{m-1}(\partial_j X)$. The geometric analogue of Theorem 9.6 of [7] is that each successive pair of maps in this sequence is a homotopy fibration. By a homotopy fibration we mean a pair of maps $A \xrightarrow{f} B \xrightarrow{g} C$ with a homotopy of the composite $g \circ f$ to a point so that the induced map of A into the fiber of g is a homotopy equivalence. In the above situation the definition gives a canonical homotopy of any composite to a point map, and Wall's theorem that π_j applied to the sequence of spaces gives a long exact sequence is equivalent to the statement that the induced map of the first space into the fiber of the second map induces isomorphisms of homotopy groups, and is thus a homotopy equivalence.

Since $\partial_j X$ and $\delta_j X$ are $(n-1)$-ads, this result is very useful in obtaining general n-ad results from the 1- or 2-ad case. As an example consider the proposition that a map $X \to Y$ of n-ads which is an isomorphism on fundamental groups induces a homotopy equivalence $\mathsf{L}_m(X) \to \mathsf{L}_m(Y)$ of L spaces. Write the diagram

$$\begin{array}{ccccc} \mathsf{L}_m(\partial_j X) & \to & \mathsf{L}_m(\delta_j X) & \to & \mathsf{L}_m(X) \\ \downarrow & & \downarrow & & \downarrow \\ \mathsf{L}_m(\partial_j Y) & \to & \mathsf{L}_m(\delta_j Y) & \to & \mathsf{L}_m(Y). \end{array}$$

If the proposition is true for $(n-1)$, then it follows for n by the 5-lemma. In the 1-ad case (no faces) it is sufficient to consider a map $f: X \to K(\pi_1 X, 1) = K$. Let (M_f, X) be the mapping cylinder; then the fibration sequence of the sequence $X \to K \to (M_f, X)$ reduces the problem to showing that $\mathsf{L}_0(M_f, X)$ is trivial. The homotopy groups π_j, $j \geq 6$ are zero by the surgery lemma [7, 3.3]. For low dimensions we construct inverses for the homomorphisms $\pi_j \mathsf{L}_0(X) \to \pi_j \mathsf{L}_0(K)$ directly. In dimension 4 surgery can be used to correct the fundamental group. In dimension 3 a surgery problem can be made an isomorphism on a disk. Removing a neighborhood of a 1- or 0-skeleton respectively reduces the homology dimension of the result to where obstruction theory can be used to pull the reference map into K back to X.

The same technique establishes the general case of the periodicity theorem from the 1-ad case. The periodicity theorem states that the map $\times CP^2 : \mathsf{L}_m(X) \to \mathsf{L}_{m+4}(X)$ which takes each surgery problem $M \to Y \to X$ to the problem $M \times CP^2 \to Y \times CP^2 \to X$, is a homotopy equivalence if $m \geq 5$. The 1-ad case is essentially a product formula for the algebraic characterization of the L groups given in [7, chaps. 5–8]. (This is the only place the algebra is necessary in this treatment.)

Next a construction generalizing the periodicity map is given which gives an example of a general philosophical point—that simple operations on spaces can correspond to very complicated operations on groups. Suppose M^m is a polyhedron which is a closed topological manifold, and form the Δ-set mapping space $\Delta(M, \mathsf{L}_k(X))$. Taking disjoint union (over common faces) of the image of a map $M \to \mathsf{L}_k(X)$ produces by classical gluing in the domain and a spectral sequence in the range, a degree 1 normal map of a manifold to a Poincaré complex. A reference map to $M \times X$ can also be assembled, but many different orientation homomorphisms may be encountered. Choose some $w: \pi_1(M \times X) \to Z_2$ which commutes with projection on X and the orientation homomorphism of X, and let this be the orientation homomorphism of $M \times X$. If $\Delta(M, \mathsf{L}_k(X))_w$ denotes the path components whose assembled images have orientation homomorphism commuting with w, we have constructed a map $\Lambda(M, \mathsf{L}_k(X))_w \to \mathsf{L}_{m+k}(M \times X)$.

$\mathsf{L}_{m+k}(M \times X)$ depends only on $\pi_1(M \times X)$, so most of the structures of $\Delta(M, \mathsf{L}_k(X))_w$ is irrelevant. This suggests investigating the case where M is a manifold $K(G, 1)$. We restrict ourselves to a special case where most of the details have already been worked out [8].

A *P-group* of rank 0 is the trivial group, and inductively a P-group of rank $r > 0$ is a group G which fits in an exact sequence $1 \to G' \to G \to Z \to 0$, where G' is a P-group of rank $r - 1$. According to [8] there is a manifold $K(G, 1)$, M_G, for every P-group G. This manifold is (essentially) a fibration over S^1 with fiber $M_{G'}$. The result is that if X is an n-ad with $\mathrm{Wh}(\pi_1(X) \times G') = 0$ for every P-group G' of rank $\leq r$, and G is a P-group of rank r, then the map $\Delta(M_G, \mathsf{L}_k(X))_w \to \mathsf{L}_{m+k}(X \times M_G)$ is a homotopy equivalence for $k - m - n \geq 5$.

To construct a homotopy inverse by induction over the skeleta of $\mathsf{L}_{m+k}(X \times M_G)$, we must take a surgery problem $N \to Y \to (X \times M_G) \times \Delta^k$, and a triangulation of M_G so that on the boundary [i.e., over $X \times M_G \times \partial(\Delta^k)$] the inverse images of the triangulation split the boundary of the problem into many smaller surgery problems, and extend this splitting over the inside. A little surgery to correct fundamental groups and an application of the Farrell–Hsiang splitting theorem to the composition $Y \to (X \times M_G) \to S^1$ splits it so that the resulting split object and its new boundary are objects over $X \times M_{G'}$. Induction on the rank of G allows us to assume that the proposition is true for G', so the desired decomposition of the G object is obtained by decomposing the split object over $M_{G'}$.

The Whitehead group hypothesis enters in the application of the Farrell–Hsiang theorem, which can also be applied to get a more complicated result with no such conditions. This proposition is essentially a generalization of the calculation of $L(G \times Z)$ by J. Shaneson. Substitution of a recent improved version of the splitting theorem due to S. Cappell would allow enlargement of the class of P-groups to allow extensions by fundamental groups of closed 2-manifolds. The Whitehead group hypothesis is satisfied by any P-group (even in the extended sense), and free groups, so in particular we have $\mathsf{L}_{m+k}(G) \simeq \Delta(M_G, \mathsf{L}_k(0))$, $k \geq 5$.

Next we consider the transfer map. Suppose π is a finite group, X an n-ad, and $p : X \to K(\pi, 1)$ a map. The induced covering space $\overline{X} \to X$ is also an n-ad with orientation homomorphism to Z_2 induced by one from X. The *transfer* $T_p : \mathsf{L}_k(X) \to \mathsf{L}_k(\overline{X})$ is defined by taking the induced cover over each object in $\mathsf{L}_k(X)$. Since these covers are finite, the result will be an object in $\mathsf{L}_k(\overline{X})$. The transfer is clearly natural on the category of n-ads with a homomorphism $\pi_1 X \to \pi$ to a finite group.

At the present time all that seems to be known about the transfer is a calculation for $Z^n \to (Z/PZ)^n$, and Wall's algebraic result that $T_1 : \mathsf{L}_5(Z_p) \to$

$\mathsf{L}_5(0)$ is onto in homotopy. We generalize the first result to P-groups, in our geometric setting.

Suppose $p_1 : \pi_1 M \to \pi, p_2 : \pi_1 X \to \pi', \pi$ and π' finite groups. Let $\varphi : \overline{M} \to M$ and $\overline{X} \to X$ be the corresponding coverings, then

$$\begin{array}{ccc} \Delta(M, \mathsf{L}_k(X))_w & \longrightarrow & \mathsf{L}_{m+k}(X \times M) \\ \downarrow{\scriptstyle \Delta(\varphi, T_{p_2})} & & \downarrow{\scriptstyle T_{p_2 \times p_1}} \\ \Delta(\overline{M}, \mathsf{L}_k(\overline{X}))_w & \longrightarrow & \mathsf{L}_{m+n}(\overline{X} \times \overline{M}) \end{array}$$

is commutative. In particular, if $M = M_G$, G a P-group, p_1 is onto, and $\pi_1 X$, $\pi_1 \overline{X}$ satisfy the Whitehead group requirements, then the horizontal arrows are homotopy equivalences, giving a calculation of $T_{p_2 \times p_1}$ in terms of T_{p_2} and mapping spaces.

Exact sequence arguments show that an extension of a P-group by a P-group is again a P-group (the ranks add), and a normal subgroup of finite index in a P-group is a P-group (of the same rank). This shows that if $M = M_G$, $\overline{G} = \pi_1(\overline{M})$ is also a P-group, and in fact $\overline{M}_G = M_{\overline{G}}$.

3. Surgery

Having defined the L spaces to realize the L groups, we now define spaces of homotopy structures, and of normal maps to obtain a fibration realizing the structure sequence [6, chap. 10].

Suppose X is a Poincaré n-ad of dimension m, and that $\partial_0 X$ is a manifold $(n-1)$-ad in the category $\mathscr{C}$ = (diff, PL, or top). Define Δ-sets $S^h_{\mathscr{C}}(X)$ and $S^s_{\mathscr{C}}(X)$ with k-simplices homotopy equivalences (simple for S^s) $M \to X \times \Delta^k$ of $(n+k+2)$-ads with $M \in \mathscr{C}$, and $\partial_{k+3} M \to \partial_0 X \times \Delta^k$ a $\mathscr{C}$-isomorphism. Under the same conditions define $NM_{\mathscr{C}}(X)$ as the Δ-set with k-simplices normal degree one maps of $(n+k+2)$-ads $M \to X \times \Delta^k$ which is a $\mathscr{C}$-isomorphism $\partial_{k+3} M \to \partial_0 X \times \Delta^k$.

Since a homotopy equivalence is a normal map there is a natural forgetful map $S(X) \to NM(X)$ (sub and superscripts are omitted when a statement holds for all $\mathscr{C}$ and s or h). Moreover since a $\mathscr{C}$-isomorphism is a simple homotopy equivalence, there is a natural map $NM(X) \to \mathsf{L}_m(\delta_0 X)$. The sequence $S(X) \to NM(X) \to \mathsf{L}_m(\delta_0 X)$ will turn out to be a fibration for $m + n \geq 4$, and thus $S(X) \to NM(X)$ will be a homotopy principal $\Omega(\mathsf{L}_m(\delta_0 X))$-fibration. To establish that it is a fibration without reference to the algebra involved in Wall's proof, we construct directly the action of $\mathsf{L}_{m+1}(\delta_0 X) \simeq \Omega(\mathsf{L}_m(\delta_0 X))$ on $S(X)$.

A construction like that of [7, 10.4] gives the general n-ad case from the "absolute" case X of a 2-ad. Thus suppose X is a 2-ad, and construct a homotopy of the projection $\mathsf{L}_{m+1}(\delta_0 X) \times S(X) \to \mathsf{L}_{m+1}(\delta_0 X)$ to a map such that the image of a k-simplex $(N^{n+k+1} \to Y \to \delta_0 X, M^{m+k} \to X \times \Delta^k)$ is an object $N' \to X \times I \times \Delta^k \to \delta_0 X$ which is a normal cobordism from $M \to X \times \{0\} \times \Delta^k$ to a homotopy equivalence $M' \to X \times \{1\} \times \Delta^k$, rel $\partial X \times I \times \Delta^k$. The action $\mathsf{L}_{m+1}(\delta_0 X) \times S(X) \to S(X)$ will then be given by taking the second homotopy equivalence in the image of each simplex. The construction will also show that any two actions gotten this way are homotopic.

Suppose as an induction step that the projection is homotopic to a map which has the desired property on the $k - 1$ skeleton. If the image of each k-simplex has a cobordism to an object of the desired type keeping the faces fixed, then these cobordisms can easily be used to define a homotopy of the map to a new one keeping the $k - 1$ skeleton fixed, and having the desired property on the k-skeleton. Thus we show how to improve a k-simplex.

Since faces are to be held fixed, we can forget the face structure of the objects, replacing Δ^k by the ball B^k. Recording the data, the k-simplex of $S(X)$ is a homotopy equivalence

$$(M^{m+k}; \partial_0 M, \partial_1 M) \to (X \times B^k; X \times S^{k-1}, \partial X \times B^k)$$

which is an isomorphism on the last face. The image of it and some k-simplex of $\mathsf{L}_{m+1}(\delta_0 X)$ is a surgery map of 4-ads,

$$(N^{m+k+1}; \partial_0 N_1 \partial, N, \partial_2 N) \to (Y; \partial_0 Y, \partial_1 Y, \partial_2 Y)$$

which is a homotopy equivalence on ∂_2, an isomorphism on $\partial_1 N \approx \partial X \times I \times B^h$, on ∂_0, its boundary as a k-simplex in $\mathsf{L}_{m+1}(\delta_0 X)$, it has the structure of a normal map

$$(\partial_0 N; \partial_{\{0,1\}} N, \partial^-_{\{0,2\}} N, \partial^+_{\{0,2\}} N) \to (\partial_0 Y; \partial_{\{0,1\}} Y, \partial^-_{\{0,2\}} Y, \partial^+_{\{0,2\}} Y)$$

$$(X \times I \times S^{k-1}; \partial X \times I \times S^{k-1}, X \times \{0\} \times S^{k-1}, X \times \{1\} \times S^{k-1}),$$

which is an isomorphism on the first face, homotopy equivalences on the other two, with

$$\begin{array}{ccc} \partial^-_{\{0,2\}} N & \to & X \times \{0\} \times S^{k-1} \\ \wr\wr & & \| \\ \partial_0 M & \to & X \times S^{k-1}. \end{array}$$

The cobordism to be constructed has a nice simple form if $k \geq 3$, so we give the argument in that case and then describe how to modify it.

Assume, either as part of the induction hypothesis or by a preliminary cobordism, that $N \to Y \to X$ and $\partial_0 N \to \partial_0 Y \to X \times S^{k-1}$ are restricted objects in the sense of Wall [7, chap. 9], i.e., that fundamental groups are mapped isomorphically. Now since the other boundaries of N are mapped by homotopy equivalence, and $\pi_1(\partial_0 Y) = \pi_1(X \times S^{k-1}) = \pi_1 X = \pi_1 Y$ for

$k \geq 3$, the surgery lemma provides a cobordism, fixing ∂_1 and ∂_2, of $N \to Y$ to a homotopy equivalence. The cobordism induced on $\partial_0 N$ gives a surgery map $\bar{N} \to \partial_0 Y \times I$ restricting to $\partial_0 N$ on one end, fixed on $\partial(\partial_0 Y) \times I$, and a homotopy equivalence on the other end. The whole cobordism is a homotopy of $N \to Y$ to $\bar{N} \to \partial_0 Y \times I$ as a k-simplex of $\mathsf{L}_{m+1}(\delta_0 X)$. Thinking of $\partial_0 Y \times I \approx (X \times I \times S^{k-1}) \times I$ as a collar of $(X \times I \times S^{k-1})$ in $X \times I \times B^k$, we must extend the cobordism over the rest of $X \times I \times B^k$. The homotopy equivalence $\partial_{\overline{\{0,2\}}} N \to X \times \{0\} \times S^{k-1} \times \{0\}$ is isomorphic to the original homotopy equivalence $\partial_0 M \to X \times S^{k-1}$. Thus a collar of this map in the face over $X \times \{0\} \times S^{k-1} \times I$ is isomorphic to the product $\partial_0 M \times I \to X \times S^{k-1} \times I$. Attach the map $M \times I \to X \times B^k \times I$ to $\bar{N} \to \partial_0 \times I$ by this isomorphism. The base is then isomorphic to $X \times I \times B^k$, and the total space is the desired cobordism of $M \to X \times B^k$ to another homotopy equivalence.

This construction may be desired as "use surgery to push the obstruction into a collar of $X \times I \times S^{k-1}$ which is a homotopy equivalence on the inside faces, and then extend over $X \times I \times B^k$ by attaching $M \times I \to X \times B^k \times I$ on a collar of the edge isomorphic to $\partial_0 M \to X \times S^k$." In case $k < 3$, and in particular if $k = 0$, we cannot push the problem near the boundary since the fundamental group is wrong. Thus we push it into some nice "neighborhood" of the boundary (which is empty if $k = 0$), glue on the homotopy equivalence as before, and do a little more gluing to make it the desired object.

First replace $X \times B^k$ in the range $M \to X \times B^k$ by the mapping cyclinder of the inverse on the boundary $Z = \mathscr{C}(X \times S^{k-1} \to M)$ (recall a mapping cylinder of a homotopy equivalence is a Poincaré h-cobordism). Let H be a handlebody on $\partial_0 M$, which is the union of the 0-, 1-, and 2-handles of $(M, \partial_0 M)$, then as the union of the boundary spheres of these handles is one-dimensional, the homotopy equivalence $X \times S^{k-1} \to \partial_0 M$ can be made an isomorphism over them, and the mapping cylinder Z then contains a copy of H. We can assume, moreover, that H is mapped isomorphically under $M \to Z$ to its copy in Z.

Now suppose the surgery problem $N \to Y$ is restricted (already adjusted on π_1 [7, chap. 9]), and that Y contains a copy of $H \times I$. The boundary then has the same fundamental group as the complement, so surgery can be used to push the obstruction over $H \times I$ (i.e., a cobordism to a surgery problem over $H \times I$, homotopy equivalence over the boundary). Now if we glue on $M \times I \to Z \times I$ as above, we can obtain $Z \times I$ in the range by identifying the copy of H in $Z \times \{1\}$ with the copy $H \times \{0\}$ contained in the range of the surgery problem. If $M \times I \to Z \times I$ and the new surgery problem $N' \to H \times I$ were isomorphisms over these copies of H, then the identification can be duplicated in the domain to give a surgery problem with the properties desired. By construction $M \times I \to Z \times I$ is an isomorphism over $H \times \{1\}$. On the other hand, $N' \to H \times I$ restricted to $\partial(H \times I) - \partial_0 H \times I$ is a homotopy equivalence, which by construction will be normally cobordant to the identity. Thus it can be made an isomorphism over low-dimensional handles, in particular $H \times \{0\}$.

To complete the construction we must obtain a cobordism of $N \to Y$ (rel ∂_0) to a restricted object which contains a copy of $H \times I$. This is easily done. Suppose first that $N \to Y$ is restricted, then we can assume that $(Y, \partial_0 Y)$ has the same 1-skeleton as $(H, \partial_0 H)$. Take a manifold neighborhood of this 1-skeleton as in [9], then to obtain a copy of H we need only do surgery on the copies of S^1 embedded in the boundary of this 1-skeleton by the handle decomposition of H, and take the handles of index ≤ 2 in the result. This surgery is done on embedded copies of S^1, and hence can be covered by surgeries of N. The resulting object is still restricted and contains $H \times I$.

Thus we have obtained an action $\mathsf{L}_{m+1}(\delta_0 X) \times S(X) \to S(X)$. The same formal considerations as in [7, chap. 10] now prove that $S(X) \to NM(X)$ is a homotopy principal $\mathsf{L}_{m+1}(\delta_0 X)$-fibration, and that

$$\mathsf{L}_{m+1}(\delta_0) \to S(X) \to NM(X) \to \mathsf{L}_m(\delta_0 X)$$

is a sequence of homotopy fibrations, $m - n \geq 4$. This is called the *structure sequence* for the Poincaré n-ad X.

The structure sequence is natural in the following sense. Let X be a Poincaré n-ad, $n \geq 2$, and let $S_0 X$ denote the $(n + 1)$-ad with $\partial_0(S_0 X) = \varnothing$. In the following commutative diagram any two consecutive horizontal or vertical maps are a homotopy fibration.

$$\begin{array}{ccccc}
S(X) & \to & NM(X) & \to & \mathsf{L}_m(\delta_0 X) \\
\downarrow & & \downarrow & & \downarrow \\
S(S_0 X) & \to & NM(S_0 X) & \to & \mathsf{L}_m(X) \\
\downarrow{\scriptstyle \partial_1} & & \downarrow{\scriptstyle \partial_1} & & \downarrow \\
S(S_0 \partial_0 X) & \to & NM(S_0 \partial_0 X) & \to & \mathsf{L}_{m-1}(\partial_0 X).
\end{array}$$

The structure of $NM(X)$ can be refined somewhat. Since part of the structure of each object of $NM(X)$ is a reduction of the stable homotopy normal bundle to the classifying space of the category, which agrees with the given one on $\partial_0 X$, we get a map

$$NM(X) \to \Delta(X, B_{\mathscr{C}}).$$

Now supposing that $NM(X) \neq \varnothing$, choose one such map and subtract it off all the others. Since they are all reductions of the same G-bundle, the difference maps to the base point in $B_{\mathscr{C}}$, and so the map lifts to the fiber,

$$NM(X) \to \Delta(X, \partial_0 X; G/\mathscr{C}, *).$$

In case $\mathscr{C}$ = top, these mapping spaces are formed with X replaced by a polyhedral n-ad of the same homotopy type. This map is a homotopy equivalence, with homotopy inverse constructed using transversality (in the topological case the dimension restriction $m - n \geq 5$ is still required). The reduction of normal bundle ν_X to $B_{\mathscr{C}}$ gives X a $\mathscr{C}$-normal bundle in the Thom space $T\nu_X$. $\mathscr{C}$-transversality applied to the canonical reduction $f : S^l \to T\nu_X^l$ representing the top homology class of $T\nu_X$ gives a map $f^{-1}(X) \to X$ which is covered by a bundle map by the definition of transversality, and is degree one by the Thom isomorphism.

Thus if $NM(X) \neq \varnothing$, the structure sequence can be written

$$S(X) \to \Delta(X, \partial_0 X; G/\mathscr{C}, *) \to \mathsf{L}_m(\delta_0 X).$$

Note however that this still does not imply $S(X) \neq \varnothing$, since it is the fiber over the *identity component* of $\mathsf{L}_m(\delta_0 X)$, and the image of $\Delta(X, \partial_0 X; G/\mathscr{C}, *)$ may be disjoint from that component.

Now we apply some of the L space results to the structure sequence. First the Poincaré conjecture for top and PL implies $S_{\mathrm{PL}}(D^5, S^4) \simeq S_{\mathrm{top}}(D^5, S^4) \simeq pt$. This calculates the "coefficient" space; $\mathsf{L}_5(0) \simeq \Omega^5(G/\mathrm{PL}) \simeq \Omega^5(G/\mathrm{top})$. Now all of these satisfy some sort of periodicity induced by cartesian product with CP^2, G/top being exactly periodic by recent calculations of Kirby-Siebenmann. Define a map $h : \mathsf{L}_0(0) \to G/\mathrm{top}$ by

$$\mathsf{L}_0(0) \xrightarrow{\times (CP^2)^2} \mathsf{L}_8(0) \xrightarrow{\sim} \Omega^8(G/\mathrm{top}) \xleftarrow{\times (CP^2)^2} G/\mathrm{top}.$$

Then we have shown that $\Omega^5 h$ is a homotopy equivalence. As far as formulas are concerned, however, h is a homotopy equivalence. For example, if G is a P-group there is a commutative diagram

$$\begin{array}{ccccc}
 & & \Delta(M_G, \mathsf{L}_k(0)) & \to & \mathsf{L}_{m+k}(G) \\
 & & \downarrow{\scriptstyle \Delta(M_G, \Omega^k h)} & & \Big| \\
 & & \Delta(M_G, \Omega^k(G/\mathrm{top})) & & \Big| \\
 & & \wr\| & & \downarrow \\
S_{\mathrm{top}}(M_G \times D^k, M_G \times S^{k-1}) & \to & NM(M_G \times D^k, M_G \times S^{k-1}) & \to & \mathsf{L}_{m+k}(G),
\end{array}$$

where the bottom row is the structure sequence, and the top row is the map defined in Section 2. If $k \geq 5$, then the top is a homotopy equivalence, and so the bottom has trivial fiber. The map $\Delta(M_G, \Omega^k(G/\mathrm{top})) \to \mathsf{L}_{m+k}(G)$ is a homotopy equivalence for $m + k \geq 5$, however, since this is the range in which periodicity holds on both sides, and the dimension may be raised by multiplying by CP^2 until $k \geq 5$ and the argument above applies. Thus $S_{\mathrm{top}}(M_G \times D^k, M_G \times D^{k-1})$ is contractible if $m + k \geq 5$.

$S_{\mathscr{C}}(M_G \times D^k)$ can also be computed this way for $\mathscr{C}$ = diff, PL. Write the structure sequence for $\mathscr{C}$ and top, with the natural forgetful transformation. Let $F : \mathscr{C} \to \mathrm{top}$ denote the map of classifying groups, then

$$\begin{array}{ccccc} S_{\mathscr{C}}(M_G \times D^k) & \to & \Delta(M_G, \Omega^k G/\mathscr{C}) & \to & \mathsf{L}_{m+k}(G) \\ \downarrow & & \downarrow{\scriptstyle \Delta(M_G, \Omega^k F)} & & \| \\ S_{\mathrm{top}}(M_G \times D^k) & \to & \Delta(M_G, \Omega^k G/\mathrm{top}) & \xrightarrow{\simeq} & \mathsf{L}_{m+k}(G). \\ \wr\!\wr & & & & \\ pt & & & & \end{array}$$

Thus $S_{\mathscr{C}}(M_G \times D^k)$ is the fiber of $\Delta(M_G, \Omega^k F)$, which is just $\Delta(M_G, \Omega^k(\mathrm{top}/\mathscr{C}))$. Since top/PL $\simeq K(Z_2, 3)$, this shows

$$S_{\mathrm{PL}}(M_G \times D^k, M_G \times S^{k-1}) \simeq \Delta(M_G, \Omega^k(K(Z_2, 3))) \simeq \Delta(M_G, K(Z_2, 3-k)).$$

$\pi_0 S_{\mathrm{PL}}(X)$ is a set classically ([6]) called the *homotopy triangulations* of X. The calculation shows that if M_G is a manifold $K(\pi, 1)$ of a P-group G, then homotopy triangulations of $M_G \times D^k$ rel the boundary are in natural bijection with $H^{3-k}(M_G; Z_2)$.

4. Remarks

A geometric formulation of non-simply connected surgery has been presented. So far it has been like category theory in that it provides a convenient way to state results gotten by other techniques, but is not useful in proving things. Hopefully this will not always be the case. One direction in which new results might be sought is in application of the theory to the fibrations problem mentioned in the introduction, and application of the results back to the L spaces.

Suppose M is a PL manifold (topological manifold with some triangulation), N is a manifold, then form the mapping spaces of M into the structure sequence of N. There is a natural transformation of this into the structure sequence of $M \times N$, which on the L spaces has been defined in Section 2.

$$\begin{array}{ccccc} \Omega A & \to & pt & \to & A \\ \downarrow & & \downarrow & & \downarrow \\ \Delta(M, S(n)) & \to & \Delta(M, \Delta(N, G/\mathscr{C})) & \to & \Delta(M, \mathsf{L}_n(N)) \\ \downarrow & & \wr\!\wr & & \downarrow \\ S(M \times N) & \to & \Delta(M \times N, G/\mathscr{C}) & \to & \mathsf{L}_{m+n}(M \times N) \end{array}$$

where $\Omega A \to pt \to A$ is the fiber of the transformation. $S(M \times N)$ can be thought of as the set of trivial homotopy fibrations over M with fiber N

and a manifold structure on the total space. $\Delta(M, S(N))$ is the set of homotopy trivial block fibrations over M with fiber N [i.e., $S(N) \simeq B_{\tilde{\mathscr{G}}(N)/G(N)}$]. Thus the problem of when a homotopy fibration is a block fibration is directly a question about L spaces.

This bears a strong likeness to Sullivan's analysis of homotopically trivial PL bundles ([6]) with a fundamental group put in the fiber. An example of an unreasonably good answer to this problem would be that $S(M \times N)$ has a classifying space, $S(M \times N) = \Delta(M, B)$. In this case $\Omega A = \Delta(M,$ fiber $(S(N) \to B))$, $\mathsf{L}_{m+n+2}(M \times N) =$ fiber $(\Omega A \to \Delta(M, \mathsf{L}_{n+1}(N))) = \Delta(M,$ fiber(fiber $(S(N) \to B)) \to \mathsf{L}_{n+1}(N))$.

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