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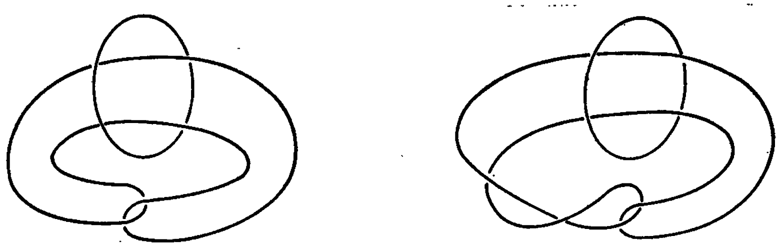
A Quick Trip Through Knot Theory

1. PROSPECTUS

Knot theory deals with a special case of the *placement problem*, but it is an important one because it is the simplest case that has an interesting theory and may therefore serve as a model for studying the problem in more complicated cases.

The general placement problem is the following: Given a space X and subsets A_1 and A_2 of it that are homeomorphic, does there exist an autohomeomorphism f of X such that $f(A_1) = A_2$? If such an f exists, the two placements A_1 and A_2 of A in X are said to be of the same *type*; the problem is to describe and classify the types. If A_1 and A_2 are of the same type, then their complementary spaces $X - A_1$ and $X - A_2$ must be homeomorphic; thus the *form invariants* of $X - A$ are all invariants of the type of placement of A in X. The form invariants that first come to mind are the homology groups $H_n(X - A)$ and the homotopy groups $\pi_n(X - A)$; it is necessary at some point, however, to construct invariants of placement that are not just form invariants of the complement. That this is so is most easily seen by the following example of placements of $A = S^1 + S^1$ in $X = S^3$; here it is easily verified that A_1 and A_2 are different types of placements of A in X, although $X - A_1$ is actually homeomorphic to $X - A_2$.



The central case of classical knot theory deals with the placements of a simple closed curve k in 3-space R (or in the 3-sphere S). The homology groups and the higher homotopy groups of $S - k$ are known to be uninteresting in this case, so we are first led to consider the fundamental group $\pi(S - k)$ of the complement, the so-called group of the knot. Generally speaking, to decide whether two given groups are isomorphic is too difficult a problem. Therefore our first step is to associate with each knot group a class of matrices, and then by a further act of bowdlerization to associate to each class of matrices a polynomial Δ. Thus we will associate to each knot a polynomial, and since it is easy to decide whether two polynomials are the same, we get a practical method for testing whether two knots are of the same type: if Δ_1 and Δ_2 are different, k_1 and k_2 are of different type; of course, if Δ_1 and Δ_2 are the same, no conclusion can be drawn.

The same procedure will apply, with variations, to other simple placement problems, for example, to *links* (unions of disjoint simple closed curves) or *graphs* (1-dimensional complexes) in 3-space, to 2-spheres in 4-space, etc. This algebraic theory will occupy us through Section 6. The arithmetic of knot types is considered in Section 7. In Sections 8 and 9, the systematic use of covering space theory is explained; much of the first six sections can be interpreted in terms of covering spaces, and in fact, a deeper understanding of the algebraic theory requires the use of covering spaces. Section 10 is devoted to the problem of finding representations of a knot group.

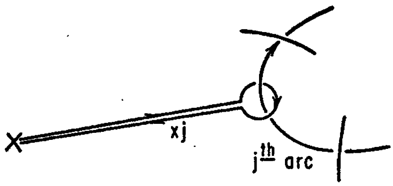
The material presented has not been systematically selected, except that it is meant to illuminate what I consider to be the core of the subject. Proofs are generally omitted or only indicated. Such proofs as do occur often constitute hitherto unpublished improvements or variations on the standard literature.†

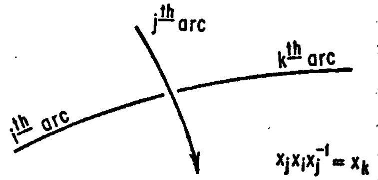
2. THE GROUP OF A KNOT

A knot type is called *tame* if it has a polygonal representative. Any simple closed polygon k can be projected in a properly chosen direction onto a plane in such a way that (a) there are no triple points and (b) no vertex of k is projected into a double point. A projection of this sort is called *regular*, and I shall now describe an algorithm for reading from a regular projection of k a set of generators and defining relations for G.

In a regular projection of k, the number n of double points is finite. Over each double point, k has an *undercrossing point* and an *overcrossing point*: the n undercrossing points divide k into n *arcs*; let x_j denote the element of G represented by a loop that circles once around the jth arc in the direction of a left-handed screw and doesn't do anything funny (in order for *left-handed screw* to mean anything I first give an orientation to k, this orientation serves no other purpose as far as we are concerned). It is intuitively clear that $x_1, \ldots, x_n$ generates G, and it is even not too

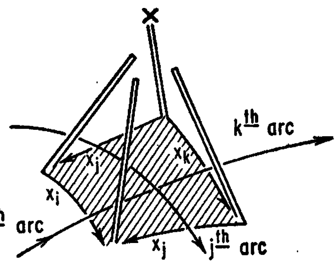
† A detailed presentation of the material touched on in the first four sections may be found in R. H. Crowell and R. H. Fox, *An Introduction to Knot Theory* (to be published soon by Ginn and Company).





difficult to prove. At each crossing a relation can be read. [Note that this depends only on the orientation of the jth arc, the orientation of the ith and the kth arc is immaterial.]

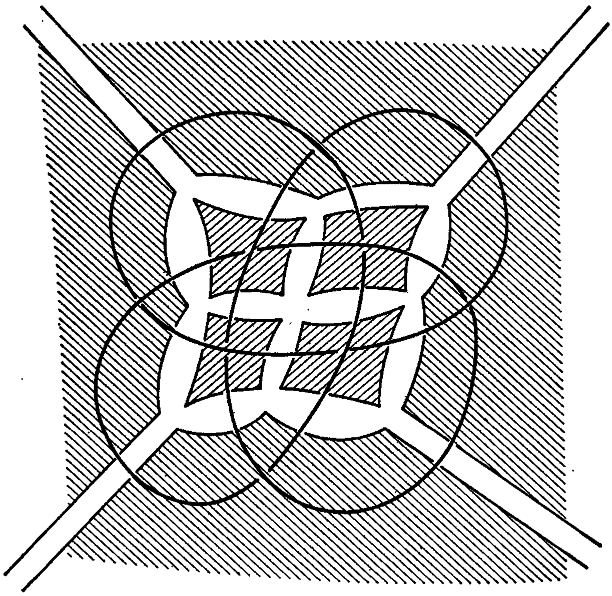
The following picture shows why this is a true relation. The n relations $r_1, \ldots, r_n$ obtained in this way form a complete system of defining rela-



tions; that is, any relation in G is a consequence of them. This may seem clear intuitively, but in fact, it is rather difficult to prove and is the most important step in the whole construction. We have now obtained a presentation $\mathfrak{P} = (x_1, \ldots, x_n : r_1, \ldots, r_n)$ of G, that is, a symbol listing the n generators $x_1, \ldots, x_n$ and the n defining relations $r_1 = 1, \ldots, r_n = 1$. The following picture should convince you that any one of the relations $r_1 = 1, \ldots, r_n = 1$ is a consequence of the others. Thus we arrive at a presentation $(x_1, \ldots, x_n : r_1, \ldots, r_{n-1})$ of G.

Two properties of a knot group G emerge from the preceding discussion: (1) the abelianized group G/G' is infinite cyclic (this can be seen from the form of the relations, or if you prefer, from the Alexander duality theorem, making use of the fact that G/G' is the first homology group of the complement); (2) the defect of G is $=1$; that is, G has a finite presentation in which there is one more generator than relator, but none in which there are two more generators than relators (since $G/G' \approx Z$).

In general, G does not determine k. In Section 4, it will be shown that the square knot and the granny knot have isomorphic groups. (By the methods of Section 9, or by the use of the peripheral structure explained below, they can be shown to be different knot types.) If $G \approx Z$, however, then the type of k is uniquely determined and is trivial—this follows from the Dehn-Papakyriakolous theorem, and a few other knot types are known to be determined by their groups.



An element of G is called *peripheral* if, for every neighborhood W of k, it is representable as a loop of the form $\gamma\alpha\gamma^{-1}$ where γ is a path from the base point to a point of $W - k$ and α is a loop in $W - k$. If k_1 and k_2 are of the same type there must be such an isomorphism of G_1 upon G_2 that peripheral elements are mapped into peripheral elements. This peripheral structure is a true *placement invariant,* in that it is not just a form invariant of the complement. Although the group of the square knot can be mapped isomorphically upon the group of the granny knot, it has been shown that no such isomorphism preserves the peripheral structure.

Among the peripheral elements there are perhaps two, determined up to conjugation and inversion, that are especially important. An element determined by the boundary of a small disk pierced once by k is a *meridian* (for example, the generators x_j^ϕ are meridians). An element determined by a curve that runs parallel to k and does not twist around it (in the sense that it is homologous to 0 in the complement of k) is called a *longitude.* Any maximal peripheral subgroup of G is generated by a meridian and a longitude, and any two maximal peripheral subgroups are conjugate. (In the group of a wild knot, the maximal peripheral subgroups may degenerate to Z, or perhaps even to 1.)

3. THE MATRICES AND POLYNOMIALS OF A KNOT

The next step involves making a careful study of the form of the *relators* $r_1, \ldots, r_n$ that appear on the left-hand side of the relations

$r_1 = 1, \ldots, r_n = 1$. Such a relator is a "word" $x_{j_1}^{\epsilon_1} x_{j_2}^{\epsilon_2} \ldots x_{j_k}^{\epsilon_k}$ in the generators $x_1, \ldots, x_n$. Since it is equal to 1 in the group G, it is not possible to discuss its form intelligently as long as we regard it as an element of G. Rather, we must form the free group $F(x_1, \ldots, x_n)$, generated by symbols $x_1, \ldots, x_n$, and make the homomorphism ϕ of F onto G that maps x_j into what we had previously called x_j and maps the elements r_i of F into 1. Thus ϕ is a homomorphism of F onto G whose kernel R is the smallest normal subgroup that contains the elements $r_1, \ldots, r_n$. A presentation $\mathfrak{P} = (x_1, \ldots, x_n: r_1, \ldots, r_m)$ is to be understood always in this sense. (Sometimes, however, it is convenient to write $r_i = 1$ or $r_i - 1 = 0$ instead of just r_i.)

Now take any word $w = x_{j_1}^{\epsilon_1} x_{j_2}^{\epsilon_2} \ldots x_{j_k}^{\epsilon_k}$ in the generators $x_1, \ldots, x_n$ and associate with it the *free derivatives*:

$$\frac{\partial w}{\partial x_1}, \frac{\partial w}{\partial x_2}, \ldots, \frac{\partial w}{\partial x_n}$$

defined as follows:

$$\frac{\partial w}{\partial x_j} = \epsilon_1 \, \delta_{jj_1} x_{j_1}^{\frac{1}{2}(\epsilon_1 - 1)} + \epsilon_2 \, \delta_{jj_2} x_{j_1}^{\epsilon_1} x_{j_2}^{\frac{1}{2}(\epsilon_2 - 1)} + \ldots$$

For example, if $w = x_2 x_1^{-1} x_2 x_2 x_2 x_1 x_2^{-1}$

$$\frac{\partial w}{\partial x_1} = -x_2 x_1^{-1} + x_2 x_1^{-1} x_2^3$$

$$\frac{\partial w}{\partial x_2} = 1 + x_2 x_1^{-1} + x_2 x_1^{-1} x_2 + x_2 x_1^{-1} x_2^2 - x_2 x_1^{-1} x_2^3 x_1 x_2^{-1}$$

The right-hand sides of the equations above are understood to be elements of the integral group ring JF of the free group F; that is, they are linear combinations of elements of F using integral coefficients. (J denotes the ring of integers.) If we take several different words representing the same element of F, we shall in fact get the same elements of JF. For instance, $x_1 x_1^{-1}$, $x_1^{-1} x_1$ and 1 all represent the same element of F and, in fact,

$$\frac{\partial (x_1 x_1^{-1})}{\partial x_1} = 1 - x_1 x_1^{-1} = 0$$

$$\frac{\partial (x_1^{-1} x_1)}{\partial x_1} = -x_1^{-1} + x_1^{-1} = 0$$

$$\frac{\partial (1)}{\partial x_1} = 0$$

Since this is so, $\partial/(\partial x_j)$ may be regarded as a mapping of F into JF. It has the characteristic property that, for $u, v \in F$,

$$\frac{\partial(uv)}{\partial x} = \frac{\partial u}{\partial x} + u\frac{\partial v}{\partial x}$$

Now we associate to any presentation

$$\mathfrak{P} = (x_1, \ldots, x_n : r_1, \ldots, r_m) \text{ of } G,$$

the matrix

$$\left(\frac{\partial(r_1, \ldots, r_n)}{\partial(x_1, \ldots, x_m)}\right)^{\phi} = \left\| \begin{matrix} \left(\frac{\partial r_1}{\partial x_1}\right)^{\phi} & \cdots & \left(\frac{\partial r_1}{\partial x_n}\right)^{\phi} \\ \cdot & \cdots & \cdot \\ \left(\frac{\partial r_m}{\partial x_1}\right)^{\phi} & \cdots & \left(\frac{\partial r_m}{\partial x_n}\right)^{\phi} \end{matrix} \right\|$$

that I call the *Jacobian* of $\mathfrak{P}$. (Actually it is not quite unique because the rows and/or the columns could appear in any order.) The entries in the Jacobian matrix are elements of the integral group ring JG of G. (The canonical homomorphism ϕ of F onto G extends in an obvious way to a homomorphism of JF onto JG which I perversely continue to denote by ϕ.)

If H is any group upon which G can be mapped by a homomorphism ψ, we can similarly extend ψ to a homomorphism ψ of JG upon JH and thus define the matrix

$$\left(\frac{\partial(r_1, \ldots, r_m)}{\partial(x_1, \ldots, x_n)}\right)^{\psi\phi} = \left\| \begin{matrix} \frac{\partial r_1}{\partial x_1} & \cdots & \frac{\partial r_1}{\partial x_n} \\ \cdot & \cdots & \cdot \\ \frac{\partial r_m}{\partial x_1} & \cdots & \frac{\partial r_m}{\partial x_n} \end{matrix} \right\|^{\psi\phi}$$

that I call *the Jacobian at* ψ.

$$F \xrightarrow{\phi} G \xrightarrow{\psi} H$$

$$JF \xrightarrow{\phi} JG \xrightarrow{\psi} JH$$

The choice of ψ ranges from ψ being the identity mapping of G onto itself to ψ being the map of G into the trivial group 1. (We are going to be most interested in choosing H to be the commutator quotient group G/G' and ψ the abelianizer.)

Call two matrices over JH *equivalent* if one can be obtained from the other by a finite sequence of the following operations (and their inverses):

0. Permute rows and/or permute columns
1. Adjoin a new row of zeros $(0, 0, \ldots, 0)$
2. Add to a row a left multiple of any other row
   Add to any column a right multiple of any other column
3. Left multiply any row by $\pm x_j^{\psi\phi}$
   Right multiply any column by $\pm x_j^{\psi\phi}$
4. Replace the $m \times n$ matrix M by the "bordered" $(m + 1) \times (n + 1)$ matrix

$$\begin{array}{c|c} M & 0 \\ \hline 0 & 1 \end{array}$$

(This generalizes the classical concept of equivalence, chiefly by admission of operations 1 and 4 that change the size of the matrix. Note that operation 1 discriminates against columns; it is a convenient trick to think of all matrices as having an infinite number of rows almost all of them filled up with zeros.) Using the so-called Tietze transformations, it is not hard to show that if two presentations define the same group G, then their Jacobian matrices at ψ are equivalent. For instance, if you adjoin the empty relation $1 = 1$ to a presentation, the matrix gets a new column $(0, \ldots, 0)$; if you adjoin a relation $x_j r_1 r_2 x_j^{-1}$ where r_1 and r_2 are relators of $\mathfrak{P}$, then you get a new row whose jth entry is

$$\left(\frac{\partial (x_j r_1 r_2 x_j^{-1})}{\partial x_i}\right)^{\psi\phi} = x_j^{\psi\phi}\left(\frac{\partial r_1}{\partial x_i} + \frac{\partial r_2}{\partial x_i}\right)^{\psi\phi}$$

and so you have applied rules 2 and 3 to the matrix. [Note that to get a new column of zeros you would have to adjoin a new generator to $\mathfrak{P}$, keeping the same relations, and this is obviously illegitimate.]

What we get out of all this is an algorithm associating to any group G an equivalence class of matrices over JH. This equivalence class is therefore an invariant of G, provided that the homomorphism ψ: $G \to H$ has an invariant significance. When ψ is the abelianizer, as it will be from now on, I call the Jacobian matrices at ψ the *Alexander matrices*. (Choosing the identity homomorphism of G on itself is self-defeating for all practical purposes; choosing $H = 1$ leads to a set of invariants characterizing G/G', which is uninteresting in the case of knots.)

If G is finitely presented (or even just finitely generated), we can apply elementary divisor theory to the Alexander matrices (since JH is a commutative ring when $H = G/G'$) to get invariants of G that are more tractable. However, JH is not usually a principal ideal ring (and if H has any elements of finite order it can even have zero-divisors) so that the classical elementary divisor theory has to be modified a little bit.

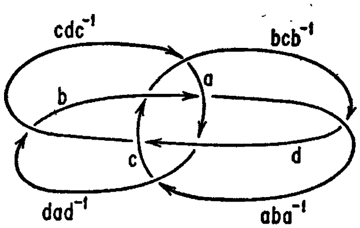
Choose any integer d smaller than the number of columns and define the dth *elementary ideal* $\mathcal{E}_d$ to be the ideal of JH generated by all the minor determinants of our matrix of order $n - d$. Clearly $\mathcal{E}_0 \subset \mathcal{E}_1 \subset \ldots \subset \mathcal{E}_{n-1}$; we can extend the range of d by defining $\mathcal{E}_d$ to be the zero ideal (0) for all $d < 0$, and to be the whole ring $(1) = JH$ for all $d \geq n$. As in the classical theory, it can be shown that equivalent matrices have the same dth elementary ideal for any d. For instance, if the matrix is subjected to the bordering operation 4, the minor determinants of order $n - d$ of the original matrix appear in the bordered matrix as minors of order $(n + 1) - d$ and all the other minors of order $n + 1 - d$ of the bordered matrix turn out to be linear combinations of minors of order $n - d$ of the original matrix.

The ideal $\mathcal{E}_0$, called the *order ideal,* is not very interesting for knots because it is equal to (0) whenever H is an infinite group. I call $\mathcal{E}_1$ the *Alexander ideal*; when G is a knot group, $\mathcal{E}_1$ is a principal ideal, and a generator $\Delta(t)$ is called the *Alexander polynomial* of the knot. Recall that when G is a knot group, H is the first homology group of the complement $R^3 - k$ hence the infinite cyclic group $(t:)$. Naturally, $\Delta(t)$ is only determined up to a factor of the form $\pm t^\lambda$.

The Alexander matrix class only depends on G modulo its second commutator subgroup G'', and furthermore the Alexander matrix class is almost surely not determined by its chain of elementary ideals $\mathcal{E}_0 \subset \ldots \subset \mathcal{E}_{n-1} \subset \ldots$. In view of this, it is remarkable how successful the Alexander polynomial $\Delta(t)$ alone is in distinguishing knot types. Of the $1 + 1 + 2 + 3 + 7 + 21 + 49 = 84$ prime knot types of 9 or fewer crossings there are exactly 3 pairs having the same polynomial. An extension of this up through 10 or 11 crossings has brought this up to about $84 + (123 + 37) + 257 = 501$ knots with some 56 pairs, 13 triplets, 2 quadruplets and 1 quintuplet.

4. EXAMPLES

Example 1



$$G = (a, b, c, d: dad^{-1} \cdot c \cdot da^{-1}d^{-1} = aba^{-1}, aba^{-1}dab^{-1}a^{-1} = bcb^{-1}, \text{etc.})$$

[Note that it is legitimate to write our relations in the group ring, and that this doesn't affect the values of the entries in the Jacobian.] The Jacobian is the 4 × 4 matrix

$$\left\| \begin{matrix} d - aba^{-1}d - 1 + aba^{-1} & -a & dad^{-1} & 1 - dad^{-1} + dad^{-1}c - aba^{-1} \\ 1 - aba^{-1} + aba^{-1}d - bcb^{-1} & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{matrix} \right\|$$

Abelianization maps G onto the infinite cyclic group $H = (t:)$ by mapping a, b, c, and d each into t. (An element of H maps into t^λ when its representative loops link the oriented knot λ times algebraically.) Hence the Alexander matrix of the given presentation is

$$\left\| \begin{matrix} -(1-t)^2 & -t & t & (1-t)^2 \\ (1-t)^2 & -(1-t)^2 & -t & t \\ t & (1-t)^2 & -(1-t)^2 & -t \\ -t & t & (1-t)^2 & -(1-t)^2 \end{matrix} \right\|$$

This is equivalent to the diagonal matrix

$$\left\| \begin{matrix} (1 \cdot t + t^2)(1 - 3t + t^2) & 0 & 0 \\ 0 & 1 - t + t^2 & 0 \end{matrix} \right\|$$

Hence $\Delta(t) = (1 - t + t^2)^2(1 - 3t + t^2)$

$$\epsilon_2(t) = (1 - t + t^2)$$

(In this case ϵ_2 is a principal ideal, but this is not always so. It should also be noted that it is not always possible to diagonalize matrices of integral polynomials.)

If k is any knot, its polynomial has the following two properties:

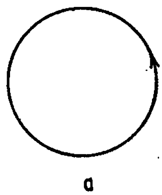
$$(1) \quad \Delta(1) = 1$$

$$(2) \quad \Delta(1/t) = \Delta(t)$$

$\Delta(t)$ having been normalized by multiplication by a suitably chosen factor $\pm t^u$.

It is known that, conversely, any polynomial that has these two properties is the polynomial of some knot (in fact of an infinite number of them). Property (1) is not deep; it is an immediate consequence of the fact that G/G' is infinite cyclic. On the other hand, Property (2) is rather difficult to prove, and it is an unsolved problem to describe the group-theoretical property that causes Property (2) to be true of knot groups. Such a property would presumably be some kind of a duality.

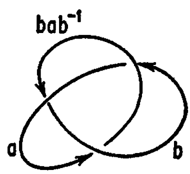
Example 2



$$G = (a:) = (a, b: b = 1)$$

The Jacobian matrix and the Alexander matrix are both $\|\, 0 \quad 1 \,\|$; $\Delta(t) = 1$.

Example 3 (trefoil)

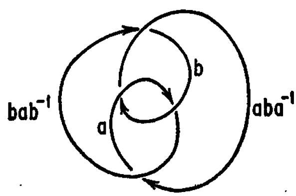


$$G = (a, b: aba - bab = 0)$$

The Jacobian matrix is $\|\, 1 - b + ab \quad -1 + a - ba \,\|$

The Alexander matrix is $\|\, 1 - t + t^2 \quad -1 + t + t^2 \,\|$ $\Delta(t) = 1 - t + t^2$

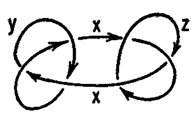
Example 4 (figure eight)



$$G = (a, b: aba^{-1}ba - bab^{-1}ab = 0)$$

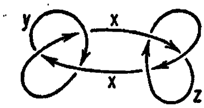
$$\Delta(t) = 1 - 3t + t^2$$

Example 5 (square knot)



$$G = (x, y, z: xyx = yxy, xzx = zxz)$$

Example 6 (granny knot)



$$G = (x, y, z\colon xyx = yxy,\ xzx = zxz)$$

In both cases, $\Delta(t) = (1 - t + t^2)^2$, $\epsilon_2(t) = (1 - t + t^2)$.

One can find deeper invariants of the same general nature by bringing representations by permutations into the act. Let's look at Example 3 and represent its group by permutations. This is easy to do in this case because the change of variable $x = aba$, $y = ab$ gives the convenient presentation $(x, y\colon x^2 = y^3)$ of G. Then we get a representation of G by mapping x for instance into any product of transpositions and y into any product of cycles of length 3. For instance let $x \to (12)(34)$, $y \to (135)$. Since $a = y^{-1}x$ and $b = x^{-1}y^2$, $a \to (15432)$ and $b \to (12534)$. Using the regular representation by permutation matrices, that is,

$$a \to \begin{Vmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{Vmatrix} \qquad b \to \begin{Vmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{Vmatrix}$$

we map the Jacobian matrix into the 5×10 matrix of integers

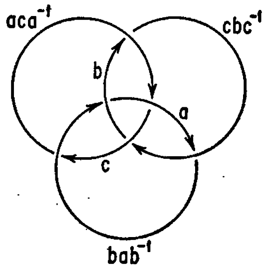
$$\begin{Vmatrix} 1 & -1 & 1 & 0 & 0 & -2 & 0 & 0 & 0 & 1 \\ 0 & 2 & 0 & 0 & -1 & 1 & -1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 & 1 & -2 & 0 & 0 \\ -1 & 0 & 0 & 2 & 0 & 0 & 0 & 1 & -1 & -1 \\ 1 & 0 & -1 & 0 & 1 & 0 & -1 & 0 & 1 & -1 \end{Vmatrix}$$

which I calculate to be equivalent to the 1×6 matrix $(3\,0\,0\,0\,0\,0)$. The resulting integers 3 and $5 - 4 = 1$ are invariants of the representation (in fact, the torsion and first Betti number of the associated covering space, as will be seen in Section 8). Of course, we get an invariant of the knot itself only by doing the same thing for every representation of G into the symmetric group $\mathfrak{S}_5$ of degree 5 that maps meridians into 5-cycles and considering the set of all integers so obtained.

5. LINKS AND GRAPHS

An example will illustrate how the theory is modified to take care of links.

Example 7 (the Borromean rings)



$$G = (a, b, c: cbc^{-1}acb^{-1}c^{-1} = bab^{-1}, \text{etc.})$$

In the case of a link, the commutator quotient group $H = G/G'$ is no longer an infinite cyclic group but is a free abelian group of rank equal to the number of components of the link, in this case, three.

$$G \xrightarrow{\psi} (x, y, z: xy = yx, xz = zx, yz = zy).$$

Thus the Alexander matrix has entries that are L-polynomials in three variables x, y, and z, where $a^{\phi} \xrightarrow{\psi} x$, $b^{\phi} \xrightarrow{\psi} y$, $c^{\phi} \xrightarrow{\psi} z$.

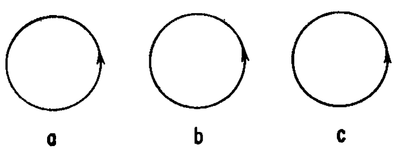
$$\left\| \begin{matrix} 0 & (z-1)(1-x) & (1-x)(1-y) \\ (1-y)(1-z) & 0 & (x-1)(1-y) \\ (y-1)(1-z) & (1-z)(1-x) & 0 \end{matrix} \right\|$$

In the case of links, ϵ_1 is the product of a certain fixed ideal, for $n = 3$ the ideal $(x - 1, y - 1, z - 1)$, and a principal ideal $(\Delta(x, y, z))$. The resulting polynomial, which is determined only up to a multiplicative factor $\pm x^f y^g z^h$, I call the *Alexander polynomial* of the link. In this case, we get

$$\Delta(x, y, z) = (x - 1)(y - 1)(z - 1)$$

$$\epsilon_2 = ((x - 1)(y - 1), \quad (x - 1)(z - 1), \quad (y - 1)(z - 1)).$$

For the link,



we get $\Delta(x, y, z) = 0$. This shows that the Borromean rings are not completely splittable.

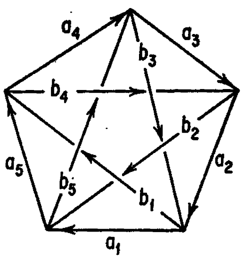
The Alexander polynomial of a link has properties analogous to Properties (1) and (2) of $\Delta(t)$. For example, for a link with two components and linking number q,

(1) $\Delta(x, 1) = [(x^q - 1)/(x - 1)]\ \Delta(x)$, where $\Delta(x)$ is the polynomial of the knot type of the first component.

(2) $\Delta\left(\frac{1}{x}, \frac{1}{y}\right) = x^{q-1}y^{q-1}\ \Delta(x, y)$

if Δ has been normalized by multiplication by a suitable factor $\pm x^u y^v$. Whether or not these properties are sufficient to assure that there is a link having a given polynomial $\Delta(x, y)$ is not known. In the case of an oriented link of μ components ($\mu \geq 1$), the meridians of the μ components determine a preferred basis for JH, and so when we compare the polynomials of two links, we need not take account of the automorphisms of H. In the case of a graph, however, it may not be possible to prescribe a preferred basis and this will cause a further complication.

Example 8



$$G = (a_1, a_2, a_3, a_4, a_5, b_1, b_2, b_3, b_4, b_5:$$

$$a_1b_1b_2 = a_2b_2b_3 = a_3b_3b_4 = a_4b_4b_5 = a_5b_5b_1)$$

Note that, in the case of a graph, further relations are to be read from each node. The group of a graph generally has defect >1—hence $\epsilon_1 = (0)$.

6. KNOTTED 2-SPHERES

A 2-sphere S^2 in 4-space R^4 has a complementary domain $R^4 - S^2$ whose 1st homology group H is infinite cyclic. Therefore we can proceed exactly as in the case of a knot in 3-space to calculate an Alexander matrix. The only thing that is different in the calculation is that the Alexander ideal is not always principal. The 2-spheres that we shall consider are going to be not only polyhedral but *locally flat*. There is no difficulty, however, in applying the same method to 2-spheres that have singularities.

If a knotted arc k in half-3-space R^3_+ is rotated in 4-space about a plane to produce a 2-sphere S^2 in R^4, that 2-sphere is said to have been obtained by *spinning*. The group $\pi(R^4 - S^2)$ of S^2 is isomorphic to $\pi(R^3_+ - k)$, which is a knot group. However there are 2-spheres in R^4 that cannot be obtained by spinning and whose groups are not knot groups, as we shall see.

The first problem is to find a method for presenting the group of S^2 in R^4. I have found the method of *hyperplane cross sections* to be the most useful. Put the polyhedral S^2 in general position in R^4 and cut it by the family of parallel hyperplanes R^3_t, $-\infty < t < \infty$, perpendicular to a properly chosen direction. If R^3_t cuts S^2 at all, the intersection will generally be a polygonal knot or link in R^3_t. There will be a finite number of t-values that are singular. A singular hyperplane may intersect S^2 in an isolated point, which may be either a maximum or a minimum for the height, or it may intersect S^2 in a graph with just one node, which is of order four. These nodes are *saddle points*. The singular hyperplanes divide R^4 into slices, and the group of the complement of S^2 in one of these slices is just the group of the knot or link that is to be found in a representative hyperplane section (unless S^2 doesn't intersect the slice, of course). The group of S^2 in R^4 is found by gluing these slices together and applying the van Kampen theorem.

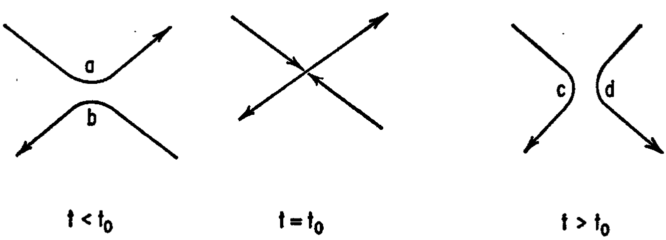
To be more precise, if the singular point is the point (0, 0, 0, 0) lying in the hyperplane section $t = 0$, one can apply the van Kampen theorem twice to the three open sets $U - S^2$, $W - S^2$, $V - S^2$, where

$$U = \{(x, y, z, t) \mid t > \max(-\epsilon, -\sqrt{x^2 + y^2 + z^2})\}$$

$$W = \{(x, y, z, t) \mid x^2 + y^2 + z^2 + t^2 < \epsilon^2\}$$

$$V = \{(x, y, z, t) \mid t < \min(\epsilon, \sqrt{x^2 + y^2 + z^2})\}$$

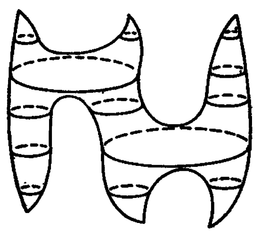
It is easily seen that at a maximum or minimum we get no new relation, and at a saddle point we get one new relation:



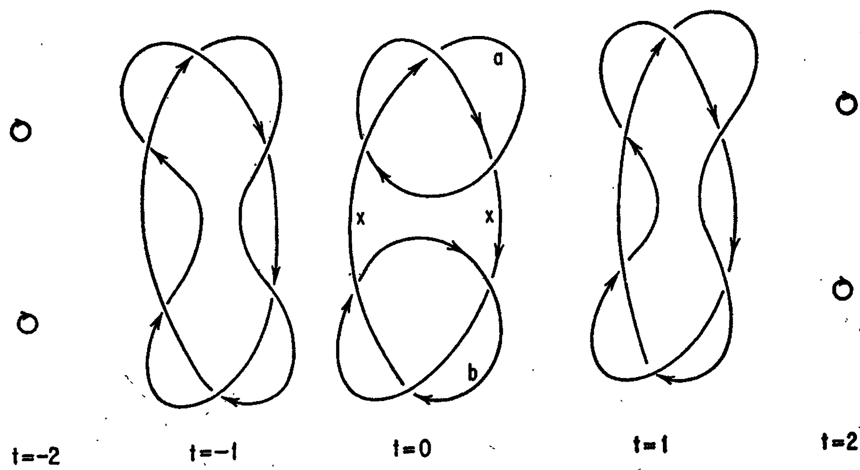
new relation adjoined to group of lower slab: $a = b$

new relation adjoined to group of upper slab: $c = d$

Note that at a saddle point the number of components changes by one. In the examples that we shall consider, the cross section at $t = 0$ will be a knot, and as t goes through a saddle point with increasing absolute value, the number of components increases. This assures that the result is a 2-sphere and not some other closed surface, as the following schematic diagram shows. Whether every locally flat type of knotted sphere is obtainable in this way is an open question (that is, it might be necessary to allow more complicated schematic diagrams like the following in which there are no connected cross sections).

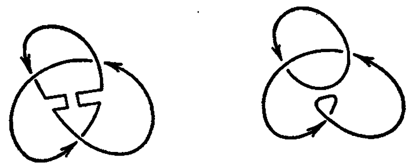


Example 9. The "equatorial" cross section is a square knot; its group is $(x, a, b\colon xax = axa, xbx = bxb)$. The group of the knotted 2-sphere is



obtained from this by adjoining the relation $a = b$ (twice, once at $t = -1$ and once at $t = 1$), resulting in the group $G = (x, a\colon xax = axa)$. Thus the group of this knotted sphere is isomorphic to the group of the trefoil (Example 3), and it has an Alexander polynomial $\Delta(t) = 1 - t + t^2$.

We see from this example that a "saddle point" transformation has to be applied to a knot with some care if we expect the resulting 2-sphere to be locally flat. For instance,



t=0 t=1

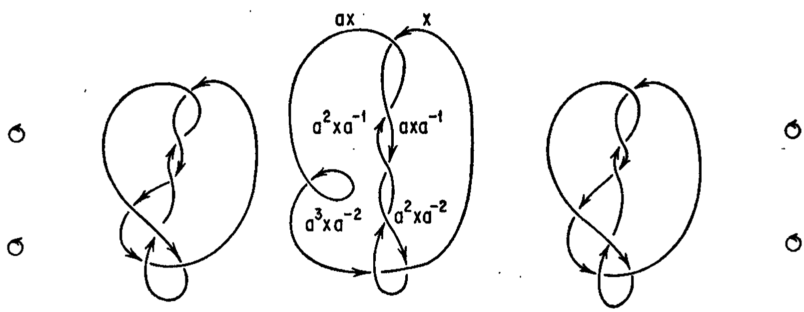
and we are stuck.

This suggests that we can't start with just any old knot, and this is in fact the case. A knot type that can be obtained from a locally flat knotted 2-sphere by slicing it with a hyperplane must have a polynomial of the form $\Delta(t) = F(t)F(1/t)$. Milnor and I called such knot types *null-equivalent*; but we are dissatisfied with this terminology which we feel may turn out to be confusing, and I would like to adopt the name *slice knot* proposed by Ed Moise. [Note that the polynomial $\Delta(t) = (1 - t + t^2)^2$ of the square knot has the required property—and the polynomial $\Delta(t) = 1 - t + t^2$ doesn't.] The converse is almost certainly false; the granny knot has the same polynomial as the square knot (in fact, it has the same group) but it is highly improbable that the granny knot is a slice knot. It would be nice to have an analogous condition for a *slice link*, but if there is one I am not aware of it.

Example 10. The equatorial cross section is a stevedore's knot; its group is

$$(x,\ a\colon x\cdot a^3xa^{-2}\cdot x^{-1} = a^2xa^{-2})$$

and its polynomial is $\Delta(t) = (1 - 2t)(2 - t)$. The group of S^2 is obtained by adjoining the relation $ax = a^2xa^{-2}$, that is, the relation $xa^2 = ax$. The



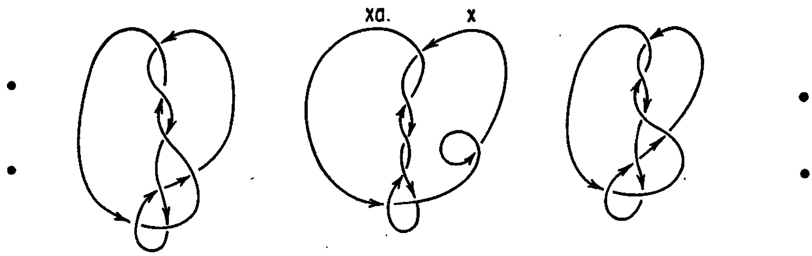
t=−2 t=−1 t=0 t=1 t=2

relation $xa^3xa^{-2}x^{-1} = a^2xa^{-2}$ is a consequence of this, hence,

$$G(x, a\colon xa^2 = ax)$$

$$\Delta(t) = 2t - 1$$

Example 11. This is quite analogous to the preceding one.

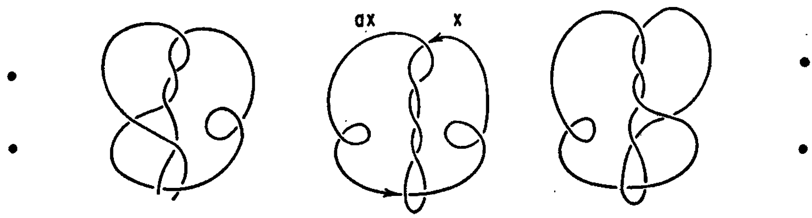


$$G = (x, a\colon a^2x = xa)$$

$$\Delta(t) = 2 - t$$

These last two examples show that the Alexander polynomial of a knotted 2-sphere need not be a reciprocal polynomial, and therefore that the group of a knotted 2-sphere need not be a knot group. Of course $|\Delta(1)| = 1$ is still true. Kinoshita has shown that any polynomial $\Delta(t)$ satisfying $|\Delta(1)| = 1$ is the polynomial of some locally flat knotted 2-sphere. (Terasaka has shown that if $\Delta(t)$ is any polynomial of the form $F(t)F(1/t)$ it is the Alexander polynomial of some slice knot.)

Example 12. This is a combination of the two preceding examples.



$$G = (x, a\colon xa^2 = ax, a^2x = xa)$$

These two relations are read from the lower and the upper saddle point respectively. The original relation of the group of the stevedore's knot is a consequence of either one of these.

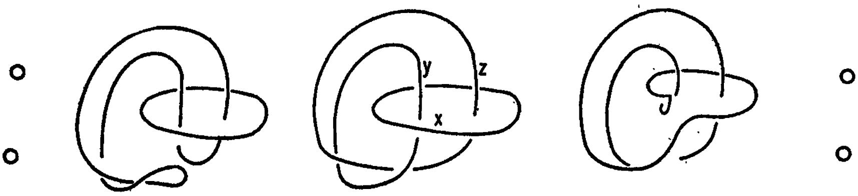
$$G = (x, a\colon a^3 = 1, xax^{-1} = a^{-1})$$

Thus the commutator subgroup G is the finite cyclic group $G' = (a\colon a^3 = 1)$, and G is an extension of this by the infinite cyclic group.

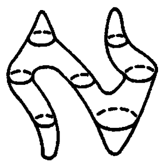
$\mathcal{E}_1 = (2t - 1, 2 - t)$, which is not a principal ideal. The representation $x \to (0 \;\; 1)$, $a \to (0 \;\; 1 \;\; 2)$ of G onto $\mathcal{S}_3$ shows that $a \neq 1$. Thus we have proved a conjecture of Morton Curtis: the group of a locally flat 2-sphere in 4-space can have an element of finite order.

The following remark explains a reason why this conjecture was reasonable. In 1957, Papakyriakopoulos proved Dehn's lemma, the asphericity of knots, and the Hopf conjecture, and the method used showed that the three problems were closely related. Then Andrews and Curtis showed that knotted 2-spheres are not always aspherical, and it has been remarked at this conference that Dehn's lemma fails to generalize, in a certain sense, to boundaried 4-manifolds. The conjecture of Morton Curtis then was just that Hopf's conjecture also fails to generalize to four dimensions.

Example 13



Schematic diagram.

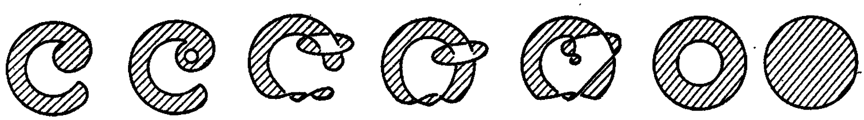
Group of link in equatorial cross section: 

$$(x, y, z: yxy^{-1} = zxz^{-1}, zxyx^{-1}z^{-1} = y)$$

Alexander polynomial: $\Delta(x, y, z) = y - zx$

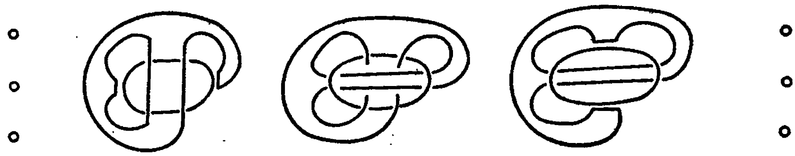
Hosokawa polynomial: $\nabla(t) = \dfrac{\Delta(t, t, t)}{1 - t} = 1$

This sphere is of trivial type because, as David Epstein pointed out, it bounds a 3-cell. The following diagram shows this 3-cell in cross section. (Each cross section is a surface of genus 0.)

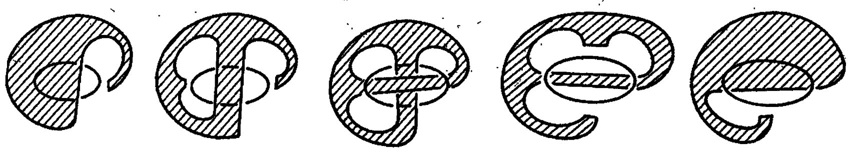


This example shows that there is a sphere of trivial type that has a cross section that is a non-trivial link. It is a little harder to construct a sphere of trivial type that has a non-trivial knot as a cross section, but John Stallings has constructed one.

Example 14. Here we have two spheres of trivial type. The Alexander polynomial of the cross section is $\Delta(x, y) = 0$, but the cross section link

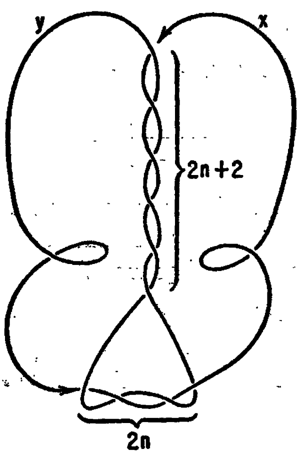


is non-trivial, because $\mathcal{E}_2 = (1 - x + xy)(1 - y + xy)$. These two spheres are actually splittable, because one can construct a 3-cell in the complement of one of them bounded by the other. (In the complement of this 3-cell, it is possible to construct another 3-cell whose boundary is the first curve, although this is not very easy to see.)



This example shows that one can have a pair of disjoint 3-cells such that a cross section of the boundary of their union is a non-trivial link.

Example 15. This example generalizes Example 12. Only its equatorial cross section—a knot of $4n + 2$ crossings—is shown. Its polynomial is



$\Delta(t) = n(n + 1) + (1 - 2n(n + 1))t + n(n + 1)t^2$. The group of the sphere is the metacyclic group

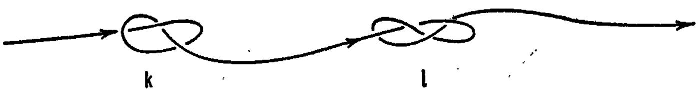
$$G = (x, a: xa^{n+1} = a^n x, a^{n+1}x = xa^n)$$
$$= (x, b: b^{2n+1} = 1, xbx^{-1} = b^{-1})$$

where $b = a^n$.

Thus for every odd integer $2n + 1$ there is a locally flat 2-sphere in 4-space whose group has an element of order $2n + 1$. I have not been able to construct any locally flat 2-sphere whose group has an element of even order.

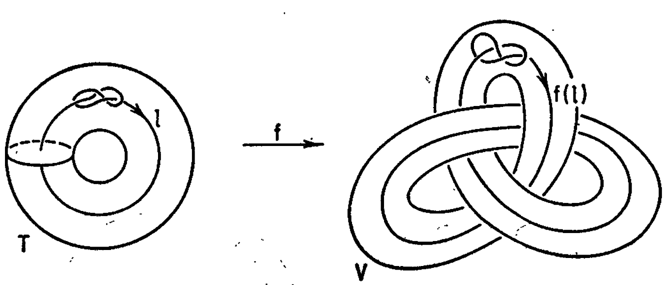
7. ARITHMETIC OF KNOTS

The knot resulting from tying two knots k and l in a piece of string one after the other is called the *composition* of the two knots and is denoted by $k \# l$. There are two other ways of expressing this.



(2) If the oriented polygonal simple closed curves k and l are represented in 3-space on opposite sides of a plane P and have in common an edge e that inherits opposite orientations from k and l, then $k \# l$ is represented by the oriented curve $(k - e) + (l - e)$.

(3) If T is a solid torus of revolution and l is represented in T as a curve that intersects a meridian cell just once, V is a solid torus representing the knot k, and f is a faithful map of T on V (that is, maps oriented longitude of $\dot{T}$ onto oriented longitude of $\dot{V}$), then the curve $f(l)$ represents $k \# l$.



In discussing composition, sometimes one representation is convenient, sometimes another. In the second representation, it is also sometimes convenient to replace P by a sphere.

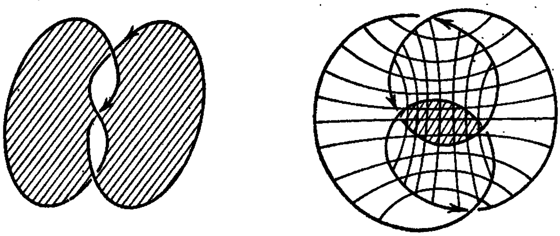
It is trivial that composition is associative and that the trivial knot type is a unit. Commutativity may be seen from the following picture.

Thus the set of all (tame, oriented) knot types form a commutative semigroup under the operation #.

Schubert has proved that, in this semigroup, factorization is unique. Just as in the proof of unique factorization of integers under multiplication, the proof may be made to depend on two fundamental lemmas: (a) finiteness of factorization, and (b) the lemma about prime divisors of a product.

To prove Lemma a, the genus of a knot may be introduced. Let me digress to define this important concept. First note that it is possible to span an orientable surface in any tame knot. This can be done in the following way: At each crossing, span a twisted rectangle as shown below on the left (and not as shown on the right).

If we remove the interiors of these rectangles and the part of their boundary that lies on the knot, what remains in the place of projection is a number of disjoint circles, that I shall call *Seifert circles*. A Seifert circle may be described by starting anywhere on the knot and following it along in the positive direction until you come to a crossing point, hopping over to the other branch and following it in the positive direction until you come to another crossing point, and so on, until you close up. The Seifert circles are disjoint but they may very well be nested. If you start with the innermost circles and work out, however; it is easy to cap each circle with a disk in such a way that their interiors are disjoint from one another and from the rectangles. The union of the disks and the rectangles clearly forms an orientable surface spanned by the knot. If d is the number of crossings and f the number of Seifert circles, what you have is an orientable surface of genus $(d - f + 1)/2$ with one boundary.



Since it is possible to span a knot k by at least one orientable surface, there is a least integer $h(k)$ such that k can be spanned by an orientable surface of genus h. This number is called the *genus* of the knot k. Obviously the trivial knot is the only knot of genus zero. The degree of $\Delta(t)$ is at most equal to $2h$, and is equal to $2h$ for any alternating knot. It would appear that it is in general difficult to calculate g; nevertheless, the current issue of *Acta Mathematicae* contains a long paper by Wolfgang Haken that gives an algorithm to calculate the genus of any knot. (In particular, this gives an algorithm for deciding whether a given projection represents the trivial type of knot.)

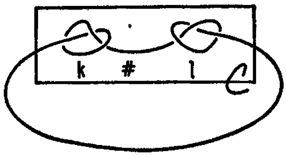
Returning to the semigroup of knots, it is not difficult to show that $h(k \# l) = h(k) + h(l)$. If F is a surface spanning $k \# l$ and P is a plane in general position separating k and l, then the intersection of F and P consists of the arc e and a number of simple closed curves. These curves can be capped to produce surfaces F_1, F_2 spanning k and l respectively, and clearly $h(F_1) + h(F_2) \leq h(F)$, thereby showing that $h(k) + h(l) \leq h(k \# l)$.

Conversely if F_1 and F_2 are surfaces spanning k and l respectively, then we can just add them together to produce a surface of genus $h(F_1) + h(F_2)$ spanning $k \# l$. Of course there may be some sheets of F_1 and/or of F_2 interfering with this project, but these can first be blown out across the point at infinity and then F_1 and F_2 can be joined at e. This shows that $h(k \# l) \leq h(k) + h(l)$ and completes the proof that genus is a homeomorphism of the semigroup of knots upon the additive semigroup of non-negative integers. It follows that no knot can be factored indefinitely; every knot has a factorization into knots that have no further factorization. These are called *prime knots*. The standard knot tables are tables of the prime knots only.

Having proved Lemma a, it is only necessary to prove Lemma b:

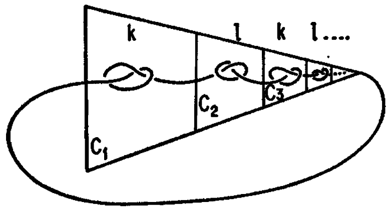
If k is a prime and k divides $l \# m$, then either k divides l or k divides m. To prove this, we start with a simple closed curve representing $l \# m$ and a plane P that cuts it in two points "separating l from m" (this is easy to make precise). Since k divides $l \# m$ there is a 2-sphere S^2 cutting the curve at two points and representing k inside it. If S^2 does not intersect P, we are finished. If not, S^2 cuts P in a number of disjoint simple closed curves. Those that do not link $l \# m$ can be removed immediately (by deforming S^2), and those that do link $l \# m$ can also be removed (also by deforming S^2) because of the hypothesis that k is prime.

A consequence of the finiteness of factorization, in particular, is the impossibility of tying two knots k and l in succession on a piece of string in such a way that they "cancel each other out." It has been stated in several popular magazines, notably in *Scientific American*, that this is an unsolved problem, but this is not so. It may be of some interest to give a short proof of this fact without using the genus.



Suppose that there were an autohomeomorphism f of space mapping $k \# l$ into 0. It may be arranged that f is the identity outside a cube C whose boundary meets $k \# l$ in two points.

Construct the following wild knot m:



Then there is an autohomeomorphism f_n of space that is the identity outside $C_{2n-1} + C_{2n}$, that replaces $k \# \mathcal{L}$ by 0 inside $C_{2n-1} + C_{2n}$. Defining f to be f_n inside $C_{2n-1} + C_{2n}$ for all n and the identity outside

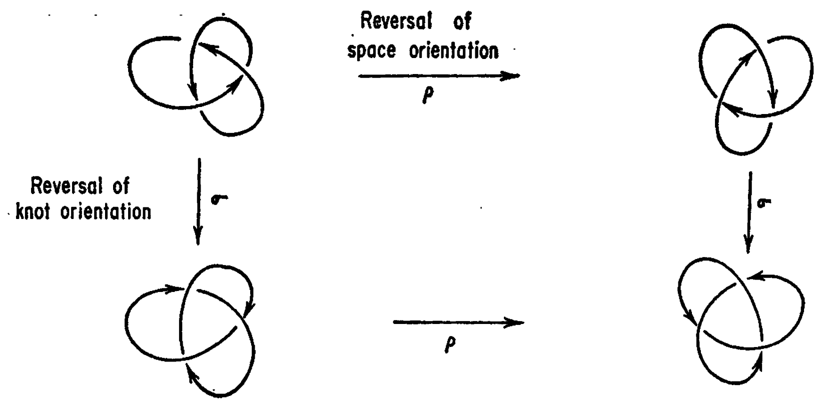
$$\sum_{i=1}^{\infty} C_i,$$

we see that $m = 0$. Repeating the same construction, using $C_{2n} + C_{2n+1}$, $n = 1, 2, 3, \ldots$, instead of $C_{2n-1} + C_{2n}$, and observing that $k \# l = l \# k$, we see that $m = k$. Consequently $k = 0$, and hence $l = 0$.

It is easy to see that k is a slice knot iff there is a locally flat 2-cell in half 4-space bounded by k. If k is the intersection of a hyperplane with a locally flat 2-sphere then either half 4-space intersects the 2-sphere in a locally flat 2-cell; conversely, if k bounds a locally flat 2-cell in the half 4-space on one side of the hyperplane containing k, then reflection about the hyperplane yields another 2-cell which, together with the first one, makes up a 2-sphere of which k is a cross section. In the definition of a locally flat 2-cell, it is, of course, necessary to require a local flatness condition at the boundary points as well as at the interior points. (Note that any knot bounds a locally flat 2-cell in all of 4-space—the restriction in the definition to half 4-space is necessary.)

Similarly, if k and l are two oriented knots, they are said to belong to the same *cobordism class* ($k \sim l$) if there is a locally flat annulus in a slab of 4-space whose boundary is $k - l$, k lying in one bounding hyperplane and l in the other. It is obvious that $\sim$ is an equivalence relation and that $k \sim 0$ iff k is a slice knot. Furthermore, it is easy to see that $k \# k^* \sim 0$, where k^* denotes the reflected inverse of k. (For example, if k is the overhand knot then $k \# k^*$ is the square knot.)

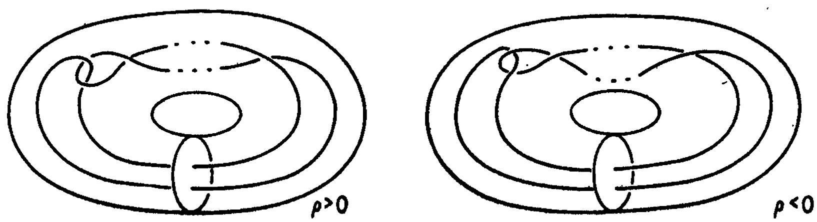
Digression. There are two orientation-reversing operations: reversal of the space orientation (this is equivalent to taking the mirror image) and reversal of the knot orientation (accomplished by reversing the direction of the arrow on the knot). This leads to the following diagram:



k^* is $\sigma\rho(k) = \rho\sigma(k)$. A knot is called *invertible* if $\sigma(k)$ is equivalent to k and *amphicheiral* if k is equivalent to either $\rho(k)$ or $\sigma\rho(k)$. (Actually the first should be called +amphicheiral and the second −amphicheiral.) The overhand knot is known to be non-amphicheiral—there is a right-handed trefoil and a left-handed trefoil; the figure eight is amphicheiral. The knot 8_{17} is obviously (!) not invertible, but this has never been proved. In fact there is no proof that there are *any* non-invertible knots; this is a very difficult problem.

A simple geometric argument shows that if $k \sim l'$, then $k \# m \sim l \# m$. Hence, the cobordism classes inherit from the semigroup of knots the operation #, and they form an abelian group thereby. In this group, the inverse of a knot is its reflected inverse. Clearly, any knot that is invertible and amphicheiral is of order 2 in this group. It is not known whether there are any elements of this group that are not of order 2. It is known that the group is not finitely generated, but these two facts are all that are known as yet about the group.

The third representation of the operation # has been generalized in the following way: Let T and V be as before and let l be any knot in T just so long as it meets every meridian cell. If V represents the type of k, the knot k is called a *companion* of the knot $f(l)$. (In other words, if a knot $f(l)$ is contained non-trivially in a knotted solid torus V, then the knot type represented by a core of V is called a *companion* of $f(l)$.) Clearly, any two knots are each companions of their composition. As another example, let l be placed in V as shown



(in each case there are $2\rho + 2$ crossings), then $f(l)$ is called a *double* of k with a twist ρ. Again, let l be a torus knot of type a, b on a torus inside T and concentric to $\dot{T}$. (A *torus knot* of type (a, b) is a knot located on the surface of a torus of revolution that runs a times around one way and b times the other way; a and b must be relatively prime integers or you get a *torus link*). In this case $f(l)$ is called a *cable* about k.

If k is a companion of $m = f(l)$ and l runs α times around T, then the genera satisfy the inequality

$$h(m) \geq \alpha h(k) + h(l)$$

and their polynomials, the equation

$$\Delta_m(t) = \Delta_k(t^\alpha) \cdot \Delta_l(t)$$

The only companions of a product knot are its prime factors and their companions. The double of any knot is of genus 1 (unless it is trivial). The companions of a non-trivial double of a knot k are all companions of k itself. The companions of a cable around a non-trivial knot k are all companions of k itself. Two cables around non-trivial knots k, k' are of the same type only if $k = k'$ and the cabling is of the "same type". Doubles of knots k and k' are of the same type only if $k = k'$ and the doubling is of the "same type" (except when $k = k' = 0$, where there is a trivial exception).

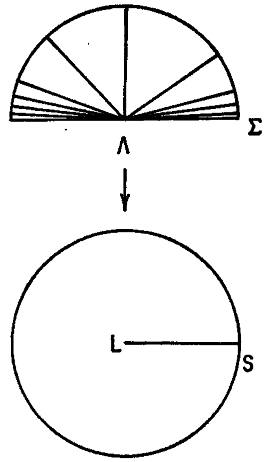
8. COVERING SPACES

Let S be an n-dimensional manifold, for example, the 3-sphere, and let L be a closed nowhere dense subset of S. To each covering space of $S - L$ there is a unique completion Σ called the *associated branched covering space.* If Λ is the set of points of Σ lying over L we have the following diagram

$$\begin{array}{ccc} \Sigma - \Lambda & \subset & \Sigma \\ \downarrow & & \downarrow \\ S - L & \subset & S \end{array}$$

Thus $\Sigma - \Lambda$ is an unbranched covering of $S - L$, and the completion Σ is the associated branched covering of S.

If S is triangulated, L a subcomplex, and the index of branching finite at each point of Λ, then Σ is triangulated and Λ is a subcomplex. If L is also a locally flat $(n - 2)$-dimensional submanifold, then Σ is an n-dimensional manifold and Λ is a locally flat $(n - 2)$-dimensional submanifold. (The condition that the branching index be everywhere finite is necessary; it is easy to see that the universal covering of a 2-cell S branched over an interior point L of it is not locally compact at the point Λ.)



It is well known that the unbranched covering $\Sigma - \Lambda$ of $S - L$ are in one-to-one correspondence with the subgroups Γ of the group $= \pi(S - L)$, if a base point $p_0 \in \Sigma - \Lambda$ lying over the base point $p \in S - L$ is specified. If the base point p_0 is unspecified within the discrete set of points $p_0, p_1, \ldots$ lying over p, the correspondence is between covering spaces $\Sigma - \Lambda$ and conjugate classes of subgroups Γ.

Each subgroup Γ of G of index $g \leq \infty$ induces a transitive representation of G into the symmetric group $\mathcal{S}_g$ of degree g. If the g symbols $0, 1, \ldots, g - 1$ permuted are identified with the right cosets $\Gamma_0, \Gamma_1, \ldots, \Gamma_{g-1}$ of $\Gamma = \Gamma_0$ this representation ρ is

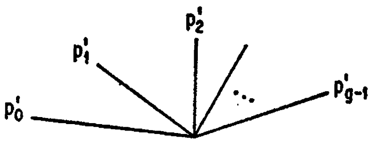
$$\text{For any } a \text{ in } G,\ \rho\colon a \to \begin{pmatrix} \Gamma_0 & \Gamma_1 & \ldots\ \Gamma_{g-1} \\ \Gamma_0\, a & \Gamma_1\, a & \ldots\ \Gamma_{g-1}\, a \end{pmatrix}$$

Conversely, if ρ is any transitive representation into $\mathcal{S}_g$, it corresponds to the subgroup Γ consisting of those elements of G for which the permutation $\rho(a)$ leaves the symbol 0 fixed. (Note that this is not generally a normal subgroup, and that it has nothing to do with the kernel of ρ. Note also that an element a of G belongs to the right coset G_k iff the permutation $\rho(a)$ sends the symbol 0 into the symbol k.) Thus there is a one-to-one correspondence between subgroups of G and transitive representations of G. To a conjugate class of subgroups corresponds a class of equivalent representations.

Covering space $\Sigma - \Lambda$ with g sheets	$\leftrightarrow$	Conjugate class of subgroups Γ of index g	$\leftrightarrow$	Equivalence class of transitive representations into the symmetric group of degree g

The inclusion map $\Sigma - \Lambda \to \Sigma$ defines a homomorphism of Γ onto $\pi(\Sigma)$. The kernel consists of those elements of $\Gamma = \pi(\Sigma - \Lambda)$ that can be represented by small loops that are arbitrarily close to Λ (that is, by loops of the form fhf^{-1} where f is a path from the base point to a given neighborhood of Λ and h is a loop in Λ), that is, by those elements of $G = \pi(S - L)$ that lie in the subgroup Γ and are represented by loops that are arbitrarily close to L and link it simply.

If a presentation $(x_1, \ldots, x_n\colon r_1, \ldots, r_m)$ is given for the group G, the Reidemeister-Schreier theorem constructs a presentation for the group Γ and from this a presentation for $\pi(\Sigma)$ can be worked out. I will now explain an algorithm for constructing a presentation of Γ, given a presentation for G and a transitive representation ρ of G by permutations. This algorithm is equivalent to the Reidemeister-Schreier theorem; however, it contains the following simplifying gimmick. In the Reidemeister-Schreier theorem, one has first to select in G a representative of each coset; if we label the points lying over the base point p of $S - L$ by the indices of the right cosets $\Gamma_0, \Gamma_1, \ldots, \Gamma_{g-1}$ of Γ, calling them $p_0, p_1, \ldots, p_{g-1}$ this amounts to selecting for each $k = 0, 1, \ldots, g - 1$ a path in $\Sigma - \Lambda$ from p_0 to p_k. (They should also satisfy the so-called Schreier condition, which says that their union should be a tree.) This selection of representatives is bound to be unsymmetric and to upset the simplicity of the algorithm. This is avoided in my algorithm by, so to speak, lifting the tree of representative paths out of the space $\Sigma - \Lambda$. To be precise, I take a g-frame Φ, with end points $p_0', p_1', \ldots, p_{g-1}'$ and identify each point p_γ' with the corresponding point p_γ of $\Sigma - \Lambda$. The fundamental group of the resulting space is obviously $\Gamma * F_{g-1}$, where F_{g-1} is the free group of rank $g - 1$. My algorithm is an algorithm for calculating a presentation for $\Gamma * F_{g-1}$ instead of for F_{g-1}. For most purposes, this is just as good, because if you know that a free factor F_{g-1} is there, it is usually easy to take account of it.

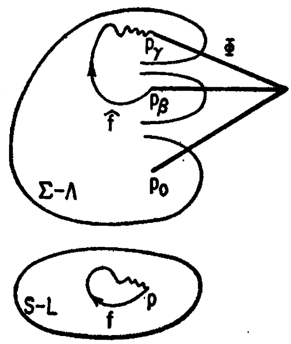


Given, then, a presentation $(x_1, \ldots, x_n : r_1, \ldots, r_m)$ for G and a transitive representation ρ of G in $\mathcal{S}_g$, the algorithm constructs for $\Gamma * F_{g-1}$ a presentation of the form

$$\begin{pmatrix} x_{10}, \ldots, x_{n0} & r_{10}, \ldots, r_{m0} \\ x_{1g-1}, \ldots, x_{ng-1} & r_{1g-1}, \ldots, r_{mg-1} \end{pmatrix}$$

The meaning of the symbols $x_{j\beta}$ and $r_{i\alpha}$ will now be explained.

If θ is any element of G, and f is a loop in $S - L$ representing it, let us denote by θ_β that element of Γ that is represented by a loop in $(\Sigma - \Lambda) + \Phi$ of the form $h_\beta \tilde{f} h_\gamma^{-1}$, where h_β is a path in Φ from p_0 to p_β, $\tilde{f}$ is the path in $\Sigma - \Lambda$ that starts at p_β and covers f, thereby ending at p_γ, say, where γ is the index into which β is sent by $\rho(\theta)$, h_γ is a path in Φ from p_0 to p_γ.



Now take the free group generated by symbols $x_{j\beta}(1 \leq j \leq n, 0 \leq \beta \leq g - 1)$, and map $x_{j\beta}$ into the element $(x_j^\phi)_\beta$ described above, and extend to a homomorphism (also called ϕ) onto Γ. Thus $x_{j\beta}^\phi = (x_j^\phi)_\beta$. This homomorphism has the following charming property: if, for any word $u = x_{j_1}^{\epsilon_1} x_{j_2}^{\epsilon_2} x_{j_3}^{\epsilon_3} \ldots (\epsilon_k = \pm 1)$ and index $\alpha \epsilon (0, 1, \ldots, g - 1)$, one defines, formally,

$$u_\alpha = x_{j_1}{}^{\epsilon_1}{}_{\alpha_1} x_{j_2}{}^{\epsilon_2}{}_{\alpha_2} x_{j_3}{}^{\epsilon_3}{}_{\alpha_3} \ldots$$

where $\alpha_1, \alpha_2, \alpha_3, \ldots$ are determined by the following rule:

$$x_{j_1}{}^{\epsilon_1} \ldots x_{j_{k-1}}{}^{\epsilon_{k-1}} \longrightarrow \begin{pmatrix} \ldots \alpha \ldots \\ \\ \ldots \alpha_k \ldots \end{pmatrix} \quad \text{if } \epsilon_k = 1$$

$$x_{j_1}{}^{\epsilon_1} \ldots x_{j_k}{}^{\epsilon_k} \longrightarrow \begin{pmatrix} \ldots \alpha \ldots \\ \\ \ldots \alpha_k \ldots \end{pmatrix} \quad \text{if } \epsilon_k = -1$$

then u_α is mapped by ϕ into the element of Γ described above and denoted there by $(u^\phi)\alpha$.

For example, if $u = x_1 x_2^2 x_1^{-1} x_2^{-1}$ and $\rho(x_1) = (034)(25)$, $\rho(x_2) = (312)(45)$, then

$$u_0 = x_{10} x_{23} x_{21} x_{15}^{-1} x_{24}^{-1}$$

$$u_1 = x_{11} x_{21} x_{22} x_{10}^{-1} x_{20}^{-1}$$

$$u_2 = x_{12} x_{25} x_{24} x_{12}^{-1} x_{21}^{-1}$$

etc.

If $w_0 = 1, w_1, \ldots, w_{g-1}$ are words in $x_1, \ldots, x_n$ such that $w_0^\phi = 1$, $w_1^\phi, \ldots, w_{g-1}^\phi$ lie in the coset $\Gamma_0, \Gamma_1, \ldots, \Gamma_{g-1}$ respectively, and if the Schreier condition is satisfied, that is, if any left segment of any word w_i is one of the other words w_k (this can always be arranged), then $w_{10}^\phi, \ldots, w_{g-1,0}^\phi$ may be selected as the generators of a free factor F_{g-1}, and hence a presentation of the group Γ can be obtained from our presentation of $\Gamma * F_{g-1}$ by adjoining the relations $w_{10} = 1, \ldots, w_{g-1,0} = 1$. This effectively recovers the Reidemeister-Schreier algorithm.

Example 11 (continued). $G = (x, a\colon a^2x = xa)$ has the representation $x \to (0 \;\; 1)$, $a \to (0 \;\; 1 \;\; 2)$ onto S_3. If Γ is the fundamental group of the corresponding 3-fold irregular covering $\Sigma - \Lambda$ of the complementary domain, then $\Gamma * F_2$ has the presentation

$$\begin{pmatrix} x_0, a_0 & & a_0 a_1 x_2 = x_0 a_1 \\ & \cdot & \\ x_1, a_1 & \cdot & a_1 a_2 x_0 = x_1 a_0 \\ & \cdot & \\ x_2, a_2 & & a_2 a_0 x_1 = x_2 a_2 \end{pmatrix}$$

$$= (a_0, a_1\colon) * (x_2, A_2\colon A_2^2 x_2 = x_2 A_2)$$

where $A_2 = a_2a_0a_1$. We may choose $w_0 = 1$, $w_1 = a$, $w_2 = a^2$, satisfying the Schreier condition, and obtain the presentation $\Gamma = (x_2, a_2: a_2^2x_2 = x_2a_2)$ by adjoining the relations $a_0 = 1$, $a_1 = 1$. Thus $\Gamma \approx G$; the homology group of $\Sigma - \Lambda$ is infinite cyclic.

To obtain the group $\pi(\Sigma)$ we must now describe the branch relators, whose consequence is the kernel of $\pi(\Sigma - \Lambda) \to \pi(\Sigma)$. If v is an element of G that can be represented by arbitrarily small loops close to L, and if $\rho(v) = (\beta_1\beta_2 \ldots \beta_\lambda)(\ldots) \ldots$, then the corresponding branch relations are $v_{\beta_1}v_{\beta_2} \ldots v_{\beta_\lambda} = 1, \ldots$, the geometrical meaning of which is easy to perceive.

Example 11 (continued). The elements x and ax of G are represented by arbitrarily small loops close to L. The corresponding branch relations are

$$x_0x_1 = 1,\ x_2 = 1,\ a_0x_1,\ a_1x_2a_2x_0 = 1$$

(the last two are redundant, of course), so that Σ is seen to be simply connected. Since Λ is a pair of 2-spheres and $H_1(\Sigma - L) \approx Z$, Σ cannot be a 4-sphere. Probably Σ is topologically $S^2 \times S^2$.

Having obtained $\Gamma = \pi(\Sigma - \Lambda)$ and $\pi(\Sigma)$, it is, of course, easy to get the homology groups of $\Sigma - \Lambda$ and Σ. However, the process can be mechanized as follows: Let J denote the Jacobian matrix

$$\left\| \frac{\partial r_i}{\partial x_j} \right\|^{\phi}$$

of G and let θ be the regular representation of $\mathcal{S}_g$ on the group of $g \times g$ matrices, that is,

$$\theta\begin{pmatrix} 0 & 1 & \ldots & g-1 \\ \gamma(0) & \gamma(1) & \ldots & \gamma(g-1)\end{pmatrix} = \| \delta_{i,\gamma(i)} \|$$

and extend to the group rings. Then a relation matrix for $H_1(\Sigma - \Lambda) \oplus A_{g-1}$, where $\oplus$ denotes direct sum and A_{g-1} denotes the free abelian group of rank $g - 1$, is the $ng \times mg$ integral matrix $J^{\theta\rho}$.

Example 11 (continued). The Jacobian of the presentation

$$(x, a: a^2x = xa) \quad \text{is} \quad J = \| a^2 - 1 \quad 1 + a - x \|^{\phi}$$

and

$$J^{\theta\rho} = \begin{Vmatrix} -1 & 0 & 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & -1 & 1 & 1 \\ 0 & 1 & -1 & 1 & 0 & 0 \end{Vmatrix} \sim \| 1 \quad 0 \quad 0 \quad 0 \|$$

A relation matrix for $H_1(\Sigma) \oplus A_{g-1}$ is obtained from $J^{\theta\rho}$ by adjoining rows corresponding to the branch relators. Thus, in our example, a relation matrix for $H_1(\Sigma) \oplus A_2$ is

$$\begin{Vmatrix} -1 & 0 & 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & -1 & 1 & 1 \\ 0 & 1 & -1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{Vmatrix} \sim \begin{Vmatrix} 0 & 0 \end{Vmatrix}$$

(If, instead of $\theta\rho$, we had used the monomial representation $u \to \| \delta_{\alpha\beta} u_\alpha \|$ we would have obtained a Jacobian matrix of $\Gamma * F_{g-1}$.)

The abelianizing homomorphism maps a knot group G onto the infinite cyclic group Z. For each positive integer g there is a unique homomorphism of Z onto Z_g, the cyclic group of order g, hence a unique homomorphism of G onto Z_g. The coverings $\Sigma - \Lambda$ and Σ that belong to the kernel G' of $G \to Z$ are called the *infinite cyclic coverings*, and those that belong to the kernel of $G \to Z_g$ the gth *cyclic coverings*. The first homology group $H_1(\Sigma - \Lambda)$ of the unbranched gth cyclic covering is the direct sum of Z and the first homology group $H_1(\Sigma)$ of the branched gth cyclic covering. The order Θ of the group $H_1(\Sigma)$ in this case can be shown to be

$$\Theta = R(t^g - 1, \Delta(t)) = \prod_{j=0}^{g-1} \Delta(\omega^j)$$

where ω denotes a primitive gth root of unity; and when $H_1(\Sigma)$ is infinite, its Betti number turns out to be just the number of roots of $\Delta(t) = 0$, properly counted, that are gth roots of unity. If $H_1(\Sigma)$ is a finite group, the commutator quotient group of $\pi(\Sigma - \Lambda)$ is infinite cyclic, and so $\Sigma - \Lambda$ has an Alexander polynomial. It can be shown that this polynomial $\tilde{\Delta}(\tau)$ is equal to $\prod_{j=0}^{g-1} \Delta(\omega^j \tau)$, where $\tau = t^g$. Clearly, then, $\tilde{\Delta}(1)$ is the order of $H_1(\Sigma)$.

Example.  $\Delta(t) = 1 - t + t^2$

$g = 2;\ \Theta = 3,\ \tilde{\Delta}(\tau) = (1 - t + t^2)(1 + t + t^2) = 1 + \tau + \tau_2$

$g = 3;\ \Theta = 4,\ \tilde{\Delta}(\tau) = (1 - t + t^2)(1 - \omega t + \omega^2 t^2)(1 - \omega^2 t + \omega t^2)$
$= 1 + 2\tau + \tau^2$

$g = 4;\ \Theta = 3,\ \tilde{\Delta}(\tau) = 1 + \tau + \tau^2$

etc.

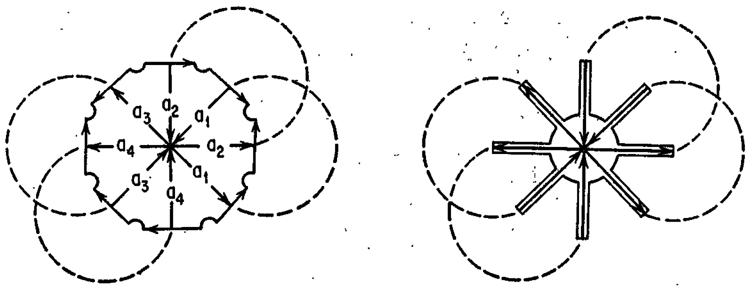
(For non-cyclic coverings, $H_1(\Sigma)$ and $H_1(\Sigma - \Lambda)$ are not related by any obvious formula, nor is there any known formula for the order of these groups.)

[There is a sense in which $\Delta(t)$ is the "order of the group $H_1(\Sigma - \Lambda)$" for the infinite cyclic covering. It is necessary to regard $H_1(\Sigma - \Lambda)$ as an operator group, the covering transformations being the operators.]

9. THE CYCLIC COVERINGS OF A KNOT

Now let us examine the finite cyclic coverings more closely, utilizing a simplifying procedure due to Seifert.

Let $\mathfrak{F}$ be a surface of genus h with one boundary curve. As the accompanying diagram shows, $\mathfrak{F}$ can be shrunk isotopically to a model consisting of a 2-cell with $2h$ bands

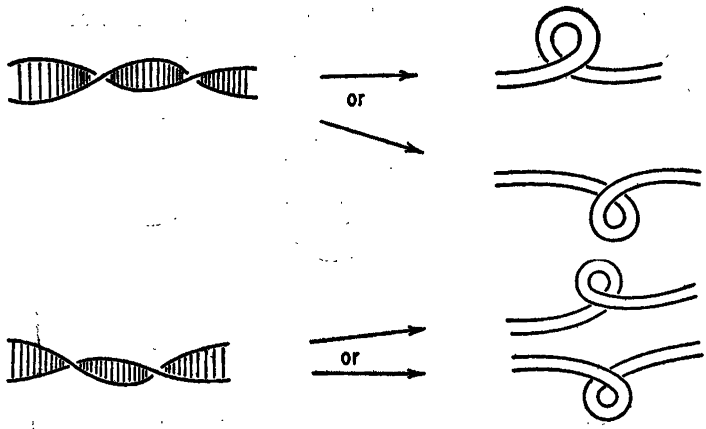


If $\mathfrak{F}$ is embedded semi-linearly in 3-space, this isotopy can be extended to an isotopy of space; hence every type of embedding of $\mathfrak{F}$ contains a representative consisting of a 2-cell with $2h$ attached bands. These bands may, however, be twisted, knotted, and linked.

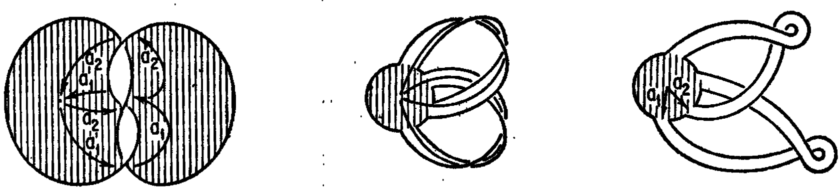
Let $a_1, a_2, a_3, a_4, \ldots, a_{2h-1}a_{2h}$ be a canonical set of curves on $\mathfrak{F}$. These are oriented closed curves through a common point but otherwise disjoint and placed as in the diagram. We shrink $\mathfrak{F}$ down to a neighborhood of $a_1 + a_2 + \ldots + a_{2h-1} + a_{2h}$ so that the bands occur around the 2-cell in the order a_1 leaving, a_2 leaving, a_1 entering, a_2 entering, a_3 leaving, etc.

If we have a knot given to us and we span an orientable surface $\mathfrak{F}$ of genus h in it by Seifert's (or any other) method, we just pick a point on $\mathfrak{F}$ and run a system $a_1, a_2, \ldots, a_{2h-1}, a_{2h}$ of canonical curves through it and shrink $\mathfrak{F}$ down onto these curves. This can always be done, although for surfaces of high genus some patience may be required.

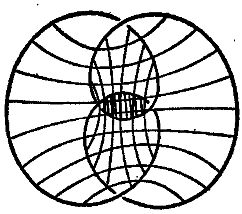
Since $\mathfrak{F}$ is orientable, the number of twists in any one band is necessarily even; hence these twists can be replaced by curls (just half as many curls as twists) as shown in the diagram.



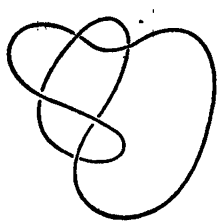
The resulting surface $\mathfrak{F}$, of the same embedding type as the original surface, may be laid down flat on the table so that only one side of it is visible. For example, for the trefoil:



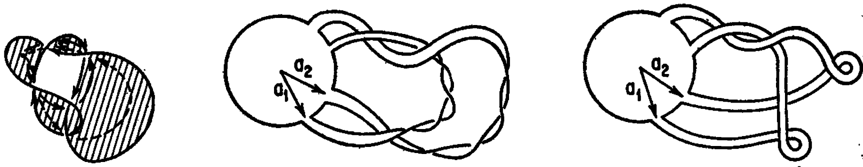
If the Seifert circles (see p. 140) are nested, the Seifert surface will appear to be in layers, and this is rather confusing. I have found that in every case that I have tried I have been able to avoid this by changing the knot diagram.



is too confusing; but another diagram of the figure eight is



which, although it is not economical as regards the number of crossings, is much more convenient as regards its Seifert surface.



From this "normalized" surface $\mathfrak{F}$ one can read off a $2h \times 2h$ integral matrix $V = (v_{ij})$. The entry v_{ij} is defined to be the algebraic number of times that the jth band crosses over the ith from left to right.

Thus, for the surface constructed above that spans the trefoil,

$$V = \begin{Vmatrix} -1 & 1 \\ 0 & -1 \end{Vmatrix}$$

and, for the one that spans the figure eight,

$$V = \begin{Vmatrix} -1 & -1 \\ -2 & -1 \end{Vmatrix}$$

Note that one always has

$$v_{12} - v_{21} = 1,\ v_{34} - v_{43} = 1, \ldots, \quad \text{and}$$

$$v_{ij} = v_{ji} \quad \text{otherwise},$$

or, in matrix form,

$$V' - V = I$$

where prime denotes transpose and I denotes the block diagonal matrix

$$\sum_{i=1}^{h} \begin{Vmatrix} 0 & -1 \\ 1 & 0 \end{Vmatrix}$$

Note also that I is the $2h \times 2h$ matrix of intersection numbers on $\mathfrak{F}$ of the canonical curves

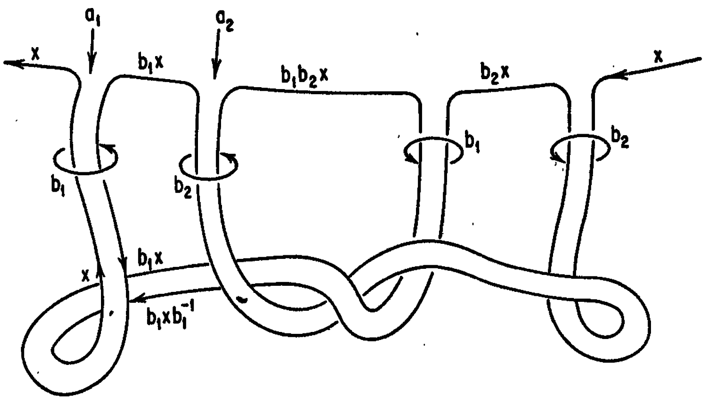
$$I = \| \mathcal{S}(a_i, a_j \|$$

Also define, with Seifert,

$$\Gamma = VI = \left\| \begin{matrix} v_{12} & -v_{11} & v_{14} & -v_{13} & \cdot \\ v_{22} & -v_{21} & v_{24} & -v_{23} & \cdot \\ v_{32} & -v_{31} & v_{34} & -v_{33} & \cdot \\ v_{42} & -v_{41} & v_{44} & -v_{43} & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{matrix} \right\|$$

The 1-cycles $a_1, a_2, \ldots, a_{2h-1}, a_{2h}$ form a base for the 1-dimensional homology group of $\mathfrak{F}$, and a change of basis will replace V by UVU' where U is a unimodular matrix. Thus the congruence class of the matrix V is an invariant of the type of embedding of $\mathfrak{F}$; but since it is possible to span a knot k by various surfaces $\mathfrak{F}$, the congruence class of V is certainly not an invariant of the knot type of k. It is possible, however, to express some of the invariants of the knot type of k in terms of the matrix V, and it is this that leads to some considerable simplification in these invariants.

Let us now calculate the Alexander matrix in terms of V. Since the Alexander matrix in an invariant of G/G'', we can begin by finding a presentation of G/G''. Let us look at the first pair of bands, where a_1, a_2 are as before, b_i is a loop that circles the ith band in the direction indicated



on the diagram, and x is a little loop circling the knot at the place indicated. Since we are working modulo the second commutator group G'', the elements $a_1, \ldots, a_{2h}, b_1, \ldots, b_{2h}$, which obviously belong to G', commute, so that it doesn't matter where on the ith band b_i does its circling. Other anomalies of the diagram, such as the appearance of x at two different places, are explained in the same way. It is not very difficult to verify that G/G'' is generated by $x, a_1, \ldots, a_{2h}, b_1, \ldots, b_{2h}$

Aside from the relations that say that commutators commute, there are two kinds of relations; that is,

$$G/G'' = (x, a_1, \ldots, a_{2h}, b_1, \ldots, b_{2h}: r_1, \ldots, r_{2h}, s_1, \ldots, s_{2h}, \ldots)$$

Relation r_i is obtained by trying to lift the loop a_i straight up. Naturally, it gets caught in the bands, so this gives an expression for a_i in terms of the loops $b_1, \ldots, b_{2h}$ that go around the bands. What you get is

$$(r_i): a_i = \prod_{j=1}^{2h} b_j^{-v_{ij}}$$

Relation s_1 is obtained by transporting the loop x around the first band (following the right edge). As you do this, it also gets caught in the various bands and this results in conjugations. For example, in the figure, x gets caught first in the first curl of the first band and the loop around the right-hand edge of the first band just after this curl is, in fact, $b_1 \times b_1^{-1}$ as shown. Having translated x clear around this band, we get two names for the little loop that goes around the right-hand edge of the band near its end, and thus we have our first relation.

$$(s_1): a_1^{-1}xa_1 = b_2x$$

Similarly we get

$$(s_2): a_2^{-1}(b_1x)a_2 = x$$

In general,

$$(s_{2i-1}): a_{2i-1}^{-1}xa_{2i-1}x^{-1} = b_{2i}$$

$$(s_{2i}): a_{2i}xa_{2i}^{-1}x^{-1} = b_{2i-1}$$

From this presentation, we can either eliminate the generators a_i and the relations (r_i) or eliminate the generators b_i and the relations (s_i). The first method leads to the matrix

$$\left\| \frac{\partial s_i}{\partial b_j} \right\|^{\psi\phi} = I + (1 - t)V$$

and the second method to the matrix

$$\left\| \frac{\partial r_i}{\partial a_j} \right\|^{\psi\phi} = E + (t - 1)\Gamma$$

(Since $I^2 = -E$, the second can be obtained formally from the first by right multiplying by $-I$.) In the Alexander matrix there was an extra column of 0's obtained by differentiating with respect to x. Following Seifert, we shall consider the second matrix

$$F(t) = E + (t - 1)\Gamma$$

Clearly $\Delta(t) = \det F(t)$. Now the gth cyclic covering belongs to the representation $x \to (0 \quad 1 \ldots g-1)$, $a_j \to$ identity. Let T_g denote the $g \times g$ matrix

$$\begin{Vmatrix} 0 & 1 & 0 \ldots 0 \\ 0 & 0 & 1 \ldots 0 \\ 0 & 0 & 0 \ldots 0 \\ \cdot & \cdot & \cdot \ldots \cdot \\ 0 & 0 & 0 \ldots 1 \\ 1 & 0 & 0 \quad\; 0 \end{Vmatrix}$$

the regular representation of the permutation $(0 \quad 1 \ldots g-1)$. Then $F(T_g)$ is a relation matrix for the first homology group of the gth cyclic branched covering Σ of 3-space branched over our knot. But, rearranging rows and columns,

$$F(T_g) = \begin{Vmatrix} E-\Gamma & \Gamma & 0 & 0 \\ 0 & E-\Gamma & \Gamma & 0 \\ \Gamma & 0 & 0 & E-\Gamma \end{Vmatrix}$$

and this is equivalent to

$$\begin{Vmatrix} E & * \ldots * & & * \\ 0 & E \ldots * & & * \\ & \cdot & & \\ & & \cdot & \\ & & & \cdot \\ 0 & 0 \quad E & & * \\ 0 & 0 \quad 0 & & \Gamma^g - (\Gamma - E)^g \end{Vmatrix}$$

Thus the $2h \times 2h$ matrix

$$F_g = \Gamma^g - (\Gamma - E)^g$$

is a relation matrix for $H_1(\Sigma)$.

For many knots, this represents an enormous simplification. It shows that $H_1(\Sigma)$ can be calculated from a $2h \times 2h$ matrix if the knot is of genus h, even though the method explained earlier leads to a matrix that is in general much larger.

For a knot of genus 1, it is only a matter of solving some difference equations to work out the entries of F_2. This leads to explicit calculation of $H_1(\Sigma)$ for all values of g, and some of the results are quite interesting. For instance, for the trefoil knot, we get that

$$\begin{aligned} H_1(\Sigma) &= Z \oplus Z && \text{for } g \equiv 0 \pmod 6 \\ &= 0 && \text{for } g \equiv \pm 1 \pmod 6 \\ &= Z_3 && \text{for } g \equiv \pm 2 \pmod 6 \\ &= Z_2 \oplus Z_2 && \text{for } g \equiv 3 \pmod 6 \end{aligned}$$

On the other hand, for the figure-eight knot we get the accompanying table:

g	H_1 (Z)
1	0
2	Z_5
3	$Z_4 \oplus Z_4$
4	$Z_{15} \oplus Z_3$
5	$Z_{11} \oplus Z_{11}$
6	$Z_{40} \oplus Z_8$
7	$Z_{29} \oplus Z_{29}$
8	$Z_{105} \oplus Z_{21}$
9	$Z_{76} \oplus Z_{76}$
10	$Z_{275} \oplus Z_{55}$

The appearance of direct doubles in the odd-numbered rows is not accidental. It is a fact that, for any knot, the homology group $H_1(\Sigma)$ of the gth cyclic branched covering is a direct double for every odd g.

Since the gth branched cyclic covering Σ is a closed oriented 3-manifold, it supports a self-linking L. L is a primitive bilinear symmetric mapping from the 1-dimensional torsion group into the rationals mod 1. It is defined as follows: if a and b are torsion cycles then there is a 2-chain A whose boundary is ma for some integer $m \neq 0$. Then

$$L(a, b) \equiv \frac{1}{m} \mathcal{S}(A, b) \pmod 1$$

where $\mathcal{S}$ denotes intersection number in Σ. If Σ is reoriented, L changes sign. Since Σ inherits an orientation from S, we see that L changes its sign when S is reoriented.

The torsion group $T_1(\Sigma)$, together with the primitive bilinear symmetric mapping,

$$L: T_1(\textstyle\sum) \oplus T_1(\textstyle\sum) \rightarrow \text{rationals mod } 1$$

is an invariant of the type of knot k in the oriented 3-sphere S. Numerical invariants can be read from it as follows: If $\tau_1, \tau_2, \ldots, \tau_n$ are the torsion numbers of Σ (they may be calculated from the matrix F_g) in the order in which τ_{r+1} divides τ_r, and p is any odd prime divisor of τ_r such that p^d divides τ_r but not τ_{r+1} (where τ_{n+1} is defined to be 1), then, for properly chosen torsion elements $g_1, \ldots, g_r$ of $H_1(\Sigma)$,

$$\tau_1 \ldots \tau_r \det \| L(g_i, g_j) \|_{i,j=1, \ldots, r}$$

is an integer prime to p, and $\chi_r(p)$, defined to be the quadratic residue character $(\tau_1 \ldots \tau_r \det \| L(g_i, g_j) \| / p)$ of this integer, is independent of the choices made, and is an invariant of the type of the knot k in the oriented 3-sphere S.

It can be shown that the matrix of intersection numbers corresponding to the matrix F_g of boundary coefficients is just

$$S_g = (\Gamma - E)^g \cdot I$$

The corresponding matrix of self-linking numbers L_g is now determined, since

$$S_g = F_g \cdot L_g$$

As an illustration, we found the matrix

$$V = \begin{Vmatrix} -1 & 1 \\ 0 & -1 \end{Vmatrix}$$

for the trefoil knot. Hence

$$\Gamma = \begin{Vmatrix} 1 & 1 \\ -1 & 0 \end{Vmatrix}$$

$$F_2 = \begin{Vmatrix} 1 & 2 \\ -2 & -1 \end{Vmatrix}, \quad S_2 = \begin{Vmatrix} -1 & 1 \\ 0 & -1 \end{Vmatrix}, \quad L_2 \equiv \begin{Vmatrix} \frac{1}{3} & \frac{1}{3} \\ -\frac{2}{3} & \frac{1}{3} \end{Vmatrix} \pmod 1$$

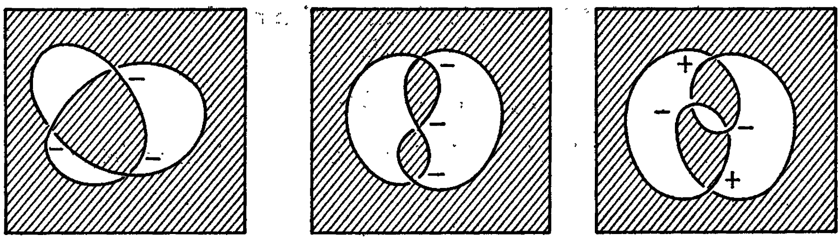
$\tau_1 = 3$, $\tau_2 = 1$, $p = 3$, $r = 1$. $\chi_1(3)$ is the quadratic residue character of $3 \cdot \frac{1}{3} = 1$. Since 1 is a quadratic residue mod 3, $\chi_1(3) = 1$. On the other hand, if the 3-sphere is reoriented we get

$$V = \begin{Vmatrix} 1 & 0 \\ -1 & 1 \end{Vmatrix}, \quad \Gamma = \begin{Vmatrix} 0 & -1 \\ 1 & 1 \end{Vmatrix}, \quad F_2 = \begin{Vmatrix} -1 & -2 \\ 2 & 1 \end{Vmatrix}, \quad S_2 = \begin{Vmatrix} 1 & 0 \\ -1 & 1 \end{Vmatrix}$$

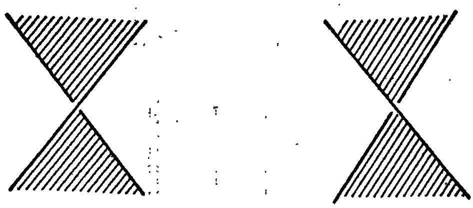
$$L_2 \equiv \begin{Vmatrix} -\frac{1}{3} & \frac{2}{3} \\ -\frac{1}{3} & -\frac{1}{3} \end{Vmatrix} \pmod 1$$

$\chi_1(3)$ is the quadratic residue character of $3 \cdot (-\frac{1}{3}) = -1$. Since -1 is a quadratic non-residue mod 3, $\chi_1(3) = -1$. This shows that the trefoil is not amphicheiral.

For $g = 2$ there is a similar algorithm based on a surface $\mathfrak{F}$ spanning the knot that is not necessarily orientable. Such a surface can be obtained by coloring the regions of the diagram alternately so as to form a "chess-board surface." A rule which would produce such a surface is the following; Color a region dark or light according as a path from that region that goes out to infinity cuts the projection of the knot an even or an odd number of times.



(At the crossings the surface twists, just as before.) To each crossing c an integer $\eta(c) = \pm 1$ is assigned, measuring the "twist" of the surface at that place.



$\eta = +1$ $\eta = -1$

Let $X_0, X_1, \ldots, X_n$ denote the shaded regions (where X_0 denotes the unbounded region, say), and define $e_{ij} = \Sigma\, \eta(c)$, summed over those crossings that are incident to both X_i and X_j. Let

$$\hat{f}(x_0, x_1, \ldots, x_n) = \sum_{i<j} e_{ij}(x_i - x_j)^2$$

and define

$$f(x_1, \ldots, x_n) = \hat{f}(0, x_1, \ldots, x_n)$$

This is called the *quadratic form* of the diagram. Clearly

$$\hat{f}(x_0, x_1, \ldots, x_n) = \sum_{i,j=0}^{n} a_{ij} x_i x_j$$

where $a_{ij} = -e_{ij}$ for $i \neq j$, and $a_{ii} = \sum_{j\neq i} e_{ij} = \sum \eta(c)$, summed over the crossings incident to X_i. To the quadratic form $\hat{f}(x_0, x_1, \ldots, x_n)$ is associated the symmetric integral matrix $\hat{A} = (a_{ij})_{i,j=0,1,\ldots,n}$ and to the quadratic form $f(x_1, \ldots, x_n)$ is associated its principal minor

$$A = (a_{ij})_{i,j=1,\ldots,n}$$

It can be shown that a relation matrix for $H_1(\Sigma)$, where Σ is the 2-fold branched cyclic covering, is this $n \times n$ matrix A, and that the corresponding matrix of self-linking numbers is A^{-1} (mod 1). [Note that A is necessarily a non-singular matrix, for det $A = \Delta(-1) \neq 0$.]

Examples: (figures above)

1. Trefoil knot (first projection)

$$f(x_1) = -3x_1^2,\ A = -3,\ A^{-1} \equiv -\tfrac{1}{3} \pmod 1$$

The non-amphicheirality very plain.

2. Trefoil knot (second projection)

$$f(x_1, x_2) = -x_1^2 - (x_1 - x_2)^2 - x_2^2$$

$$A = \begin{Vmatrix} -2 & 1 \\ 1 & -2 \end{Vmatrix},\quad A^{-1} \equiv \tfrac{1}{3}\begin{Vmatrix} -2 & -1 \\ -1 & -2 \end{Vmatrix} \pmod 1$$

3. Figure-eight knot

$$f(x_1, x_2) = x_1^2 - 2(x_1 - x_2)^2 + x_2^2$$

$$A = \begin{Vmatrix} -1 & 2 \\ 2 & -1 \end{Vmatrix},\quad A^{-1} \equiv -\tfrac{1}{5}\begin{Vmatrix} -1 & -2 \\ -2 & -1 \end{Vmatrix} \pmod 1$$

The figure eight is obviously amphicheiral, and, in fact, if the crossings are all reversed, so as to get the mirror image, what results is

$$f(x_1, x_2) = -x_1^2 + 2(x_1 - x_2)^2 - x_2^2$$

$$A = \begin{Vmatrix} 1 & -2 \\ -2 & 1 \end{Vmatrix},\quad A^{-1} \equiv -\tfrac{1}{5}\begin{Vmatrix} 1 & 2 \\ 2 & 1 \end{Vmatrix} \pmod 1$$

In either case, $\chi_1(5) = (\pm\tfrac{1}{5}) = 1$.

10. FINDING REPRESENTATIONS

In 3- and 4-dimensional topology, one often faces the problem of finding representations of a presented group G. For example, you may want to show that a certain arc is wild, for which purpose you will be perfectly happy to find *any* representation onto *any* non-trivial group. Or you may want to distinguish between two different embeddings; in this case, the problem is more subtle, and it may be necessary to examine *all* the representations onto a selected non-trivial group.

The range of groups upon which representations may be made is limited only by one's imagination; most frequently permutation groups have been used, but mathematicians have also used groups of matrices, for example, groups of motions in the hyperbolic plane, or even knot groups themselves.

The most obvious way to find representations of a group G, when you have a presentation of G before you, is to adjoin some relations, and if you are lucky, the resulting homomorph is sufficiently simplified so that you can recognize it, but not so oversimplified that it becomes trivial. If G is perfect ($G = G'$), it becomes a little delicate to handle this properly; if G happens to be simple, it becomes impossible. (There is, however, a different technique, which I shall mention later, that sometimes allows one to deal even with simple groups.)

Since many of the groups G that come up are knot groups, or like knot groups, let us consider first the representations of the group G of a tame knot k. The first thing to observe is that abelianization maps G onto Z, and that there is a unique homeomorphism of Z onto Z_g for any g. Thus there is a unique representation of G onto Z_g. If we think of the elements of Z_g as the powers of the permutation $(0 \ \ 1 \ \ 2 \ldots g - 1)$, this representation is defined by mapping each of the generators x_j of a given over-presentation into the cycle $(0 \ \ 1 \ \ 2, \ldots, g - 1)$. These are the only finite abelian representations.

(Changing the names of the symbols $0, 1, \ldots, g - 1$ say, by permuting them, amounts to following the representation by an inner automorphism of the symmetric group $\mathcal{S}_g$ of degree g; representations into $\mathcal{S}_g$ are always determined only up to these inner automorphisms.)

Of the non-abelian representations, the only ones that can be obtained at all systematically are the metabelian ones, in particular the metacyclic ones. Here is how metacyclic representations can be found:

Look at the relation at a typical crossing,

$$x_k = x_j x_i x_j^{-1} \quad \text{(see figure p. 122)}$$

The non-zero entries in the corresponding row of the Alexander matrix are

x_i	x_j	x_k
t	$1 - t$	-1

so if we multiply the Alexander matrix on the right by the column matrix

$$\begin{Vmatrix} \cdot \\ \cdot \\ \cdot \\ \xi_i \\ \cdot \\ \cdot \\ \cdot \\ \xi_j \\ \cdot \\ \cdot \\ \cdot \\ \xi_k \\ \cdot \\ \cdot \\ \cdot \end{Vmatrix}$$

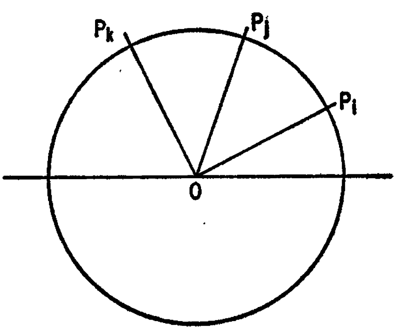
we will get the 0 matrix if the system of n linear homogeneous equations in n unknowns.

$$\begin{array}{c} \cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ t\xi_i + (1 - t)\xi_j - \xi_k = 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{array}$$

has a solution. Since the sum of the coefficients in any one of the equations is 0, and since any of the n relations is a consequence of the others, any one of these equations may be thrown away and one of the variables may be given the value 0 arbitrarily. Since the determinant of the resulting $(n - 1) \times (n - 1)$ system of homogeneous equations is $\Delta(t)$, the system has at least one non-trivial solution, if we reinterpret the equations to be congruences modulo $|\Delta(t)|$. Here, I am thinking of t as, say, an integer.

Now our congruences can be given an interesting interpretation. The typical one written above may be rewritten

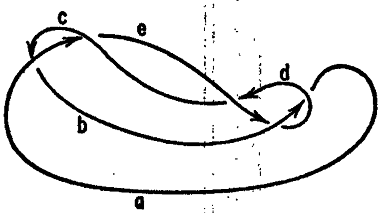
$$\xi_j - \xi_k \equiv t(\xi_j - \xi_i) \pmod{\Delta(t)}$$



Think of n points $P_1, P_2, \ldots, P_n$ on the unit circle of the plane and the radii $OP_1, OP_2, \ldots, OP_n$, and interpret $\xi_1, \xi_2, \ldots, \xi_n$ as the inclinations of these radii divided by $2\pi/|\Delta(t)|$. Then our typical congruence has the interpretation that $\angle P_jOP_k$ is t times $\angle P_jOP_i$. (For instance, if $t = -1$, this says that

OP_j bisects the angle $\angle P_iOP_k$, if $t = 3$, OP_i trisects the angle $\angle P_jOP_k$.) Thus, each solution of our system of linear equations results in a distribution of the n points $P_1, \ldots, P_n$ on the unit circle. The group G may now be represented onto the groups of motions of a regular polygon of $|\Delta(t)|$ sides, in case $t = -1$ by representing x_i by the reflection across the line OP_i, and, in the general case, by dilation of the angle by a factor of t. This can, of course, be written as a permutation.

Example 16



$$G = (a, b, c, d, e: b = dad^{-1}, c = aba^{-1}, d = ece^{-1}, e = bdb^{-1}, a = cec^{-1}).$$

The Alexander matrix is

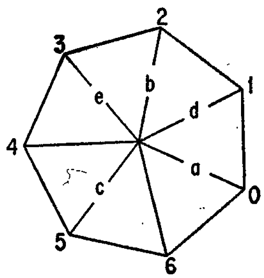
$$\left\| \begin{matrix} t & -1 & 0 & 1-t & 0 \\ 1-t & t & -1 & 0 & 0 \\ 0 & 0 & t & -1 & 1-t \\ 0 & 1-t & 0 & t & -1 \\ -1 & 0 & 1-t & 0 & t \end{matrix} \right\| \cdot \Delta(t) = 2 - 3t + 2t^2$$

$\Delta(-1) = 7$. The homogeneous $n \times n$ system is

$$\begin{aligned} -\alpha - \beta \quad + 2\delta \quad &\equiv 0 \\ 2\alpha - \beta - \gamma \quad &\equiv 0 \\ -\gamma - \delta + 2\epsilon &\equiv 0 \quad (\text{mod } 7) \\ 2\beta - \delta - \epsilon &\equiv 0 \\ -\alpha + 2\beta - \epsilon &\equiv 0 \end{aligned}$$

Choose $\alpha \equiv 0 \pmod 7$, say, and throw away one of the congruences. A solution is

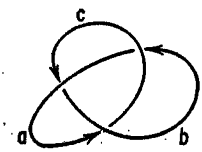
$$\alpha \equiv 0, \beta \equiv 2, \gamma \equiv 5, \delta \equiv 1, \epsilon \equiv 3$$



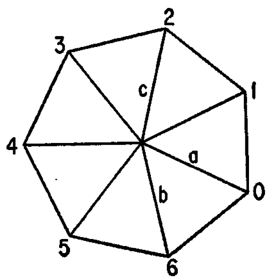
Hence, we can mark a, b, c, d, e on a regular heptagon as shown and obtain the representation

$$a \to (16)(25)(34)$$
$$b \to (13)(04)(56)$$
$$c \to (46)(03)(12)$$
$$d \to (02)(36)(45)$$
$$e \to (24)(15)(06)$$

Example 17. $G = (a, b, c\colon b = cac^{-1}, c = aba^{-1}, a = bcb^{-1})$ $\Delta(t) = 1 - t + t^2$, $\Delta(-2) = 7$



$$-2\alpha - \beta + 3\gamma \equiv 0$$
$$3\alpha - 2\beta - \gamma \equiv 0 \quad (\text{mod } 7)$$
$$-\alpha + 3\beta - 2\gamma \equiv 0$$



and we get the representation

$$a \to (132645)$$
$$b \to (021534)$$
$$c \to (354160)$$

A metacyclic representation that is ancestral to all other metacyclic representations may be obtained as follows: Let $M(n, t)$ denote the metacyclic group $(y, u\colon u^n = 1, yuy^{-1} = u^t)$. Map x_i into $u^{c_i}y$ and try to determine the integers c_i so that a homeomorphism is defined. Our typical relation maps into

$$u^{c_k}y = u^{c_i}yu^{c_j}yy^{-1}u^{-c_i}$$

so that we must have $tc_i + (1 - t)c_j - c_k \equiv 0 \pmod{n}$. This is the same system of congruences, and there will be non-trivial solutions whenever n divides $\Delta(t)$.

Example 18. $(a, b, c\colon b = cac^{-1}, c = aba^{-1}, a = bcb^{-1})$ is represented on $M(7, -2)$ by

$$a \to y \qquad \text{(compare the previous example)}$$
$$b \to u^6y$$
$$c \to u^2y$$

In my paper "A remarkable simple closed curve" the group

$$\Gamma = (b_0, b_1, b_2, \ldots : b_1 b_0 b_1^{-1} = b_2 b_1 b_2^{-1} = \ldots)$$

was obtained, and it was required to show that Γ was non-abelian, that is, different from $\Gamma/\Gamma' \approx Z$. If one adjoins the relations $b_0 = b_2 = b_4 = \ldots$, $b_1 = b_3 = b_5 = \ldots$, the group is mapped homeomorphically onto the finitely presented group $(b_0, b_1 : b_1 b_0 b_1^{-1} = b_0 b_1 b_0^{-1})$ and the methods discussed above would have led to the representation

$$\begin{aligned} b_n &\to (12) \quad \text{for } n \text{ even} \\ &\to (23) \quad \text{for } n \text{ odd} \end{aligned}$$

that was used there. As a matter of fact, I didn't know any method then and just guessed. Example 1.4 of the paper "Some wild cells . . ." could have been handled similarly. The other representations of that paper, however, were into the alternating group A_5 of degree 5. Since A_5 is far from being metacyclic or even metabelian, these methods would fail for these examples. As a matter of fact, A_5 is a simple group, so that I know of no method for finding representations on A_5 other than just trying.

In the paper, "A mildly wild imbedding . . ." it was difficult to find a representation of the group. Fortunately the relations of an obvious homomorph were $x_\mu y_\mu x_\mu = y_\mu x_\mu y_\mu$, $x_{\mu+1} y_\mu x_{\mu+1} = y_\mu M_{\mu+1} y_\mu$, $x_\mu^n = 1$, which suggests the well-known presentation:

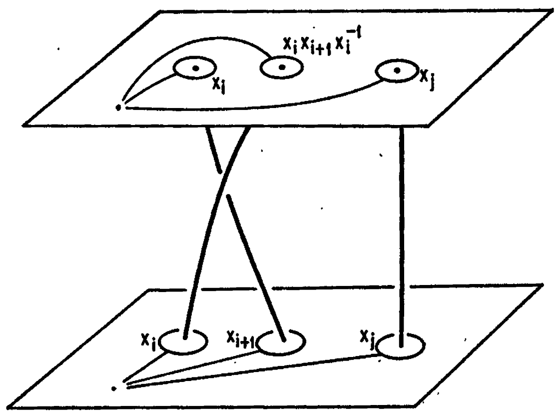
$$\begin{aligned} (\sigma_1, \sigma_2, \ldots, \sigma_d : \sigma_i \sigma_{i+1} \sigma_i &= \sigma_{i+1} \sigma_i \sigma_{i+1} \\ \sigma_i \sigma_j &= \sigma_j \sigma_i, \; |i - j| \geq 2) \end{aligned}$$

of the braid group B_{d+1}, and this suggested the use of the Burau matrices that did in fact turn the trick.

These Burau matrices are of considerable interest in themselves, and, as the above remarks show, they can do the work of permutation groups of infinite degree. Burau considers the group of infinite matrices whose entries are almost all the same as those of an infinite identity matrix, and he represents σ_i by the matrix

$$\left\| \begin{array}{c|cc|c} E & 0 & & 0 \\ \hline 0 & 1-t & t & 0 \\ & 1 & 0 & \\ \hline 0 & 0 & & E \end{array} \right\| \begin{array}{l} \\ i \\ i+1 \\ \end{array}$$

This representation has an enlightening explanation in terms of the free calculus. As the accompanying diagram shows, σ_i is associated with the



automorphism T_1

$$x_i \rightarrow x_i x_{i+1} x_i^{-1}$$

$$x_{i+1} \rightarrow x_i$$

$$x_j \rightarrow x_j \quad \text{for all } j \neq i, i+1$$

of the fundamental group of the compactified plane punctured at $d + 1$ points (that is, the free group of rank d). This representation of B_{d+1} by the automorphism group of the free group of rank d was fundamental in Artin's study of the braid groups. Now the entries in the critical 2×2 minor of Burau's matrix are just

$$\left\| \begin{matrix} \dfrac{\partial T_i(x_i)}{\partial x_i} & \dfrac{\partial T_i(x_i)}{\partial x_{i+1}} \\[2ex] \dfrac{\partial T_i(x_{i+1})}{\partial x_i} & \dfrac{\partial T_i(x_{i+1})}{\partial x_{i+1}} \end{matrix} \right\|^{\psi} \quad \text{where } \psi(x_k) = t$$

so that Burau's representation is just

$$\sigma_\mu \rightarrow \left\| \frac{\partial T_\mu(x_i)}{\partial x_j} \right\|^{\psi} \qquad i, j = 1, 2, \ldots$$

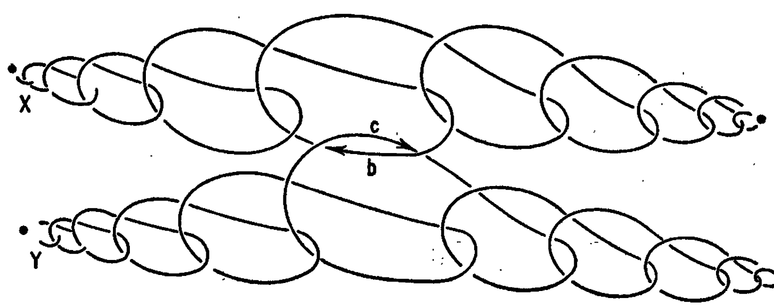
From elementary formulas of the free calculus, it follows that the Burau matrix representing any braid word $w = w(\sigma_1, \sigma_2, \ldots, \sigma_d)$ is just

$$w \rightarrow \left\| \frac{\partial T(x_i)}{\partial x_j} \right\|^{\psi}$$

where T is the automorphism of the free group of rank d that is associated with the braid w.

The *free product with amalgamation* is often a more powerful method than representation. A presentation defines a free product $A \times B$ if the generators split into two sets, say, $x_1, x_2, \ldots$ and $y_1, y_2, \ldots$ and the relations also into two sets, say, $r_1, r_2, \ldots$ and $s_1, s_2, \ldots$, where each r_i is a word in $x_1, x_2, \ldots$ alone and each s_i is a word in $y_1, y_2, \ldots$ alone. If there are further relations of the form $u_1 = v_1$, $u_2 = v_2$, $\ldots$ where each u_i is a word in $x_1, x_2, \ldots$ and each v_i is a word in $y_1, y_2, \ldots$, this is not necessarily a free product with amalgamation $AC{*}B$; you have to prove somehow that the subgroup C_1 of A generated by $u_1, u_2, \ldots$ is isomorphic to the subgroup C_1 of B generated by $v_1, v_2, \ldots$ under an isomorphism that makes u_1 correspond to v_1, u_2 to v_2, and so on. If one can show this, however, the rewards may be tremendous. For example, A and B are contained isomorphically in $A \underset{C}{*} B$, so that if either A or B is known to be non-trivial, $A \underset{C}{*} B$ is immediately seen to be non-trivial. (This may be used, for example, to prove that $k \# l$ is not trivial unless k and l are both trivial.) (Since A and B are not normal in $A \underset{C}{*} B$, finding a non-trivial representation may be difficult or even impossible.) Further pleasant properties are, for example: if an element of $A \underset{C}{*} B$ is of finite order it must be conjugate to an element of A or to an element of B; if two elements of a free product $A * B$ commute, they must both belong to zAz^{-1} or to zBz^{-1} for some z.

As a simple illustration of the use of free products with amalgamation (in this case free products), I shall prove that there exist a pair of unsplittable arcs; that is, a pair of disjoint arcs X and Y such that any 3-cell that contains X intersects Y.



X and Y are just two copies of example 1.1 of "Some wild cells . . ." that are hooked together. If X and Y were splittable, the group $G = \pi(S^3 - (X + Y))$ would be the free product of $\pi(S^3 - X)$ and $\pi(S^3 - Y)$, and hence no conjugate of b could commute with any conjugate of c (since neither b nor c is trivial, as is shown by the representation, *loc. cit.*). Since b and c obviously commute, X and Y must be unsplittable.

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