Flaming swords and hermaphrodite monsters – Peter Guthrie Tait and the promotion of quaternions, part II

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In the first part of this paper [1] we traced the early development of quaternions in the hands of Hamilton's successor, Peter Guthrie Tait, Professor of Natural Philosophy in the University of Edinburgh from 1860 to 1900. Tait had neither the intuitive feel for physical concepts that Maxwell possessed nor the entrepreneurial talent of Thomson (Lord Kelvin) and yet he fulfilled a pivotal role in nineteenth century British physics through his correspondence with the former, with whom he went to school, and his collaboration with the latter. Though ambivalent towards quaternions, Maxwell was chivvied by Tait into drafting his 1873 *Treatise on Electricity and Magnetism* in both Cartesian and quaternion form.

At this point we need to backtrack a little and introduce Arthur Cayley with whom Tait was to disagree over the relative merits of Cartesians and quaternions. Cayley had graduated as Senior Wrangler in 1842 but later practised law at Lincoln's Inn for almost twenty years before returning to Cambridge in 1863 as the first Sadleirian Professor of Pure Mathematics. By the time of his death in 1895 this gaunt, modest intellectual had written nigh on a thousand papers. Famed for establishing the theory of matrices he also founded the theory of invariants, the geometry of *n*-dimensional space and the enumerative geometry of plane curves, and made lasting contributions to the theory of elliptic functions, non-Euclidean geometry and group theory [2].

Contact between Cayley and Tait went back at least as far as the mathematical tripos of 1852 when Cayley, as one of the two examiners, designated Tait the Senior Wrangler. Cayley had a long interest in quaternions. In fact, he was the first person after Hamilton to make use of them in a published paper. Yet, he was highly sceptical of their supposed merits. When Tait published his *Elementary Treatise on Quaternions* in 1867 [3], Cayley must have been greatly taken aback by comments in the preface. Not only were Cartesians downgraded, even trivialised, but there was strong criticism of the tripos, the very examination system which had set up both Tait and Cayley and with which Cayley, as Sadleirian Professor, had an intimate concern:

> ' It must always be remembered that Cartesian methods are mere particular cases of Quaternions, where most of the distinctive features have disappeared. ... Nothing, therefore, is ever lost, though much is generally gained, by employing Quaternions in preference to ordinary methods. In fact, even when Quaternions degrade to scalars, they give the solution of the most general statement of the problem they are applied to ...
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> ' ... The University of Cambridge, while seeking to supply a real want (the deficiency of subjects of examination for mathematical honours, and the consequent frequent introduction of the wildest extravagance in the

> shape of data for "Problems"), is in danger of making too much of such elegant trifles as Trilinear Coordinates. ... One grand step to the supply of this want is, of course, the introduction into the scheme of examinations of such branches of mathematical physics as the Theory of Heat and Electricity. But it appears to me that the study of a mathematical method like Quaternions, which, while of immense power and comprehensiveness, is of extraordinary simplicity, and yet requires constant thought in its applications, would also be of great benefit. With it there can be no "shut your eyes and write down your equations," for mere technical dexterity of analysis is certain to lead at once to error on account of the novelty of the processes employed.'

While Cayley recoiled, Tait set about freeing quaternions, if not physics itself, from the shackles of coordinate geometry. He began in 1868 by recasting some formulas of Cayley in quaternions and elicited results much more directly. Two years later he introduced a new definition of the differential operator, without resorting to Cartesian representation. It was these two papers that landed Tait the Keith Prize (see [1]); but, coming as they did on the back of the controversial preface they forced Cayley to make his position clear. He freely accepted that quaternions were amazingly effective devices for contracting a large amount of information into a relatively small number of symbols. All the requisite information was contained within these 'pocket maps', as he dubbed them, but it was the full scale Cartesian map which offered up its information the more readily. There the matter rested for fourteen years. Tait and Cayley acknowledged that there was a gulf between them but there was no animosity.

Maxwell, meanwhile, was fully supportive of Tait. In a letter of November 1871 he characterised the quaternion as a 'flaming sword which turns every way and the Cartesian method as a ram, pushing westward and northward (and downward?)'. He believed there to be a distinct possibility that quaternions would supersede Cartesian methods in time but noted that progress was slow. A degree of pragmatism was required during the period of transition. In September 1878 he asked Tait:

> ' May one plough with an ox and an ass together? The like of you may write everything in pure 4^{nions}, but in the transition period the bilingual method may help to introduce and explain the more perfect system ...'

And this in the face of a severe practical problem:

> ' Now in the bilingual treatise it is troublesome, to say the least, to find that the square of AB is always positive in Cartesians and always negative in 4nions, and that when the thing is mentioned incidentally you do not know which language is being spoken ... It is also awkward when you are discussing, say, kinetic energy to find that to ensure its being $+^{\text{ve}}$ you must stick a – sign to it, and that when you are proving a minimum in certain cases the whole appearance of the proof should be trending towards a maximum.'

Sadly, Maxwell's influence was lost in 1879 when he succumbed to cancer and Tait was no pragmatist. When, in the late 1880s, Tait came to prepare the third edition of his *Elementary Treatise on Quaternions* he first solicited material from Cayley and then refused to incorporate it except as a free-standing chapter. He wrote to Cayley arguing that 'no problem or subject is a fit one for the introduction of Quaternions if it necessitates the introduction of Cartesian Machinery'. What Cayley called the 'Analytical Theory of Quaternions' Tait viewed more as an extension of matrices, too closely associated with coordinates. His tone in intimating this to Cayley was somewhat condescending. In the supplementary chapter, Tait wrote, there 'will naturally assemble all the disaffected or lob-sided members, which are not capable of pure quaternionic treatment but which are nevertheless valuable, like the occipital ribs and the anencephalous heads in an anatomical museum'.

There was more to come the following year. In November 1889 Tait gave an address to the Physical Society of Edinburgh on the importance of quaternions in physics, an address which was published in the *Philosophical Magazine* two months later. It was only four years since the appearance of the second volume of a massive, fourteen hundred page biography of Hamilton in which Graves had bolstered the Irishman's reputation. On that November evening Tait was not slow in introducing the name of Hamilton into the discussion. Yet, giant though he was, argued Tait, in binding his new quaternions to the Cartesian system he had inadvertently hampered their growth and acceptance. Beginning by paraphrasing Horace he suggested that:

> ' The highest art is the absence of artifice. This commends itself to reason as well as to experience; but nowhere more forcibly than in the application of mathematics to physical science. The difficulties of physics are sufficiently great, in themselves, to tax the highest resources of human intellect; to mix them up with avoidable difficulties is unreason little short of crime. ... The intensely artificial system of Cartesian coordinates, splendidly useful as it was *in its day*, is one of the wholly avoidable encumbrances which now retard the progress of mathematical physics. Let any of you take up a treatise on the higher branches of hydrokinetics, or of stresses and strains, and then let him examine the twofold notation in Maxwell's *Electricity*. He will see at a glance how much expressiveness as well as simplicity is secured by an adoption of the mere notation, as distinguished from the processes, of quaternions.'

Tait went on to give his audience several remarkable examples of the conciseness achieved by adopting quaternion notation: among them, this Cartesian expression of Gauss for the number of interlinkings of two endless curves in space:

$$\frac{1}{4\pi}\int\int\frac{(x'-x)(dy\,dz'-dz\,dy')+(y'-y)(dz\,dx'-dx\,dz')+(z'-z)(dx\,dy'-dy\,dx')}{\{(x'-x)^2+(y'-y)^2+(z'-z)^2\}^{3/2}}.$$

Tait gave the equivalent result in quaternions, noting its 'simplicity and intelligibility':

$$\frac{1}{4\pi}\int S.d\rho\int\frac{V(\rho-\rho_1)\,d\rho_1}{T(\rho-\rho_1)^3}$$

Cayley offered counterexamples, Tait responded by countering them. With the temperature rising in 1894, Cayley questioned whether quaternions constituted a mathematical method at all and Tait sought to challenge Cayley's priority in discovering matrices.

The two remained 'poles asunder' in the words of Tait. They agreed that each should make out a case for their respective algebraic forms at a meeting of the Royal Society of Edinburgh on 2 July 1894 and present them for publication side by side in the *Proceedings*. In his paper, 'On Coordinates versus Quaternions', Cayley differentiated between quaternions in pure and in applied mathematics, concluding that they are structurally interesting but, even applied to geometry, offer nothing but an abbreviated notation. They are neither useful nor natural. Tait in a paper entitled, 'On the Intrinsic Nature of the Quaternion Method', replied that quaternions 'exist in space, and we have only to recognize them:- but we have to *invent* or *imagine* co-ordinates of all kinds.' Cayley's criticism, he argued, was misdirected, aimed as it was at the original quaternions of Hamilton which were defined in terms of coordinates and had little relevance for the revised formulation of Tait. On the pocket-map jibe Tait said that there was a better analogy. He likened coordinate geometry to a steam-hammer, quaternions to an elephant's trunk. The steam-hammer needs an expert to manage it and then it can only cope with routine work. The elephant's trunk can be put to good use by an unskilled native and is flexible. Cayley died the following year with the two men still completely at loggerheads. Meanwhile, with the advent of vectors, the debate moved off in another direction and Tait, always a colourful character, was at his most strident.

Josiah Willard Gibbs was professor of Mathematical Physics at Yale from 1873 to 1903. Probably in the late 1870s he came to the view that a vector calculus could be developed without using imaginaries. Since the scalar part of the quaternion was being used sparingly it too could be dispensed with to leave just the vector part. He tried out his new algebra on his students before making it available to a wider audience in privately published pamphlets of 1881 and 1884. They were later expanded by E. B. Wilson and published in book form in 1901 as the *Vector Analysis of J. W. Gibbs*. Gibbs departed from the quaternionists by using a dot to denote a scalar product ($\mathbf{u} \cdot \mathbf{v}$) and a cross to indicate a vector product ($\mathbf{u} \times \mathbf{v}$). Quaternions and vector analysis differ fundamentally in respect of the definitions which underpin them. In quaternions, the square of the unit vector is -1 whereas in the algebra of Gibbs it is a dyad with unit dot product and zero vector product. Scalar and vector products are used without the concept of the quaternion being evoked at all.

Gibbs was not alone in producing a vector analysis in the 1880s. Oliver Heaviside was well versed in quaternions but like Gibbs considered their form unnecessarily elaborate for the purposes they were designed to fulfil. Between 1885 and 1887 he wrote a number of papers for the journal *The Electrician* happily retaining the scalar and vector products in Hamilton's notation but with no role for quaternions as such. In his book, *Electromagnetic Theory*, Heaviside even had a section entitled 'Abstrusity of Quaternions and Comparative Simplicity Gained by Ignoring Them'. Here he did not mince his words:

> ' Clearly then, the quaternionic is an undesirable way of beginning the subject, and impedes the diffusion of vectorial analysis in a way which is as vexatious and brain-wasting as it is unnecessary.'

Tait, meanwhile, had become somewhat disillusioned by the fact that a majority of scientists remained unconvinced of the benefits of adopting quaternions. The seeds had been sown by Tait in papers, books and lectures but the stony ground would not bear wheat. Worse still, shoots of a plant with an apparently simpler structure began to break the ground and it threatened to put the quaternion in the shade. Tait vented his frustration in the preface to the third edition of *Elementary Treatise on Quaternions*:

> ' It is disappointing to find how little progress has recently been made with the development of Quaternions ... Even Professor Gibbs must be ranked as one of the retarders of Quaternion progress, in view of his pamphlet on Vector Analysis, a sort of hermaphrodite monster, compounded of the notations of Hamilton and Grassmann.'

This excess of language sparked off a bitter dispute which was fought out mainly in the journal *Nature*, largely between Tait and his junior colleague Knott, the promoters of quaternions, and Gibbs and Heaviside, the champions of vector analysis. It was a dispute in which the vector analysts offered detailed arguments couched in moderate language while the quaternionists gave careless responses laced with condescension, and crucially it resulted in Tait's taking quite a mauling. To look in detail at the arguments put by the protagonists would take too long so we must make do with a flavour of the early skirmishes and a résumé of the rest. (Further details may be found in [4], [5], [6].) Gibbs began by replying to the charge that vectors were hermaphrodite monsters, with restraint and not a little charm in April 1891:

> ' It seems to be assumed that a departure from quaternionic usage in the treatment of vectors is an enormity. If this assumption is true, it is an important truth; if not, it would be unfortunate if it should remain unchallenged, especially when supported by so high an authority. The criticism relates particularly to notations, but I believe there is a deeper question of notions underlying that of notations. Indeed, if my offence had been solely in the matter of notation, it would have been less accurate to describe my production as a monstrosity, than to characterize its dress as uncouth ...'

Gibbs then compared the two competing algebras, concentrating on the felicity with which sums, scalar products and vector products describe physical quantities. As fundamental notions there is a triviality and artificiality about the quaternionic product and quotient, he argued. Furthermore, quaternions cannot be used to analyse problems in four or more dimensions while vectors admit of extension. Finally the notation used for the two forms of multiplication is simpler in vector analysis. Superior in both its notions and notations the vector algebra of Gibbs is anything but a monster.

As fate would have it the appearance of Gibbs's letter coincided with Edinburgh University's main diet of examinations and Tait's time was at a premium. Rather than considering at length the points raised and dismissing them with a detailed argument he responded with a relatively brief letter to *Nature* within a month. Whilst stressing that in Hamilton's notation there is a reduced need for brackets he also conceded that the notations of the new algebra are 'very ingenious, and well calculated to furnish short cuts to various results already obtained.' On the question of extension he asked, 'What have students of physics, as such, to do with space of more than three dimensions?' This question would surely have haunted Tait had he lived to hear Herman Minkowski's Cologne address of 1908 in which Einstein's mathematical foil explained that 'henceforth, space by itself, and time by itself, are doomed to fade away into mere shadows, and only a kind of union of the two will preserve an independent reality.'

Only a few weeks passed before Gibbs again wrote to *Nature*. This time he gently took Tait to task over comments he had made about Grassmann's algebra in the previous letter, in the *Encyclopædia Britannica* and in the third edition of his book. Gibbs extolled the virtues of Grassmann's algebra and pointed out that it was the first algebra to incorporate the linear vector function. To some extent this was by the by, he argued, because virtually anyone can debate priority issues whereas few can discuss the respective merits of various algebras. Gibbs knew that priority issues were Tait's Achilles' heel. Tait found them compelling but he rarely researched them well. The paper was carefully constructed and reasonable and it must have appealed greatly to the uncommitted and the neutrals [5].

From Tait's brief rejoinder it is clear that Gibbs had caught him off-guard. This was not the old territory; he was being made to fight on new ground and his response was brief and inadequate. This is Tait at his weakest, unwilling or unable to parry the criticism and apologetic to boot. 'Since 1860, when I ceased to be a Professor of Mathematics, I have paid no special attention to general systems of *Sets, Matrices, or Algebras* ...'

And so it went on with the quaternionists emphasising algebraic simplicity and mathematical elegance and the vector analysts giving weight to naturalness and ease of comprehension. In all twelve scientists from England, Scotland, America and Australia made three dozen contributions to eight scientific journals in a little over three years. Tait wrote with the authority of a distinguished scientist but he also wrote with bitterness. Furthermore he wrote for and spoke to an Edinburgh audience already sympathetic to the cause. It was parochial and it was safe. Gibbs wrote dispassionately and constructed, in the words of Crowe, 'masterpieces of mathematical rhetoric.'

Regrettably, from Tait's point of view the public divisions between the two camps had the effect of bolstering vector analysis at the expense of quaternions. Yet, as he endured his final illness, still scribbling his quaternion notes he could consider with pride his contribution to the development of the family of vector methods. Firstly, from the time he took up his professorial post to his death at the beginning of the twentieth century Tait was the leading authority on quaternions. Secondly, he extended the work of Hamilton by promoting the use of quaternions in the physical sciences. Thirdly, he spurred Maxwell into mastering quaternions and thereby influenced the form in which his electrical researches were brought to the notice of the scientific community. He would have derived less comfort, perhaps, from the realisation that he had also demonstrated many new results in quaternions which were open to translation into the emerging vector analysis.

Note

In the first part of this article, *Tendril of the hop and tendril of the vine*, I said that the story of quaternions began on the banks of the River Liffey. In fact, Hamilton's breakthrough came when he was walking along the towpath of the Royal Canal in Dublin. I am indebted to J. D. Weston for pointing out this mistake. His article [7] contains more details of Hamilton's discovery.

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