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L. Pontrjagin, A classification of mappings of the threedimensional complex into the two-dimensional sphere, Rec. Math. [Mat. Sbornik] N.S., 1941, Volume 9(51), Number 2, 331363

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# A classification of mappings of the three-dimensional complex into the two-dimensional sphere 

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Let $K$ and $L$ be two complexes. The family $f_{t}$, where $t$ is a real number $(0 \leqslant t \leqslant 1)$, of continuous mappings of the complex $K$ into the complex $L$ is called a continuous deformation of mappings of the complex $K$ into the complex $L$ if the function $f_{t}(x)(x \in K)$ is a continuous function of the pair of arguments $x$, $t$. Two continuous mappings $g$ and $h$ of the complex $K$ into the complex $L$ are said to be homotopic or equivalent if there exists a continuous deformation $f_{t}$ transforming the mapping $g$ into the mapping $h$, i. e. such that $g=f_{0}$, $h=f_{1}$. In virtue of this criterium of equivalency all continuous mappings of the complex $K$ into the complex $L$ fall into classes of equivalent mappings. A classification of mappings from this point of view, i. e. the determination of more or less effective criteria of equivalency, forms one of the fundamental problems of topology.

The present state of topology leaves no hopes for the solution of the formulated problem in the near future. At present only certain particular cases have been investigated and solved.

Hopf ${ }^{1}$ gave the classification of mappings of the $n$-dimensional complex $K^{n}$ into the $n$-dimensional sphere $S^{n}$. The necessary and sufficient conditions of equivalency are given by him in this case in terms of homologies, which is the best way to solve the problem, since homologies admit of a rather effective computation. Hopf ${ }^{2}$ has also shown that there exists an enumerable number of classes of the mappings of the ( $4 n-1$ )-dimensional sphere $S^{4 n-1}$ into the $2 n$-dimensional sphere $S^{2 n}$; the criterium of non-equivalency in this case has been given by him in terms related to homologies. I ${ }^{3}$ have given a classification of mappings of the $(n+k)$-dimensional sphere $S^{n+k}$ into the $n$-dimensional sphere $S^{n}$ in the case $k=1,2$. Freudenthal ${ }^{4}$, knowing my results, but not knowing my proof, gave for a part of them a new proof and established, moreover, that for

[^0]$k=3,7$ there are at least two classes of mappings. I ${ }^{5}$ gave, further, a classification of mappings of an $(n+1)$-dimensional complex into the $n$-dimensional sphere - these results have been published only in a brief exposition. This is, so far I know, all that has been done in the question on the classification of mappings.

The first and the most important question in the general problem of classification of mappings is undoubtedly the question on the classification of mappings of the $(n+k)$-dimensional sphere into the $n$-dimensional sphere. Having solved it, it would be possible to attempt a classification of mappings of an $(n+k)$ dimensional complex $K^{n+k}$ into the $n$-dimensional sphere $S^{n}$, as well as a classification of mappings of $S^{n+k}$ into $K$. It may be surmised that on the way to the solution of these two questions new invariants of complexes of the type of homologies and intersections will arise. It is possible to approach the classification of mappings of $K^{n+k}$ into $S^{n}$ and of $S^{n+k}$ into $K$ in a different way, without classifying firstly the mappings of $S^{n+k}$ into $S^{n}$, but simply assuming that the classification of mappings of $S^{n+k}$ into $S^{n}$ is already carried out, or, more exactly, that the group ${ }^{6}$ of mappings of $S^{n+k}$ into $S^{n}$ is known.

In the present paper is given a complete exposition of my earlier published results on mappings of the three-dimensional complex $K^{3}$ into the two-dimensional sphere $S^{2}{ }^{5}$; besides, in this paper is partly touched the question on mappings of the four-dimensional complex $K^{4}$ into the two-dimensional sphere $S^{2}$. So particular a question as the classification of mappings of $K^{3}$ into $S^{2}$ represents a certain interest due to the fact that in its solution we obtain certain indications as to how should be solved the question on the classification of mappings of $K^{n+k}$ into $S^{k}$. Moreover, we give here for the first time an application of the theory of products (intersections) in complexes ${ }^{7}$ to the solution of a purely geometrical question, in the formulation of which homologies, not to speak of products, are not even mentioned.

In $\S \S 3,4$ and 5 are essentially used the results of my preceding paper „Products in complexes"; I shall refer to this paper in the sequel as to P.C. ${ }^{7}$.

In the whole of the present paper we shall consider only continuous mappings and continuous deformations of mappings, and therefore the word continuous will be omitted in the sequel.

## § 1. Mappings of the three-dimensional sphere into the two-dimensional one

In the present paragraph is given a classification of mappings of the threedimensional sphere into the two-dimensional one. The classification is based on the invariant introduced by Hopf ${ }^{2}$; we shall see that this invariant uniquely determines the class of mappings. The fundamental role in the proof is played

[^1]by Lemma 3. The results of W. Hurewicz ${ }^{6}$, in particular Lemma 1, do not play here an essential role, but they are important by themselves and form an essential complement to Lemma 3.

The fundamental role for all further constructions of this paragraph is played by the standard mapping $\vartheta$ of the three-dimensional sphere $S^{3}$ on the two-dimensional sphere $S^{2}$. We proceed now in the first place to construct this mapping.
A) Let us construct the mapping $\vartheta$ of the three-dimensional sphere $S^{3}$ on the two-dimensional sphere $S^{2}$. We shall consider the sphere $S^{3}$ as the set of all quaternions equal to one in modulus, i. e. every point $z \in S^{3}$ we shall write in the form

$$
z=a+b i+c j+d k
$$

where $i, j, k$ are quaternion units and $a, b, c, d$ are real numbers connected by the relation

$$
a^{2}+b^{2}+c^{2}+d^{2}=1
$$

The set $S^{3}$ of quaternions forms a group with respect to multiplication. Denote by $H$ the subgroup composed of all quaternions of the form

$$
\begin{equation*}
\cos \alpha+\sin \alpha \cdot i \tag{1}
\end{equation*}
$$

The aggregate $S^{3} / H$ of all right co-sets of the group $S^{3}$ with respect to the subgroup $H$ forms naturally a certain manifold; it turns out that this manifold is homoeomorphic to the two-dimensional sphere $S^{2}$. Correlating to every element $z \in S^{3}$ the co-set $Z \in S^{3} \mid H$, to which $z$ belongs, we obtain the mapping $\vartheta$.

Let us show that $S^{3} / H$ is homoeomorphic to the two-dimensional sphere and let us consider the mapping $\vartheta$ more detailed.

We introduce in the metrical sphere $S^{2}$ polar coordinates. To this end denote by $p$ its north and by $q$ its south pole and choose a certain fixed meridian $p m q$; the centre of the sphere $S^{2}$ we denote by $o$. For the radius vector of the point $y \in S^{2}$ we take the angle poy divided by $\pi$ and for the amplitude the angle mpy between the meridians $p m$ and $p y$. Then the point $q$ will have an indefinite amplitude.

The set of all quaternions from $S^{2}$ of the form

$$
\begin{equation*}
a+c j+d k, \tag{2}
\end{equation*}
$$

where $a \geqslant 0$, we denote by $A$. Since every quaternion from $S^{3}$ has a modulus equal to one, every element from $A$ may be represented in the form

$$
\begin{equation*}
\sigma+\rho(\cos \beta \cdot j+\sin \beta \cdot k), \tag{3}
\end{equation*}
$$

where $0 \leqslant \rho \leqslant 1, \sigma=+\sqrt{1-\rho^{2}}$. Here $\rho$ and $\beta$ may be interpreted as polar coordinates introduced in $A$, from which immediately follows that $A$ is homoeomorphic to a circle, the boundary $\dot{A}$ of which is composed of all elements of the form

$$
\begin{equation*}
\cos \beta \cdot j+\sin \beta \cdot k \tag{4}
\end{equation*}
$$

Further we have

$$
\begin{gather*}
(\cos \alpha+\sin \alpha \cdot i)(\sigma+\rho(\cos \beta \cdot j+\sin \beta \cdot k))= \\
=\sigma(\cos \alpha+\sin \alpha \cdot i)+\rho(\cos (\alpha+\beta) \cdot j+\sin (\alpha+\beta) \cdot k) . \tag{5}
\end{gather*}
$$

From this we see that every $z \in S^{3}$ is representable in the form (5), i. e. that

$$
\begin{equation*}
z=x \cdot y \tag{6}
\end{equation*}
$$

where $x \in H, y \in A$. From the same relation (5) follows that the decomposition (6) is unique for every $z$ not belonging to $\dot{A}$; if, however, $z \in \dot{A}$, then $y$ becomes an arbitrary element from $\dot{A}$, and $\alpha$ is determined from the relation (5). Correlate now to every point $z \in S^{3}$, represented in the form (5), the point from $S^{2}$ with polar coordinates $\rho, \beta$. Then we obtain the mapping $\boldsymbol{v}$ transforming every co-set f:om $S^{3}$ into a point from $S^{2}$.

The following propositions $B$ ) and $C$ ), as well as Lemma 1 , belong to W. Hurewicz ${ }^{6}$. I give them with full proofs.
B) Let $K$ be a compact metrical space and $f_{0}$ and $f_{1}$ two of its mappings into $S^{3}$. If

$$
\begin{equation*}
\left.\vartheta f_{0}=\vartheta f_{1} \quad[\mathrm{cf} . \mathrm{A})\right] \tag{7}
\end{equation*}
$$

then the mappings $f_{0}$ and $f_{1}$ are equivalent.
For the proof we consider the sphere $S^{3}$ as the group of quaternions [cf. A)] and use the possibility of multiplication of its points.

From relation (7) follows that for every $x \in K$ the elements $f_{0}(x)$ and $f_{1}(x)$ belong to one and the same co-set of the group $S^{3}$ with respect to the subgroup $H$. Thus

$$
\begin{equation*}
h(x)=f_{0}(x) f_{1}(x)^{-1} \in H \tag{8}
\end{equation*}
$$

Since the mapping $h$ transforms the whole space $K$ into the circumference $H \subset S^{3}$, there exists a continuous deformation $h_{t}$ of mappings of $K$ into $S^{3}$ such that $h_{0}=h, h_{1}(K)=\{e\}$, where $e$ is the unit of the group $S^{3}$. From this and the relation (8) follows that

$$
f_{0}(x)=h_{0}(x) f_{1}(x), \quad f_{1}(x)=h_{1}(x) f_{1}(x)
$$

hence it is natural to put $f_{t}(x)=h_{t}(x) f_{1}(x)$, and $f_{t}$ gives a continuous deformation of the mapping $f_{0}$ into the mapping $f_{1}$.
C) Let $f_{0}$ be a mapping of a compact metrical space $K$ into the sphere $S^{3}$ and $\varphi_{t}$ a continuous deformation of the mappings of the space $K$ into the sphere $\mathcal{S}^{2}$ such that $\varphi_{0}=\vartheta f_{0}$. Then there exists such a continuous deformation $f_{t}$ of mappings of the space $K$ into $S^{3}$ that

$$
\begin{equation*}
\left.\varphi_{t}=\vartheta f_{t} \quad[\mathrm{cf} . \mathrm{A})\right] \tag{9}
\end{equation*}
$$

For the proof we consider again the sphere $S^{3}$ as a group of quaternions and interprete the points of the sphere as right co-sets of the group $S^{3}$ with respect to the subgroup $H[\mathrm{cf}$. A) $]$. We recall that the set $A$ constructed in A) intersects with every right co-set only in one point, with the only exception of the co-set $\dot{A}$, which coincides with the boundary of the topological circle $A$.

Let $\varepsilon$ be a positive number so small that if $u \in S^{3}, v \in S^{3}$, and the distance between the points $\vartheta(u)$ and $\vartheta(v)$ in the sphere $S^{2}$ is less than $\varepsilon$, then $u v^{-1}$ does not belong to $\dot{A}$. The existence of such an $\varepsilon$ is easily established. By $n$ we denote a natural number so great that for $\left|t^{\prime}-t^{\prime \prime}\right| \leqslant \frac{1}{n}$ the distance between
the points $\varphi_{t^{\prime \prime}}(x)$ and $\varphi_{t^{\prime \prime}}(x)$ is less than $\varepsilon$, where $x$ is an arbitraty element from $K$.
We shall carry out the construction of the mapping $f_{t}$ inductively. Suppose that the mapping $f_{t}$, satisfying the condition (9), is already constructed for $0 \leqslant t \leqslant \frac{m}{n}$. Starting from this assumption, let us construct the mapping $f_{t}$, sa-tisfying the condition (9), for $\frac{m}{n} \leqslant t \leqslant \frac{m+1}{n}$.

Let $\frac{m}{n} \leqslant t \leqslant \frac{m+1}{n}$; then $\varphi_{t}(x)$ is a definite right co-set. Since the mapping $f_{\frac{m}{n}}$ is already constructed, $\frac{f_{\frac{m}{n}}}{}(x)$ is a definite element from $S^{3}$. Thus $\varphi_{t}(x) \frac{f_{m}^{n}}{}(x)^{-1}$ is a definite co-set. In virtue of the choice of the numbers $\varepsilon$ and $n$ this co-set does not coincide with $\dot{A}$, and hence the point of its intersection with $A$ is determined; we shall denote it by $h_{t}(x)$. Put $f_{t}(x)=h_{t}(x) \frac{f_{\frac{m}{n}}}{}(x)$. Then $f_{t}(x)$ enters into the co-set $\varphi_{t}(x)$ and, consequently, $\varphi_{t}(x)=\vartheta\left(f_{t}(x)\right)$. Thus the mapping $f_{t}$ is constructed. The continuity of the deformation $f_{t}$ is obvious.

Lemma 1. Let $f$ and $g$ be two mappings of a compact metrical space $K$ into the sphere $S^{3}$. The mappings $f$ and $g$ are then and only then equivalent, when the mappings $\vartheta f$ and $\vartheta g$ are equivalent [cf. A)].

Proof. Suppose that the mappings $\varphi_{0}=\vartheta f$ and $\varphi_{1}=\vartheta g$ are equivalent. Then there exists a continuous deformation $\varphi_{t}$ connecting them. Put $f_{0}=f$. Thus we can apply the proposition C ), i. e. there exists a continuous deformation $f_{t}$ of mappings of the space $K$ into the sphere $S^{3}$, and $\vartheta f_{t}=\varphi_{t}$. In virtue of this last, $f_{1}$ and $g$ are equivalent [cf. B)]. Since, moreover, $f=f_{0}$ and $f_{1}$ are equivalent, $f$ and $g$ are equivalent.

Suppose that the mappings $f$ and $g$ are equivalent. Then there exists a continuous deformation $g_{t}$ such that $g_{0}=f, g_{1}=g$. The continuous deformation $\forall g_{t}$ connects $\forall f$ and $\forall g$. The lemma is thus proved.

The following Lemma 2 forms the base for the proof of Lemma 3.
Lemma 2. Let $K$ be a certain complex of arbitrary dimensionality and $f$ its simplicial mapping into the n-dimensional complex $L$. Denote by $V$ a certain open n-dimensional simplex of the complex $L$ and put $U=f^{-1}(V)$. Then $U$ naturally falls into the topological product of the simplex $V$ and a certain. complex $P, U=V \cdot P, i$. e. every point $z \in U$ is uniquely and continuously, representable in the form of a pair $z=x \cdot y$, where $x \in V, y \in P$, and $f(z)=$ $=f(x \cdot y)=x$. Further, it turns out that if $K$ is a manifold, then $P$ is also. a manifold.

Proof. Denote by $\Omega$ the set of all simplexes from $K$ which are mapped under $t$ on $V$. Let $p$ be an inner point of the simplex $V$, and denote the complete original of the point $p$ in a certain $r$-dimensional simplex $T^{r} \in \Omega$ by $\psi\left(T^{r}\right)$. It is easily seen that $\psi\left(T^{r}\right)$ is a convex body of dimensionality $r$ - $n$. Further, if $T^{r-1} \in \Omega$ is an $(r-1)$-dimensional face of the simplex $T^{r}$, then $\psi\left(T^{r-1}\right)$ is an $(r-n-1)$-dimensional face of the convex body $\psi\left(T^{r}\right)$. Conversely, every $(r-n-1)$-dimensional face of the convex body $\psi\left(T^{r}\right)$ is obtained as $\psi\left(T^{r-1}\right)$, where $T^{r-1}$ is an $(r-1)$-dimensional face of the simplex $T^{r}$ entering into $\Omega$.

Put $P=f^{-1}(p)$; then $P$ is composed of all convex bodies of the form $\psi(T)$,
where $T \in Q$, and we shall interprete $P$ as the geometrical complex composed of these convex bodies. From the just established relation between $T^{r}, T^{r-1}$ and $\psi\left(T^{r}\right), \psi\left(T^{r-1}\right)$ follows that the relations of incidences in $P$ are the same as in $\Omega$, only their dimensionalities are reduced by $n$. Taking into account that if $T^{r-1} \in \Omega$ and $T^{r} \in K$, and $T^{r-1}$ is incidentic with $T^{r}$, then $T^{r} \in \Omega$; we conclude that if $K$ is a manifold, then $P$ is also a manifold.

Denote by $a_{0}, a_{1}, \ldots, a_{n}$ the vertices of the simplex $V$, and let $Y_{i}$ be the complete original of the point $a_{i}$ in the simplex $T \in \Omega$ under the mapping $f$. Then $Y_{i}$ is a certain face of the simplex $T$. Let $y_{i} \in Y_{i}, i=0,1, \ldots, n$, be a system of points from the simplexes $Y_{0}, Y_{1}, \ldots, Y_{n}$. Denote by

$$
\begin{equation*}
V\left(y_{0}, y_{1}, \ldots, y_{n}\right) \tag{10}
\end{equation*}
$$

the open simplex from $T$ with vertices $y_{0}, y_{1}, \ldots, y_{n}$. From elementary geometrical considerations follows that through every point $z \in T$, satisfying the condition $f(z) \in V$, passes one and only one simplex of the form (10). In particular, if $z=y \in \psi(T)$, then we denote the simplex of the form (10) passing through $y$ by $V(y)$. Since the mapping $f$ transforms the simplex $V(y)$ on the whole simplex $V$, every simplex of the form (10) may be written in the form $V(y)$, where $y \in \Psi(T)$. From this, in particular, follows that $\psi(T)$ is the topological product of simplexes $Y_{0}, Y_{1}, \ldots, Y_{n}$.

Let $z$ be an arbitraty point from $U$. Put $f(z)=x$ and denote by $T$ such a simplex of the system $\Omega$ that $z \in T$. Then there exists a simplex $V(y)$ of the form (10) containing $z$, such that $y \in P$ and the point $y$ is uniquely determined. Thus to every $z \in U$ uniquely corresponds a pair $x, y$, where $x \in V, y \in P$. Conversely, to every pair $x, y$ uniquely corresponds a $z$. Thus $U$ falls into the topological product $V \cdot P$ and $f(z)=f(x \cdot y)=x$. Thus the lemma is proved.

In connection with the proof of Lemma 2 it is convenient to formulate the following remark $C^{\prime}$ ), which is not necessary for the proof of Lemma 3, but will be used later.
$\mathrm{C}^{\prime}$ ) Let $f$ be a simplicial mapping of the complex $K$ into the $n$-dimensional complex $L, V$ a certain open orientated $n$-dimensional simplex from $L$ and $p$ a point belonging to $V$. If $T^{r}$ is an $r$-dimensional orientated simplex from $K$, then the complete original $\psi\left(T^{r}\right)$ of the point $p$ under the mapping $f$, which is a convex body of dimensionality $r-n$ or the void set, may be naturally considered as an orientated element. The orientation of the body $\psi\left(T^{r}\right)$ we define inductively. If $r=n$, then we assign to the point $\psi\left(T^{n}\right)$ the sign coinciding with the sign of the power of the mapping $f$ of the simplex $T^{n}$ on the simplex $V$. If now the orientated original $\psi\left(T^{r}\right)$ is defined for every simplex $T^{r}$ with $r<s$, then the function $\psi$ may be by additivity extended to an arbitrary algebraical complex $C$, which is a linear form of orlentated simplexes from $K$ of dimensionalities less than $s$. Thus the function $\psi$ is defined also for the boundary $\dot{T}^{s}$ of the simplex $T^{s}$. The orientation of the element $\psi\left(T^{s}\right)$ we define by the condition $\psi\left(T^{s}\right)^{\cdot}=\psi\left(\dot{T}^{s}\right)$. If we now assume that the function $\psi$ is already defined for all simplexes from $K$ and, by additivity, for all algebraical complexes from $K$, then we have $\psi(C)^{\cdot}=\Psi(\dot{C})$.

The following propositions D), E) and F) serve for the proof of Lemma 3. I formulate them without proof with reference to the corresponding literature.
D) Let $G$ be a commutative group with a finite number of generators taken in the additive notation. Under the integral character $\chi$ of the group $G$ we understand its homoeomorphic mapping into the additive group of all integers. Let $u_{1}, u_{2}, \ldots, u_{n}$ be an arbitrary finite system of elements from $G$ and $h_{1}, h_{2}, \ldots$ $\ldots, h_{k}$ an arbitrary system of integers. We ask under what conditions there exists an integral character $\chi$ of the group $G$ satisfying the conditions

$$
\chi\left(u_{i}\right)=h_{i}, \quad i=1,2, \ldots, k .
$$

It turns out that the character $\chi$ exists then and only then, when the following condition is satisfied:

Whatever be the system of integers $a_{1}, a_{2}, \ldots, a_{k}, m$, where $m \geqslant 2$, such that $a_{1} u_{1}+a_{2} u_{2}+\ldots+a_{k} u_{k}=m u$, where $u \in G$, the integer $a_{1} h_{1}+a_{2} h_{2}+\ldots$ $\ldots+a_{k} h_{k}$ is divisible by $m$.

This assertion is easily proved. For its proof see ${ }^{1}$.
E) Let $S^{1}$ be an orientated circumference, $K$ a complex of arbitrary dimensionality, $G$ its one-dimensional Betti group and $f$ a mapping of the complex $K$ into $S^{1}$. Let, further, $u \in G$ and $z$ be a cycle from the class of homologies $u$; then $f(z)=\chi_{f}(u) S^{1}$, where $\chi_{f}(u)$ is an integer equal to the power of the mapping of the cycle $z$, which depends only on $u$ and not on the incidental choice of $z \in u$. It is easy to show that $\chi_{f}$ is an integral character of the group $G$. It turns out that two mappings $f$ and $g$ of the complex $K$ into $S^{1}$ are then and only then equivalent, when the characters $\chi_{f}$ and $\chi_{g}$ corresponding to them coincide, $\chi_{f}=\chi_{g}$. Further, for every given character $\chi$ of the group $G$ there exists such a mapping $h$ that $\chi=\chi_{h}$.

For the proof see ${ }^{8}$.
F) Let $K$ be a certain complex and

$$
\begin{equation*}
u_{1}^{r}, u_{2}^{r}, \ldots, u_{p_{r}}^{r} \tag{11}
\end{equation*}
$$

the $r$-dimensional basis of its weak homologies, i. e. such a system of $r$-dimensional cycles that every cycle of dimensionality $r$ from $K$ is weakly homologic to a linear form of cycles of the system (11) and, moreover, only to one such form. Similarly, let $L$ be another complex, and $v_{1}^{s}, v_{2}^{s}, \ldots, v_{q_{s}}^{s}$ its $s$-dimensional basis of weak homologies. Then the $t$-dimensional basis of weak homologies of the topological product $K \cdot L$ is composed of all cycles of the form $u_{i}^{r} \cdot v_{j}^{s}$, where $r+s=t$.

For the proof cf. ${ }^{9}$.
For us only the case $t=1$ will be important.
Definition 1. A mapping $f$ of the complex $K$ into the $n$-dimensional sphere $S^{n}$ is called homologically unessential, if whatever be the integer $m \geqslant 2$ and whatever be the $n$-dimensional cycle $z$, taken from $K$ to the modulus $m$, its image under the mapping $f$ is equal to zero to the modulus $m$.

[^2]Observe that if the mapping $f$ is homologically unessential, then every mapping equivalent to it will be also homologically unessential.

Lemma 3. Let $K$ be a complex of arbitrary dimensionality and $\varphi$ its $h$ mologically unessential mapping into the two-dimensional sphere $S^{2}$. Then there exists such a mapping $f$ of the complex $K$ into the three-dimensional sphere $S^{3}$ that $\varphi=\vartheta f[\mathrm{cf}$. A)].

Proof. We triangulate the sphere $S^{2}$ so thiat not one of the edges of triangulation passes through the north pole $p$ of the sphere $S^{2}$ [cf. A)] and choose an open circle $V$ so small that its boundary $\dot{V}$ does not intersect with the edges of triangulation; then $\bar{V}=V+\dot{V}$ will also not intersect with the edges of triangulation. Let us further approximate the mapping $\varphi$ by a simplicial mapping $\psi$ and put

$$
U=\Psi^{-1}(V), \quad \dot{U}=\Psi^{-1}(\dot{V}), \quad \bar{U}=\Psi^{-1}(\bar{V}) ;
$$

then $\dot{U}$ is the boundary of the domain $U$ in $K$ and $\bar{U}=U+\dot{U}$ is its closure.
Since the closed circle $\bar{V}$ lies inside a simplex of triangulation, the set $\bar{U}$ falls, in virtue of Lemma 2, into the topological product $\bar{V} \cdot P$, so that if $x \cdot y=$ $=z \in \bar{U}$, where $x \in \bar{V}, y \in P$, then $\psi(z)=x$.

We shall assume that the circumference $\dot{V}$ is orientated in accordance with the angle coordinate introduced in it [cf. A)]. Choose in every component of the complex $P$ one point and denote these points by $p_{1}, p_{2}, \ldots, p_{k}$. The orientated circumferences

$$
\begin{equation*}
\dot{V} \cdot p_{1}, \quad \dot{V} \cdot p_{2}, \ldots, \quad \dot{V} \cdot p_{k} \tag{12}
\end{equation*}
$$

are cycles in the complex $K-U$. Let us show that there exists a mapping $\lambda$ of the complex $K-U$ into the orientated circumference $S^{1}$ mapping every cycle of the system (12) with the power one.

In virtue of the propositions D) and E) for the proof of the stated assertions it is sufficient to show that from every relation of the form

$$
\begin{equation*}
a_{1} \dot{V} \cdot p_{1}+a_{2} \dot{V} \cdot p_{2}+\ldots+a_{k} \dot{V} \cdot p_{k} \sim m W \tag{13}
\end{equation*}
$$

in $K-U$ follows that $a_{1}+a_{2}+\ldots+a_{k}$ is divisible by $m$. Suppose that the relation (13) has place. Then

$$
m V-\left(a_{1} \dot{V} \cdot p_{1}+a_{2} \dot{V} \cdot p_{2}+\ldots+a_{k} \dot{V} \cdot p_{k}\right)=\dot{c},
$$

where $\dot{c}$ is the boundary of a certain complex $c$ from $K-U$. The complex $c+a_{1} V \cdot p_{1}+a_{2} V \cdot p_{2}+\ldots+a_{k} V \cdot p_{k}$ is evidently a cycle to the modulus $m$ from $K$, and the power of the mapping $\psi$ on it is equal to $a_{1}+a_{2}+\ldots+a_{k}$. Since the mapping $\psi$ is homologically unessential, the number $a_{1}+a_{2}+\ldots+a_{k}$ is divisible by $m$, and our assertion on the existence of the mapping $\lambda$ is proved.

On the circumference $\dot{V}$ there is an angle coordinate, and in the sequel we shall not distinguish between the point from $\dot{V}$ itself and its angle coordinate. On the circumference $S^{1}$ we shall also introduce the angle coordinate and shall make no distinction between the point from $S^{1}$ and its angle coordinate.

In virtue of this agreement $o \in \dot{V}$ is the point from $\dot{V}$ with the angle coordinate $O$. The point $O \cdot y$, where $y \in P$, belongs to $\dot{U}=\dot{V} \cdot P$, and hence is determined the function

$$
\begin{equation*}
\mu(y)=\lambda(O \cdot y) \tag{14}
\end{equation*}
$$

where $\mu(y) \in S^{1}$, or the angle coordinate of the point $\mu(y)$. For $x \in \dot{V}, y \in P$ put

$$
\begin{equation*}
\nu(x \cdot y)=x+\mu(y) \tag{15}
\end{equation*}
$$

Thus we have defined a mapping $\nu$ of the complex $\dot{U}$ into the circumference $S^{1}$. Let us show that this mapping is equivalent to the mapping $\lambda$ of the complex $\dot{U} \subset K-U$.

For the proof we use the proposition E). Since every weakly homologic to zero one-dimensional cycle from $\dot{U}$ is transformed under any mapping of the complex $\dot{U}$ into $S^{1}$ into the zero cycle, it is sufficient to show that the mappings $\lambda$ and $\nu$ are algebraically equal on a certain basis of weak homologies from $\dot{U}$. For the construction of the basis of weak homologies of the complex $\dot{U}$ we use the proposition F). Let $z_{1}, z_{2}, \ldots, z_{l}$ be an one-dimensional basis of weak homologies of the complex $P$. Then the basis of weak homologies of the complex $\dot{U}$ is

$$
0 \cdot z_{1}, 0 \cdot z_{2}, \ldots, 0 \cdot z_{l}, \quad \dot{V} \cdot p_{1}, \quad \dot{V} \cdot p_{2}, \ldots, \dot{V} \cdot p_{k}
$$

On every cycle of the form $0 \cdot z_{j}$ the mappings $\lambda$ and $\nu$ simply coincide. On every cycle of the form $\dot{V} \cdot p_{i}$ both of them have the power one. Thus the mappings $\lambda$ and $\nu$ of the complex $\dot{U}$ are equivalent.

Since the mappings $\lambda$ and $\nu$ of the complex $\dot{U}$ are equivalent, there exists a continuous deformation transforming the mapping $\lambda$ of the complex $\dot{U}$ into the mapping $\nu$. This deformation may be extended into a deformation of the mapping $\lambda$ of the whole complex $K-U$, transforming the mapping $\lambda$ into a certain new mapping $\eta$, and this latter coincides on the complex $\dot{U}$ with $\nu$.

Thus we have constructed a mapping $\eta$ of the complex $K-U$ into $S^{1}$, coinciding with $\nu$ on $\dot{U}$ [cf. (15)].

The circumference $\dot{V}$ is given in polar coordinates, which are on the sphere $S^{2}$ [cf. A)], by the equation $\rho=\varepsilon$, where $\varepsilon$ is a constant and $\rho$ the radius vector.

The mapping $g$ of the complex $\bar{U}=\bar{V} \cdot P$ into $S^{3}$ we define by putting

$$
g(x \cdot y)=(\cos (\mu(y))+\sin (\mu(y)) i)(\sigma+\rho(\cos \beta \cdot j+\sin \beta \cdot k)) \quad[\mathrm{cf} . \mathrm{A})],(16)
$$

where $x \in V$ and has the coordinates $\varepsilon \rho, \beta$ and $y \in P$. For $x \in \dot{V}$ we have

$$
\begin{equation*}
g(x \cdot y)=\cos (\mu(y)+\beta) \cdot j+\sin (\mu(y)+\beta) \cdot k \tag{17}
\end{equation*}
$$

The mapping $g$ of the complex $K-U$ into $S^{3}$ we define by putting

$$
\begin{equation*}
g(z)=\cos \eta(z) \cdot j+\sin \eta(z) \cdot k \tag{18}
\end{equation*}
$$

From the relations (15), (17) and (18) directly follows that the so constructed on two parts of the complex $K$ mapping $g$ is coordinated on the intersec-
tion $\dot{U}$ of these parts, and hence we have a mapping $g$ of the whole complex $K$ into the sphere $S^{3}$ [cf. A)].

Let us show that the mappings $\psi$ and $\mathcal{V} g$ of the complex $K$ into the sphere $S^{2}$ are equivalent.

Let us, in the first place, investigate the structure of the mapping $\vartheta g$. If $x$ is a point from $\bar{V}$ with coordinates $\varepsilon \rho, \beta$, and $y \in P$, then $\boldsymbol{v}(g(x \cdot y))$ is the point from $S^{2}$ with coordinates $\rho, \beta$. The mapping $\psi$ transforms the same point $x \cdot y$ into the point from $S^{2}$ with coordinates $\varepsilon \rho, \beta$. The mapping $\forall g$ is thus obtained from the mapping $\psi$ for the point from $\bar{U}$ by a simple elongation of the radius vector. If the point $z \in K-U$, then $\vartheta(g(z))=q$ [cf. A)]. Thus, in order to obtain the mapping $\vartheta g$ from the mapping $\psi$ for $z \in K-U$, it is necessary to make the point $\psi(z)$ slide from its original position along the radius vector into the point $q$. From what has been said we sêe that the mappings $\forall g$ and $\psi$ are equivalent.

Since the mappings $\varphi$ and $\psi$ are equivalent, we conclude, by what has been just proved, that the mappings $\varphi$ and $\mathfrak{\vartheta} g$ are equivalent. Hence, in virtue of C ), follows that there exists a mapping $f$ satisfying the condition $\varphi=\vartheta f$. The lemma is thus proved.

In addition to Lemma 3 we make the following obvious remark.
G) If $f$ is a mapping of the complex $K$ into $S^{3}$, then the mapping $\varphi=\vartheta f$ of the complex $K$ into $S^{2}$ is homologically unessential.

From Lemmas 1 and 3 and the remark G) we can deduce now the following important

Theorem 1. Let $D$ be a class of mappings of the complex $K$ into the three-dimensional sphere $S^{3}$. If $t$ is a mapping of the class $D$, then denote by $\Delta$ the class of mappings of the complex $K$ into the two-dimensional sphere $S^{2}$, which contains the mapping it [cf. A)]. Then the class $\Delta$ is determined by the class $D$ and not by the incidental choice from $D$ of the mapping $f$; hence we may put $\Delta=\vartheta(D)$. It turns out that so obtained correspondence $\vartheta$ is an one-to-one correspondence between all classes of mappings of the complex $K$ into $S^{3}$ and all homologically unessential classes of mappings of the complex $K$ into $S^{2}$.

Since we have already a classification of mappings of the three-dimensional sphere $\Sigma^{3}$ into the three-dimensional sphere $S^{3}$, Theorem 1 gives us a classification of mappings of the sphere $\Sigma^{3}$ into the sphere $S^{2}$.

If $f$ is a mapping of the sphere $\Sigma^{3}$ into $S^{3}$, denote by $\omega_{0}\left(f, \Sigma^{3}\right)$ the power of this mapping. As is known, two mappings $f$ and $g$ of the sphere $\Sigma^{3}$ into the sphere $S^{3}$ are then and only then equivalent, when

$$
\begin{equation*}
\omega_{0}\left(f, \Sigma^{3}\right)=\omega_{0}\left(g, \Sigma^{3}\right) \tag{19}
\end{equation*}
$$

In order to be able to give a more concrete classification of the mappings of the sphere $\Sigma^{3}$ into $S^{2}$, we recall the following definition due to Hopf (cf. ${ }^{2}$ ):

Definition 2. Let $o$ be a simplicial mapping of the orientated sphere $\Sigma^{3}$ into the orientated sphere $S^{2}$. Choose in the sphere $S^{2}$ two points $a \neq b$ not betonging to the edges of triangulation. Then $\varphi^{-1}(a)$ and $\varphi^{-1}(b)$ are naturally one-
dimensional cycles from $\Sigma^{3}$ [cf. $\left.\left.C^{\prime}\right)\right]$. Denote the linkage coefficient of these cycles by $\omega_{1}\left(\varphi, \Sigma^{3}\right)$. Let, further, $c$ be a certain two-dimensional algebraical complex from $\Sigma^{3}$ with the boundary $\rho^{-1}(a)$; then the power of its mapping under $\varphi$ on $S^{2}$ is equal to $\omega_{1}\left(\varphi, \Sigma^{3}\right)$.
H. Hopf, to whom this construction belongs, has shown (cf. ${ }^{2}$ ) that for two equivalent mappings $\varphi$ and $\psi$ we have $\omega_{1}\left(\varphi, \Sigma^{3}\right)=\omega_{1}\left(\psi, \Sigma^{3}\right)$. He has also shown that if $f$ is a mapping of the sphere $\Sigma^{3}$ into the sphere $S^{3}$ then

$$
\begin{equation*}
\left.\omega_{1}\left(\vartheta f, \Sigma^{3}\right)=\omega_{0}\left(f, \Sigma^{3}\right) \quad[\mathrm{cf} . \mathrm{A})\right] \tag{20}
\end{equation*}
$$

From this we deduce on ground of Lemma 3 and the condition of equivalency (19) the following

Theorem 2. Two mappings $\varphi$ and $\psi$ of the three-dimensional sphere $\Sigma^{3}$ into the two-dimensional sphere $S^{2}$ are then and only then equivalent, when
(cf. Definition 2).

$$
\omega_{1}\left(\varphi, \Sigma^{3}\right)=\omega_{1}\left(\psi, \Sigma^{3}\right)
$$

## § 2. Preliminary notions and remarks

For a classification of mappings of a three-dimensional complex $K^{3}$ into the two-dimensional sphere $S^{2}$ (cf. Theorem 3 ) we have to introduce certain invariants of the mappings of $K^{3}$ into $\left.S^{2}[\mathrm{cf} . \S 2, \mathrm{~F})\right]$, as well as certain invariants of pairs of mappings of $K^{3}$ into $S^{2}$ [cf. $\left.\left.§ 2, A^{\prime}\right)\right]$. The present paragraph is devoted to the introduction of these invariants necessary for the formulation of Theorem 3 itself, as well as of invariants necessary for its proof.

In the first place let us introduce certain denotations and terms. Let $\left\{f_{t}\right\}$ be a family of continuous mappings of the space $F$ into the space $R$, where $t$ is an arbitrary element of the topological space $\Delta$. Denote by $F \cdot \Delta$ the topological product of the spaces $F$ and $\Delta$. Then every element $z \in F \cdot \Delta$ is representable in the form of a pair $z=x \cdot t$, where $x \in F, t \in \Delta$. We define the mapping $f_{\Delta}$ of the space $F \cdot \Delta$ into $R$ by putting $f_{\Delta}(z)=f_{\Delta}(x \cdot t)=f_{t}(x)$. If this mapping is continuous, then we shall call the family $\left\{f_{t}\right\}$ also continuous. If, conversely, a certain continuous mapping $f_{\Delta}$ of the product $F \cdot \Delta$ is given, then it generates a continuous family $\left\{f_{t}\right\}$ of mappings of the space $F$.

In the sequel two cases will be essential for us:
a) $\Delta$ is composed of all real numbers $0 \leqslant t \leqslant 1$,
b) $\Delta$ assumes two values 0 and 1 .

If the question requires an algebraical interpretation and $F$ is an algebraical complex, then in both cases we shall consider the product $F \cdot \Delta$ also as an algebraical complex. In the case a) we orientate $F \cdot \Delta$ so that $F \cdot 0$ should enter into the boundary of $F \cdot \Delta$ with the negative sign. In the case b) we orientate $F \cdot \Delta$ so that $F \cdot \Delta=F \cdot 1-F \cdot 0$.

If two mappings $f$ and $g$ of the space $F$ coincide on a closed subset $E \subset F$ and there exists a continuous deformation of the mapping $f$ into the mapping $g$ not changing the mapping on $E$, we shall say that the mappings $f$ and $g$ are equivalent with respect to $E$.

Let $f_{t}$ and $g_{t}$ be two continuous deformations of the mappings of the space $F$, coinciding on a closed subset $E \subset F$, such that $f_{0}$ and $g_{0}$ coincide, as well as $f_{1}$
and $g_{1}$. It is easily seen that the mappings $f_{\Delta}$ and $g_{\Delta}$ [cf. a)] coincide then on the set $E^{*}=F \cdot 0 \vee F \cdot 1 \vee E \cdot \Delta \subset F \cdot \Delta$. We shall say that the continuous deformations $f_{t}$ and $g_{t}$ are equivalent with respect to $E$, if the mappings $f_{\Delta}$ and $g_{\Delta}$ of the space $F \cdot \Delta$ are equivalent with respect to $E^{*}$.

If $f$ is a continuous mappping of the orientated $n$-dimensional ( $n=2,3$ ) sphere $\Sigma^{n}$ into the orientated two-dimensional sphere $S^{2}$, then by $\omega_{n-2}\left(f, \Sigma^{n}\right)$ we shall denote for $n=2$ the power and for $n=3$ Hopf's number of the mapping $f$ (cf. Definition 2).
A) Let $f_{0}$ and $f_{1}$ be two mappings of the $n$-dimensional ( $n=2,3$ ) orientated element $T^{n}$ into the two-dimensional orientated sphere $S^{2}$, coinciding on the boundary $\dot{T}^{n}$ of the element $T^{n}$. Let us introduce the index $\omega_{n-2}\left(f_{0}, f_{1}, T^{n}\right.$ ) estimating the difference of the mappings $f_{0}$ and $f_{1}$. Denote by $\Delta$ the aggregate of two numbers 0 and 1. Identify in the space $T^{n} \cdot \Delta$ in one point every pair of points $x \cdot 0$ and $x \cdot 1$, where $x \in \dot{T}^{n}$; then we obtain from the complex $T \cdot \Delta$ the orientated sphere $\Sigma^{n}$. Since the mappings $f_{0}$ and $f_{1}$ coincide on the boundary $\dot{T}^{n}$, the mapping $f_{\Delta}$ may be interpreted as a mapping of the sphere $\Sigma^{n}$. Put

$$
\omega_{n-2}\left(f_{0}, f_{1}, T^{n}\right)=\omega_{n-2}\left(f_{\Delta}, \Sigma^{n}\right)
$$

It is easily seen that in order that the mappings $f_{0}$ and $f_{1}$ should be equivalent with respect to $T^{n}$, it is necessary and sufficient that $\omega_{n-2}\left(f_{0}, f_{1}, T^{n}\right)=0$. It is as easily seen that the index introduced above does not vary at a simultaneous deformation of the mappings $f_{0}$ and $f_{1}$, if they remain coinciding on the boundary $\dot{T}^{n}$.
B) Let $f_{t}$ and $g_{t}$ be two continuous deformations of the mappings of the ( $n-1$ )-dimensional ( $n=2,3$ ) orientated element $E^{n-1}$ into the orientated twodimensional sphere $S^{2}$, coinciding on the boundary $\dot{E}^{n-1}$ of the element $E^{n-1}$, such that the mappings $f_{0}$ and $g_{0}$ coincide, as well as the mappings $f_{1}$ and $g_{1}$. Introduce the index $\omega_{n-2}\left(f_{t}, g_{t}, E^{n-1}\right)$ estimating the difference of deformations $f_{t}$ and $g_{t}$. Let $\Delta$ be the set of all numbers $0 \leqslant t \leqslant 1$. It is easily seen that the mappings $f_{\Delta}$ and $g_{\Delta}$ of the element $T^{n}=E^{n-1} \cdot \Delta$ coincide on the boundary $\dot{T}^{n}$ of the element $T^{n}$. Put

$$
\omega_{n-2}\left(f_{t}, g_{t}, E^{n-1}\right)=\omega_{n-2}\left(f_{\Delta}, g_{\Delta}, T^{n}\right)
$$

It is easily seen that the deformations $f_{t}$ and $g_{t}$ are equivalent with respect to $\dot{E}^{n-1}$ then and only then, when the index is equal to zero.
C) Let $f_{t}$ and $g_{t}$ be two continuous deformations of mappings of the orientated $n$-dimensional ( $n=2,3$ ) simplex $T^{n}$ into the orientated two-dimensional sphere $S^{2}$. Denote by $\dot{T}^{n}$ the boundary of $T^{n}$ and by $T^{\prime}$ the aggregate of ( $n-2$ )dimensional faces of the simplex $T^{n}$. Suppose that the mappings $f_{0}$ and $g_{0}$ coincide on $\dot{T}^{n}$, as well as the mappings $f_{1}$ and $g_{1}$. Suppose, further, that the deformations $f_{t}$ and $g_{t}$ coincide on $T^{\prime}$. Denote by $E_{i}^{n-1}, i=0, \ldots, n$, the faces of the simplex $T^{n}$, properly orientated. Then we have

$$
\omega_{n-2}\left(f_{1}, g_{1}, T^{n}\right)=\omega_{n-2}\left(f_{0}, g_{0}, T^{n}\right)+\sum_{i=0}^{n} \omega_{n-2}\left(f_{t}, g_{t}, E_{i}^{n-1}\right)
$$

Let us prove the assertion C ). The set of all numbers $0 \leqslant t \leqslant 1$ denote by $\Delta$. We shall consider the complex $T^{n} \cdot \Delta$ in two copies; the first we shall denote by $T^{n} \cdot \Delta$, the second by $\left[T^{n} \cdot \Delta\right]$. Similarly we shall distinguish between all possible subsets and algebraical subcomplexes of the complexes $T^{n} \cdot \Delta$ and $\left[T^{n} \cdot \Delta\right]$. By $P$ we shall denote the complex consisting of the two components $T^{n} \cdot \Delta$ and $\left[T^{n} \cdot \Delta\right]$. Define the mapping $\psi$ of the complex $P$ as coinciding with $f_{\Delta}$ on $T^{n} \cdot \Delta$ and with $g_{\Delta}$ on $\left[T^{n} \cdot \Delta\right]$. The aggregate of all $(n-1)$-dimensional faces of the prism $T^{n} \cdot \Delta$ we denote by $A$. If $x \in A$, then it is easily seen that $\psi(x)=\Psi([x])$. Let us identify in the complex $P$ every pair of points $x,[x]$, where $x \in A$. Then we obtain a complex $Q$. The mapping $\Psi$ of the complex $P$ may be now interpreted as a mapping of the complex $Q$.

Observe that the following algebraical complexes from $Q$ are orientated spheres:

$$
\begin{gathered}
T^{n} \cdot 0-T^{n} \cdot 1+\sum_{i=0}^{n} E_{i}^{n-1} \cdot \Delta=U,\left[T^{n} \cdot 0\right]-\left[T^{n} \cdot 1\right]+\sum_{i=0}^{n}\left[E_{i}^{n-1} \cdot \Delta\right]=[U] \\
{\left[T^{n} \cdot 0\right]-T^{n} \cdot 0=V_{0},\left[T^{n} \cdot 1\right]-T^{n} \cdot 1=V_{1}} \\
{\left[E_{i}^{n-1} \cdot \Delta\right]-E_{i}^{n-1} \cdot \Delta=W_{i}}
\end{gathered}
$$

Hence we have

$$
[U]-U=V_{0}-V_{1}+\sum_{i=0}^{n} W_{i}
$$

Observe that the sphere $U$ is the boundary of the prism $-T^{n} \cdot \Delta$, and the sphere $[U]$ - the boundary of the prism - $\left[T^{n} \cdot \Delta\right]$.

Consider now the case $n=2$. Since $U$ and $[U]$ are homotopic to zero in $Q$,

$$
\omega_{0}\left(\psi, V_{0}-V_{1}+\sum_{i=0}^{2} W_{i}\right)=\omega_{0}(\psi,[U]-U)=0 .
$$

But

$$
\begin{gathered}
\omega_{0}\left(\Psi, V_{0}\right)=\omega_{0}\left(f_{0}, g_{0}, T^{2}\right), \quad \omega_{0}\left(\Psi, V_{1}\right)=\omega_{0}\left(f_{1}, g_{1}, T^{2}\right) \\
\omega_{0}\left(\Psi, W_{i}\right)=\omega_{0}\left(f_{t}, g_{t}, E_{i}^{1}\right)
\end{gathered}
$$

Thus, for $n=2$ the assertion is proved.
Consider the case $n=3$. It is easily seen that in this case every two-dimensional cycle from $Q$ is homologic to zero, and hence the mapping $\psi$ is homologically unessential; therefore, there exists a mapping $\chi$ of the complex $Q$ into the three-dimensional sphere $S^{3}$ such that $\psi=\vartheta \chi$ (cf. § 1, Lemma 3). Since $\omega_{0}\left(\chi, \Sigma^{3}\right)=\omega_{1}\left(\psi, \Sigma^{3}\right)$ for an arbitrary sphere $\Sigma^{3}$ from $Q$, we obtain the required result by applying to the mapping $\chi$ the same argument, as we applied above to $\psi$.
D) Let $T^{n}$ be an $n$-dimensional $(n=2,3)$ orientated element and $f, g, h$ three of its mappings into the orientated two-dimensional sphere $S^{2}$ such that all these mappings coincide on the boundary $T^{n}$ of the element $T^{n}$. Then we have

$$
\omega_{n-2}\left(f, g, T^{n}\right)+\omega_{n-2}\left(g, h, T^{n}\right)=\omega_{n-2}\left(f, h, T^{n}\right)
$$

In the case $n=2$ the proof follows directly by computation of the powers of the mappings. Consider the case $n=3$. Take three copies of the element $T^{3}$ and denote them by $T^{3},\left[T^{3}\right],\left\{T^{3}\right\}$. Compose the complex $P$ of the three com-
ponents $T^{3},\left[T^{3}\right],\left\{T^{3}\right\}$ and define the mapping $\psi$ of the complex $P$ as coinciding with $f$ on $T^{3}$, with $g$ on $\left[T^{3}\right]$ and with $h$ on $\left\{T^{3}\right\}$. Identify in the complex $P$ in one point every triple of points $x,[x],\{x\}$, where $x \in \dot{T}^{3}$; the so obtained complex denote by $Q$. The mapping $\psi$ may be obviously interpreted as a mapping of the complex $Q$. Since in the complex $Q$ every two-dimensional cycle is homologic to zero, there exists a mapping $\chi$ of the complex $Q$ into the three-dimensional sphere $S^{3}$ such that $\psi=\vartheta \chi$ (cf. $\S 1$, Lemma 3). Thus the question is again, as in C), reduced to consideration of the power of the mapping.

We apply now the established definitions and results to the mappings of the $n$-dimensional complex $K^{n}(n=2,3)$ into the two-dimensional orientated sphere $S^{2}$. By $K^{r}$ we shall denote hereby the aggregate of all simplexes of the complex $K^{n}$ of dimensionalities less than or equal to $r$.

A') Let $f_{0}$ and $f_{1}$ be two mappings of the complex $K^{n}(n=2,3)$ into $S^{2}$, coinciding on $K^{n-1}$. We introduce the $n$-dimensional $\nabla$-complex estimating the difference between the mappings $f_{0}$ and $f_{1}$ and denote it by $\omega_{n-2}\left(f_{0}, f_{1}, K^{n}\right)$.

If $T^{n}$ is an orientated $n$-dimensional simplex from $K^{n}$, then we define the value of the $n$-dimensional $\nabla$-complex to be introduced as
on $T^{n}$.

$$
\omega_{n-2}\left(f_{0}, f_{1}, T^{n}\right)
$$

It is easily seen that mappings $f_{0}$ and $f_{1}$ of the complex $K^{n}$ are then and only then equivalent with respect to $K^{n-1}$, when $\omega_{n-2}\left(f_{0}, f_{1}, K^{n}\right)=0$. It is as easily seen that if the mappings $f_{0}$ and $f_{1}$ are subjected to one and the same simultaneous deformation, while they remain coinciding on $K^{n-1}$, then the complex $\omega_{n-2}\left(f_{0}, f_{1}, K^{n}\right)$ does not vary.
$\mathrm{B}^{\prime}$ ) Let $f_{t}$ and $g_{t}$ be two continuous deformations of the mappings of the complex $K^{n-1}$ into $S^{2}$, coinciding on $K^{n-2}$, such that the mappings $f_{0}$ and $g_{0}$ coincide, as well as $f_{1}$ and $g_{1}$. Introduce the $(n-1)$-dimensional $\nabla$-complex $\omega_{n-2}\left(f_{t}, g_{t}, K^{n-1}\right)$ estimating the difference of the deformations $f_{t}$ and $g_{t}$. If $T^{n-1}$ is an ( $n-1$ )-dimensional orientated simplex from $K^{n-1}$, then the value on it of the $\nabla$-complex to be introduced we define as

$$
\omega_{n-2}\left(f_{t}, g_{t}, T^{n-1}\right)
$$

It is easily seen that continuous deformations $f_{t}$ and $g_{t}$ are then and only then equivalent with respect to $K^{n-2}$, when $\omega_{n-2}\left(f_{t}, g_{t}, K^{n-1}\right)=0$.
$\mathrm{C}^{\prime}$ ) Let $f_{t}$ and $g_{t}$ be two continuous deformations of the mappings of the complex $K^{n}$ into $S^{2}$, coinciding on $K^{n-2}$, such that the mappings $f_{0}$ and $g_{0}$, as well as $f_{1}$ and $g_{1}$, coincide on $K^{n-1}$. Then we have

$$
\omega_{n-2}\left(f_{1}, g_{1}, K^{n}\right)=\omega_{n-2}\left(f_{0}, g_{0}, K^{n}\right)+\nabla \omega_{n-2}\left(f_{t}, g_{t}, K^{n-1}\right),
$$

where the sign $\nabla$ denotes the $\nabla$-boundary.
$\mathrm{D}^{\prime}$ ) Let $f, g$ and $h$ be three mappings of the complex $K^{n}$ into $S^{2}$, coinciding on $K^{n-1}$. Then we have

$$
\omega_{n-2}\left(f, g, K^{n}\right)+\omega_{n-2}\left(g, h, K^{n}\right)=\omega_{n-2}\left(f, h, K^{n}\right)
$$

E) In addition to A) we observe that if $f_{0}$ is a certain mapping of the $n$-dimensional ( $n=2,3$ ) element $T^{n}$ into $S^{2}$, then there exists a mapping $f_{1}$ of the same element into $S^{2}$ such that the number $\omega_{n-2}\left(f_{0}, f_{1}, T^{n}\right)$ is defined and has
a given value. Hence, in addition to $\mathrm{A}^{\prime}$ ), follows that if $f_{0}$ is a mapping of the $n$-dimensional ( $n=2,3$ ) complex $K^{n}$ into $S^{2}$, then there exists a mapping $f_{1}$ of the same complex into $S^{2}$ such that the $\nabla$-complex $\omega_{n-2}\left(f_{0}, f_{1}, K^{n}\right)$ is defined and coincides with the given one. In precisely the same manner we observe, in addition to B ), that if $f_{t}$ is a deformation of the mappings of the ( $n-1$ )-dimensional ( $n=2,3$ ) element $E^{n-1}$ into $S^{2}$, then there exists a continuous deformation $g_{t}$ of mappings of the same eiement into $S^{2}$ such that the number

$$
\omega_{n-2}\left(f_{t}, g_{t}, E^{n-1}\right)
$$

is defined and has a given value. In addition to $\mathrm{B}^{\prime}$ ) hence follows that if $f_{t}$ is a deformation of the mappings of the ( $n-1$ )-dimensional ( $n=2,3$ ) complex $K^{n-1}$ into $S^{2}$, then there exists a continuous deformation $g_{t}$ of the mappings of the same complex into $S^{2}$ such that the $\nabla$-complex $\omega_{n-2}\left(f_{t}, g_{t}, K^{n-1}\right)$ is defined and coincides with the given one.

Let us prove E). Let $f_{0}$ be a given mapping of the element $T^{n}$ and let $R^{n}$ be an element from $T^{n}$ not intersecting with the boundary $\dot{T}^{n}$ of the element $T^{n}$. Let us deformate the mapping $f_{0}$, not changing it on the boundary, into sucin a mapping $f_{0}^{\prime}$ that $f_{0}^{\prime}\left(R^{n}\right)$ contains only one point $p \in S^{2}$. Identify now in one point $q$ all points of the boundary $\dot{R}^{n}$ of the element $R^{n}$ and denote the so obtained sphere from $R^{n}$ by $\Sigma^{n}$. Let us now determine the mapping $f^{\prime}$ of the sphere $\Sigma^{n}$ into $S^{2}$ such that $f^{\prime}(q)=p$ and $\omega_{n-2}\left(f^{\prime}, \Sigma^{n}\right)$ has a given value. The mapping $f^{\prime}$ of the sphere $\Sigma^{n}$ we shall interprete as a mapping of the element $R^{n}$. The mapping $f_{1}$ of the element $T^{n}$ we define as coinciding with $f_{0}^{\prime}$ on $T^{n}-R^{n}$ and as coinciding with $f^{\prime}$ on $R^{n}$. The continuous deformation $g_{t}$ is constructed in precisely the same way by starting from the mapping $f_{\Delta}$ ( $\Delta$ being the set of all numbers $0 \leqslant t \leqslant 1$ ) of the element $T^{n}=E^{n-1} \cdot \Delta$.
F) Let $K^{2}$ be a certain two-dimensional complex, $K^{1}$ - the complex composed from all nul-dimensional and one-dimensional simplexes of the complex $K^{2}$, and $f$ a mapping of $K^{2}$ into $S^{2}$. Suppose that there exists a point $p \in S^{2}$ such that $f\left(K^{1}\right)$ does not contain $p$. Let us now define the $\nabla$-complex $\omega_{0}\left(f, K^{2}\right)$ characterizing the mapping $f$. The value of the complex $\omega_{0}\left(f, K^{2}\right)$ on the simplex $T^{2}$ from $K^{2}$ we define as the power of the mapping $f$ of the simplex $T^{2}$ at the point $p$. It turns out that two mappings $f$ and $g$ are equivalent then and only then, when

$$
\omega_{0}\left(f, K^{2}\right) \widetilde{\nabla} \omega_{0}\left(g, K^{2}\right) .
$$

It is evident that if $f_{0}$ and $f_{1}$ are two mappings of the complex $K^{2}$ into $S^{2}$, coinciding on $K^{1}$, then $\omega_{0}\left(f_{0}, f_{1}, K^{2}\right)=\omega_{0}\left(f_{1}, K^{2}\right)-\omega_{0}\left(f_{0}, K^{2}\right)$.

The proposition F ) which is a particular case of Whitney's theorem ${ }^{10}$, is given here without proof; it follows also very easily from what has been proved already in the present paragraph.
G) Let $K^{3}$ be a three-dimensional complex and $K^{2}$ the complex composed of all simplexes of the complex $K^{3}$ of dimensionality not greater than 2. The mapping $f$ of the complex $K^{2}$ into $S^{2}$ may be then and only then extended to the whole complex $K^{3}$, when $\omega_{0}\left(f, K^{2}\right)$ is a $\nabla$-cycle in $K^{3}[$ cf. F)].

[^3]Suppose that the mapping $f$ is already defined on the whole complex $K^{3}$, and that $T^{3}$ is a three-dimensional orientated simplex from $K^{3}$. Then the power of the mapping $f$ of the boundary $\dot{T}^{3}$ of the simplex $T^{3}$ is equal to zero. On the other hand this power is obviously equal to the sum of powers of the mappings of the faces of the simplex $T^{3}$. Thus $\nabla \omega_{0}\left(f, K^{2}\right)=0$.

Suppose that $f$ is given on $K^{2}$ and $\nabla \omega_{0}\left(f, K^{2}\right)=0$. Then the power of the mapping $f$ of the boundary $\dot{T}^{3}$ of a certain simplex $T^{3}$ from $K^{3}$ is equal to zero, and consequently the mapping $f$ may be extended to $T^{3}$, and we obtain an extension of the mapping $f$ to the whole complex $K^{3}$.
§ 3. The mappings of a three-dimensional complex into the two-dimensional sphere
By $K^{n}$ we shall, as above, denote an $n$-dimensional complex and by $K^{r}$ the aggregate of all simplexes from $K^{n}$, whose dimensionality does not exceed $r$. By $S^{r}$ we shall denote the $r$-dimensional orientated sphere.

If $f$ and $g$ are two mappings of the complex $K^{3}$ into $S^{2}$, then for the solution of the question on their equivalency we have first of all to solve the question on the equivalency of these mappings on $K^{2}$. In fact, if it turns out that the mappings $f$ and $g$ are not equivalent already on $K^{2}$, then the question on their equivalency on $K^{3}$ is by this answered in the negative. The criterium of equivalency of the mappings $f$ and $g$ on $K^{2}$ has been already given in § 2 [cf. § 2, F)]. Thus it remains to consider the question on the equivalency of the mappings $f$ and $g$ of the complex $K^{3}$ in the case, when these mappings are equivalent on $K^{2}$. Under this assumption we can transform the mapping $g$ of the complex $K^{2}$ by a continuous deformation into the mapping $f$ and then extend this deformation $t_{0}$ the whole complex $K^{3}$. We come so to the case, when the mappings $f$ and $g$ of the complex $K^{3}$ simply coincide on $K^{2}$. The question on the equivalency in this case is completely answered by the following theorem.

Theorem 3. Let $f$ and $g$ be two mappings of the complex $K^{3}$ into $S^{2}$, coinciding on $K^{2}$. Put $\omega_{0}\left(f_{1}, K^{2}\right)=\omega_{0}\left(g, K^{2}\right)=z^{2}[c \mathrm{cf}$. §2, E)]. Put, further, $\left.\omega_{1}\left(f, g, K^{3}\right)=z^{3}\left[\mathrm{cf} . \S 2, \mathrm{~A}^{\prime}\right)\right] . z^{2}$ and $z^{3}$ are a two-dimensional and threedimensional $\nabla$-cycles from $K^{3}$ [cf. § 2, G)]. The mappings $f$ and $g$ of the complex $K^{3}$ are equivalent then and only then, when there exists in $K^{3}$ an one-dimensional $\nabla$-cycle $x^{1}$ such that

$$
\begin{equation*}
z^{3} \widetilde{\nabla}_{2} x^{1} \times z^{2} \quad \text { (cf. P. C.) } \tag{1}
\end{equation*}
$$

Before we proceed to the proof of Theorem 3, we prove Lemma 4, which in substance solves already the question.

Lemma 4. Let $f_{t}$ be a continuous deformation of mappings of the complex $K^{3}$ into the sphere $S^{2}$ such that the mappings $f_{0}$ and $f_{1}$ coincide on $K^{2}$, and the mapping $f_{t}$ coincides on $K^{0}$ with the mapping $f_{0}$ for arbitrary $t$. Put

$$
\begin{gather*}
\omega_{0}\left(f_{0}, K^{2}\right)=\omega_{0}\left(f_{1}, K^{2}\right)=z^{2},  \tag{2}\\
\omega_{1}\left(f_{0}, f_{1}, K^{3}\right)=z^{3} . \tag{3}
\end{gather*}
$$

Then $z^{2}$ and $z^{3}$ are a two-dimensional and a three-dimensional $\nabla$-cycles from $K^{3}$. Denote, further, by $e_{t}$ such a continuous deformation of the mappings of
the complex $K^{3}$ that the mapping $e_{t}$ coincides with $f_{0}$ for arbitrary $t$ and put

$$
\begin{equation*}
\omega_{0}\left(e_{t}, f_{t}, K^{1}\right)=x^{1} \tag{4}
\end{equation*}
$$

Then it turns out that $x^{1}$ is a $\nabla$-cycle from $K^{3}$ and

$$
\begin{align*}
& z^{3} \widetilde{\nabla} 2 \varepsilon x^{1} \times z^{2} \quad \text { (cf. §2), }  \tag{5}\\
& \varepsilon= \pm 1
\end{align*}
$$

Proof. Let $\Delta$ be the set of all numbers $0 \leqslant t \leqslant 1$; then the mapping $f_{\Delta}$ of the complex $K^{3} \cdot \Delta$ into $S^{2}$ is defined (cf. $\S 2$ ). The complex $K^{3} \cdot \Delta$ is not simplicial, but without limiting the generality we may suppose that $f_{\Delta}$ is a simplicial mapping of a certain simplicial subdivision of the complex $K^{3} \cdot \Delta$.

By $p^{0}$ and $p^{1}$ we denote two points of the sphere $S^{2}$ lying inside one of the simplexes of the taken triangulation of the sphere $S^{2}$. Let $T^{r}$ be an orientated simplex of the complex $K^{3}$. By $P^{i}\left(T^{r} \cdot \Delta\right)$ we denote the complete orientated original of the point $p^{i}$ in $T^{r} \cdot \Delta$ under the mapping $f_{\Delta}, i=0,1\left[c \mathrm{cf}\right.$ § $\left.\left.1, \mathrm{C}^{\prime}\right)\right]$. By $P_{t}^{i}\left(T^{r}\right)$ denote the complete orientated original of the point $p^{i}$ in $T^{r}$ under the mapping $\left.f_{t}, i=0,1, t=0,1\left[\mathrm{cf}. \S 1, \mathrm{C}^{\prime}\right)\right]$. In view of the fact that the mappings $f_{0}$ and $f_{1}$ coincide on $K^{2}$, we have

$$
P_{0}^{i}\left(T^{2}\right)=P_{1}^{i}\left(T^{2}\right)=P^{i}\left(T^{2}\right)
$$

In the sequel we shall for shortness use the denotations introduced in my preceding paper [cf. P. C., A)]. For computation of the relations of bounding we shall use the relation obtained in the present paper [cf. § $\left.1, \mathrm{C}^{\prime}\right)$ ].

We note the following relations of bounding:

$$
\begin{gather*}
P^{i}\left(T^{2} \cdot \Delta\right)=P^{i}\left(T^{2}\right) \cdot 1-P^{i}\left(T^{2}\right) \cdot 0-P^{i}\left(\dot{T}^{2} \cdot \Delta\right),  \tag{6}\\
P^{i}\left(\dot{T}^{3} \cdot \Delta\right)^{\cdot}=P^{i}\left(\dot{T}^{3}\right) \cdot 1-P^{i}\left(\dot{T}^{3}\right) \cdot 0 \quad[\mathrm{cf.}(6)],  \tag{7}\\
P_{0}^{i}\left(T^{3}\right)^{\cdot}=P_{1}^{i}\left(T^{3}\right)^{\cdot}=P^{i}\left(\dot{T^{3}}\right) . \tag{8}
\end{gather*}
$$

If $C$ is a certain nul-dimensional complex, then under the index $I(C)$ of this complex we shall understand the algebraical number of points entering into it. In virtue of the very definition of $\nabla$-complexes $z^{2}$ and $x^{1}$ we have

$$
\begin{align*}
& z^{2}\left(T^{2}\right)=I\left(P^{i}\left(T^{2}\right)\right)  \tag{9}\\
& x^{1}\left(T^{1}\right)=I\left(P^{i}\left(T^{1} \cdot \Delta\right)\right) . \tag{10}
\end{align*}
$$

- Let now every point $x \cdot t$ ( $x \in T^{r}, t$ a number) from the complex $P^{i}\left(T^{r} \cdot \Delta\right)$ slide along a straight line and uniformly in time into the point $x \cdot 0$ (the straightness and uniformness is understood in the sense of affine geometry which is in the prism $T^{r} \cdot \Delta$ ). The complex, situated in $T^{r} \cdot \Delta$, described by this motion of the whole complex $P^{i}\left(T^{r} \cdot \Delta\right)$ we denote by $Q^{i}\left(T^{r} \cdot \Delta\right)$. The complex from $T^{r} \cdot 0$, into which the complex $P^{i}\left(T^{r} \cdot \Delta\right)$ passes at the end of the motion, we denote by $Q^{i}\left(T^{r}\right) \cdot 0$, where the complex $Q^{i}\left(T^{r}\right)$ belongs to $T^{r}$.

Observe that the complex $Q^{i}\left(T^{r}\right)$ is a projection of the complex $P^{i}\left(T^{r} \cdot \Delta\right)$; in particular, for $r=1$ both these nul-dimensional complexes have an equal index, and consequently

$$
\begin{equation*}
x^{1}\left(T^{1}\right)=I\left(Q^{i}\left(T^{1}\right)\right) \quad[\text { cf. (10) }] . \tag{11}
\end{equation*}
$$

Note the following relations of bounding:

$$
\begin{align*}
Q^{i}\left(T^{2} \cdot \Delta\right)= & P^{i}\left(T^{2} \cdot \Delta\right)-P^{i}\left(T^{2}\right) \cdot \Delta-Q^{i}\left(T^{2}\right) \cdot 0-Q^{i}\left(\dot{T}^{2} \cdot \Delta\right)  \tag{12}\\
Q^{i}\left(\dot{T}^{3} \cdot \Delta\right)^{\cdot}= & P^{i}\left(\dot{T}^{3} \cdot \Delta\right)-P^{i}\left(\dot{T}^{3}\right) \cdot \Delta-Q^{i}\left(\dot{T}^{3}\right) \cdot 0 \quad[\text { cf. (12) }],  \tag{13}\\
& Q^{i}\left(T^{2}\right)^{\cdot}=-Q^{i}\left(\dot{T}^{2}\right)  \tag{14}\\
Q^{i}\left(\dot{T}^{3}\right)^{-}=0 & {[\text { cf. (6) }(6)] } \tag{15}
\end{align*}
$$

From the relations (14) and (11) follows that $x^{1}$ is a $\nabla$-cycle in $K^{3}$. This fact could have been also established in a more direct way on ground of the proposition C'), § 2. Similarly, from the relations (8) and (9) follows that $z^{2}$ is a $\nabla$-cycle in $K^{3}$.

From the relation (8) follows that the nul-dimensional cycle $P^{i}\left(\dot{T}^{3}\right)$ situated in $\dot{T}^{3}$ has the index zero and, consequently, bounds a certain one-dimensional complex $A^{i}\left(\dot{T}^{3}\right)$ also situated in $\dot{T}^{3}$,

$$
\begin{equation*}
A^{i}\left(\dot{T}^{3}\right)^{\cdot}=P^{i}\left(\dot{T}^{3}\right) . \tag{16}
\end{equation*}
$$

In virtue of (8) and (16) the one-dimensional complex $P_{t}^{i}\left(T^{3}\right)-A^{i}\left(\dot{T}^{3}\right)$ is, for $t=0,1$, a cycle in the simplex $T^{3}$ and, consequently, bounds in it a certain two-dimensional complex $B_{t}^{i}\left(T^{3}\right), t=0,1$,

$$
\begin{equation*}
B_{t}^{i}\left(T^{3}\right)^{\cdot}=P_{t}^{i}\left(T^{3}\right)-A^{i}\left(\dot{T}^{3}\right) \quad(t=0,1) \tag{17}
\end{equation*}
$$

In virtue of the relations (7) and (16) the one-dimensional complex

$$
P^{i}\left(\dot{T}^{3} \cdot \Delta\right)+A^{i}\left(\dot{T}^{3}\right) \cdot 0-A^{i}\left(\dot{T}^{3}\right) \cdot 1
$$

situated in $\dot{T}^{3} \cdot \Delta$ is a cycle and, consequently, bounds a certain two-dimensional complex $B^{i}\left(\dot{T}^{3} \cdot \Delta\right)$ situated in $\dot{T}^{3} \cdot \Delta$,

$$
\begin{equation*}
B^{i}\left(\dot{T}^{3} \cdot \Delta\right)=P^{i}\left(\dot{T}^{3} \cdot \Delta\right)+A^{i}\left(\dot{T}^{3}\right) \cdot 0-A^{i}\left(\dot{T}^{3}\right) \cdot 1 \tag{18}
\end{equation*}
$$

Consider in the complex $K^{3} \cdot \Delta$ the orientated three-dimensional sphere

$$
\begin{equation*}
\Sigma=\left(T^{3} \cdot \Delta\right)^{\cdot}=T^{3} \cdot 1-T^{3} \cdot 0-\dot{T}^{3} \cdot \Delta . \tag{19}
\end{equation*}
$$

Since this sphere bounds in $K^{3} \cdot \Delta$ the element $T^{3} \cdot \Delta$, the mapping $f_{\Delta}$ is for it unessential, i. e. $\omega_{1}\left(f_{\Delta}, \Sigma\right)=0$. Let us calculate this vanishing invariant by means of the introduced algebraical complexes; to this end denote by $U^{i}$ the complete original of the point $p^{i}$ in $\Sigma$ under the mapping $f_{\Delta}$. It is easily seen that

$$
\begin{equation*}
U^{i}=P_{1}^{i}\left(T^{3}\right) \cdot 1-P_{0}^{i}\left(T^{3}\right) \cdot 0-P^{i}\left(\dot{T}^{3} \cdot \Delta\right) \tag{20}
\end{equation*}
$$

Thus the linkage coefficient of the cycles $U^{0}$ and $U^{1}$ in $\Sigma$ is equal to zero, and we have

$$
\begin{gathered}
V_{\Sigma}\left(U^{0}, U^{1}\right)= \\
=I_{\Sigma}\left(B_{1}^{0}\left(T^{3}\right) \cdot 1-B_{0}^{0}\left(T^{3}\right) \cdot 0-B^{0}\left(\dot{T}^{3} \cdot \Delta\right), P_{1}^{1}\left(T^{3}\right) \cdot 1-P_{0}^{1}\left(T^{3}\right) \cdot 0-P^{1}\left(\dot{T}^{3} \cdot \Delta\right)\right)= \\
=I_{T^{8}}\left(B_{1}^{0}\left(T^{3}\right), P_{1}^{1}\left(T^{3}\right)\right)-I_{T^{3}}\left(B_{0}^{0}\left(T^{3}\right), P_{0}^{1}\left(T^{3}\right)\right)-I_{\dot{T}_{3} \cdot \Delta}\left(B^{0}\left(\dot{T}^{3} \cdot \Delta\right), P^{1}\left(\dot{T}^{3} \cdot \Delta\right)\right)=0 .
\end{gathered}
$$

Hence we obtain

$$
\begin{gather*}
w\left(T^{3}\right)=I_{T^{3}}\left(B_{1}^{0}\left(T^{3}\right), P_{1}^{1}\left(T^{3}\right)\right)-I_{T^{3}}\left(B_{0}^{0}\left(T^{3}\right), P_{0}^{1}\left(T^{3}\right)\right)= \\
=I_{\dot{T}^{3} \cdot \Delta}\left(B^{0}\left(\dot{T}^{3} \cdot \Delta\right), P^{1}\left(\dot{T}^{3} \cdot \Delta\right)\right) \tag{21}
\end{gather*}
$$

Let us now show that

$$
\begin{equation*}
w\left(T^{3}\right)=z^{3}\left(T^{3}\right) \quad[\mathrm{cf} .(3)] \tag{22}
\end{equation*}
$$

To this end denote by $\Delta^{\prime}$ the aggregate of two numbers 0 and 1 . Then is defined the mapping $f_{\Delta^{\prime}}$ of the complex $T^{3} \cdot \Delta^{\prime}(\mathrm{cf} . \S 2)$. The complex $T^{3} \cdot \Delta^{\prime}$ consists of two elements $T^{3} \cdot 1$ and $-T^{3} \cdot 0$, and $f_{\Delta^{\prime}}(x \cdot 0)=f_{\Delta^{\prime}}(x \cdot 1)$ for every $x \in \dot{T}^{3}$. Identify in one point every pair of points $x \cdot 0$ and $x \cdot 1$; then the complex $T^{3} \cdot \Delta^{\prime}$ will be transformed into the sphere $\Sigma^{\prime}$, and the mapping $f_{\Delta^{\prime}}$ may be considered as a mapping of the sphere $\Sigma^{\prime}$. In virtue of the very definition of the $\nabla$-cycle $z^{3}$ we have $z^{3}\left(T^{3}\right)=\omega_{1}\left(f_{\Delta^{\prime}}, \Sigma^{\prime}\right)$. Let us calculate $\omega_{1}\left(f_{\Delta^{\prime}}, \Sigma^{\prime}\right)$. To this end denote by $w^{i}$ the original of the point $p^{i}$ in $\Sigma^{\prime}$ under the mapping $f_{\Delta^{\prime}}$. It is easily seen that

$$
w^{i}=P_{1}^{i}\left(T^{3}\right) \cdot 1-P_{0}^{i}\left(T^{3}\right) \cdot 0 .
$$

Thus we have

$$
\begin{gathered}
z^{3}\left(T^{3}\right)=V_{\Sigma^{\prime}}\left(w^{0}, w^{1}\right)= \\
=I_{\Sigma^{\prime}}\left(B_{1}^{0}\left(T^{3}\right) \cdot 1-B_{0}^{0}\left(T^{3}\right) \cdot 0, P_{1}^{1}\left(T^{3}\right) \cdot 1-P_{0}^{1}\left(T^{3}\right) \cdot 0\right)= \\
=I_{T^{3}}\left(B_{1}^{0}\left(T^{3}\right), P_{1}^{1}\left(T^{3}\right)\right)-I_{T^{3}}\left(B_{0}^{0}\left(T^{3}\right), P_{0}^{1}\left(T^{3}\right)\right) .
\end{gathered}
$$

So the relation (22) is proved and we obtain

$$
\begin{equation*}
z^{3}\left(T^{3}\right)=I_{\dot{T}^{3} \cdot \Delta}\left(B^{0}\left(\dot{T}^{3} \cdot \Delta\right), P^{1}\left(\dot{T}^{3} \cdot \Delta\right)\right) \tag{23}
\end{equation*}
$$

Let us now calculate $z^{3}\left(T^{3}\right)$ starting from the relation (23). To this end identify in the complex $\dot{T}^{3} \cdot \Delta$ in one point every pair of points $x \cdot 0$ and $x \cdot 1$ and denote the obtained manifold by $M$; this manifold is, as may be easily seen, homoeomorphic to the product of the two-dimensional sphere and the circumference. The one-dimensional complex $P^{i}\left(\dot{T}^{3} \cdot \Delta\right)$ represents in $M$ a cycle [cf. (7)] and bounds in $M$ the two-dimensional complex $B^{i}\left(\dot{T}^{3} \cdot \Delta\right)$ [cf. (18)]. Thus the linkage coefficient

$$
\begin{equation*}
V_{M}\left(P^{0}\left(\dot{T}^{3} \cdot \Delta\right), P^{1}\left(\dot{T}^{3} \cdot \Delta\right)\right) \tag{24}
\end{equation*}
$$

is defined, since both involved cycles are homologic to zero in $M$. The relation (23), on the other hand, shows that $z^{3}\left(T^{3}\right)$ is nothing else but the linkage coefficient (24),

$$
\begin{equation*}
z^{3}\left(T^{3}\right)=V_{M}\left(P^{0}\left(\dot{T}^{3} \cdot \Delta\right), P^{1}\left(\dot{T}^{3} \cdot \Delta\right)\right) \tag{25}
\end{equation*}
$$

Let us now calculate this linkage coefficient.
Observe that the complexes $P^{i}\left(\dot{T}^{3}\right) \cdot \Delta$ and $Q^{i}\left(\dot{T}^{3}\right) \cdot 0$ are cycles in $M[\mathrm{cf}$. (15)]. In virtue of the relation (13) we have

$$
\begin{gather*}
V_{M}\left(P^{0}\left(\dot{T}^{3} \cdot \Delta\right)-P^{0}\left(\dot{T}^{3}\right) \cdot \Delta-Q^{0}\left(\dot{T}^{3}\right) \cdot 0, P^{1}\left(\dot{T}^{3} \cdot \Delta\right)\right)= \\
=I_{M}\left(Q^{0}\left(\dot{T}^{3} \cdot \Delta\right), P^{1}\left(\dot{T}^{3} \cdot \Delta\right)\right)= \\
=\sum_{T^{2} \in \dot{T}^{3}} I_{T^{2} \cdot \Delta}\left(Q^{0}\left(T^{2} \cdot \Delta\right), P^{1}\left(T^{2} \cdot \Delta\right)\right)=\sum_{T^{2} \in \dot{T}^{3}} y^{2}\left(T^{2}\right) \tag{26}
\end{gather*}
$$

where $y^{2}$ is a function defined on an arbitrary two-dimensional simplex $T^{2}$, i. e. a two-dimensional $\nabla$-complex from $K^{3}$ depending on the deformation $f_{t}$. Put now

$$
\begin{equation*}
z^{* 3}\left(T^{3}\right)=V_{M}\left(P^{0}\left(\dot{T}^{2}\right) \cdot \Delta+Q^{0}\left(\dot{T}^{3}\right) \cdot 0, P^{1}\left(\dot{T}^{3} \cdot \Delta\right)\right) \tag{27}
\end{equation*}
$$

In virtue of the relations (25), (26) and (27) we have

$$
z^{3}-z^{* 3}=\nabla y^{2}
$$

and this means that

$$
\begin{equation*}
z^{3} \widetilde{\nabla} z^{* 3} \tag{28}
\end{equation*}
$$

Now we shall calculate $z^{* 3}\left(T^{3}\right)$ on ground of the relation (27).
In virtue of the relation (27) we have

$$
\begin{equation*}
z^{* 3}\left(T^{3}\right)=V_{M}\left(P^{0}\left(\dot{T}^{3}\right) \cdot \Delta, P^{1}\left(\dot{T}^{3} \cdot \Delta\right)\right)+V_{M}\left(Q^{0}\left(\dot{T}^{3}\right) \cdot 0, P^{1}\left(\dot{T}^{3} \cdot \Delta\right)\right) \tag{29}
\end{equation*}
$$

Since the cycle $Q^{0}\left(\dot{T}^{3}\right) \cdot 0$ lies totally in $\dot{T}^{3} \cdot 0$, and the cycle $P^{1}\left(\dot{T}^{3} \cdot \Delta\right)$ intersects $T^{3} .0$ along the nul-dimensional complex $P^{1}\left(\dot{T}^{3}\right) \cdot 0$,

$$
\begin{equation*}
V_{M}\left(Q^{0}\left(\dot{T}^{3}\right) \cdot 0, \quad P^{1}\left(\dot{T}^{3} \cdot \Delta\right)\right)=V_{\dot{T}^{3}}\left(Q^{0}\left(\dot{T}^{3}\right), P^{1}\left(\dot{T}^{3}\right)\right) \tag{30}
\end{equation*}
$$

Further,

$$
\begin{equation*}
V_{M}\left(P^{0}\left(\dot{T}^{3}\right) \cdot \Delta, P^{1}\left(\dot{T}^{3} \cdot \Delta\right)\right)=V_{M}\left(P^{1}\left(\dot{T}^{3} \cdot \Delta\right), P^{0}\left(\dot{T}^{3}\right) \cdot \Delta\right) \tag{31}
\end{equation*}
$$

In virtue of the relation (13) we have

$$
\begin{gather*}
V_{M}\left(P^{1}\left(\dot{T}^{3} \cdot \Delta\right)-P^{1}\left(\dot{T}^{3}\right) \cdot \Delta-Q^{1}\left(\dot{T}^{3}\right) \cdot 0, P^{0}\left(\dot{T}^{3}\right) \cdot \Delta\right)= \\
=I_{M}\left(Q^{1}\left(\dot{T}^{3} \cdot \Delta\right), P^{0}\left(\dot{T}^{3}\right) \cdot \Delta\right)=0, \tag{32}
\end{gather*}
$$

since the complex $Q^{1}\left(\dot{T}^{3} \cdot \Delta\right)$ does not intersect with $P^{0}\left(\dot{T^{3}}\right) \cdot \Delta$. Thus, from (32) we obtain

$$
\begin{gather*}
V_{M}\left(P^{1}\left(\dot{T}^{3} \cdot \Delta\right), P^{0}\left(\dot{T}^{3}\right) \cdot \Delta\right)= \\
=V_{M}\left(P^{1}\left(\dot{T}^{3}\right) \cdot \Delta, P^{0}\left(\dot{T}^{3}\right) \cdot \Delta\right)+V_{M}\left(Q^{1}\left(\dot{T}^{3}\right) \cdot 0, P^{0}\left(\dot{T}^{3}\right) \cdot \Delta\right)= \\
V_{M}\left(P^{1}\left(\dot{T}^{3}\right) \cdot \Delta, P^{0}\left(\dot{T}^{3}\right) \cdot \Delta\right)+ \\
+V_{\dot{T}^{3}}\left(Q^{1}\left(\dot{T}^{3}\right), P^{0}\left(\dot{T}^{3}\right)\right) \tag{33}
\end{gather*}
$$

The cycle $P^{1}\left(T^{3}\right) \cdot \Delta$ bounds in $M$ the complex $-A^{1}\left(T^{3}\right) \cdot \Delta$, which does not intersect with $P^{0}\left(\dot{T}^{3}\right) \cdot \Delta$, and hence

$$
\begin{equation*}
V_{M}\left(P^{1}\left(\dot{T}^{3}\right) \cdot \Delta, P^{0}\left(\dot{T}^{3}\right) \cdot \Delta\right)=0 \tag{34}
\end{equation*}
$$

Thus we finally obtain

$$
\begin{equation*}
z^{* 3}\left(T^{3}\right)=V_{\dot{T}^{3}}\left(Q^{0}\left(\dot{T}^{3}\right), P^{1}\left(\dot{T}^{3}\right)\right)+V_{\dot{T}^{3}}\left(Q^{1}\left(\dot{T}^{3}\right), \quad P^{0}\left(\dot{T}^{3}\right)\right) \tag{35}
\end{equation*}
$$

Taking into account the relations (9), (11) and (14), we obtain from (35)

$$
\begin{equation*}
z^{* 3}=\varepsilon 2 x^{1} \times z^{2} \tag{36}
\end{equation*}
$$

where $\varepsilon= \pm 1$ has a quite determined value, but is not calculated here (cf. P. C., theorem 2). From (36) and (28) follows (5), i. e. the assertion of Lemma 4. The Lemma 4 is thus proved.

Proof of Theorem 3. Let $g_{t}$ be such a deformation of mappings of the complex $K^{3}$ into the sphere $S^{2}$ that the mappings $g_{0}$ and $g_{1}$ coincide on $K^{2}$.

Put

$$
\begin{gathered}
\left.z^{3}=\omega_{1}\left(g_{0}, g_{1}, K^{3}\right) \quad\left[\text { cf. § } 2, \mathrm{~B}^{\prime}\right)\right] \\
\left.z^{2}=\omega_{0}\left(g_{0}, K^{2}\right)=\omega_{0}\left(g_{1}, K^{2}\right) \quad[\text { cf. § } 2, \mathrm{~F})\right]
\end{gathered}
$$

Let us show that in $K^{3}$ there exists an one-dimensional $\nabla$-cycle $u^{1}$ satisfying the condition

$$
\begin{equation*}
z^{3} \widetilde{\nabla} 2 u^{1} \times z^{2} \tag{37}
\end{equation*}
$$

Denote by $\Delta$ the set of all numbers $0 \leqslant t \leqslant 1$. Then the mapping $g_{\Delta}$ of the complex $K^{3} \cdot \Delta$ into $S^{2}$ is defined (cf. § 2). Denote by $L$ the subcomplex of the complex $K^{3} \cdot \Delta$, composed of $K^{3} \cdot 0, K^{3} \cdot 1$ and all segments of the form $x \cdot \Delta$, where $x \in K^{0}$. The mapping $g_{\Delta}$ is defined on $L$, and every segment of the form $x \cdot \Delta$ is mapped in such a way that $g_{\Delta}(x \cdot 0)=g_{\Delta}(x \cdot 1)$. Let us define a continuous deformation $\chi_{t}$ of every segment $x \cdot \Delta$ such that $\chi_{0}$ coincides with $g_{\Delta}$ and the mapping $\chi_{1}$ transforms the whole segment $x \cdot \Delta$ into the point $g_{\Delta}(x \cdot 0)=$ $=g_{\Delta}(x \cdot 1)$, where $\chi_{t}(x \cdot 0)=\chi_{t}(x \cdot 1)=g_{\Delta}(x \cdot 0)$ for every $t$. On $K^{3} .0$ and $K^{3} \cdot 1$ we define the deformation $\chi_{t}$ so that the mapping $\chi_{t}$ should coincide with $g_{\Delta}$ for every $t$. The so obtained deformation $\chi_{t}$ of the mappings of the complex $L$ is continuous and it may be continuously extended to the whole complex $K^{3} \cdot \Delta$. We denote the mapping $\chi_{1}$ by $f_{\Delta}$; it is defined on $K^{3} \cdot \Delta$. The mapping $f_{\Delta}$ determines a deformation $f_{t}$ of mappings of the complex $K^{3}$, satisfying the condition of Lemma 4. We thus obtain $z^{3} \widetilde{\nabla} 2 \varepsilon x^{1} \times z^{2}$, and the existence of the $\nabla$-cycle $u^{1,}$ satisfying the condition (37) is proved.

Let $f_{0}$ and $l$ be two mappings of the complex $K^{3}$ coinciding on $K^{2}$. Put

$$
z^{3}=\omega_{1}\left(f_{0}, l, K^{3}\right), \quad z^{2}=\omega_{0}\left(f_{0}, K^{2}\right)=\omega_{0}\left(l, K^{2}\right)
$$

We shall show that if in $K^{3}$ there exists a $\nabla$-cycle $u^{1}$ satisfying the condition $z^{3} \nabla 2 u^{1} \times z^{2}$, then the mappings $f_{0}$ and $l$ are equivalent. This will complete the proof of Theorem 3.

Let $u^{1}=\varepsilon x^{1}$ (cf. Lemma 4). Denote, further, by $e_{t}, 0 \leqslant t \leqslant 1$, a mapping of the complex $K^{3}$ into $S^{2}$, coinciding with $f_{0}$. In virtue of the remark E ), $\S 2$, there exists a continuous deformation $f_{t}$ of mappings of the complex $K^{1}$ into $S^{2}$ such that $\left.\omega_{0}\left(e_{t}, f_{t}, K^{1}\right)=x^{1}\left[\mathrm{cf} . \S 2, \mathrm{~B}^{\prime}\right)\right]$. Extend the deformation $f_{t}$ to the complex $K^{2}$. In virtue of the remark $\left.\mathrm{C}^{\prime}\right), \S 2$, we have

$$
\omega_{0}\left(e_{1}, f_{1}, K^{2}\right)-\omega_{0}\left(e_{0}, f_{0}, K^{2}\right)=\nabla \omega_{0}\left(e_{t}, f_{t}, K^{1}\right)=\nabla x^{1}=0
$$

and since, moreover, $\omega_{0}\left(e_{0}, f_{0}, K^{2}\right)=0$,

$$
\omega_{0}\left(e_{1}, f_{1}, K^{2}\right)=0
$$

and, consequently, in virtue of $\left.\mathrm{A}^{\prime}\right), \S 2$, the mappings $f_{0}=e_{1}$ and $f_{1}$ of the: complex $K^{2}$ are equivalent with respect to $K^{1}$. The continuous deformation of mappings the complex $K^{2}$ realizing this equivalence we denote by $g_{t}$. The deformation of the mappings of the complex $K^{2}$ obtained in the result of successive applications of the deformations $f_{t}$ and $g_{t}$ we denote by $h_{t}$. The mappings $h_{0}=f_{0}$ and $h_{1}=g_{1}$ of the complex $K^{2}$ coincide, and $\omega_{0}\left(e_{t}, h_{t}, K^{1}\right)=x^{1}$. We extend the deformation $h_{t}$ to the whole complex $K^{3}$ and applying to it Lemma 4 we obtain.

$$
\omega_{1}\left(f_{0}, h_{1}, K^{3}\right) \widehat{\nabla} \varepsilon 2 x^{1} \times z^{2}
$$

Thus $\left.\omega_{1}\left(h_{1}, l, K^{3}\right) \widetilde{\nabla} 0\left[\mathrm{cf} . \S 2, \mathrm{D}^{\prime}\right)\right]$, or, in other words, $\omega_{1}\left(h_{1}, l, K^{3}\right)=\nabla y^{2}$.

In virtue of the remark E ), § 2 , there exists a continuous deformation $k_{t}$ of the complex $K^{2}$, transforming the mapping $h_{1}=k_{0}$ into the mapping $k_{1}$ and not changing the mapping $h_{1}$ on $K^{1}$, such that $\omega_{1}\left(e_{t}^{\prime}, k_{t}, K^{2}\right)=y^{2}$, where $e_{t}^{\prime}=h_{1}$. Extend the deformation $k_{t}$ to the whole complex $K^{3}$. Then, in virtue of $\left.\mathrm{C}^{\prime}\right), \S 2$, we shall have

$$
\omega_{1}\left(h_{1}, k_{1}, K^{3}\right)=\omega_{1}\left(e_{1}^{\prime}, k_{1}, K^{3}\right)+\nabla y^{2}
$$

Hence we conclude that $\left.\omega_{1}\left(k_{1}, l, K^{3}\right)=0\left[c f . \S 2, \mathrm{D}^{\prime}\right)\right]$. Thus, in virtue of $\left.\mathrm{A}^{\prime}\right)$, § 2, the mappings $k_{1}$ and $l$ are equivalent, and consequently so are also the mappings $f_{0}$ and $l$. Theorem 3 is thus proved.

In addition to Theorem 3 we shall prove the following proposition on the existence of mappings.
A) Let $z^{2}$ be a two-dimensional cycle from $K^{3}$; then there exists a mapping $f$ of the complex $K^{3}$ into $S^{2}$ such that

$$
\begin{equation*}
\omega_{0}\left(f, K^{2}\right)=z^{2} . \tag{38}
\end{equation*}
$$

Furher, if $f_{0}$ is a certain mapping of the complex $K^{3}$ into $S^{2}$ and $z^{3}$ a certain three-dimensional $\nabla$-cycle from $K^{3}$, then there exists a mapping $f_{1}$ of the complex $K^{3}$ into $S^{2}$ coinciding with $f_{0}$ on $K^{2}$ and satisfying the condition

$$
\begin{equation*}
\omega_{1}\left(f_{0}, f_{1}, K^{3}\right)=z^{3} \tag{39}
\end{equation*}
$$

Choose on $S^{2}$ a certain fixed point $q$ and map every two-dimensional simplex $T^{2}$ from $K^{2}$ on $S^{2}$ with the power $z^{2}\left(T^{2}\right)$ so that its boundary $\dot{T}^{2}$ is transformed into the point $q$. The so arising mapping $f$ of the complex $K^{2}$ satisfies the condition (38) [cf. § 2, F)]. Since $z^{2}$ is a $\nabla$-cycle, the mapping $f$ may be extended to the whole complex $K^{3}$. The existence of the mapping $f_{1}$ satisfying the condition (39) follows from the proposition E), § 2.

Theorem 3 has the following defect: it does not establish the complete system of invariants of the mappings of the complex $K^{3}$ into $S^{2}$, but only enables us to establish the equivalency or non-equivalency of two mappings. Moreover, in order to establish the equivalency of two mappings already equivalent on $K^{2}$, it is necessary to subject one of the mappings to a continuous deformation so as to make it coinciding with the other on $K^{2}$.

The contents of Theorem 3 is exposed in more detail by the following proposition, which follows from Theorem 3 and the proposition A):
B) Let $B_{\nabla}^{r}$ be the $r$-dimensional $\nabla$-Betti group of the complex $K^{3}$. If $Z^{2} \in B_{\nabla}^{2}$, then there exists at least one mapping $f$ of the complex $K^{3}$ into $S^{2}$, satisfying the condition

$$
\begin{equation*}
\omega_{0}\left(f, K^{2}\right)=z^{2} \in Z^{2} . \tag{40}
\end{equation*}
$$

Denote, further, by $2 B_{\nabla}^{1} \times Z^{2}$ the set of all elements of the group $B_{\nabla}^{3}$, containing cycles of the form $2 u^{1} \times z^{2}$, where $u^{1}$ is an arbitrary $\nabla$-cycle of dimensionality one. Then to every co-set of the group $B_{\nabla}^{3}$ with respect to the subgroup $2 B_{\nabla}^{1} \times Z^{2}$ corresponds one and only one class of mappings satisfying the condition (40). In order to determine this correspondence it is, however, necessary to choose arbitrarily one definite mapping satisfying the condition (40), to which shall then correspond the co-set $2 B_{\nabla}^{1} \times Z^{2}$.

## § 4. Application to manifolds

In the case when the three-dimensional complex $K^{3}$ is an orientable manifold the results of the foregoing paragraph may be formulated by means of the usual homologies, which presents a certain advantage.

We shall understand here under $K^{3}$ a three-dimensional, in a definite manner orientated manifold, somehow subdivided into simplexes. By $B^{r}$ we shall denote the $r$-dimensional usual Betti group of the manifold $K^{3}$ and by $B_{\nabla}^{r}$-its $r$-dimensional $\nabla$-Betti group. It is known that between the groups $B^{r}$ and $B_{\nabla}^{3-r}$ there is a quite definite natural isomorphic correspondence (cf. P. C., theorems 3, 4). By $S^{2}$ we shall, as above, denote the two-dimensional orientated sphere. By $K^{r}$ we shall denote the complex composed of all simplexes of the complex $K^{3}$, whose dimensionality does not exceed $r$.
A) Let $f$ be a simplicial mapping of a certain subdivision of the complex $K^{3}$ into $S^{2}$ and $p$ a certain point from $S^{2}$ lying inside a simplex of the assumed triangulation of the sphere $S^{2}$. By $f^{-1}(p)$ we denote the complete orientated original of the point $p$ in $K^{3}$ under the mapping $f\left[\right.$ cf. § $\left.\left.1, \mathrm{C}^{\prime}\right)\right]$. Then $f^{-1}(p)$ is an one-dimensional cycle from $K^{3}$; denote the index of its intersection with an arbitrary simplex $T^{2}$ of the complex $K^{3}$ by $z^{2}\left(T^{2}\right)$. Then $z^{2}$ is a $\nabla$-cycle, and the classes of homologies containing respectively $f^{-1}(p)$ and $z^{2}$ correspond to each other (cf. P. C., theorem 3). It is easily seen that $\left.z^{2}=\omega_{0}\left(f, K^{2}\right)[\mathrm{cf} . \S 2, \mathrm{~F})\right]$. Thus the class of homologies containing $f^{-1}(p)$, as well as the class of homologies containing $\omega_{0}\left(f, K^{2}\right)$, determines the mapping $f$ up to the equivalency on the complex $\left.K^{2}[\mathrm{cf} . \S 2, \mathrm{~F})\right]$.

In view of the fact that the class of homologies, containing $\omega_{0}\left(f, K^{2}\right)$, may be taken arbitratily [cf. §3, A)], the class of homologies, containing $f^{-1}(p)$, may be also taken arbitrarily.

Now arises the question of establishing of equivalency or non-equivalency of two mappings $f_{0}$ and $f_{1}$ in the case, when $f_{0}^{-1}(p)$ and $f_{1}^{-1}(p)$ are homologic to each other in $K^{3}$.
B) Let $f$ be a mapping considered in A) and $p^{0}$ and $p^{1}$ two points lying inside the simplexes of triangulation of the sphere $S^{2}$. It is easily seen that $f^{-1}\left(p^{0}\right)$ and $f^{-1}\left(p^{1}\right)$ are homological cycles from $K^{3}$; suppose that they are weakly homologic to zero. Thus, there exists a complex $C$ from $K^{3}$ with the boundary $k f^{-1}\left(p^{0}\right)$, where $k$ is a natural number. We define the linkage coefficient $V_{K^{3}}\left(f^{-1}\left(p^{0}\right), f^{-1}\left(p^{1}\right)\right)$ by putting

$$
\begin{equation*}
V_{K^{3}}\left(f^{-1}\left(p^{0}\right), f^{-1}\left(p^{1}\right)\right)=\frac{1}{k} I_{K^{3}}\left(C, f^{-1}\left(p^{1}\right)\right)=\omega_{1}\left(f, K^{3}\right) . \tag{1}
\end{equation*}
$$

Thus this linkage coefficient may be also a fraction. Its fractional part is, as is known, an invariant of the class of homologies, containing $f^{-1}\left(p^{0}\right)$, while the linkage coefficient itself is an invariant of the class of mappings, containing $f$. This is proved in the same way as in Hopf's paper, ${ }^{2}$. It turns out that for a given class of homologies, to which the cycle $f^{-1}\left(p^{0}\right)$ belongs, the number $\omega_{1}\left(f, K^{3}\right)$ is the only invariant of the class of mappings; moreover, the integral part of the number $\omega_{1}\left(f, K^{3}\right)$ may assume arbitrary values, while the cycle $j^{-1}\left(p^{0}\right)$ belongs to the given class of homologies.

Let us prove the proposition B ). Let $f_{0}$ and $f_{1}$ be two mappings of the complex $K^{3}$ into $S^{2}$ coinciding on $K^{2}$. Denote by $P_{j}^{i}\left(T^{3}\right)$ the complete orientated original of the point $p^{i}$ in the simplex $T^{3}$ from $K^{3}$ under the mapping $f_{j}$. The complex $P_{1}^{i}\left(T^{3}\right)-P_{0}^{i}\left(T^{3}\right)$ is a cycle in the simplex $T^{3}$ and hence bounds in it a certain complex $Q^{i}\left(T^{3}\right)$,

$$
\begin{equation*}
Q^{i}\left(T^{3}\right)^{\cdot}=P_{1}^{i}\left(T^{3}\right)-P_{0}^{i}\left(T^{3}\right) . \tag{2}
\end{equation*}
$$

Since the mappings $f_{0}$ and $f_{1}$ coincide on $K^{2}$, the cycles $f_{j}^{-1}\left(p^{i}\right)(i=0,1$, $j=0,1$ ) belong all to one class of homologies [cf. A)]; suppose that this class of homologies has a finite order $k$. Thus there exists a complex $C_{j}^{i}$ from $K^{3}$ with the boundary $k f_{j}^{-1}\left(p^{i}\right)$.

Denote the intersection of the complex $C_{j}^{i}$ with the simplex $T^{3}$ by $C_{j}^{i}\left(T^{3}\right)$. Then

$$
\begin{align*}
C_{j}^{i} & =\sum_{T^{3} \in K^{3}} C_{j}^{i}\left(T^{3}\right),  \tag{3}\\
f_{j}^{-1}\left(p^{i}\right) & =\sum_{T^{3} \in K^{3}} P_{j}^{i}\left(T^{3}\right) . \tag{4}
\end{align*}
$$

Put

$$
\begin{equation*}
\left.z^{3}=\omega_{1}\left(f_{0}^{\prime}, f_{1}, K^{3}\right) \quad\left[\mathrm{cf} . \S 2, \mathrm{~A}^{\prime}\right)\right] \tag{5}
\end{equation*}
$$

and compute $z^{3}\left(T^{3}\right)$ by means of the introduced complexes. To this end denote by $\Delta$ the pair of numbers 0 and 1 . Then is defined the mapping $f_{\Delta}$ of the complex $T^{3} . \Delta$ (cf. § 2). This mapping possesses the property that for every $x \in \dot{T}^{3}$ we have $f_{\Delta}(x \cdot 0)=f_{\Delta}(x \cdot 1)$. Indentify in one point every pair of points of the form $x \cdot 0$ and $x \cdot 1$ and the sphere so obtained from $T^{3} \cdot \Delta$ denote by $\Sigma^{3}$.

The mapping $f_{\Delta}$ may be now considered as the mapping of the sphere $\Sigma^{3}$. It is easily seen that the complete original of the point $p^{i}$ in $\Sigma^{3}$ under the mapping $f_{\Delta}$ is equal to $P_{1}^{i}\left(T^{3}\right) \cdot 1-P_{0}^{i}\left(T^{3}\right) \cdot 0$. Thus

$$
\begin{gather*}
z^{3}\left(T^{3}\right)=V_{\Sigma^{3}}\left(P_{1}^{0}\left(T^{3}\right) \cdot 1-P_{0}^{0}\left(T^{3}\right) \cdot 0, P_{1}^{1}\left(T^{3}\right) \cdot 1-P_{0}^{1}\left(T^{3}\right) \cdot 0\right)= \\
=\frac{1}{k} I_{\Sigma^{3}}\left(k Q^{0}\left(T^{3}\right) \cdot 1+C_{0}^{0}\left(T^{3}\right) \cdot 1-C_{0}^{0}\left(T^{3}\right) \cdot 0, P_{1}^{1}\left(T^{3}\right) \cdot 1-P_{0}^{1}\left(T^{3}\right) \cdot 0\right)= \\
=\frac{1}{k} I_{T^{3}}\left(k Q^{0}\left(T^{3}\right)+C_{0}^{0}\left(T^{3}\right), P_{1}^{0}\left(T^{3}\right)\right)-\frac{1}{k} I_{T^{3}}\left(C_{0}^{0}\left(T^{3}\right), P_{0}^{0}\left(T^{3}\right)\right) . \tag{6}
\end{gather*}
$$

Since the manifold $K^{3}$ is orientated, we may assume that every its three-dimensional simplex has a definite orientation, coordinated with the orientation of the whole manifold $K^{3}$. Put

$$
\begin{equation*}
\sum_{T^{3} \in K^{3}} z^{3}\left(T^{3}\right)=z^{3}\left(K^{3}\right), \quad \sum_{T^{3} \in K^{3}} Q^{0}\left(T^{3}\right)=Q^{0} . \tag{7}
\end{equation*}
$$

With these denotations we have

$$
\begin{equation*}
\left(k Q^{0}+C_{0}^{0}\right)^{\cdot}=k f_{1}^{-1}\left(p^{0}\right) \quad[\mathrm{cf.}(2),(3),(4),(7)], \quad\left(C_{0}^{0}\right)^{-}=k f_{0}^{-1}\left(p^{0}\right) \tag{8}
\end{equation*}
$$

Summing the equality (6) over all $T^{3} \in K^{3}$, we obtain

$$
z^{3}\left(K^{3}\right)=\frac{1}{k} I_{K^{3}}\left(k Q^{0}+C_{0}^{0}, f_{1}^{-1}\left(p^{1}\right)\right)-\frac{1}{k} I_{K^{3}}\left(C_{0}^{0}, f_{0}^{-1}\left(p^{1}\right)\right),
$$

i. e.

$$
z^{3}\left(K^{3}\right)=V_{K^{3}}\left(f_{1}^{-1}\left(p^{0}\right), f_{1}^{-1}\left(p^{1}\right)\right)-V_{K^{3}}\left(f_{0}^{-1}\left(p^{0}\right), f_{0}^{-1}\left(p^{1}\right)\right)
$$

[cf. (1), (8)], or, which is the same thing,

$$
\begin{equation*}
z^{3}\left(K^{3}\right)=\omega_{1}\left(f_{1}, K^{3}\right)-\omega_{1}\left(f_{0}, K^{3}\right) \tag{9}
\end{equation*}
$$

The group $B_{\nabla}^{3}$ is for the three-dimensional orientable manifold $K^{3}$ the free cyclic group. If $\stackrel{*}{T}^{3}$ is a certain simplex from $K^{3}$ and $\stackrel{*}{z}^{3}$ is the $\nabla$-cycle from $K^{3}$ assuming the value 1 on $\stackrel{*}{T}^{3}$ and the value 0 on all other three-dimensional simplexes from $K^{3}$, then $\stackrel{*}{z}^{3}$ may be taken for the basis of $\nabla$-homologies in $K^{3}$. It is easy to see that

$$
\begin{equation*}
z^{3} \widetilde{\nabla} z^{3}\left(K^{3}\right) \stackrel{*}{z}^{3} \tag{10}
\end{equation*}
$$

and consequently the number $z^{3}\left(K^{3}\right)$ determines the class of homologies, to which belongs the $\nabla$-cycle $z^{3}$. Thus, in the case, when $\omega_{1}\left(f_{1}, K^{3}\right)=\omega_{1}\left(f_{0}, K^{3}\right)$, we have $z^{3} \widetilde{\nabla}^{0}$, and hence the mappings $f_{0}$ and $f_{1}$ are equivalent (cf. Theorem 3).

Let us now show that the integral part of the number $\omega_{1}\left(f, K^{3}\right)$ may be made arbitrary, the class of homologies, to which $f^{-1}\left(p^{0}\right)$ belongs, being given. In fact, let $f_{0}$ be an arbitrary mapping such that $f_{0}^{-1}\left(p^{0}\right)$ belongs to the given class of homologies; then, in virtue of A$), \S 3$, there exists such a mapping $f_{1}$ that $z^{3}=\omega_{1}\left(f_{0}, f_{1}, K^{3}\right)$ is an arbitrary $\nabla$-cycle from $K^{3}$. Thus for a given $f_{0}$ we may choose $f_{1}$ such that the number $z^{3}\left(K^{3}\right)$ should have an arbitrary integral value [cf. (10)], and this means that the integral part of the number $\omega_{1}\left(f_{1}, K^{3}\right)$ may be chosen arbitrarily [cf. (9)]. The proposition B) is thus completely proved.

Consider now the case when the cycle $f^{-1}(p)$ [cf. A)] is weakly not homologic to zero.
C) Let $Z^{1}$ be a fixed free element of the group $B^{1}$ and $U^{2}$ an arbitrary element of the group $B^{1}$. Denote by $\lambda$ the smallest positive value which the number $I_{K^{3}}\left(U^{2}, Z^{1}\right)$ may assume for a given $Z^{1}$ and arbitrary $U^{2}$. From PoincaréVeblen's theorem follows that $I_{K^{3}}\left(U^{2}, Z^{1}\right)$ admits of positive values, since $Z^{1}$ is a free element of the group $B^{1}$. It turns out that among the mappings $f$ satisfying the condition

$$
\begin{equation*}
\left.f^{-1}(p) \in Z^{1} \quad[\mathrm{cf} . \mathrm{A})\right] \tag{11}
\end{equation*}
$$

there are exactly $2 \lambda$ pairwise non-equivalent.
In every class of mappings satisfying the condition (11) we may choose one, so that all chosen mappings should coincide on $K^{2}$ [cf. A)]. Let $f_{0}$ and $f_{1}$ be two mappings satisfying the condition (11) and coinciding on $K^{2}$. Put

$$
z^{3}=\omega_{1}\left(f_{0}, f_{1}, K^{3}\right), \quad z^{2}=\omega_{0}\left(f_{0}, K^{2}\right)=\omega_{0}\left(f_{1}, K^{2}\right)
$$

In virtue of Theorem 3 the mappings $f_{0}$ and $f_{1}$ are equivalent then and only then, when

$$
\begin{equation*}
z^{3} \widetilde{\nabla}^{2} u^{1} \times z^{2} \tag{12}
\end{equation*}
$$

The class of $\nabla$-homologies, to which the $\nabla$-cycle $Z^{3}$ belongs, is determined by the integer $z^{3}\left(K^{3}\right)$ [cf. (7)]. The class of usual homolozies $Z^{0}$, corresponding to this class, is also determined by an integer, namely by the index of the nul8*
dimensional complexes entering into $Z^{0}$. Thus we may simply take it that $Z^{0}$ is an integer and that it coincides with $Z^{3}\left(K^{3}\right)$. We denote the class of usual homologies, corresponding to the class of $\nabla$-homologies containing the $\nabla$-cycle $u^{1}$, by $U^{2}$; since $u^{1}$ is an arbitrary $\nabla$-cycle, $U^{2}$ is an arbitrary element of the group $B^{2}$. Finally, to the class of $\nabla$-homologies containing $z^{2}$ corresponds, the class of usual homologies $Z^{1}$ containing $f_{i}^{-1}(p)$ [cf. A)]. In usual homologies the relation (12) may be thus written in the form

$$
Z^{0}=2 I_{K^{3}}\left(U^{2}, Z^{1}\right)
$$

In virtue of the arbitrariness of the class of homologies $U^{2}$ the right-hand side of the last relation is an arbitrary number divisible by the number $2 \lambda$. Thus the mappings $f_{0}$ and $f_{1}$ are equivalent then and only then, when the number $z^{3}\left(K^{3}\right)$ is divisible by $2 \lambda$; at the same time the number $z^{3}\left(K^{3}\right)$ may assume, for a given $f_{0}$, an arbitrary integral value and, consequently, there exist exactly $2 \lambda$ non-equivalent among each other mappings $f$ satisfying the condition (11). The proposition C) is thus proved.

Example. Let $S^{2}$ be the metrical two-dimensional sphere and $p$ and $q$ two its diametrically opposite points. Denote by $\varphi_{\alpha}$ the mapping of the sphere $S^{2}$ on itself, obtained by means of a rotation of the sphere $S^{2}$ by the angle $\alpha$ about the axis $p q$. Denote by $S^{1}$ the circumference with the parameter $t, 0 \leqslant t \leqslant 2 \pi$, introduced on it. The topological product of $S^{2}$ and $S^{1}$ denote by $K^{3}$. Every point $y \in K^{3}$ is given by a pair $y=x \cdot t, x \in S^{2}, 0 \leqslant t \leqslant 2 \pi$. Define the mapping $\varphi$ of the manifold $K^{3}$ on itself by putting $\varphi(x \cdot t)=\varphi_{t}(x) \cdot t$. Define, further, the mapping $f_{0}$ of the manifold $K^{3}$ on the sphere $S^{2}$ by putting $f_{0}(x \cdot t)=x$. Define a second mapping $f_{1}$ by putting $f_{1}=f_{0}$. Then we obtain two mappings $f_{0}$ and $f_{1}$ of the manifold $K^{3}$ on $S^{2}$ such that the complete original of the point $p$ under both mappings is $p \cdot S^{1}$. The number $\lambda$ for the cycle $p \cdot S^{1}$ is easily seen to be equal to 1 [cf. C)], and hence there exist exactly two non-equivalent mappings satisfying the condition $f^{-1}(p) \sim p \cdot S^{1}$. These two mappings are precisely $f_{0}$ and $t_{1}$.

The mappings $f_{0}$ and $f_{1}$ are completely equipollent, since the mapping $\varphi$ is an homoeomorphism. Thus there is no possibility to establish in a natural manner a correspondence between the classes of mappings and the co-sets of the group $B_{\nabla}^{3}$ with respect to the subgroup $2 B_{\nabla}^{1} \times Z^{2}$ [cf. § 3, C)], but it is necessary to choose arbitrarily that mapping, which corresponds to the co-set $2 B_{\nabla}^{1} \times Z^{3}$.

From the fact that the mappings $f_{0}$ and $f_{1}$ are not equivalent follows that the mapping $\varphi$ is not equivalent to the identical mapping, whereas the mapping $\varphi$ and the identical mapping are homologically equivalent.

## § 5. The mappings of the four-dimensional complex into the two-dimensional sphere

By $K^{4}$ we shall denote here the four-dimensional complex and by $K^{r}$ the subcomplex of the complex $K^{4}$ composed of all simplexes of the complex $K^{4}$ of dimensionalities not exceeding $r$. By $S^{2}$ we shall denote the two-dimensional sphere.

In § 2 was shown that the mapping $t$ of the complex $K^{2}$ into $S^{2}$ may be then and only then extended to the complex $K^{3}$, when $\omega_{0}\left(f, K^{2}\right)$ is a $\nabla$-cycle in $\left.K^{3}[c f . \S 2, G)\right]$. Here we shall solve the question on the extension of the mapping $t$, defined on $K^{2}$, to the whole complex $K^{4}$.

Theorem 4. Let $f$ be a certain mapping of the complex $K^{2}{ }^{\text {a }}$ into the sphere $S^{2}$. The mapping $f$ may be extended to the whole complex $K^{4}$ then and only then, when $\omega_{0}\left(f, K^{2}\right)$ is a $\nabla$-cycle in $K^{4}$, satisfying the condition

$$
\begin{equation*}
\left.\omega_{0}\left(t, K^{2}\right) \times \omega_{0}\left(f, K^{2}\right) \widetilde{\nabla}^{0} \quad[\mathrm{cf.} \S 2, \mathrm{~F})\right] . \tag{1}
\end{equation*}
$$

Proof. Suppose that the mapping $f$ is already defined on the complex $K^{4}$. Without reducing the generality we may suppose that $f$ is a simplicial mapping of a certain simplicial subdivision of the complex $K^{4}$. Let $p^{0}$ and $p^{1}$ be two inner points of a certain simplex from the assumed triangulation of the sphere $S^{2}$. If $T^{r}$ is a certain orientated $r$-dimensional simplex of the complex $K^{4}$, then we denote by $P^{i}\left(T^{r}\right)$ the complete original of the point $p^{i}$ in the simplex $T^{r}$ under the mapping $f$. Put, for shortness,

$$
\begin{equation*}
\sum_{T^{r-1} \in \dot{T}^{r}} P^{i}\left(T^{r-1}\right)=P^{i}\left(\dot{T}^{\prime}\right) \tag{2}
\end{equation*}
$$

From the fact that the three-dimensional sphere $\dot{T}^{4}$ bounds in $K^{4}$ the simplex $T^{4}$ follows $\omega_{1}\left(f, \dot{T}^{4}\right)=0$, or, in other words,

$$
\begin{equation*}
V_{\dot{T}^{4}}\left(P^{0}\left(\dot{T}^{4}\right), P^{1}\left(\dot{T}^{4}\right)\right)=0 \tag{3}
\end{equation*}
$$

Put

$$
\begin{equation*}
\omega_{0}\left(f, K^{2}\right)=z^{2} \tag{4}
\end{equation*}
$$

In virtue of the very definition of the $\nabla$-complex $z^{2}$ we have

$$
\begin{equation*}
z^{2}\left(T^{2}\right)=I\left(P^{i}\left(T^{2}\right)\right) \tag{5}
\end{equation*}
$$

Further,

$$
\begin{equation*}
P^{i}\left(T^{3}\right)^{\cdot}=P^{i}\left(\dot{T}^{3}\right) \tag{6}
\end{equation*}
$$

From the relations (3)-(6) follows the relation (1) (cf. P. C., theorem 2).
Suppose now that $f$ is defined on $K^{2}$ and satisfies the conditions of the theorem, i. e. that $\omega_{0}\left(f, K^{2}\right)=z^{2}$ is a $\nabla$-cycle in $K^{4}$ satisfying the condition (1). Since $\omega_{0}\left(f, K^{2}\right)$ is a $\nabla$-cycle in $K^{3}$, the mapping $f$ may be extended to $K^{3}$ [cf. §2, G)], but it may be done in different ways. Suppose that $f_{0}$ and $f_{1}$ are two such extensions, i. e. two mappings of the complex $K^{3}$, coinciding on $K^{2}$ with $f$ and, consequently, also one with another. Put

$$
\begin{equation*}
\omega_{0}\left(f_{0}, f_{1}, K^{3}\right)=y^{3}, \quad \omega_{1}\left(f_{i}, \dot{T}^{4}\right)=y_{i}^{4}\left(T^{4}\right) \tag{7}
\end{equation*}
$$

We shall show that

$$
\begin{equation*}
y_{1}^{4}-y_{0}^{4}=\nabla y^{3} . \tag{8}
\end{equation*}
$$

Let $\Delta$ be the aggregate of the two numbers 0 and 1 . Then is defined the mapping $f_{\Delta}$ of the complex $K^{3} \cdot \Delta$ into $S^{2}$ (cf. § 2 ). Let $T^{4}$ be a certain fourdimensional simplex from $K^{4}$. Consider the mapping $f_{\Delta}$ in application to the complex $\dot{T}^{4} \cdot \Delta$. If $x \in \dot{T}^{4}, x \in K^{2}$, then we have $f_{\Delta}(x \cdot 0)=f_{\Delta}(x \cdot 1)$. Identify in $K^{4} \cdot \Delta$ in one point every pair of points $x \cdot 0$ and $x \cdot 1$ and denote the com-
plex, so obtained from $\dot{T}^{4} \cdot \Delta$, by $L^{3}$. The mapping $f_{\Delta}$ may be now considered as defined on the complex $L^{3}$. It is easily seen that every two-dimensional cycle from $L^{3}$ is homologic to zero in $L^{3}$ to every modulus, and hence, in virtue of Lemma 3, there exists such a mapping $g$ of the complex $L^{3}$ into the three-dimensional sphere $S^{3}$ that $f_{\Delta}=\boldsymbol{v} g$ [cf. § 1, A)].

Denote by $E_{k}^{3}, k=0,1,2,3,4$, the faces of the simplex $T^{4}$. Consider now in the complex $L^{3}$ the following orientated three-dimensional spheres:

$$
\begin{gathered}
S_{k}^{3}=E_{k}^{3} \cdot 1-E_{k}^{3} \cdot 0 \\
S_{5}^{3}=\sum_{k=0}^{4} E_{k}^{3} \cdot 0=\dot{T}^{4} \cdot 0, \quad S_{6}^{3}=\sum_{k=0}^{4} E_{k}^{3} \cdot 1=\dot{T}^{4} \cdot 1
\end{gathered}
$$

Evidently we have

$$
S_{6}^{3}-S_{5}^{3}=\sum_{k=0}^{4} S_{k}^{3} .
$$

Thus we have

$$
\omega_{0}\left(g, S_{6}^{3}\right)-\omega_{0}\left(g, S_{5}^{3}\right)=\sum_{k=0}^{4} \omega_{0}\left(g, S_{k}^{4}\right)
$$

further,

$$
\omega_{0}\left(g, S_{n}^{3}\right)=\omega_{1}\left(f_{\Delta}, S_{n}^{3}\right), \quad n=0,1, \ldots, 6 \quad[c f . \S 1,(20)] .
$$

Consequently,

$$
\omega_{1}\left(f_{\Delta}, S_{6}^{3}\right)-\omega_{1}\left(f_{\Delta}, S_{5}^{3}\right)=\sum_{k=0}^{4} \omega_{1}\left(f_{\Delta}, S_{k}^{3}\right)
$$

and this is the relation (8), written explicitly.
The complete original of the point $p^{i}$ in the simplex $T^{r}$ under the mapping $f_{j}$ we denote by $P_{j}^{i}\left(T^{r}\right)$. For shortness introduce the notations:

$$
\sum_{T^{r-1} \in \dot{T}^{r}} P_{j}^{i}\left(T^{r-1}\right)=P_{j}^{i}\left(\dot{T}^{r}\right) .
$$

Then we have

$$
\begin{align*}
P_{j}^{i}\left(T^{3}\right)^{\cdot} & =P_{j}^{i}\left(\dot{T}^{3}\right)  \tag{9}\\
z^{2}\left(T^{2}\right) & =I\left(P_{j}^{i}\left(T^{2}\right)\right) \tag{10}
\end{align*}
$$

From the relations (9), (10) and (7) follows that

$$
\begin{align*}
\left(z^{2} \times z^{2}\right)_{0}\left(T^{4}\right) & =V_{\dot{T}^{*}}\left(P_{0}^{0}\left(\dot{T}^{4}\right), P_{0}^{1}\left(\dot{T}^{4}\right)\right)=y_{0}^{4}\left(T^{4}\right)  \tag{11}\\
\left(z^{2} \times z^{2}\right)_{1}\left(T^{4}\right) & =V_{\dot{T}^{*}}\left(P_{1}^{0}\left(\dot{T}^{4}\right), P_{1}^{1}\left(\dot{T}^{4}\right)\right)=y_{1}^{4}\left(T^{4}\right) \tag{12}
\end{align*}
$$

In the relation (11) the product $z^{2} \times z^{2}$ is calculated by means of the auxiliary complexes $P_{0}^{i}\left(T^{F}\right)$, and in the relation (12) the product $z^{2} \times z^{2}$ is calculated by means of the auxiliary complexes $P_{1}^{i}\left(T^{i}\right)$. Although the so obtained products are homologic, they need not coincide.

Suppose now that the mapping $f$ defined on $K^{2}$ and satisfying the conditions of the theorem is in some way extended by means of the mapping $f_{0}$ to $K^{3}$.

Since $\left(z^{2} \times z^{2}\right)_{0} \widetilde{\nabla}^{0}$,

$$
y_{0}^{4}=\left(z^{2} \times z^{2}\right)_{0}=\nabla u u^{3} .
$$

Choose now the mapping $f_{1}$, coinciding with $f_{0}$ on $K^{2}$ and such that $\omega_{1}\left(f_{0}, f_{1}, K^{3}\right)=$ $\left.=-u^{3}\left[\mathrm{cf} . \S 2, \mathrm{~A}^{\prime}\right)\right]$. Then, in virtue of (8), we have $z_{1}^{4}=0$, and this means that the mapping $f$ of each sphere $\dot{T}^{4}$ may be extended to the simplex $T^{4}[\mathrm{cf} .(7)]$, i. e. we obtain an extension of the mapping $t$ to the whole complex $K^{4}$. Theorem 4 is thus proved.

Let us now formulate the obtained result in terms of usual homologies in the case when $K^{4}$ is a four-dimensional orientable manifold.
A) Let $f$ be a mapping of the orientated four-dimensional manifold $K^{4}$ into the two-dimensional orientated sphere $S^{2}$. Without reducing the generality we may assume that the mapping $f$ is a simplicial mapping of a certain simplicial subdivision of the complex $K^{4}$. Let $p$ be a certain inner point of a simplex from the assumed triangulation of the sphere $S^{2}$ and $f^{-1}(p)$ its complete original in $K^{4}$ under the mapping $f$. It is easily shown that for two equivalent mappings $f_{0}$ and $f_{1}$ we have $f_{0}^{-1}(p) \sim f_{1}^{-1}(p)$. If $C^{2}$ is a class of usual two-dimensional homologies, then for the existence of a mapping $f$ satisfying the condition $f^{-1}(p) \in C^{2}$ it is necessary and sufficient that $I_{K^{4}}\left(C^{2}, C^{2}\right)=0$.

Let $c^{2}$ be a certain cycle from $C^{2}$. Denote by $z^{2}\left(T^{2}\right)$ its index of intersection with $T^{2}$. Then the class of homologies $C^{2}$ corresponds to the class of $\nabla$-homologies containing $z^{2}$. In order that there should exist a mapping $f$ satisfying the condition $f^{-1}(p) \in C^{2}$, it is necessary and sufficient that there should exist a mapping $f$ satisfying the condition $\omega_{0}\left(t, K^{2}\right) \widetilde{\nabla} z^{2}$. But for this it is necessary and sufficient that $z^{2} \times z^{2} \widetilde{\nabla}^{0}$. This last condition has in terms of usual homologies the form: $I_{K^{4}}\left(C^{2} \times C^{2}\right)=0$. The assertion A$)$ is thus proved.

It is of interest to note the following:
Let $K^{n+2}$ be a complex of dimensionality $n+2$. Denote by $K^{r}$ the aggregate of all its simplexes of dimensionalities not exceeding $r$. Denote by $S^{n}$ the $n$-dimensional orientated sphere. If $f$ is a mapping of the complex $K^{n}$ into $S^{n}$, then we introduce, in the same way as in G), § 2 , a $\nabla$-complex $\omega_{0}\left(f_{1}, K^{n}\right)$ of dimensionality $n$, characterizing the mapping $f$. Namely, two mappings $f_{0}$ and $f_{1}$ are then and only then equivalent, when $\omega_{0}\left(f_{0}, K^{n}\right) \widetilde{\nabla}_{0}\left(f_{1}, K^{n}\right)$. Further, in order that the mapping $f$, defined on $K^{n}$, could be extended to $K^{n+1}$, it is necessary and sufficient that $\omega_{0}\left(f, K^{n}\right)$ should be a $\nabla$-cycle in $K^{n+1}$. If this condition is satisfied, then there arises the question on the possibility of extension of the mapping $f$ from the complex $K^{n}$ to the whole complex $K^{n+2}$. It turns out that the condition of such an extension is expressed by the demand that a certain $(n+2)$-dimensional $\nabla$-cycle from $K^{n+2}$ should be homologic to zero to the modulus 2 ; this cycle is determined up to homologies by the cycle $\omega_{0}\left(f, K^{n}\right)$, but its construction can not be carried out by means of the product operations in the complex $K^{n+2}$, which we possess. Thus we are lead to a new operation of a homological type.

# Классификация отображений трехмерного комплекса в двумерную сферу 

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Пусъ $\left\{f_{t}\right\}(0 \leqslant t \leqslant 1)$ - семейство отображений комплекса $K$ в комплексе $L$ такое, что функция $f_{t}(x)(x \in K)$ является непрерывной функцией пары аргументов $x$ и $t$. Говорят, что $f_{t}$ есть непрерывная деформация отображений комплекса $K$ в комплекс $L$. Два непрерывных отображения $g$ и $h$ комплекса $K$ в $L$ называются гомотопными или эквивалентными, если существует непрерывная деформация $f_{t}$ такая, что $f_{0}=g, f_{1}=h$. Классификация непрерывных отображений, с этой точки зрения, принадлежит к числу наиболее существенных задач современной топологии, она, однако, разрешена лишь в немногих частных случаях. Дана классификация отображений $n$-мерного комплекса $K^{n}$ в $n$-мерную сферу $S^{n}$ и, следовательно, в частности, классификация отображений $S^{n}$ в $S^{n},{ }^{1}$ и ${ }^{10}$. Имеется, далее, классификация отображений $(n+k)$-мерной сферы $S^{n+k}$ в $n$-мерную сферу $S^{n}$, при $k=1,2,{ }^{2}$ и $^{3}$. Имеется также классификация отображений $(n+1)$-мерного комплекса $K^{n+1}$ в $n$-мерную сферу $S^{n}{ }^{5}$.

В настоящей работе полностью излагаются мои результаты ${ }^{5}$ относительно классификации $K^{3}$ в $S^{2}$.

Пусть $f$ - симплициальное отображение ориентированной сферы $S^{3}$ в ориентированную сферу $S^{2}$ и $p^{i}(i=0,1)$ - две точки, выбранные внутри треугольников триангуляции сферы $S^{2}$. Тогда полный прообраз $f^{-1}\left(p^{i}\right)$ точки $p^{i}$ при отображении $f$, естественно, оказывается одномерным циклом в $S^{3}$. Коэфициент зацепления циклов $f^{-1}\left(p^{0}\right)$ и $f^{-1}\left(p^{1}\right)$ обозначим через

$$
\begin{equation*}
\omega_{1}\left(f, S^{3}\right) \tag{1}
\end{equation*}
$$

Hopf ${ }^{2}$, которому принадлежит описанная конструкция, показал, что число $\omega_{1}\left(f, S^{3}\right)$ не зависит от выбора точек $p^{i}(i=0,1)$ и является инвариантом класса отображений, т. е. для двух эквивалентных отображений $f$ и $g$

$$
\omega_{1}\left(f, S^{3}\right)=\omega_{1}\left(g, S^{3}\right)
$$

В предлагаемой работе мною доказано следующее:
Теорема 1. Для того, чтобы два отображения $f$ и $g$ трехмерной сферьи $S^{3}$ в двумерную сфлеру $S^{2}$ были эквивалентны, достаточно, чтобы

$$
\begin{equation*}
\omega_{1}\left(f, S^{3}\right)=\omega_{1}\left(g, S^{3}\right) \tag{2}
\end{equation*}
$$

Таким образом, в силу результата Hopf'а и моего, равенство (2) является необходимым и достаточным условием для эквивалентности отображений $f$ и $g$.

Для изложения результатов о классификации отображений $K^{3}$ в $S^{2}$ введем следующие обозначения:

Через $K^{3}$ будем обозначать трехмерный комплекс, а через $K^{r}$ - совокупность всех его симплексов размерности не выше $r$.

Пусть $p$ - некоторая точка из сферы $S^{2}$ и $f$ — такое отображение $K^{2}$ в $S^{2}$, что $f\left(K^{1}\right)$ не содержит точку $p$. Тогда определена степень отображения $f$ произвольного ориентированного двумерного симплекса $T^{2}$ из $K^{2}$ в точке $p$, ее мы обозначим через $z^{2}\left(T^{2}\right)$. Функция $z^{2}$ является $\nabla$-комплексом ${ }^{7}$ в $K^{2}$, этот $\nabla$-комплекс обозначим через

$$
\begin{equation*}
\omega_{0}\left(f, K^{2}\right) \tag{3}
\end{equation*}
$$

Пусть $f$ и $g$ - два отображения комплекса $K^{3}$ в сферу $S^{2}$, совпадающие на $K^{2}$. Пусть, далее, $T^{3}$ - произвольный ориентированный симплекс из $K^{3}$ размерности 3. Рассмотрим комплекс $P$, составленный из двух экземпляров [ $T^{3}$ ] и $\left\{T^{3}\right\}$ симплекса $T^{3}$, и определим отображение $\varphi$ комплекса $P$ в $S^{2}$, считая его совпадающим с $f$ на $\left[T^{3}\right]$ и с $g$ на $\left\{T^{3}\right\}$. Так как отображения $f$ и $g$ совпадают на $K^{2}$, то для каждой точки $x$, принадлежащей границе $\dot{T}^{3}$ симплекса $T^{3}$, имеем

$$
\varphi([x])=\varphi(\{x\})
$$

Отождествим теперь в одну каждую пару точек $[x],\{x\}$ при $x \in \dot{T}$, тогда комплекс $P$ превратится в трехмерную сферу $S^{3}$. Отображение $\varphi$ теперь можно рассматривать как отображение сферы $S^{3}$ в $S^{2}$. Положим

$$
z^{3}\left(T^{3}\right)=\omega_{1}\left(\varphi, S^{3}\right)
$$

[см. (1)]. Функция $z^{2}$ является трехмерным $\nabla$-комплексом ${ }^{7}$ в $K^{3}$, ее мы обозначим через

$$
\begin{equation*}
\omega_{1}\left(f, g, K^{3}\right) \tag{4}
\end{equation*}
$$

Whitney ${ }^{10}$ доказал, что два отображения $f$ и $g$ комплекса $K^{2}$ в $S^{2}$ тогда и только тогда эквивалентны, когда

$$
\begin{equation*}
\omega_{0}\left(f, K^{2}\right) \widetilde{\nabla}_{0}\left(g, K^{2}\right) \tag{5}
\end{equation*}
$$

[см. (3)]. Доказывается, далее, без труда, что отображение $f$ комплекса $K^{2}$, в $S^{2}$ тогда и только тогда можно распространить в отображение всего комплекса $K^{3}$, когда

$$
\begin{equation*}
\omega_{0}\left(f, K^{2}\right) \tag{6}
\end{equation*}
$$

является $\nabla$-циклом в $K^{3}$.
Для того, чтобы решить вопрос об эквивалентности отображений $f$ и $g$. комплекса $K^{3}$ в сферу $S^{2}$, его нужно прежде всего решить для тех же отображений, рассматриваемых на $K^{2}$ [см. (5)]. Если отображения $f$ и $g$ не эквивалентны уже на $K^{2}$, то, тем более, они не эквивалентны на $K^{3}$. Если же отображения $f$ и $g$ эквивалентны на $K^{2}$, то отображение $g$ можно заменить. эквивалентным ему и совпадающим с $f$ на $K^{2}$. Таким образом, вопрос сводится. к выяснению, эквивалентны или нет два отображения $f$ и $g$, совпадающие на $K^{2}$. Вопрос этот решается следующей теоремой:

Теорема 2. Пусть $f u g$-два отображения комплекса $K^{3}$ в соберу $S^{2}$, совпадаюющие на $K^{2}$. Положим

$$
z^{2}=\omega_{0}\left(g, K^{2}\right)=\omega_{0}\left(f, K^{2}\right)[\text { см. (3) }], z^{3}=\omega_{1}\left(f, g, K^{3}\right)[\text { см. (4) }]
$$

Тогда $z^{2}$ есть $\nabla$-иикл в $K^{3}$ [см. (6)], $z^{3}$ есть также $\nabla$-иикл в $K^{3}$, так как $K^{3}$ не имеет симплексов размерности 4. Оказывается, что отображе-

ния $f$ и $g$ тогда и только тогда эквивалентны, когда в $K^{3}$ существует $\nabla$-иикл $x^{1}$ размерности 1, удовлетворяющий условию

$$
z^{3} \widetilde{\nabla} 2 x^{1} \times z^{2}
$$

В случае, когда $K^{3}$ есть ориентируемое многообразие, теорема 2 может быть формулирована в форме теоремы 3. Для формулировки ее напомним следующие известные факты.

Пусть $x$ и $y$-два одномерных слабо гомологичных нулю цикла из трехмерного ориентированного многообразия $K^{3}$. Пусть, далее, $c$ - некоторый двумерный комплекс из $K^{3}$, граница которого есть $\alpha x$, где $a$ - натуральное число. Число $\frac{1}{a} I(c, y)$ называется коэфициентом зацепления циклов $x$ и $y$ (здесь $I$ означает индекс пересечения). Коэфициент зацепления, так определенный, является инвариантом циклов $x$ и $y$, а дробная его часть- инвариантом классов гомологий $X$ и $Y$, к которым циклы эти принадлежат.

Если $u$ есть некоторый слабо не гомологичный нулю одномерный цикл из трехмерного ориентированного многообразия $K^{3}$, то существует в $K^{3}$ двумерный цикл $v$ такой, что $I(u, v)$ есть положительное число. Индекс пересечения $I(u, v)$ является инвариантом классов гомологий $U$ и $V$, к которым принадлежат взятые циклы:

$$
I(u, v)=I(U, V)
$$

Теорема 3. Пусть $K^{3}$ - ориентированное трехмерное многообразие и $f$ - его симплициальное отображение в ориентированную двумерную сбкру $S^{2}$. Выберем две точки $p^{i}(i=0,1)$, принадлежащие внутренностям треугольников триангуляции сблеры $S^{2}$. Тогда полный прообраз $f^{-1}\left(p^{i}\right)$ точки $p^{i}$, естественно, оказывается одномерным циклом из $K^{3}$. Оба так полученных цикла принадлежат одному и тому же классу гомологий $Z^{1}$. Оказывается, что класс $Z^{1}$ является инвариантом класса $F$ отображений, к которому принадлежит отображение $f$. Оказывается, далее, что при заданном классе гомологий $Z^{1}$ всегда можно найти отображение $f$ такое, чтоб́ы оба возникающих из него цикла принадлежали $Z^{1}$.

Далее, будем различать два случая:

1) Если класс $Z^{1}$ имеет конечный порядок, т. е. циклы $f^{-1}\left(p^{i}\right)$ слабо гомологичны нулю, то коэфициент зацепления этих циклов является инвариантом класса отображений $F$, и при заданном классе $Z^{1}$ этот коэфициент зацепления является единственным инвариантом, т. е. определяет $F$. Целую часть этого инварианта при заданном $Z^{1}$ можно выбирать произвольно, подбирая надлежащим образом $F$, в то время как дробная его часть является инвариантом класса $Z^{1}$.
2) Если класс гомологий $Z^{1}$ свободный, т. е. циклы $f^{-1}\left(p^{i}\right)$ слабо не гомологичны нулю, то обозначим через $\lambda$ минимальное положительное значение, которое может принимать число $I\left(f^{-1}\left(p^{0}\right), x^{2}\right)$, где $x^{2}$ - произвольный двумерный цикл из $K^{3}$. Тогда при заданном классе $Z^{1}$ существует ровно $2 \lambda$ классов отображений $f$, для которых $f^{-1}\left(p^{0}\right) \in Z^{1}$.

В работе решается также один вопрос об отображениях четырехмерного комплекса $K^{4}$ в двумерную сферу $S^{2}$. Через $K^{r}$ попрежнему будем обозначать комплекс, составленный из всех симплексов комплекса $K^{4}$ размерности не больше $r$.

Если $f$ есть отображение комплекса $K^{2}$ в $S^{2}$, то для распространения этого отображения на комплекс $K^{3}$, как ранее было отмечено, необходимо и достаточно, чтобы $\nabla$-комплекс $\omega_{0}\left(f, K^{2}\right)$ [см. (3)] был циклом в $K^{3}$. Теорема 4 решает вопрос о возможности распространения отображения $f$ на весь комплекс $K^{4}$.

Теорема 4. Пусть $K^{4}$ - четырехмерный комплекс и $f$-отображение комплекса $K^{2}$ в сфјеру $S^{2}$. Для того, чтобы отображение $f$ можно было распространить на весь комплекс $K^{4}$, необходимо и достаточно, чтобы $\nabla$-комплекс $z^{2}=\omega_{0}\left(f, K^{2}\right)\left[\right.$ см. (3)] был $\nabla$-циклом в $K^{4}$ и чтобы

$$
z^{2} \times z^{2} \widetilde{\nabla}^{0}
$$

Для случая, когда $K^{4}$ есть ориентируемое многообразие, теорема 4 приобретает вид:

Теорема 5. Пусть $K^{4}$ - ориентированное многообразие и $f$-его симплициальное отображение в ориентированную двумерную соберу $S^{2}$. Выберем некоторую точку $p$ из внутренности какого-либо треугольникп триангуляии сбқеры $S^{2}$. Тогда $f^{-1}(p)$, естественно, оказывается циклом размерности 2 из $K^{4}$. Если $Z^{2}$ есть некоторый класс двумерных гомологичных между собой циклов из $K^{4}$, то для того, чтобы существовало отображение $f$, удовлетворяющее условию $f^{-1}(p) \in Z^{2}$, необходимо и достаточно, чтобы

$$
I\left(Z^{2}, Z^{2}\right)=0
$$


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    ${ }^{7}$ L. Pontrjagin, Products in complexes, see the preceding paper, p. 321.

[^2]:    8 N. Bruszlinsky, Stetige Abbildungen und Bettische Gruppen, Mathematische Annalen, 109, (1934).
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