PIECEWISE LINEAR TOPOLOGY

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1. Introduction

The piecewise linear category offers a rich structural setting in which to study many of the problems that arise in geometric topology. The first systematic accounts of the subject may be found in [2] and [63]. Whitehead's important paper [63] contains the foundation of the geometric and algebraic theory of simplicial complexes that we use today. More recent sources, such as [30], [50], and [66], together with [17] and [37], provide a fairly complete development of *PL* theory up through the early 1970's. This chapter will present an overview of the subject, drawing heavily upon these sources as well as others with the goal of unifying various topics found there as well as in other parts of the literature. We shall try to give enough in the way of proofs to provide the reader with a flavor of some of the techniques of the subject, while deferring the more intricate details to the literature. Our discussion will generally avoid problems associated with embedding and isotopy in codimension 2. The reader is referred to [12] for a survey of results in this very important area.

2. Basic Definitions and Terminology.

Simplexes. A *simplex of dimension* p (a p-*simplex*) σ is the convex closure of a set of $(p+1)$ geometrically independent points $\{v_0, \ldots, v_p\}$ in euclidean n-space $\mathbb{R}^n$. That is, each point x of σ can be expressed uniquely as $\sum t_i v_i$, where $0 \le t_i \le 1$ for $0 \le i \le p$ and $\sum t_i = 1$. (This is equivalent to requiring linear independence of the set of vectors $\{v_1 - v_0, \ldots, v_p - v_0\}$.) The v_i's are the *vertices* of σ; the t_i's are the *barycentric coordinates* of x in σ. We say that σ is *spanned* by its vertices, and write $\sigma = v_0 v_1 \cdots v_p$. The point $\beta(\sigma) = \sum \frac{1}{p+1} v_i$ is the *barycenter* of σ. A simplex τ spanned by a subset of the vertices is called a *face* of σ, written $\tau < \sigma$.

Simplicial Complexes. A collection K of simplexes in $\mathbb{R}^n$ is called a *(simplicial) complex* provided

(1) if $\sigma \in K$ and $\tau < \sigma$, then $\tau \in K$,
(2) if $\sigma, \tau \in K$, then $\sigma \cap \tau < \sigma$ and $\sigma \cap \tau < \tau$, and
(3) K is *locally finite*; that is, given $x \in \sigma \in K$, then some neighborhood of x meets only finitely many τ in K. [1]

Simplicial complexes K and L are *isomorphic*, $K \cong L$, if there is a face-preserving bijection $K \leftrightarrow L$. The subset $|K| = \bigcup\{\sigma : \sigma \in K\}$ of $\mathbb{R}^n$ is called the *polyhedron* of K. Property (3) ensures that a subset of A of $|K|$ is closed in $|K|$ iff $A \cap \sigma$ is closed in σ for all $\sigma \in K$. That is, the weak topology on $|K|$ with respect to the collection K of simplexes coincides with the subspace topology on $|K|$. A complex L is a *subcomplex* of a complex K, $L < K$, provided $L \subseteq K$ and L satisfies (1) − (3). If $L < K$, then $|L|$ is a closed subset of $|K|$. If $A \subseteq |K|$ and $A = |L|$, for some $L < K$, we shall occasionally write $L = K|A$. For any complex K and any $p \ge 0$, we have the subcomplex $K^{(p)} = \{\sigma \in K : \dim \sigma \le p\}$ called the *p-skeleton of* K. For a simplex σ, the *boundary subcomplex* of σ is the subcomplex $\dot{\sigma} = \{\tau < \sigma : \tau \neq \sigma\}$. The *interior* of σ, $\mathring{\sigma} = \sigma - |\dot{\sigma}|$.

[1] This is not a standard requirement, but we shall find it convenient for the purposes of this exposition. The astute reader, however, may notice an occasional lapse in our adherence to this restriction.

Subdivisions. A complex K_1 is a *subdivision of* K, $K_1 \prec K$, provided $|K_1| = |K|$ and each simplex τ of K_1 lies in some simplex σ of K. We write $(K_1, L_1) \prec (K, L)$ to denote that K_1 is a subdivision of K inducing a subdivision L_1 of L. If σ is a simplex, $L \prec \dot\sigma$, and $x \in \overset{\circ}{\sigma}$, then the subdivision $K = L \cup \{xw_0w_1 \cdots w_k : w_0w_1 \cdots w_k \in L\}$ is obtained from L by *starring* σ at x over L. A *derived* subdivision of K is one that is obtained by the following inductive process: assuming $K^{(p-1)}$ has been subdivided as a complex L and σ is a p-simplex of K, choose a point $\hat\sigma$ in $\overset{\circ}{\sigma}$ and star σ at $\hat\sigma$ over $L||\dot\sigma|$, thereby obtaining a subdivision of $K^{(p)}$. If we choose each $\hat\sigma = \beta(\sigma)$, the resulting derived subdivision is called the *first barycentric subdivision of* K and is denoted by K^1. More generally, K^r will note the rth-barycentric subdivision of K: $K^r = (K^{r-1})^1$ (where $K^0 = K$). There are relative versions of this process: if L is a subcomplex of K, inductively choose points $\hat\sigma \in \operatorname{int}\sigma$ for $\sigma \notin L$. The result is a *derived subdivision of* K mod L.

A subcomplex L of a complex K is *full* in K, $L \lhd K$, if a simplex σ of K belongs to L whenever all of its vertices are in L. If L is a subcomplex of K and K' is a derived subdivision of K, then the subcomplex L' of K' subdividing L is full in K'.

Any two subdivisions $L \prec K$ and $J \prec K$ have a common subdivision. The set $\mathcal{C} = \{\sigma \cap \tau : \sigma \in L, \tau \in J\}$ is a collection of convex linear cells that forms a *cell complex*: given $C,\ D \in \mathcal{C}$, $C \cap D \in \mathcal{C}$ is a face of each. $\mathcal{C}$ can be subdivided into simplexes by induction using the process described above. $\mathcal{C}$ can also be subdivided into simplexes without introducing any additional vertices, other than those in the convex cells of $\mathcal{C}$, by a similar process: order the vertices of $\mathcal{C}$ and, assuming the boundary of a convex cell C of $\mathcal{C}$ has been subdivided, choose the first vertex v of C and form simplexes $vw_0 \cdots w_p$ where $w_0 \cdots w_p$ is a simplex in the boundary of C not containing v. A consequence of the latter construction is that if subdivisions $L \prec K$ and $J \prec K$ share a common subcomplex M, then a common subdivision of L and J can be found containing M as a subcomplex. Finally, if $L < K$ and $L' \prec L$, then there is a subdivision $K' \prec K$ such that $K'||L| = L'$: proceed inductively starring a p-simplex σ of K not in L at an interior point x over ${K^{(p-1)}}'||\dot\sigma|$.

Stars and Links. Given a complex K and a simplex $\sigma \in K$, the *star* and *link* of σ in K are the subcomplexes $\operatorname{St}(\sigma, K) = \{\tau \in K :$ for some $\eta \in K, \sigma, \tau < \eta\}$, and $\operatorname{Lk}(\sigma, K) = \{\tau \in \operatorname{St}(\sigma, K) : \tau \cap \sigma = \emptyset\}$, respectively. We let $\operatorname{st}(\sigma, K) = |\operatorname{St}(\sigma, K)|$ and $\operatorname{lk}(\sigma, K) = |\operatorname{Lk}(\sigma, K)|$. The *open star* of σ in K, $\overset{\circ}{\operatorname{st}}(\sigma, K) = \operatorname{st}(\sigma, K) - \operatorname{lk}(\sigma, K)$. One can easily show that the collection $\{\overset{\circ}{\operatorname{st}}(v, K^r) : v \in (K^r)^{(0)}, r = 0, 1, \ldots\}$ forms a basis for the open sets in $|K|$.

Simplicial and Piecewise Linear Maps. Given complexes K and L, a *simplicial map* $f\colon K \to L$ is a map (we still call) $f\colon |K| \to |L|$ such that for each $\sigma \in K$, $f|\sigma$ maps σ linearly onto a simplex of L. A simplicial map $f\colon K \to L$ is *nondegenerate* if $f|\sigma$ is injective for each $\sigma \in K$. A simplicial map is then determined by its restriction to the vertices of K. A map $f\colon |K| \to |L|$ is *piecewise linear* or *PL* if there are subdivisions $K' \prec K$ and $L' \prec L$ such that $f\colon K' \to L'$ is simplicial. Polyhedra $|K|$ and $|L|$ are *piecewise linearly* (or *PL*) *homeomorphic*, $|K| \cong |L|$, if they have subdivisions $K' \prec K$ and $L' \prec L$ such that $K' \cong L'$.

Simplicial Approximation Theorem. *If K and L are complexes and $f\colon |K| \to |L|$ is a continuous function, then there is a subdivision $K' \prec K$ and a simplicial map $g\colon K' \to L$ homotopic to f. Moreover, if $\epsilon\colon |L| \to (0, \infty)$ is continuous, then* *there are subdivisions $K' \prec K$ and $L' \prec L$ and a simplicial map $g\colon K' \to L'$ such that g is ϵ-homotopic to f; that is, there is a homotopy $H\colon |K| \times [0,1] \to |L|$ from f to g such that $\mathrm{diam}(H(x \times [0,1]) < \epsilon(f(x))$ for all $x \in |K|$.*

The proof of this theorem is elementary and can be found in a number of texts. (See, for example, [45] or [54].) The idea of the proof is to get an r such that for each vertex v of K^r, $f(\mathrm{st}(v, K^r)) \subseteq \overset{\circ}{\mathrm{st}}(w, L)$ for some vertex w of L. The assignment of vertices $v \mapsto w$, defines a vertex map $g\colon (K^r)^{(0)} \to L^{(0)}$ that extends to a simplicial map $g\colon K^r \to L$ homotopic to f (by a straight line homotopy). This can be done whenever K is finite. When K is not finite, one may use a *generalized barycentric subdivision* of K, constructed inductively as follows. Assuming J is a subdivision of $K^{(p-1)}$, and σ is a p-simplex of K, let K_σ be the subdivision of σ obtained by starring σ at $\beta(\sigma)$ over $J||\dot{\sigma}|$. Let $n = n_\sigma$ be a non-negative integer, and let K'_σ be the nth-barycentric subdivision of K_σ mod $L||\dot{\sigma}|$. It can be shown that any open cover of $|K|$ can be refined by $\{\mathrm{st}(v, K') : v \in (K')^{(0)}\}$ for some generalized barycentric subdivision K' of K.

To get the "moreover" part, start with a (generalized) rth barycentric subdivision L' of L such that vertex stars have diameter less than ϵ.

Generalized barycentric subdivisions can also be used to show that if U is an open subset of the polyhedron $|K|$ of a complex K, then U is the polyhedron of a complex J each simplex of which is linearly embedded in a simplex of K.

Combinatorial Manifolds. A *combinatorial n-manifold* is a complex K for which the link of each p-simplex is *PL* homeomorphic to either the boundary of an $(n-p)$-simplex or to an $(n-p-1)$-simplex. If there are simplexes of the latter type, they constitute a subcomplex ∂K of K, the *boundary* of K, which is, in turn, a combinatorial $(n-1)$-manifold without boundary. If K is a combinatorial n-manifold, then $|K|$ is a topological n-manifold, possibly with boundary $|\partial K|$.

Triangulations. A *triangulation* of a topological space X consists of a complex K and a homeomorphism $t\colon |K| \to X$. Two triangulations $t\colon |K| \to X$ and $t'\colon |K'| \to X$ of X are *equivalent* if there is a *PL* homeomorphism $h\colon |K| \to |K'|$ such that $t' \circ h = t$. A *PL* n-manifold is a space (topological n-manifold) M, together with a triangulation $t\colon |K| \to M$, where K is a combinatorial n-manifold. Such a triangulation will be called a *PL* triangulation of M or a *PL structure* on M. $\partial M = |\partial K|$ and $\mathrm{int}\, M = M - \partial M$. M is *PL n-ball* (respectively, *PL n-sphere*) if we can choose K to be an n-simplex (respectively, the boundary subcomplex of an $(n+1)$-simplex). In a similar manner we may define a triangulation $K > L$ of a pair $X \supseteq Y$, where Y is closed in X (or for a triad $X \supseteq Y, Z$, or n-ad, etc.).

A *PL* structure on a topological n-manifold M can also be prescribed by an *atlas* Σ on M, consisting of a covering $\mathcal{U}$ of open sets (*charts*) in M together with embeddings $\phi_U : U \to \mathbb{R}^n$, $U \in \mathcal{U}$, such that if $U, V \in \mathcal{U}$, then

$$\phi_V(\phi_U)^{-1}\colon \phi_U(U \cap V) \to \phi_V(U \cap V)$$

is piecewise linear. Here we assume that open subsets of $\mathbb{R}^n$ inherit triangulations from linear triangulations of $\mathbb{R}^n$ as described above. Two atlases Σ and Σ' are *equivalent* if there is a (topological) homeomorphism $h\colon M \to M$ such that the union of Σ and $h(\Sigma')$ forms an atlas, where $h(\Sigma')$ is the atlas consisting of the cover $\{h(U') : U' \in \mathcal{U}'\}$ and embeddings $\phi_{U'} h^{-1}$. An atlas Σ on M determines a *PL* triangulation of M as follows. If M is compact, cover M by a finite number of compact polyhedra obtained from a finite cover of open sets in Σ, and triangulate inductively. If M is not compact, then dimension theory provides a cover $\mathcal{X} = \mathcal{X}_0 \cup \mathcal{X}_1 \cup \ldots \cup \mathcal{X}_n$, subordinate to $\mathcal{U}$, such that the members of $\mathcal{X}_i$, $i = 0, 1, \ldots, n$, are mutually exclusive, compact polyhedra. One can then proceed as in the compact case. It is not difficult to show that atlases Σ and Σ' are equivalent if, and only if, the induced triangulations of M are equivalent.

One may also consider the problem of "triangulating" a diagram of polyhedra and *PL* maps; that is, subdividing all spaces so that each of the mappings in the diagram is simplicial. If the diagram forms a "one-way tree" in which each polyhedron is compact and is the domain of at most one mapping, then it is possible to use an inductive argument, based on the following construction, to triangulate the diagram. Given a simplicial mapping $f\colon K \to L$ and a subdivision $L' \prec L$, form the cell complex $\mathcal{C} = \{\sigma \cap f^{-1}(\tau) : \sigma \in K,\ \tau \in L'\}$, and subdivide $\mathcal{C}$ as a simplicial complex K' without introducing any new vertices, as above. Then $f\colon K' \to L'$ is simplicial.

If a diagram does not form a one-way tree, then it may not be triangulable, as a simple example found in [66] illustrates. Let $|K| = [-1, 1]$, $|L| = |J| = [0, 1]$, let $f\colon |K| \to |L|$ be defined by $f(x) = |x|$, and let $g\colon |K| \to |J|$ be defined by $g(x) = x$, if $x \geq 0$, and $g(x) = -x/2$, if $x \leq 0$. The problem is that there is a sequence $\{1/2, 1/4, 1/8, \ldots\}$ in $|L|$ such that $gf^{-1}(\frac{1}{2^i}) \cap gf^{-1}(\frac{1}{2^{i+1}}) \neq \emptyset$. In [9] it is shown that a two-way diagram $|J| \xleftarrow{g} |K| \xrightarrow{f} |L|$ can be triangulated provided it does not admit such sequences. (See [9] for a precise statement of the theorem and its proof.)

The *PL* Category. The *piecewise linear category*, *PL*, can now be described as the category whose objects are triangulated spaces, or, simply, polyhedra, and whose morphisms are *PL* maps. The usual cartesian product and quotient constructions can be carried out in *PL* with some care: the cartesian product of two polyhedra doesn't have a well-defined triangulation (since the product of two simplexes is rarely a simplex), and a complex obtained by an identification on the vertices of another complex may not give a complex with the expected (or desired) polyhedron. For example, identifying the vertices of a 1-simplex will not produce a complex with polyhedron homeomorphic to S^1, since the only simplicial map from a 1-simplex making this identification is a constant map. One must first subdivide the simplex (it takes two derived subdivisions). Either of the two processes described above for finding a common subdivision of two subdivisions of a complex may be used to triangulate the cartesian product of two complexes K and L. For example, one can inductively star the convex cells $\sigma \times \tau$, $(\sigma \in K,\ \tau \in L)$. Alternatively, one can order $K^{(0)} \times L^{(0)}$, perhaps using a lexicographic ordering resulting from an ordering of $K^{(0)}$ and $L^{(0)}$ separately, and inductively triangulate the convex linear cells $\sigma \times \tau$ $(\sigma \in K,\ \tau \in L)$ without introducing any new vertices.

Joins: Cones and Suspensions. The join operation is a more natural operation in *PL* than are products and quotients. Disjoint subsets A and B in $\mathbb{R}^n$ are *joinable* provided any two line segments joining points of A to points of B meet in at most a common endpoint (or coincide). If A and B are joinable then the *join* of A and B, $A * B$, is the union of all line segments joining a point of A to a point of B. We can always "make" A and B joinable: if $A \subseteq \mathbb{R}^m$ and $B \subseteq \mathbb{R}^n$, then $A \times 0 \times 0$ and $0 \times B \times 1$ are joinable in $\mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R} = \mathbb{R}^{m+n+1}$. We assume the convention that $A*\emptyset=\emptyset*A=A$. If $A\cap B=C\neq\emptyset$, then A and B are *joinable relative to* C if $A-C$ and $B-C$ are joinable and every line segment joining a point of $A-C$ and $B-C$ misses C. Then $A*B$ (rel C) $=[(A-C)*(B-C)]\cup C$ denotes the *reduced join of* A *and* B *relative to* C. For example, given a simplex $\sigma=v_0\cdots v_p$ and faces $\tau=v_0\cdots v_i$ and $\eta=v_j\cdots v_p$ with $j\leq i+1$, then $\sigma=\tau*\eta$ (rel $\tau\cap\eta$). Likewise, if K and L are finite complexes in $\mathbb{R}^n$ such that $|K|$ and $|L|$ are joinable, then we can define the *join complex*, $K*L=\{\sigma*\tau\subseteq\mathbb{R}^{m+n}:\sigma\in K \text{ and } \tau\in L\}$. For example, if σ is a simplex in a complex K, then $\mathrm{St}(\sigma,K)=\sigma*\mathrm{Lk}(\sigma,K)$. Unlike the case for products and quotients, triangulations of compact spaces X and Y induce a canonical triangulation of $X*Y$. An important artifact of the join construction is that the join of two spaces $A*B$ comes equipped with a *join parameter* obtained from a natural map $s\colon A*B\to[0,1]$ that maps each line segment in $A*B$ from a point of A to a point of B linearly onto $[0,1]$. When K and L are finite complexes, the map s is a simplicial map from $K*L$ onto the simplex $[0,1]$. With the aid of the join parameter, one can easily extend simplicial maps $f\colon H\to K$ and $g\colon J\to L$ between finite complexes to their joins, $f*g\colon H*J\to K*L$.

Two special cases of the join construction are the cone and suspension. Given a compact set X and a point v, the *cone* on X with vertex v, $C(X,v)=v*X$. We may also write $C(X)$ to denote $C(X,v)$. The *suspension* of X, $\Sigma(K)=S^0*K$, where S^0 is the 0-sphere. One defines cone and suspension complexes of a (finite) complex K similarly. As observed above, if v is a vertex of a complex K, then $\mathrm{St}(v,K)\cong v*\mathrm{Lk}(v,K)$. Using the join construction for simplicial maps, one can easily prove PL equivalence for stars of vertices.

Theorem 2.1. *Suppose that* X *is a polyhedron,* $x\in X$*, and* K_1 *and* K_2 *are equivalent triangulations of* X *containing* x *as a vertex. Then* $st(x,K_1)\cong st(x,K_2)$.

Proof. Without loss of generality we may assume that K_2 is a subdivision of K_1 so that $\mathrm{st}(x,K_2)\subseteq\mathrm{st}(x,K_1)$. Hence, for each point y of $\mathrm{lk}(x,K_2)$, there is a unique point $z\in\mathrm{lk}(x,K_1)$ such that $y\in x*z\subseteq x*\mathrm{lk}(x,K_1)=\mathrm{st}(x,K_1)$. Conversely, for each $z\in\mathrm{lk}(x,K_1)$ there is a unique $y\in\mathrm{lk}(x,K_2)$ such that $x*z\cap\mathrm{lk}(x,K_2)=y$. Moreover, if z is a vertex of $\mathrm{Lk}(x,K_1)$, then y is a vertex of $\mathrm{Lk}(x,K_2)$. Thus, we can get a simplicial isomorphism f from $\mathrm{Lk}(x,K_2)$ to a subdivision $\mathrm{Lk}(x,K_1)'$ of $\mathrm{Lk}(x,K_1)$ by extending the map above from the vertices of $\mathrm{Lk}(x,K_2)$ into $\mathrm{lk}(x,K_1)$. Extending further to the cones over x gives the desired equivalence.

As pointed out in [66] and [50] , the natural projection $\mathrm{lk}(x,K_2)\to\mathrm{lk}(x,K_1)$ along cone lines is not linear on the simplexes of K_2, although it does match up the simplexes of $\mathrm{Lk}(x,K_2)$ with those of the subdivision $\mathrm{Lk}(x,K_1)'$ of $\mathrm{Lk}(x,K_1)$. (This is the "Standard Mistake".)

The proof of the following important theorem can be found in [50] .

Theorem 2.2. *Suppose* B^p *(respectively,* S^p*) denotes a PL ball (respectively, sphere) of dimension* p*, then*

(1) $B^p*B^q=B^{p+q+1}$,
(2) $B^p*S^q=B^{p+q+1}$, *and*
(3) $S^p*S^q=S^{p+q+1}$.

For example, if K is a combinatorial n-manifold and σ is a p-simplex of K, then $\mathrm{st}(\sigma,K)\cong\sigma*\mathrm{lk}(\sigma,K)\cong B^n$.

An elementary argument shows that the join operation is associative. This implies, for example, that a k-fold suspension $\Sigma^k(X) = \Sigma(\Sigma(\cdots(\Sigma(X))\cdots))$ of a compact polyhedron X is *PL* homeomorphic to $S^{k-1} * X$. The proof of the following proposition is a pleasant exercise in the use of some of the ideas presented so far.

Proposition 2.3. *If X is a compact polyhedron, then*

$$C(X)\times[-1,1]\cong C((X\times[-1,1])\cup(C(X)\times\{-1,1\}))$$

by a homeomorphism that preserves $C(X)\times[-1,0]$ and $C(X)\times[0,1]$. In particular, if $J > J_+, J_-, J_0$ is a triangulation of

$$C(X)\times[-1,1]\supseteq C(X)\times[0,1], C(X)\times[-1,0], C(X)\times\{0\},$$

then

$$\begin{aligned}&(st(v,J), st(v,J_+), st(v,J_-), st(v,J_0))\cong\\ &(C(X)\times[-1,1], C(X)\times[0,1], C(X)\times[-1,0], C(X)\times\{0\})\end{aligned}$$

(where v is the vertex of $C(X)$).

Proposition 2.3 in turn may be applied to give a proof of a *PL* version of Morton Brown's Collaring Theorem [7] . A subpolyhedron Y of a polyhedron X is *collared* in X if Y has a neighborhood in X *PL* homeomorphic to $Y \times I$. Y is *locally collared* in X if each $x \in Y$ has a neighborhood pair (U, V) in (X, Y) such that $(U, V) \cong (V \times I, V \times \{0\})$.

Theorem 2.4. *If the subpolyhedron Y of X is locally collared in X, then Y is collared in X.*

Proof. Let $K > L$ be a triangulation of $X \supseteq Y$, and assume that for each vertex $v \in L$, $\mathrm{st}(v, L)$ lies in a collared neighborhood. That is, v has a neighborhood pair (U, V) *PL* homeomorphic to $(\mathrm{st}(v, L)\times I, \mathrm{st}(v, L)\times\{0\})$ $(I = [0, 1])$. By Proposition 2.3, we may assume that $U = \mathrm{st}(v, K)$. Let $X^+ = X \cup_{Y\times\{0\}} (Y \times [-1, 0])$. Then $U\cup_{V\times\{0\}}(V\times[-1,0])\cong V\times[-1,1]$ is a neighborhood of $v = (v, 0)$ in X^+, and $V \times [-1, 1] \cong v*(\mathrm{lk}(v, L)\times[-1, 1]\cup V\times\{-1, 1\})$. Let $\Sigma = \mathrm{lk}(v, L)\times[-1, 1]\cup V\times\{-1, 1\}$, and let $v' = (v, -\frac{1}{2})$. Then there is a homeomorphism $h_v : V\times[-1, 1] \to V\times[-1, 1]$ such that $h_v(v) = v'$, $h_v|\Sigma = \mathrm{id}$, and h_v sends each $v * z$, $z \in \Sigma$ "linearly" onto $v' * z$. In particular, h_v commutes with the projection map $V \times [-1, 1] \to V$.

Now let $K' > L'$ be a derived subdivision of $K > L$. Write $(L')^{(0)} = V_0 \cup V_1 \cup \cdots V_m$, where $V_i = \{\hat\sigma \in L' : \dim\sigma = i\}$. Then for $v_1, v_2 \in V_i$ $\mathrm{st}(v_1, K')\cap \mathrm{st}(v_2, K') \subseteq \mathrm{lk}(v_1, K')\cap\mathrm{lk}(v_2, K')$ so that h_{v_1} is the identity on $\mathrm{st}(v_1, K')\cap \mathrm{st}(v_2, K')$. Let $h_i : X^+ \to X^+$ be the *PL* homeomorphism satisfying $h_i = h_v$ on $\mathrm{st}(v, K')\cup_{\mathrm{st}(v,L')\times\{0\}}(\mathrm{st}(v, L')\times[-1, 0])$ for $v \in V_i$ and $h_i = \mathrm{id}$, otherwise. Then $h = h_m\circ\cdots h_1\circ h_0 : X^+ \to X^+$ is a homeomorphism that takes $(Y\times[-1, 0], Y\times\{0\})$ to $(Y \times [-1, -\frac{1}{2}], Y \times \{-\frac{1}{2}\})$. Hence, h^{-1} takes $Y \times [-\frac{1}{2}, 0]$ onto a neighborhood of Y in X.

Corollary 2.5. *Suppose that X is a PL n-manifold with boundary Y. Then Y is collared in X.*

Proof. Each $x \in Y$ has a neighborhood N such that $N \cong B^n$ and $N \cap Y \cong B^{n-1}$. Since S^{n-1} is collared in B^n, x has a neighborhood *PL* homeomorphic to $B^{n-1} \times [0, 1]$.

Join structures play an essential role in PL theory. They lie at the heart of many constructions and much of the structure theory. We conclude this section with three important examples.

Simplicial Mapping Cylinders. Suppose $f\colon K \to L$ is a simplicial map. (If K is not finite, assume additionally that $f^{-1}(v)$ is a finite complex for each vertex v of L.) Choose first derived subdivisions K' of K and L' of L such that $f\colon K' \to L'$ is still simplicial. For example, we can choose $L' = L^1$, the first barycentric subdivision of L, and for each $\sigma \in K$, choose a point $\hat\sigma \in \overset{\circ}{\sigma} \cap f^{-1}(\beta(f(\sigma)))$ at which to star σ. The *simplicial mapping cylinder of* f is the subcomplex of $L' * K'$,

$$C_f = \{\hat\tau_1\hat\tau_2\cdots\hat\tau_j * \hat\sigma_1\hat\sigma_2\cdots\hat\sigma_i |\ \tau_1 < \cdots \tau_j < f(\sigma_1), \sigma_1 < \cdots < \sigma_i \in K\} \cup L'.$$

Thus, a simplex of C_f is either in L' or is of the form $\alpha * \beta \in L' * K'$, where, for some $\tau \in L$ and $\sigma \in K$, $\alpha \subseteq \tau$, $\beta \subseteq \sigma$, and $\tau < f(\sigma)$. There is a natural projection $\gamma\colon C_f \to L$ defined by

$$\gamma(\hat\tau_1\hat\tau_2\cdots\hat\tau_j * \hat\sigma_1\hat\sigma_2\cdots\hat\sigma_i) = \hat\tau_1\hat\tau_2\cdots\hat\tau_j f(\hat\sigma_1)f(\hat\sigma_2)\cdots f(\hat\sigma_i).$$

Figure 2.1 illustrates the simplicial mapping cylinder of a simplicial map $f\colon \sigma \to \tau$ from a 2-simplex σ to a 1-simplex τ.

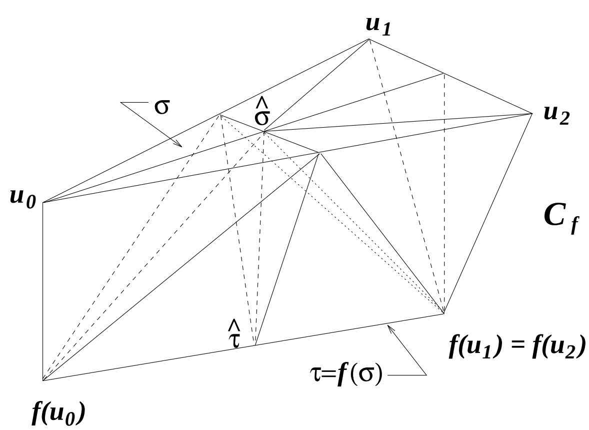


Fig. 2.1

As is shown in [15], the simplicial mapping cylinder C_f is topologically homeomorphic to the topological mapping cylinder $|K| \times I \cup_{f\times\{1\}} |L|$. If f is degenerate, however, any PL map $(|K| \times I) \coprod |L| \to C_f$ restricting to f on $|K| \times 1$ will fail to be one-to-one on $|K| \times [0, 1)$.

If $f\colon K \to L$ is the identity on a complex $H < K \cap L$, then one can also define the *reduced* simplicial mapping cylinder, a subcomplex of $L' * K'$ rel H, where L' and K' are first derives mod H:

$$\begin{aligned} C_f \text{ rel } H \quad = \{&\alpha * \hat\tau_1\hat\tau_2\cdots\hat\tau_j * \hat\sigma_1\hat\sigma_2\cdots\hat\sigma_i |\ \alpha < \tau_1 < \tau_2 < \cdots \tau_j < f(\sigma_1),\ \alpha \in H,\\ &\tau_1 \in L - H,\ \sigma_1 < \sigma_2 < \cdots < \sigma_i\} \cup L'.\end{aligned}$$

Suppose $f\colon X \to Y$ is a PL mapping between polyhedra. In light of the comment above, we may refer to "the" PL mapping cylinder M_f of f, obtained from triangulations K of X and L of Y under which $f\colon K \to L$ is simplicial. M_f is well-defined topologically, but its combinatorial structure will depend on K and L. If $f|A$ is an embedding for some subpolyhedron A of X, we may also define the reduced PL mapping cylinder M_f rel A.

Dual Subcomplexes. Given complexes $L < K$, let K' be the first barycentric subdivision of K mod L, and let $J = \{\sigma \in K' |\ \sigma \cap |L| = \emptyset\}$. Then J is the *dual* of L in K. In particular, if K is an n-complex and if $L = K^{(p)}$ is the p-skeleton of K, then J is called the *dual* $(n-p-1)$-*skeleton* of K, and is denoted by $\tilde{K}^{(n-p-1)}$. Whenever J is the dual of L in K, K' is isomorphic to a subcomplex of $L * J$, since every simplex of K' is either in L, in J, or is the join of a simplex of L and a simplex of J. It is occasionally useful to consider relative versions of duals. For example, if K is a combinatorial n-manifold with boundary ∂K, then the dual $(n-p-1)$-skeleton of K rel ∂K is the dual of $K^{(p)} \cup \partial K$.

Dual Cell Structures. Suppose K is a combinatorial n-manifold (possibly with boundary), and K' is a first derived subdivision. Given a p-simplex σ in K, $K'|\mathrm{lk}(\sigma, K)$ is naturally isomorphic to the subcomplex $\tilde{K}_\sigma = \{\hat{\tau}_1\hat{\tau}_2 \cdots \hat{\tau}_m : \sigma < \tau_1 < \cdots \tau_m \in K,\ \sigma \neq \tau_1\}$ of K'. Thus, $|\tilde{K}_\sigma| \cong S^{n-p-1}$ or B^{n-p-1}, and, hence, $\tilde{B}_\sigma = \hat{\sigma} * |\tilde{K}_\sigma|$ is a *PL* $(n-p)$-ball. $\tilde{B}_\sigma$ is the *dual cell* to σ, and the collection $\tilde{K}$ of dual cells is called the *dual cell complex* of K. $\tilde{K}$ satisfies the conditions:

(1) $\sigma < \tau$, whenever $\tilde{B}_\tau \subseteq \partial \tilde{B}_\sigma$, and

(2) $\tilde{B}_\sigma \cap \tilde{B}_\tau = \tilde{B}_\eta$, if $\eta = \sigma * \tau$ (rel $\sigma \cap \tau$) is a simplex of K, (and $= \emptyset$, otherwise).

Figure 2.2 illustrates cell-dual cell pairs for a 1-dimensional face σ of a 2-simplex τ.

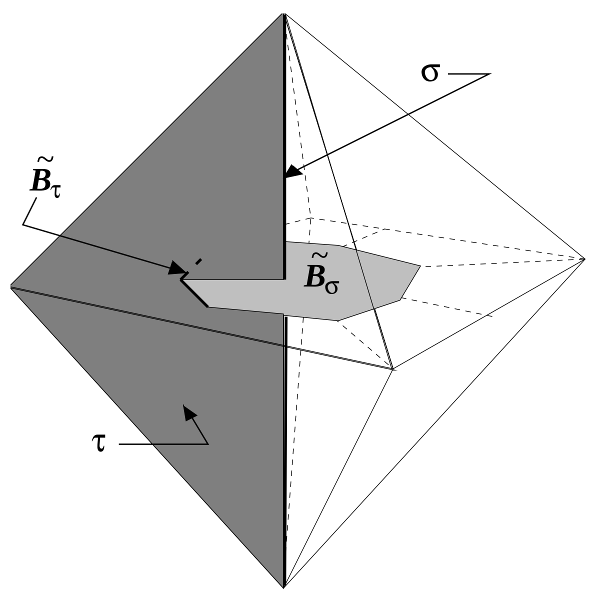


Fig 2.2

3. Regular Neighborhoods

Derived Neighborhoods. Given a subcomplex L of a complex K, the *simplicial neighborhood* of L in K is the subcomplex

$$\begin{aligned} N(L, K) &= \{\sigma : \sigma \in K, \sigma < \tau, \tau \cap |L| \neq \emptyset\} \\ &= \bigcup \{\mathrm{St}(v, K) : v \in L^{(0)}\}. \end{aligned}$$

Suppose $L \triangleleft K$. Let $C(L,K) = \{\sigma \in\ K : \sigma \cap |L| = \emptyset\}$, the *simplicial complement* of L in K, and let K' be a derived subdivision of K mod $L \cup C(L,K)$. Then $N(L,K')$ is a *derived neighborhood* of L in K. Any two derived neighborhoods corresponding to derived subdivisions K_1 and K_2 of K mod $L \cup C(L,K)$ are canonically isomorphic via an isomorphism $\phi : K_1 \to K_2$ that is the identity on $L \cup C(K,L)$. The *boundary* of $N(L,K')$ is the subcomplex $\dot{N}(L,K') = \{\sigma \in\ N(L,K') : \sigma \cap |L| = \emptyset\}$. Given $\epsilon > 0$, the ϵ-neighborhood of L in K is a derived neighborhood constructed as follows. Since $L \triangleleft K$, the simplicial map $f \colon K \to [0,1]$ defined by the vertex map

$$f(v) = \begin{cases} 0, & \text{if } v \in L, \\ 1, & \text{if } v \notin L, \end{cases}$$

has the property that $f^{-1}(0) = L$. For any simplex σ of K such that $\sigma \notin L \cup C(L,K)$ choose $\hat{\sigma} \in \overset{\circ}{\sigma} \cap f^{-1}(\epsilon)$. Let K' be the resulting derived subdivision of K mod $L \cup C(L,K)$, and set $N_\epsilon(L,K) = N(L,K')$.

Example. Given a complex K and $p \geq 0$, let $L = K^{(p)}$, let K' be the first barycentric subdivision of K mod L, let $\tilde{L} < K'$ be the dual of L, and let K'' be a derived subdivision of K' mod $L \cup \tilde{L}$. Then $N(L,K'') \cup N(\tilde{L},K'') = K''$ and $\dot{N}(L,K'') = \dot{N}(\tilde{L},K'')$.

Proposition 3.1. *Suppose $L \triangleleft K$ and $(K_1, L_1) \prec (K,L)$. Then there are derived neighborhoods $N(L,K')$ and $N(L_1,K_1')$ such that $|N(L,K')| = |N(L_1,K_1')|$.*

Proof. Given $f \colon K \to [0,1]$ as above, choose $\epsilon > 0$ so that $f^{-1}((0,\epsilon))$ contains no vertex of K or K_1. For each simplex σ of K (respectively, K_1) that meets $|L|$ $(= |L_1|)$, choose $\hat{\sigma} \in \overset{\circ}{\sigma} \cap f^{-1}(\epsilon)$.

Regular Neighborhoods. Given polyhedra $Y \subseteq X$, choose a triangulation K of X containing a subcomplex L triangulating Y. By passing to a derived subdivision of K mod L, we may assume that $L \triangleleft K$. The polyhedron $N = |N(L,K')|$ is called a *regular neighborhood* of Y in X. Proposition 3.1 can be applied to prove the following uniqueness theorem.

Theorem 3.2. *Suppose N_1 and N_2 are regular neighborhoods of Y in X. Then there is a PL homeomorphism $h : X \to X$ such that $h|Y = id$ and $h(N_1) = h(N_2)$. If Y is compact, then we can choose h so that h is the identity outside a compact subset of X.*

Proof. Suppose $N_1 = |N(L_1,K_1')|$ and $N_2 = |N(L_2,K_2')|$, where $K_i > L_i$ triangulates $X \supseteq Y$, and $L_i \triangleleft K_i$. Let $K_0 > L_0$ be a triangulation of $X \supseteq Y$ subdividing both K_1 and K_2. By Proposition 3.1 there are derived subdivisions K_i'' of K_i mod $L_i \cup C(L_i,K_i)$, $i = 0,1,2$, such that $|N(L_0,K_0'')| = |N(L_1,K_1'')| = |N(L_2,K_2'')|$. By the canonical uniqueness, there are isomorphisms $\phi_i \colon K_i' \to K_i''$, fixed on $L_i \cup C(L_i,K_i)$, $i = 1,2$, taking $N(L_i,K_i')$ to $N(L_i,K_i'')$. The composition $\phi_2^{-1} \circ \phi_1$ is a *PL* homeomorphism of X that is the identity on $Y \cup [C(L_1,K_1) \cap C(L_2,K_2)]$ and takes N_1 to N_2.

Theorem 3.3. *Suppose that N is a regular neighborhood of Y in X. Then a regular neighborhood of $\dot{N}$ in X is PL homeomorphic to $\dot{N} \times I$.*

Proof. Suppose $N = |N(L,K')|$ where $K > L$ triangulates $X \supseteq Y$ and $L \triangleleft K$. Without loss of generality, $N = |N(L,K')| = |N_{\frac{1}{2}}(L,K')| = f^{-1}([0,1/2])$, where $f \colon K \to [0,1]$ is the simplicial map described above. For any simplex $\sigma \in K -$

$(L \cup C(L,K))$, $f^{-1}([1/4,3/4]) \cap \sigma$ is canonically *PL* homeomorphic to $f^{-1}(\frac{1}{2}) \times [1/4,3/4]$. These homeomorphisms fit together naturally to give the desired result.

The next theorem follows easily from Theorem 2.4 and Theorem 3.2.

Theorem 3.4. *Suppose Y is a subpolyhedron of a polyhedron X such that Y is locally collared in X. Then a regular neighborhood of Y in X is a collar.*

Theorem 3.5. *Suppose that N is a regular neighborhood of Y in X. Then N is PL homeomorphic to the mapping cylinder C_ϕ of a PL map $\phi\colon \dot{N} \to Y$.*

Proof. As above, we suppose $K > L$ triangulates $X \supseteq Y$ with $L \triangleleft K$ and $N = |N(L,K')| = |N(L',K')| = |N_{\frac{1}{2}}(L',K')| = f^{-1}([0,1/2])$, where $K' > L'$ is a first derived subdivision of $K > L$ mod $C(L,K)$. Any simplex $\sigma \in K - (L \cup C(L,K))$ is a join, $\sigma = \tau * \eta$, with $\tau \in L$ and $\eta \in C(L,K)$. The vertex assignment $\widehat{\tau * \eta} \mapsto \hat{\tau}$ defines a simplicial map $\phi\colon \dot{N}(L',K') \to L'$, and $C_\phi = N(L',K')$.

The proof of the following theorem is left as an exercise. (Use Theorem 3.3.)

Theorem 3.6. *Suppose X is a subpolyhedron of a PL manifold M. Then a regular neighborhood N of X in M is a PL manifold. If X is in the interior of M and $N = |N(L,K')|$ for some triangulation $K > L$ of $M \supseteq X$, then $\partial N = |\dot{N}(L,K')|$.*

A converse of Theorem 3.6 is contained in the following Simplicial Neighborhood Theorem. We state the theorem along with a selection of some of its more important corollaries. A proof may be found in [50].

Theorem 3.7. (Simplicial Neighborhood Theorem) *Suppose X is a subpolyhedron in the interior of a PL manifold M, and N is a neighborhood of X in M. Then N is a regular neighborhood of X if and only if*

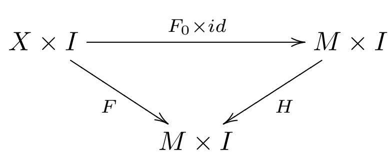
(1) *N is a PL manifold with boundary, and*
(2) *there is a triangulation $K > L, J$ of $N \supseteq X, \partial N$ with $L \triangleleft K$, $K = N(L,K)$ and $J = \dot{N}(L,K)$.*

Corollary 3.8. *If $B^n \subseteq S^n$ is a PL ball in a PL sphere, then $\mathcal{C}\ell(S^n - B^n) \cong B^n$.*

Corollary 3.9. *If $N_1 \subseteq \operatorname{int} N_2$ are two regular neighborhoods of X in $\operatorname{int} M$, then $\mathcal{C}\ell(N_2 - N_1) \cong \partial N_1 \times I$.*

Corollary 3.10. (Combinatorial Annulus Theorem) *If B_1 and B_2 are PL n-balls with $B_1 \subseteq \operatorname{int} B_2$, then $\mathcal{C}\ell(B_2 - B_1) \cong S^{n-1} \times I$.*

The Regular Neighborhood Theorem. The Regular Neighborhood Theorem provides a strong isotopy uniqueness theorem for regular neighborhoods of X in M. Given a subpolyhedron X of a polyhedron M, an *isotopy* of X in M is a level-preserving, closed, *PL* embedding $F\colon X \times I \to M \times I$. (This term will also be used for a (closed) *PL* map $F\colon X \times I \to M$ whose restriction to each $X \times \{t\}$, $t \in I$, is an embedding.) An *isotopy* of M is a level-preserving *PL* homeomorphism $H\colon M \times I \to M \times I$ such that $H_0 = \text{id}$. An isotopy F of X in M is *ambient* if there is an isotopy H of M making the following diagram commute.



We compose isotopies F and G of M by "stacking":

$$F \circ G(x,t) = \begin{cases} F(x,2t), & \text{if } 0 \le t \le 1/2; \\ G(F(x,1),2t-1), & \text{if } 1/2 \le t \le 1. \end{cases}$$

Proposition 3.11. (Alexander Isotopy) *If $h_0, h_1 \colon B^n \to B^n$ are PL homeomorphisms that agree on S^{n-1}, then h_0 and h_1 are ambient isotopic by an isotopy that fixes S^{n-1}.*

Proof. As $B^n \cong v * S^{n-1}$, use Proposition 2.3 to get $B^n \times [-1,1] \cong v * (S^{n-1} \times [-1,1] \cup B^n \times \{-1,1\})$. Define $H \colon S^{n-1} \times [-1,1] \cup B^n \times \{-1,1\} \to S^{n-1} \times [-1,1] \cup B^n \times \{-1,1\}$ by $H|S^{n-1} \times [-1,1] \cup B^n \times \{-1\} = \text{id}$ and $H|B^n \times \{1\} = h_1 \circ h_0^{-1}$. Extend linearly over the cone to get $H \colon B^n \times [-1,1] \to B^n \times [-1,1]$. (The Alexander Isotopy is the isotopy $Hh_0|B^n \times I \colon B^n \times I \to B^n \times I$.)

Proposition 3.12. *If X is collared in M, then any isotopy of X extends to an isotopy of M supported on a collar of X in M.*

The proof of this proposition as well as the following corollary to 3.11 and 3.12 are left as exercises.

Corollary 3.13. *If $\mathcal{C}$ is a cell complex and $f \colon \mathcal{C} \to \mathcal{C}$ is a homeomorphism that carries each cell of $\mathcal{C}$ onto itself, then f is ambient isotopic to the identity.*

Theorem 3.14. (Regular Neighborhood Theorem) *Suppose X is a subpolyhedron in the interior of a PL manifold M and N_1 and N_2 are regular neighborhoods of X in $\operatorname{int} M$. Then there is an isotopy of M, fixed on X and outside an arbitrary neighborhood of $N_1 \cup N_2$ taking N_1 to N_2.*

Proof. Let $N_0 \subseteq \operatorname{int} N_1 \cap \operatorname{int} N_2$ be a regular neighborhood of X. Then $\mathcal{C}\ell(N_i - N_0) \cong \partial N_0 \times I$ for $i = 1,\ 2$. For a given neighborhood U of $N_1 \cup N_2$, choose regular neighborhoods N_i^+ of $\mathcal{C}\ell(N_i - N_0)$ in $U - X$, $i = 1,\ 2$. Then there is a *PL* homeomorphism $h_i \colon N_i^+ \to \partial N_0 \times [0,3]$ such that $h_i(\mathcal{C}\ell(N_i - N_0), \partial N_0, \partial N_i) = (\partial N_0 \times [1,2], \partial N_0 \times \{1\}, \partial N_0 \times \{2\}$. There is an obvious ambient isotopy of $\partial N_0 \times [0,3]$, fixing $\partial N_0 \times \{0,3\}$, taking $\partial N_0 \times \{2\}$ to $\partial N_0 \times \{1\}$. An appropriate composition does the job.

Collapsing and Shelling. Theorem 3.5 leads toward another important characterization of regular neighborhoods, because of the very special way in which a simplicial mapping cylinder deforms to its range. If $X \supseteq Y$ are polyhedra such that, for some $n \ge 0$,

$$(\mathcal{C}\ell(X - Y), \mathcal{C}\ell(X - Y) \cap Y) \cong (B^{n-1} \times I, B^{n-1} \times \{0\}),$$

then we say that there is an *elementary collapse* from X to Y, $X \searrow_e Y$. We say that X *collapses to* Y, $X \searrow Y$, if there is a sequence of elementary collapses $X = X_0 \searrow_e X_1 \searrow_e X_2 \searrow_e \cdots \searrow_e X_k = Y$. If $X \searrow Y$, then Y *expands* to X, $Y \nearrow X$. A (compact) polyhedron X is *collapsible*, $X \searrow 0$, if X collapses to a point.

If $M \supseteq Q$ are *PL* n-manifolds and $M \searrow_e Q$, then we call the elementary collapse an *elementary shelling*. If we set $(B^n, B^{n-1}) = \mathcal{C}\ell(M - Q), \mathcal{C}\ell(M - Q) \cap Q)$, then $B^{n-1} \subseteq \partial Q$, and, hence, there is a homeomorphism $h \colon M \to Q$, fixed outside any preassigned neighborhood of $\operatorname{int} B^{n-1}$ in ∂Q. We say that M *shells* to Q if there is a sequence of elementary shellings starting with M and ending with Q.

Proposition 3.15. If $f \colon K \to L$ is a simplicial map with K finite, then $|C_f| \searrow |L|$.

Proof. A quick way to see this is to apply a result of M.H.A. Newman (see [66], Ch. 7, Lemma 46, also [15], Proposition 9.1), which says that if σ and τ are simplexes, $\dim \sigma = n$, and if $f\colon \sigma \to \tau$ is a linear surjection, then $(|C_f|, \sigma) \cong (B^n \times I, B^n \times \{0\})$. (The proof of this assertion is not as immediate as one might like.) One then proceeds by induction downward through the skeleta of K.

It is also possible to prove this directly, using induction and the fact that, if X is a compact polyhedron, $C(C(X)) \searrow C(X)$.

Clearly, if $X \searrow Y$, then X deformation retracts to Y, but the converse may fail be true in a very strong sense. Polyhedra X and Y are *simple homotopy equivalent* if there is a sequence $X = X_0 \searrow X_1 \nearrow X_2 \searrow X_3 \nearrow \cdots \searrow Y_k = Y$. In particular, if $f\colon K \to L$ is a simplicial map (of finite complexes), which is also a homotopy equivalence from $|K|$ to $|L|$, then $|C_f|$ deformation retracts to $|K|$, but the equivalence may not be simple. There is an obstruction $\tau_f \in \mathrm{Wh}(\pi_1(|K|)$, the Whitehead group of the fundamental group of $|K|$: for a homotopy equivalence $f\colon |K| \to |L|$, $\tau_f = 0$ if, and only if, the inclusion of $|K|$ in $|C_f|$ is a simple homotopy equivalence. We refer the reader to [17] for a comprehensive treatment of this topic.

We state the collapsibility criteria for regular neighborhoods. They depend upon the fact that if $X \searrow Y$, then a regular neighborhood of X shells to a regular neighborhood of Y. Complete proofs may be found in [50] or [66] .

Theorem 3.16. *Suppose X is a compact polyhedron in the interior of a PL manifold M. A polyhedral neighborhood N of X in $\operatorname{int} M$ is a regular neighborhood of X if and only if*

(1) *N is a compact manifold with boundary,*
(2) *$N \searrow X$.*

Corollary 3.17. *If $X \searrow 0$, then a regular neighborhood of X in a PL manifold is a ball.*

There are analogues for these results in the case of noncompact polyhedra and "proper" maps. The reader is referred to [51] and [53] for more details.

Regular Neighborhoods of Pairs. The Simplicial and Regular Neighborhood Theorems can be generalized to the "proper" inclusion of polyhedral pairs $(Y, Y_0) \subseteq (X, X_0)$, meaning $Y \cap X_0 = Y_0$. The simplicial model is constructed as before: Let $(K, K_0) > (L, L_0)$ be triangulations of $(X, X_0) \supseteq (Y, Y_0)$, with $L \lhd K$. Then $L_0 \lhd K_0$ and polyhedra $N_0 \subseteq N$ of the derived neighborhoods $N(L_0, K_0') < N(L, K')$ are regular neighborhoods of Y_0 in X_0 and Y in X, respectively. Call (N, N_0) a *regular neighborhood of the pair* (Y, Y_0) in (X, X_0).

We will mostly be interested in the case in which X and X_0 are *PL* manifolds. Suppose Q is a q-dimensional submanifold of a *PL* n-manifold M. We say that Q is *proper* in M if $Q \cap \partial M = \partial Q$, and, if $Q \subseteq M$ is proper, we call the pair (M, Q) an *(n, q)-manifold pair.* A proper ball pair (B^n, B^q) is *unknotted* if $(B^n, B^q) \cong (J^n, J^q \times \{0\})$, where $J = [-1, 1]$. Similarly, a sphere pair (S^n, S^q) is *unknotted* if $(S^n, S^q) \cong (\partial J^n, \partial J^q \times \{0\})$. A manifold pair (M, Q) is *locally flat at $x \in Q$* if there is a triangulation $K > L$ of $M \supseteq Q$, containing x as a vertex, such that the pair $(\mathrm{st}(v, K), \mathrm{st}(v, L))$ is an unknotted ball pair. (In the case that $Q \subseteq M$ is not proper and $x \in \partial Q - \partial M$, require instead that $(\mathrm{st}(v, K), \mathrm{st}(v, L)) \cong (J^n, J^{q-1} \times [0, 1] \times \{0\})$.) (M, Q) is a *locally flat manifold pair* if it is locally flat at every point. It is an exercise to see that (M, Q) is a locally flat manifold pair if there is a triangulation

$K > L$ of $M \supseteq Q$ such that $(\mathrm{st}(v, K), \mathrm{st}(v, L))$ is an unknotted ball pair for each vertex v of L.

We state the Regular Neighborhood Theorem for Pairs. The proof follows that of that of the Regular Neighborhood Theorem with the obvious changes.

Theorem 3.18. (Regular Neighborhood Theorem for Pairs) *Suppose (X, Y) is a polyhedral pair in a locally flat manifold pair (M, Q), with $X \cap Q = Y$, and suppose $(N_1, N_{1,0})$ and $(N_2, N_{2,0})$ are regular neighborhoods of (X, Y) in (M, Q). Then there is an isotopy H of (M, Q), fixed on X and outside a neighborhood of $N_1 \cup N_2$ with $H_1(N_1, N_{1,0}) = (N_2, N_{2,0})$.*

If (M, Q) is a locally flat manifold pair, then the pair $(\partial M, \partial Q)$ is locally collared *as pairs* in (M, Q). That is, if $x \in \partial Q$, then x has a neighborhood pair $(X, Y) \subseteq (M, Q)$ and $(X_0, Y_0) \subseteq (\partial M, \partial Q)$ such that $(X, X_0, Y, Y_0) \cong (X_0 \times [0, 1], X_0 \times \{0\}, Y_0 \times [0, 1], Y_0 \times \{0\})$ The proof of Theorem 2.4 generalizes immediately to provide a collaring theorem for pairs.

Theorem 3.19. *If $(Y, Y_0) \subseteq (X, X_0)$ is a proper inclusion of polyhedral pairs and (Y, Y_0) is locally collared in (X, X_0) at each point of Y_0, then (Y, Y_0) is collared in (X, X_0).*

Corollary 3.20. *If (M, Q) is a locally flat manifold pair, then $(\partial M, \partial Q)$ is collared in (M, Q).*

One may define collapsing and shelling for pairs. For example, $(X, X_0) \searrow (Y, Y_0)$ means that $X_0 \cap Y = Y_0$, $X \searrow Y$, $X_0 \searrow Y_0$, and the collapse preserves X_0. For example, if $X \overset{e}{\searrow} Y$ so that $X = Y \cup B$, where B is a cell meeting Y in a face C, then $B \cap X_0$ must be a cell meeting Y_0 in a face that lies in C. In particular one can arrange that $X \searrow X_0 \cup Y \searrow Y \searrow Y_0$.

Theorem 3.21. *If $(Y, Y_0) \subseteq (X, X_0) \subseteq (M, Q)$ are proper inclusions of pairs and (M, Q) is a locally flat manifold pair, and if $(X, X_0) \searrow (Y, Y_0)$, then a regular neighborhood pair of (X, X_0) in (M, Q) shells to one of (Y, Y_0).*

Corollary 3.22. *If $(X, X_0) \subseteq (M, Q)$ is a proper inclusion, where (M, Q) is a locally flat manifold pair, and if $(X, X_0) \searrow 0$, then a regular neighborhood pair of (X, X_0) in (M, Q) is an unknotted ball pair.*

Cellular Moves. Two q-dimensional, locally flat submanifolds Q_1, Q_2 of a PL n-manifold M differ by a *cellular move* if there is a $(q+1)$-ball $B^{q+1} \subseteq \mathrm{int}\, M$ meeting Q_1 and Q_2 in complementary q-balls B_1^q and B_2^q, respectively, in ∂B^{q+1} such that $Q_1 \cap Q_2 = Q_1 - \mathrm{int}\, B_1^q = Q_2 - \mathrm{int}\, B_2^q$.

Theorem 3.23. *If Q_1, $Q_2 \subseteq M$ differ by a cellular move across a $(q+1)$-ball B^{q+1}, then there is an isotopy H of M, fixed outside an arbitrary neighborhood of B^{q+1}, such that $H_1(Q_1) = Q_2$.*

Proof. Using derived neighborhoods, we can get a regular neighborhood N of B^{q+1} in M such that if $N_i = N \cap Q_i$, then (N, N_i), $i = 1, 2$, is a regular neighborhood pairs of (B^{q+1}, B_i^q) in (M, Q_i). Since, by Corollary 3.22, each (N, N_i) is an unknotted ball pair there is a homeomorphism $h\colon (N, N_1) \to (N, N_2)$, fixed on the boundary. The Alexander Isotopy provides the isotopy H.

Corollary 3.24. *A locally flat sphere pair (S^n, S^q) is unknotted iff S^q bounds a $(q+1)$-ball in S^n.*

Proof. If $S^q = \partial B^{q+1}$, choose a triangulation $K > L$ of $S^n \supseteq B^{q+1}$ and a $(q+1)$-simplex σ of L such that $\sigma \cap S^q$ is a q-dimensional face of σ. Then $(\mathrm{st}(\sigma, K), \sigma)$ is an unknotted ball pair, and $\partial\sigma$ and S^q differ by a cellular move.

Relative Regular Neighborhoods. If $Z \supseteq X \supseteq Y$ are polyhedra, then one can define a relative regular neighborhood of X mod Y in Z. The simplicial model is constructed much as above: Choose a triangulation $J > K > L$ of $Z \supseteq X \supseteq Y$, let J'' be a second derived subdivision of J mod K, and set $N(K - L, J'') = \{\sigma \in J'' : \text{for some } \tau \in J'',\ \sigma < \tau \text{ and } \tau \cap |K| - |L| \neq \emptyset\}$. We recommend [16] for a complete treatment, including recognition and uniqueness theorems. As an example result from the theory we have the following

Theorem 3.25. *Suppose that* $(B^q, \mathrm{int}\, B^q) \subseteq (M, \mathrm{int}\, M)$ *and* $(\mathrm{int}\, M, \mathrm{int}\, B^q)$ *is a locally flat pair. If* N *is a regular neighborhood of* B^q *mod* ∂B^q *in* M*, then* (N, B^q) *is an unknotted ball pair.*

Structure of Regular Neighborhoods. We have commented on the fact that a regular neighborhood N of a polyhedron Y in a polyhedron X has the structure of a mapping cylinder of a mapping $\phi\colon \dot{N} \to Y$. In [16], Section 5, Cohen analyzes the fine structure of the mapping cylinder projection $\gamma\colon N \to Y$.

Theorem 3.26. [16] *If* N *is a regular neighborhood of* Y *in* X*, then for each* $y \in Y$*,* $\gamma^{-1}(y) \cong y * \phi^{-1}(y)$*. Moreover, if* (X, Y) *is a locally unknotted* (n, q)*-manifold pair, then* $\phi^{-1}(y) \cong S^{n-q-1} \times B^i$*, where* i *is an integer depending on* y*.*

Suppose now that (M, Q) is a locally flat (n, q)-manifold pair. It is not generally true that we can get the integer i in Theorem 3.26 to be 0 for all $y \in Q$. Whenever that is possible the regular neighborhood N of Q in M has the structure of an $(n - q)$-disk bundle over Q. There are, however, examples [25] , [48] of locally flat *PL* embeddings without disk-bundle neighborhoods (although, they acquire disk-bundle neighborhoods after stabilizing the ambient manifold). Rourke and Sanderson show [49] that it is possible, however, to give N the structure of a block bundle. Given polyhedra E, F, and X, a *PL* mapping $\phi\colon E \to X$ is a (*PL*) *block bundle with fiber* F if there are *PL* cell complex structures $\mathcal{K}$ and $\mathcal{L}$ on E and X, respectively, such that $\phi\colon \mathcal{K} \to \mathcal{L}$ is cellular and for each cell $C \in \mathcal{L}$, $\phi^{-1}(C)$ is *PL* homeomorphic to $C \times F$. If $\phi\colon E \to X$ is a block bundle with fiber F, then the mapping cylinder retraction $\gamma\colon C_\phi \to X$ is also a block bundle with fiber the cone $x * F$, and for each cell $C \in \mathcal{L}$, $(\gamma^{-1}(C), C) \cong (C \times (x * F), C \times \{x\})$. If $F = S^{m-1}$ and C is a p-cell in $\mathcal{L}$, then $(\gamma^{-1}(C), C) \cong (J^{p+m}, J^p \times \{0\})$. A *PL* retraction $\gamma\colon E \to X$ satisfying this property is called an *m-block bundle* over X.

Theorem 3.27. [49] *Suppose that* (M, Q) *is a locally flat* (n, q)*-manifold pair. Then a regular neighborhood* N *of* Q *in* M *has the structure of an* $(n - q)$*-block bundle over* Q*.*

Proof. We only consider the case in which $\partial Q = \emptyset$. Let $K > L$ be a triangulation of $M \supseteq Q$ with $L \lhd K$, let K_1 be a first derived subdivision of K mod L, and let $N = |N(L, K_1)|$. Since (M, Q) is a locally flat, for any p-simplex $\sigma \in L$, $(\mathrm{lk}(\sigma, K_1), \mathrm{lk}(\sigma, L))$ is an unknotted $(n - p - 1, q - p - 1)$-sphere pair; hence,

$$(\mathrm{lk}(\sigma, K_1), \mathrm{lk}(\sigma, L)) \cong (S^{q-p-1} * S^{n-q-1}, S^{q-p-1}).$$

Let $K' > L'$ be a first derived subdivision of $K > L$ extending K_1, and let σ be a p-simplex of L. Let $\tilde{K}_\sigma < K'$ and $\tilde{L}_\sigma < L'$ denote the dual $(n - p - 1)$-

and $(q-p-1)$-spheres to σ in K' and L', respectively, and let $\tilde{C}_\sigma = \hat{\sigma} * \tilde{K}_\sigma$ and $\tilde{D}_\sigma = \hat{\sigma} * L_\sigma$ denote the respective dual cells. Then

$$(\tilde{K}_\sigma, \tilde{L}_\sigma) \cong (\mathrm{lk}(\sigma, K_1), \mathrm{lk}(\sigma, L)) \cong (S^{q-p-1} * S^{n-q-1}, S^{q-p-1}),$$

so that

$$(\tilde{C}_\sigma, \tilde{D}_\sigma) \cong (J^{n-p}, J^{q-p} \times \{0\}).$$

These dual cell pairs fit together nicely to give the neighborhood N the structure of an $(n-q)$-block bundle over Q with respect to the dual cell structures on M and Q obtained from K and L. The mapping $\gamma\colon N \to Q$ is obtained by induction on the dual cells of L; it is not, in general, the same as the natural projection defined above.

4. General Position

General position is a process by which two polyhedra X and Y in a *PL* manifold M may be repositioned slightly in order to minimize the dimension of $X \cap Y$. It is also a process by which the dimension of the singularities of a *PL* map $f\colon X \to M$ may be minimized by a small adjustment of f. A combination of general position and join structure arguments form the underpinnings of nearly every result in *PL* topology. We start with definitions of "small adjustments."

Given metric spaces X and M and $\epsilon > 0$ (ϵ may be a continuous function of X), An *ϵ-homotopy* (*isotopy*) of X in M is a homotopy (isotopy) $F\colon X \times I \to M$ such that $\operatorname{diam} F(x \times I) < \epsilon$ for every $x \in X$. An *ϵ-isotopy* of M is an isotopy H of M that is also an ϵ-homotopy. If $X, Y \subseteq M$, then an *ϵ-push* of X in M, rel Y, is an ϵ-isotopy of M that is fixed on Y and outside the ϵ-neighborhood of X.

Suppose $f\colon X \to M$ is a (continuous) function. The *singular set* of f, is the subset $S(f) = \mathcal{C}\ell\{x \in X : f^{-1}f(x) \neq x\}$. If X and M are polyhedra and f is *PL*, then f is *nondegenerate* if $\dim f^{-1}(y) \leq 0$ for each $y \in M$. If f is a *PL* map, and $f^{-1}(C)$ is compact for every compact subset C of M, then $S(f)$ is a subpolyhedron of X.

Let us start with a (countable) discrete set S of points in $\mathbb{R}^n$. We say that S is in *general position* if every subset $\{v_0, v_1, \ldots, v_p\}$ of S spans a p-simplex, whenever $p \leq n$. Since the set of all hyperplanes of $\mathbb{R}^n$ of dimension $< n$ spanned by points of S is nowhere dense, it is clear that if $\epsilon\colon S \to (0, \infty)$ is arbitrary, then there is an isotopy H of $\mathbb{R}^n$, fixed outside an ϵ-neighborhood of S such that $H_1(S)$ is in general position and $\operatorname{diam} H(v \times I) < \epsilon(v)$ for all $v \in S$. Moreover, if S_0 is a subset of S that is already in general position, then we can require that H fixes S_0 as well. We can also approximate any map $f\colon S \to \mathbb{R}^n$ by map g such that $g(S)$ is in general position, insisting that $g|S_0 = f|S_0$ if $f(S_0)$ is already in general position.

General position properties devolve from the following elementary fact from linear algebra.

Proposition 4.1. *Suppose that E_1, E_2 and E_0 are hyperplanes in $\mathbb{R}^n$ of dimensions p, q and r, respectively, spanned by $\{u_0, u_1, \ldots, u_p\}$, $\{v_0, v_1, \ldots, v_q\}$, and $\{w_0, w_1, \ldots, w_r\}$ with $u_i = v_i = w_i$ for $0 \leq i \leq r$ and $u_i \neq v_j$ for $i, j > r$. If the set $S = \{u_0, u_1, \ldots, u_p, v_{r+1}, v_{r+2}, \ldots, v_q\}$ is in general position, then $\dim((E_1 - E_0) \cap (E_2 - E_0)) \leq p + q - n$.*

As usual, we interpret $\dim(A \cap B) < 0$ to mean that $A \cap B = \emptyset$. Proposition 4.1 motivates the definition of general position for polyhedra X and Y embedded in a

PL manifold M. If $\dim X = p$, $\dim Y = q$, and $\dim M = n$, we say that X and Y are in *general position* in M if $\dim(X \cap Y) \leq p + q - n$.

Theorem 4.2. *Suppose that $X \supseteq X_0$ and Y are polyhedra in the interior of a PL n-manifold M with $\dim(X - X_0) = p$ and $\dim Y = q$ and $\epsilon\colon M \to (0, \infty)$ is continuous. Then there is an ϵ-push H of X in M, rel X_0, such that $\dim[H_1(X - X_0) \cap Y] \leq p + q - n$.*

Proof. Let $J > K, K_0$ be a triangulation of $M \supseteq X, X_0$ with $K_0 \triangleleft K$. Let v be a vertex of $K - K_0$. Let $g\colon \mathrm{lk}(v, J) \to S^{n-1}$ be a *PL* homeomorphism that is linear on each simplex of $\mathrm{Lk}(v, J)$. Extend g conewise to a *PL* homeomorphism $h\colon \mathrm{st}(v, J) \to B^n$ such that $h(v) = 0$. Apply Proposition 4.1 to get a point $x \in \overset{\circ}{B}{}^n$ such that $\dim(((x * \tau) - \tau) \cap h(Y)) \leq p + q - n$ for every simplex $\tau = h(\sigma)$, $\sigma \in \mathrm{Lk}(v, K)$. Let F be an isotopy of B^n, fixed on S^{n-1}, with F_1 the conewise extension of $\mathrm{id}_{S^{n-1}}$ that takes x to 0. Let F^v be the isotopy of M, fixed outside $\mathrm{st}(v, J)$, obtained by conjugating F with h. By choosing x sufficiently close to $0 \in B^n$, we can assume that F^v is a δ-push of Y in M, rel $(M - \overset{\circ}{\mathrm{st}}(v, J))$. Then $\dim((\mathrm{st}(v, K) - \mathrm{lk}(v, K)) \cap F_1^v(Y)) \leq p + q - n$, and for any $\delta > 0$.

Assume now that K is a derived subdivision of a triangulation of X so that the vertices of K can be partitioned: $K^{(0)} = V_0 \cup V_1 \cup \cdots \cup V_k$, $k = \dim K$, where $\mathrm{st}(v, K) \cap \mathrm{st}(w, K) \subseteq \mathrm{lk}(v, K) \cap \mathrm{lk}(w, K)$ when $v, w \in V_i$, $v \neq w$. (See the proof of Theorem 2.4.) For $0 \leq i \leq k$, define an isotopy F^i of M by $F^i = F^v$ on $\mathrm{st}(v, J)$, $v \in V_i$, and $F^i = \mathrm{id}$ on $M - \bigcup_{v \in V_i} \overset{\circ}{\mathrm{st}}(v, J)$. We can easily make F^i an $\frac{\epsilon}{2(k+1)}$-push of X in M, rel $(M - \bigcup_{v \in V_i} \overset{\circ}{\mathrm{st}}(v, J))$. If we construct the F^i's inductively we can ensure that the composition $G = F^0 \circ \ldots \circ F^k$ is an $\frac{\epsilon}{2}$-push of X in M, rel X_0, and that $\dim((X - X_0) \cap G_1(Y)) \leq p + q - n$. The inverse H of G is then an ϵ-push of X in M, rel X_0, such that $\dim(H_1(X - X_0) \cap Y) \leq p + q - n$. (The inverse of an ϵ-push H of X is only a 2ϵ-push of $H(X)$.)

A similar type of argument can be used to prove a general position theorem for mappings.

Theorem 4.3. *Suppose $X \supseteq X_0$ are polyhedra with $\dim(X - X_0) = p$, M is a PL n-manifold, $p \leq n$, and $f\colon X \to M$ is a continuous map with $f|X_0$ PL and nondegenerate on some triangulation of X_0. Then for every continuous $\epsilon\colon X \to (0, \infty)$ there is an ϵ-homotopy, rel X_0, of f to $f'\colon X \to M$ such that $\dim(S_f - X_0) \leq 2p - n$. Moreover, if $X_1 \subseteq X$ and $\dim(X_1 - X_0) = q$, then we can arrange to have $\dim((S_f \cap (X_1 - X_0)) \leq p + q - n$.*

A mapping satisfying this last condition is said to be *in general position with respect to* X_1 rel X_0.

Corollary 4.4. *Suppose $X \supseteq X_0$ is a p-dimensional polyhedron, $f\colon X \to M$ is a continuous mapping of X into a PL n-manifold M, $2p + 1 \leq m$, such that $f|X_0$ is a PL embedding, and if $\epsilon\colon X \to (0, \infty)$ is continuous, then f is ϵ-homotopic, rel X_0, to a PL embedding.*

This is the best one can expect in such full generality. There is a p-dimensional polyhedron X, namely the p-skeleton of a $(2p + 2)$-simplex, that does not embed in $\mathbb{R}^{2p}$ [20] . Shapiro [52] has developed an obstruction theory for embedding p-dimensional polyhedra in $\mathbb{R}^{2p}$.

General position and regular neighborhood theory can be used to establish an unknotting theorem for sphere pairs.

Theorem 4.5. *A sphere pair (S^n, S^q) is unknotted, if*

(1) $q = 1$ *and* $n = 4$, *or*
(2) $n \geq 2q + 1$ *and* $n \geq 5$.

Corollary 4.6. *An (n, q)-manifold pair (M, Q) is locally flat provided $q = 1,\ n \geq 1$, or $q = 2,\ n \geq 5$, or $q > 2,\ n \geq 2q$.*

Proof. [50] (i) If $n \geq 5$, then general position gives an embedding of the cone on S^1, so that S^1 is unknotted by 3.24.

If $n = 4$, then there is a point $x \in \mathbb{R}^4$ such that x and S^1 are joinable: Let $V = \bigcup\{E(u, v) : u, v \in S^1\}$, where $E(u, v)$ is the line determined by u and v. V is a finite union of hyperplanes, each of dimension at most 3 in $\mathbb{R}^4$. Hence, if $x \notin V$, then $x * S^1$ is the cone on S^1. Thus S^1 bounds a 2-ball in $\mathbb{R}^4$.

(ii) Assume as above that $S^q \subseteq \mathbb{R}^n$. By induction, using Corollary 4.6, we may assume that (S^n, S^q) is locally flat. Since $2q \leq n - 1$, we may assume that the restriction of the projection $\pi\colon \mathbb{R}^n \to \mathbb{R}^{n-1}$ to S^q has a singular set consisting of double points $\{a_1, b_1', \dots, a_r, b_r'\}$, where a_i lies "above" b_i'. Choose a point x near infinity "above" S^q, and let $f\colon x * S^q \to \mathbb{R}^n$ be the natural linear extension to the cone $x * S^q$, so that the singularities of f lie in $\bigcup_{i=1}^r x * \{a_i, b_i\}$, where b_i is close to b_i'. Since $q \geq 2$, there is a *PL* q-cell B in $S^q - \{b_1, \dots, b_r\}$ containing $\{a_1, \dots, a_r\}$. Then $f|x * \partial B$ is an embedding as is $f|x * (S^q - \operatorname{int} B)$. The $(q+1)$-ball $f(x * B)$ provides a cellular move from S^q to $\partial f(x * (S^q - \operatorname{int} B))$. But $\partial f(x * (S^q - \operatorname{int} B))$ is unknotted, by 3.24.

Theorem 4.7. [6] *Suppose $X \supseteq X_0$ is a p-dimensional polyhedron, M is a PL n-manifold, $2p + 2 \leq n$, and $f, g\colon X \to M$ are PL embeddings such that $f|X_0 = g|X_0$ and $f \simeq g$, rel X_0. Then f and g are ambient isotopic, rel X_0, by an isotopy supported on an arbitrary neighborhood of the image of a homotopy of f to g.*

Proof. Let $K > K_0$ be a triangulation of $X \supseteq X_0$ and assume, inductively, that $f|K_0 \cup K^{(p-1)} = g|K_0 \cup K^{(p-1)}$. As the isotopy will be constructed by moving across balls with disjoint interiors, we assume further that $f \simeq g$, rel $|K_0| \cup |K^{(p-1)}|$. Let $Z = X \times I \bmod (|K_0| \cup |K^{(p-1)}|)$, and let $F\colon Z \to M$ be a relative homotopy from f to g. Assume F is *PL* and in general position, so that $\dim S(F) \leq 2(p+1) - n \leq 0$ and $S(F)$ consists of double points $\{a_i, b_i\}$ lying in the interiors of cells $\sigma \times I \bmod \partial\sigma$, where σ is a p-simplex of K. For each $a_i \in \operatorname{int}\sigma \times (0, 1)$, get a *PL* arc A_i in $\operatorname{int}\sigma \times [0, 1)$ joining a_i to a point $c_i \in \operatorname{int}\sigma \times \{0\}$, chosen so that the A_i's are disjoint and contain none of the b_j's. Get a regular neighborhood pair $(D_i, D_{i,0})$ of (A_i, c_i) in $(\operatorname{int}\sigma \times [0, 1), \operatorname{int}\sigma \times \{0\})$, chosen so that the D_i's are disjoint (and contain none of the b_j's). Let C_i be the face of ∂D_i complementary to $D_{i,0}$. Then there is a cellular move across D_i taking $\sigma_i \times \{0\}$ to $(\sigma_i \times \{0\}) - \operatorname{int} D_{i,0}$. The net effect of these moves is to get a homotopy of F to an embedding.

Assume now that we have an embedding $F\colon Z \to M$. For p-simplexes $\sigma \in K$, choose relative regular neighborhoods N_σ of $F(\sigma \times I) \bmod F(\partial\sigma)$ so that $N_\sigma \cap F(Z) = F(\sigma \times I)$ and the N_σ's have disjoint interiors. Then $(N_\sigma, F(\sigma \times \{i\}))$, $i = 0, 1$, is an unknotted ball pair. Hence, there is an isotopy H of M, fixed outside the union of the N_σ's, such that on N_σ, $H_1 \circ f|\sigma = g|\sigma$.

5. Embeddings, Engulfing

In this section we address the following question, which arises naturally from Corollary 4.4. Suppose X is a p-dimensional polyhedron and M is a *PL* n-manifold. When is a map $f\colon X \to M$ homotopic to a *PL* embedding? The first theorem takes a small but important step in reducing the codimension restriction of Corollary 4.4.

Theorem 5.1. *Suppose that Q is a connected PL q-manifold, M is a properly simply connected PL n-manifold, and $n \geq 2q \neq 4$. Then every closed mapping $f\colon Q \to M$ is homotopic to a PL embedding.*

Proof. To say that M is *properly* simply connected means that M is simply connected and *simply connected at infinity*. That is, for every compact set C in M there is a compact set $D \supseteq C$ such that any loop in $M - D$ is null-homotopic in $M - C$. We consider only the case $n \geq 6$. The proof exploits the now famous "Whitney Trick" [64], [62]. Given $f\colon Q \to M$, use general position to get a *PL* mapping $g\colon Q \to \operatorname{int} M$ homotopic to f such that $S(g)$ is a closed set consisting only of "double points": $S(g) = \{a_1, b_1, a_2, b_2, \ldots\} \subseteq \operatorname{int} Q$, where the indicated points are distinct, $g(a_i) = g(b_i)$, $i = 1, 2, \ldots$, and $g(a_i) \neq g(a_j)$ if $i \neq j$. Since $q \geq 3$, we can get a closed family of mutually exclusive *PL* arcs $A_1, A_2, \ldots$ joining a_i to b_i, respectively. The images $g(A_i)$ are *PL* simple closed curves in $\operatorname{int} M$. Since M is properly simply connected and $n \geq 6$, we can use general position to get a closed family of mutually exclusive *PL* 2-cells $D_1, D_2, \ldots$ in $\operatorname{int} M$ such that $\partial D_i = g(A_i)$. Using suitable triangulations we can get mutually exclusive regular neighborhoods N_i of D_i in $\operatorname{int} M$ such that $V_i = g^{-1}(N_i)$ is a regular neighborhood of A_i in $\operatorname{int} Q$, $i = 1, 2, \ldots$. By Corollary 3.17 N_i and V_i are *PL* balls of dimensions n and q, respectively. Using the cone structures on N_i and V_i, we can redefine $g|V_i$ to get an embedding $h_i\colon V_i \to N_i$, agreeing with g on ∂V_i, and homotopic to $g|V_i$ rel ∂V_i. Then $g \simeq h$, where $h|V_i = h_i$ and $h|Q - \bigcup_i V_i = g|Q - \bigcup_i V_i$.

Generalizations of the Whitney Trick may be used to reduce the codimension, $n-q$, provided compensating assumptions are made on the connectivity of Q and M. One approach uses engulfing techniques, introduced by Stallings [56] and Zeeman [66] , which have proved useful in other contexts as well.

Engulfing. The engulfing problem: Given a closed set Y (polyhedron) and a compact set C in a *PL* manifold M, with $C \subseteq \operatorname{int} M$, and an arbitrary neighborhood U of Y in M, find an ambient isotopy H of M, fixed on $Y \cup \partial M$ and outside a compact set, such that $H_1(U) \supseteq C$. If such an isotopy of M exists, we say that C *can be engulfed from* Y. Obvious homotopy conditions must be met, but they are not sufficient in general to find H. One need only look at the Whitehead link as C in the torus $S^1 \times B^2$, with $Y = pt$. (See, e.g., [66], Ch. 7.)

Theorem 5.2. *Suppose Y is a compact polyhedron of dimension $\leq n - 3$ in a PL n-manifold M, such that (M, Y) is k-connected. A compact, k-dimensional polyhedron X in* $\operatorname{int} M$ *can be engulfed from Y provided*

(1) *$n \geq 6$ and $n - k \geq 3$, or*
(2) *$n = 4$ or 5 and $k = 1$, or*
(3) *$n = 5$, $k = 2$.*

Proof. The proof uses the collapsing techniques of [56] and [66] . We shall first give an argument for (i) in the case $n - k \geq 4$, deferring the case $n - k = 3$ of (i) and (iii). We leave the proof of (ii) as an exercise. An elegant alternative proof of Theorem 5.2, using handle theory, may be found in [50].

The proof uses the fact that a simplicial mapping cylinder collapses to its range. Suppose $A \supseteq B$ are polyhedra and $A \searrow B$. For a subset C of A, define the *trail* of C, $\mathrm{tr}(C) \supseteq C$, under the collapse as follows. Let $A = A_0 \searrow_e A_1 \searrow_e \cdots \searrow_e A_r = B$ be a sequence of elementary collapses, so that

$$(\mathcal{C}\ell(A_{i-1} - A_i), \mathcal{C}\ell(A_{i-1} - A_i) \cap A_i) \overset{h_i}{\cong} (B^{m-1} \times I, B^{m-1} \times \{0\}).$$

Suppose $\mathrm{tr}_i(C) = \mathrm{tr}(C) \cap \mathcal{C}\ell(A - A_i)$ has been defined for $0 \le i < k$ (where $\mathrm{tr}_0(C) = \emptyset$). Let $D = h_k((\mathrm{tr}_{k-1}(C) \cup C) \cap (\mathcal{C}\ell(A_{k-1} - A_k)) \subseteq B^{m-1} \times I$, and let $E = \{(x,t) \in B^{m-1} \times I : (x,s) \in D, \text{ for some } s \ge t\}$. Define $\mathrm{tr}_k(C) = \mathrm{tr}_{k-1}(C) \cup h_k^{-1}(E)$. Finally, define $\mathrm{tr}(C) = \mathrm{tr}_r(C) \cup C$. If C is a polyhedron of dimension p in A, then elementary arguments show that

(a) $A \searrow B \cup \mathrm{tr}(C) \searrow B$, and
(b) $\dim \mathrm{tr}(C) \le p + 1$.

Suppose now that $Y, X \subseteq M$, as in (i), with $n - k \ge 4$. We shall actually prove the stronger

Assertion 5.3. There is a polyhedron $Q \subseteq M$ such that $X \subseteq Q$, $Q \searrow Y$, and $\dim(Q - Y) \le k + 1$.

Given the assertion, one may apply the Regular Neighborhood Theorem to obtain the desired isotopy.

Proof of Assertion 5.3. Fix $k (\le n - 4)$, and suppose inductively that, for $0 \le i \le k$, we have the following:

(1) a polyhedron $Q \supseteq X \cup Y$ in M with $\dim Q \le n - 3$, such that
(2) $Q \searrow Y \cup P$, where
(3) $\dim P \le k - i$.

Start the induction at $i = 0$ with $Q = X \cup Y$ and $P = X$.

Since (M, Y) is k-connected, there is a homotopy of the inclusion of P in M, rel $P \cap Y$, to a map $f \colon P \to Y$, which we may assume to be *PL*. Choose triangulations K and L of P and Y, respectively, such that $H = K \cap L$ triangulates $P \cap Y$ and $f \colon K \to L$ is simplicial. Let $Z = |C_f \text{ rel } H|$. Then $\dim(Z - Y) \le k - i + 1$ and the homotopy provides a map $F \colon Z \to M$ such that $F|P \cup Y = \mathrm{id}$. We may assume that F is in general position (with respect to Y) so that $\dim S(F) \le (n-3) + (k-i+1) - n \le k - i - 2$ (Theorem 4.3). Let $T = \mathrm{tr}(S(F))$ under the collapse $Z \searrow Y$. Then $\dim T \le k - i - 1$, and $Z \searrow Y \cup T \searrow Y$; hence, $F(Z) \searrow Y \cup F(T)$. Let $R = \mathrm{tr}(F(Z) \cap Q)$ under the collapse $Q \searrow Y \cup P$. Then $\dim R \le k - i - 1$, and $Q \searrow Y \cup P \cup R \searrow Y \cup P$.

Set $Q_1 = Q \cup S(Z)$ and $P_1 = F(T) \cup R$. Then $Q_1 \searrow Y \cup F(Z) \cup R \searrow Y \cup F(T) \cup R = Y \cup P_1$, and $\dim P_1 \le k - i - 1$.

When $i = k + 1$, the process stops, since the set $P = \emptyset$.

The inductive argument given above does not work in the case $n - k = 3$. (Check the dimension of the polyhedron T in the proof.) To argue this case we shall use Zeeman's Piping Lemma, which we paraphrase next. A proof may be found in [66], Ch. 7, Lemma 48.

Lemma 5.4. (Piping Lemma [66]) *Suppose M is a PL n-manifold, K is a finite complex of dimension $k \le n - 3$, $f \colon K \to L$, $\dim L \le n - 3$, is a simplicial* *mapping that restricts to an embedding on a subcomplex* $H < K$, $Z = |C_f \text{ rel } H|$, $Z_0 = |C_{f|K^{(k-1)}} \text{ rel } H|$, *and* $F\colon Z \to M$ *is a PL mapping that is in general position with respect to* Z_0. *Then* F *is homotopic rel* $|K| \cup Z_0$ *to a PL mapping* $G\colon Z \to M$ *such that*

(a) $Z \searrow Z_1 \searrow |L|$,
(b) $S(G) \cup Z_0 \subseteq Z_1$,
(c) $\dim(Z_1 - |L|) \leq k - 1$, *and*
(d) $\dim \mathcal{C}\ell(Z_1 - |L|) \cap Z_0) \leq k - 2$.

We indicate the proof of Assertion 5.3 when $k = n - 3$, $n \geq 5$. The inclusion of X in M is homotopic rel $X \cap Y$ to a mapping $f\colon X \to Y$, which we may assume to be *PL*. Let K, L, H triangulate X, Y, $X \cap Y$ so that $f\colon K \to L$ is simplicial and $f|H = \text{id}$. Let $Z = |C_f \text{ rel } H|$, and let $F\colon Z \to M$ be a *PL* mapping with $F|X \cup Y = \text{id}$, guaranteed by the connectivity, in general position with respect to $Z_0 = |C_{f|K^{(n-4)}} \text{ rel } H|$. By Lemma 5.4 F is homotopic, rel $|K| \cup Z_0$, to a *PL* mapping $G\colon Z \to M$ satisfying (a), (b). and (c) of 5.4. Then $G(Z) \searrow G(Z_1)$. In the proof of Assertion 5.3, set $Q = G(Z)$ and $P = \mathcal{C}\ell(G(Z_1) - Y)$. Then $\dim Q \leq n - 2$ and $\dim P \leq n - 4$, and the inductive argument proceeds without a problem.

Generalizations of the Whitney Trick for eliminating double point singularities of a mapping $f\colon Q^q \to M^{2q}$, as in Theorem 5.1, can be obtained from the engulfing techniques just described. Irwin's embedding theorem, which we now state, can be thought of as the generalization to codimension 3 of the process of removing one pair of double points.

Theorem 5.5. ([33]) *Suppose* Q *is a compact PL* q*-manifold,* M *is a PL* n*-manifold,* $n - q \geq 3$, *such that* Q *is* $(2q - n)$*-connected and* M *is* $(2q - n + 1)$*-connected. Then every map* $f\colon (Q, \partial Q) \to (M, \partial M)$ *for which* $f|\partial Q\colon \partial Q \to \partial M$ *is a PL embedding is homotopic rel* ∂Q *to a PL embedding.*

Proof. Since general position works when $n \leq 5$, we assume that $n \geq 6$. By playing with the collar structures on ∂Q and ∂M, one may assume that $f(\text{int}\, Q) \subseteq \text{int}\, M$ and $f|N$ is a *PL* embedding for some collar neighborhood N of ∂Q in Q and that a general position approximation $g\colon Q \to M$ satisfies $g|N = f|N$, $S(g) \subseteq \mathcal{C}\ell(Q - N)$, and $\dim S(g) \leq 2q - n$. We will find collapsible polyhedra $C \subseteq \text{int}\, Q$ and $D \subseteq \text{int}\, M$ such that $S(g) \subseteq C = g^{-1}(D)$. Once we have C and D, we can proceed as in the proof of Theorem 5.1: Get regular neighborhoods U of D in $\text{int}\, M$ and V of C in $\text{int}\, Q$ such that $g^{-1}(U) = V$. Then U and V are *PL* n- and q-balls, respectively, and $g|\partial V\colon \partial V \to \partial U$ is a *PL* embedding. We redefine g on V to get a *PL* embedding homotopic to g.

We shall assume, initially, that $n - q \leq 4$. We construct C and D by induction. Assume that $f\colon Q \to M$ is a *PL* mapping in general position with $S(f) \subseteq \text{int}\, Q$ and $f(S(f)) \subseteq \text{int}\, M$. Suppose, inductively, we have the following:

(a) polyhedra $C \subseteq Q$, $D \subseteq M$ such that
(b) $S(f) \subseteq C \searrow 0$, $f(C) \subseteq D \searrow 0$,
(c) $f^{-1}(D) = C \cup C_1$,
(d) $\dim C \leq q - 3$, $\dim D \leq q - 2$, and
(e) $\dim C_1 \leq q - 3 - i$, and $\dim(C_1 \cap C) \leq q - 4 - i$.

We start the induction at $i = 3$. Since $\dim S(g) \leq 2q - n \leq q - 4$, we can use the connectivity conditions and Assertion 5.3, with $Y = \text{pt}$, to get a collapsible polyhedron $C \subseteq Q$ with $\dim C \leq 2q - n + 1 \leq q - 3$. Apply the connectivity conditions and Assertion 5.3 again, with $Y = \text{pt}$, to get a collapsible polyhedron $D \supseteq f(C)$, with $\dim D \leq 2q - n + 2 \leq q - 2$. Use general position to get $\dim(f(Q - C) \cap D) \leq q + (q - 2) - n \leq q - 6$, and $\dim(\mathcal{C}\ell(f(Q - C) \cap D)) \cap D) \leq q - 7$. Set $C_1 = f^{-1}(\mathcal{C}\ell(f(Q - C) \cap D))$; $\dim C_1 \leq q - 6$.

Suppose we are given (a) – (e), for $1 \leq i \leq k$, so that $\dim C_1 \leq q - 3 - k$, and $\dim(C_1 \cap C) \leq q - 4 - k$. Let $S = C_1 \cap C$, and let $T = \operatorname{tr} S$ under the collapse $C \searrow 0$; $\dim T \leq q - 3 - k$. Then $C \searrow T \searrow 0$. The connectivity conditions, together with the homotopy extension theorem, imply there is a homotopy, rel $C_1 \cap T$, of id_{C_1} to a mapping $g\colon C_1 \to T \subseteq C$. Assertion 5.3 then provides a polyhedron $A \supseteq C_1 \cup C$ such that $A \searrow C$ ($\searrow 0$), $\dim(A - C) \leq q - 2 - k$, and $\mathcal{C}\ell(A - C) \cap C \leq q - 3 - k$.

Let $A_1 = \mathcal{C}\ell(A - C)$ and let $B_1 = f(A_1)$. Then $\dim B_1 \leq q - 2 - k$ and $\dim B_1 \cap D \leq q - 3 - k$. Let $S_1 = B_1 \cap D$ and let $T_1 = \operatorname{tr} S_1$ under the collapse $D \searrow 0$. Then $\dim T_1 \leq q - 2 - k$ and $T_1 \searrow 0$. Repeat the argument above: use the connectivity conditions, together with the Homotopy Extension Theorem, to get a homotopy, rel $B_1 \cap T_1$, of id_{B_1} to a mapping $h\colon B_1 \to T_1 \subseteq D$. Assertion 5.3 then provides a polyhedron $P \supseteq B_1 \cup D$ such that $P \searrow D$ ($\searrow 0$), $\dim(P - D) \leq q - 1 - k$, and $\mathcal{C}\ell(P - D) \cap D \leq q - 2 - k$. Use general position to get $\dim((P - D) \cap f(Q)) \leq (q - 1 - k) + q - n \leq q - 5 - k$. Set $P_1 = f^{-1}(\mathcal{C}\ell(P - D)$. Then A and P_1 replace C and C_1 to complete the inductive step.

The case $n - q = 3$ requires Lemma 5.4 to get the induction going, very much as in the proof of Assertion 5.3. We shall leave the details to the reader.

Corollary 5.6. *If Q is a compact k-connected q-manifold, $q - k \geq 3$, then Q embeds in $\mathbb{R}^{2q-k}$.*

Corollary 5.7. *If $f\colon S^{q-1} \to \partial M$ is a PL embedding of the $(q-1)$-sphere into the boundary of a $(q-1)$-connected n-manifold M, $n - q \geq 3$, then f extends to a PL embedding $\bar{f}\colon B^q \to M$.*

To get a generalization of the Whitney Trick analogous to the removal of *all* of the double point singularities of Theorem 5.1, one must impose a connectivity condition on the mapping f. Recall that a mapping $f\colon Q \to M$ is k-connected if $\pi_i(f) = \pi_i(M_f, Q) = 0$, for $0 \leq i \leq k$.

Theorem 5.8. [29, 57, 60] *Suppose Q and M are PL manifolds of dimensions q and n, respectively, $n - q \geq 3$, and $f\colon (Q, \partial Q) \to (M, \partial M)$ is a $(2q - n + 1)$-connected map such that $f|\partial Q$ is a PL embedding. Then f is homotopic, rel ∂Q, to a PL embedding.*

This theorem was first proved by Hudson [29], with an extra connectivity hypothesis on Q, using a generalization of the techniques of the proof of Theorem 5.5. This condition later proved to be superfluous as a consequence of the argument in the proof of following theorem of Stallings [57]. We include Stallings' argument, since it has only appeared in preprint form.

Theorem 5.9. [57] *Suppose X is a compact k-dimensional polyhedron, M is a PL manifold of dimension n, $n - k \geq 3$, and $f\colon X \to M$ is $(2k - n + 1)$-connected. Then there is a k-dimensional polyhedron $X_1 \subseteq M$ and a simple homotopy equivalence $f_1\colon X \to X_1$ such that f_1 and f are homotopic as maps to M.*

Proof. [57] It is not difficult to see that a map $f\colon X \to M$ is i-connected if, and only if, any map $\alpha\colon (P,Q) \to (C_f, X)$ of a polyhedral pair (P,Q) into the mapping cylinder of f, with $\dim(P-Q) \le i$, is homotopic, rel $\alpha|Q$, to a map into X.

Suppose that $f\colon X \to M$ is a $(2k-n+1)$-connected map and that f is in general position, so that $\dim S_f \le 2k-n$. Suppose, inductively, that we have a k-dimensional polyhedron Y, a simple homotopy equivalence $h\colon X \to Y$, and a PL map $g\colon Y \to M$ such that

(a) $gh = f$,
(b) $\dim S(g) \le 2k-n-j$, for some j, $0 \le j \le 2k-n$.

Then g is $(2k-n+1)$-connected. We start the induction by setting $Y = X$, $h = \mathrm{id}$, and $g = f$.

Set $S = S(g)$, $T = g(S(g))$, and let C be the mapping cylinder of $g|S\colon S \to T$ with projection $\gamma\colon C \to T$. Then C is a submapping cylinder of C_g and $\dim C \le 2k-n-j+1 \le 2k-n+1$. Our hypotheses imply that the inclusion $(C,S) \subseteq (C_g, Y)$ is homotopic, rel S, to a map of C into Y. Let $H\colon C \times I \to C_g$ be such a homotopy. That is, $H_0(y) = y$, for all $y \in C$, $H_t(y) = y$, for all $y \in S$, and $H_1(C) \subseteq Y$. Let $\beta = H_1\colon C \to Y$ (keep in mind that $\beta(y) = y$, if $y \in S$), and form the reduced mapping cylinder D_β rel S. Then $\dim(D_\beta - Y) \le 2k-n-j+2$ and $D_\beta \searrow Y$, so that the inclusion $Y \subseteq D_\beta$ is a simple homotopy equivalence. Since $C \subseteq D_\beta$ and $C \searrow T$, the adjunction space $Y_1 = D_\beta \cup_\gamma T$ is simple homotopy equivalent to D_β. (See (5.9) of [17].) Hence, each of the maps $X \to Y \to D_\beta \to Y_1$ is a simple homotopy equivalence. Denote the composition by $h_1\colon X \to Y_1$.

Observe that the composition $g'\colon Y \to D_\beta \to Y_1$ induces the same identifications on Y that g does, so that Y is sent to a subset of Y_1 homeomorphic to $g(Y) \subset M$. Thus, the composition $\gamma_g \circ H\colon C \times I \to M$, where $\gamma_g\colon C_g \to M$ is the projection, induces a map $g_1\colon Y_1 \to M$ such that $g_1 g' = g$ and $g_1|g'(Y)(= g(Y))$ is an embedding. Assume g_1 is in general position rel $g(Y)$. Then we have

(a) $g_1 h_1 = f$, and
(b) $\dim S(g_1) \le (2k-n-j+2) + k - n \le 2k-n-j-1$,

since $\dim(Y_1 - g(Y)) \le 2k-n-j+2$, and $n-k \ge 3$. The inductive process stops after at most $2k-n$ iterations.

Using surgery theory, Wall [60] obtains the following embedding theorem, which was proved first for Q simply connected by Casson and Sullivan and by Browder and Haefliger [24]. One can easily see that Theorem 5.8 follows from Theorems 5.9 and 5.10.

Theorem 5.10. [60] *Suppose Q and N are compact PL manifolds of dimensions q and n, respectively, $n-q \ge 3$, and $f\colon (Q,\partial Q) \to (N,\partial N)$ is a homotopy equivalence such that $f|\partial Q$ is a PL embedding. Then f is homotopic, rel ∂Q, to a PL embedding.*

Perhaps the first important application of engulfing due to Stallings is his proof of the higher dimensional Poincaré Conjecture [56].

Theorem 5.11. (Weak Poincaré Conjecture) [56] *Suppose that M is a k-connected, closed PL n-manifold, $n \ge 5$, $k = \lfloor n/2 \rfloor$. Then M is topologically homeomorphic to S^n.*

Proof. Let M_1 be M minus the interior of an n-ball. Then M_1 is also k-connected. Let $K > K_0$ be a triangulation of $M_1 \supseteq \partial M_1$, let $L = K^{(k)} \cup K_0$, and let $\tilde{L} < K'$

be the dual $(n-k-1)$-skeleton of K rel K_0. Let N be a regular neighborhood of ∂M_1 in M_1 (a collar), and let B be a small n-ball in $\operatorname{int} M_1$ containing a point $p \notin N$ in its interior. Use 5.2 to get isotopies H^1 and H^2 of M_1, fixed on ∂M_1, such that $H_1^1(N) \supseteq |L|$ and $H_1^2(B) \supseteq |\tilde{L}|$. Using the Example of Section 3 and the Regular Neighborhood Theorem, we may assume that $M_1 = H_1^1(N) \cup H_1^2(B)$. The composition $H = H_2^{-1} \circ H_1$ is an isotopy of M_1, fixed on ∂M_1, such that $H(N) \supseteq M_1 - \operatorname{int} B$. Without loss of generality, $p \notin H(N)$.

Set $M_2 = \mathcal{C}\ell(B - H(N))$, and repeat the construction for M_2, obtaining $M_2 = N_2 \cup B_2$, where N_2 is a collar on ∂M_2, $p \notin N_2$, and B_2 is a small n-ball in $\operatorname{int} M_2$ containing p in its interior. After an infinite repetition we obtain $M_1 - \{p\} \cong \partial M_1 \times [0,\infty)$, so that $M - \{p\} \cong \mathbb{R}^n$. Thus M is the one point compactification of $\mathbb{R}^n$, which is topologically homeomorphic to S^n.

A stronger version of the Poincaré Conjecture, concluding that M is *PL* homeomorphic to S^n, can be proved for $n \geq 6$ using handle theory and the h-cobordism theorem [55], [3] . We shall discuss these topics in the next section.

6. Handle Theory

Suppose that K is a combinatorial n-manifold with polyhedron M. The combinatorial structure of K provides M with a nice decomposition into *PL* n-balls, stratified naturally by their "cores", called a *handle decomposition.* Given *PL* n-manifolds W_1 and W_0, and $J = [-1,1]$, we say that W_1 *is obtained from* W_0 *by adding a handle of index* p, if $W_1 = W_0 \cup H^{(p)}$, where $(H^{(p)}, H^{(p)} \cap W_0) = (H^{(p)}, \partial H^{(p)} \cap \partial W_0) \cong (J^p \times J^{n-p}, \partial J^p \times J^{n-p})$. Given a *PL* homeomorphism $h\colon (J^p \times J^{n-p}, \partial J^p \times J^{n-p}) \to (H^{(p)}, H^{(p)} \cap \partial W_0)$, we call $h(J^p \times \{0\})$ the *core* of the handle $H^{(p)}$, $h(\partial J^p \times \{0\})$ is the *attaching sphere*, $h(\{0\} \times J^{n-p})$ is the *cocore*, and $h(\{0\} \times \partial J^{n-p})$ is the *belt sphere*. We call h the *characteristic map*, and $f = h|(\partial J^p) \times J^{n-p}$ the *attaching map.*

For example, suppose M is a *PL* n-manifold without boundary, and K is a combinatorial triangulation of M with first and second derived subdivisions $K' \succ K''$. If σ is a p-simplex of K, then $\operatorname{lk}(\sigma, K) \cong S^{n-p-1}$ so that

$$\begin{aligned}(\operatorname{st}(\sigma,K),\sigma) &\cong (\sigma * \operatorname{lk}(\sigma,K),\sigma)\\ &\cong (B^p * S^{n-p-1}, B^p)\\ &\cong (J^p \times J^{n-p}, J^p \times \{0\}).\end{aligned}$$

Using a cone construction one in turn sees that

$$(\operatorname{st}(\sigma,K),\sigma) \cong (\operatorname{st}(\hat{\sigma},K''), \operatorname{st}(\hat{\sigma},\sigma'')),$$

where $\sigma'' = K''|\sigma$.

Denote the *PL* n-ball, $\operatorname{st}(\hat{\sigma}, K'')$, by B_σ. It is not difficult to see that $B_\sigma \cap B_\tau = \emptyset$ whenever $\dim\sigma = \dim\tau$. Setting $H_p = \bigcup\{B_\sigma : \dim\sigma = p\}$, we see that $M = H_0 \cup H_1 \cup \cdots \cup H_n$, where each H_p is a disjoint union of n-balls. (See Fig. 6.1, where B_p denotes a B_σ, $\dim\sigma = p$.) If we set $W_{p-1} = \bigcup_{i<p} H_i$, then for $\dim\sigma = p$, $B_\sigma \cap W_{p-1} = \partial B_\sigma \cap \partial W_{p-1}$ is a regular neighborhood of $\partial\sigma''$ in ∂B_σ; hence, $B_\sigma \cap W_{p-1} \cong S^{p-1} \times J^{n-p}$. Thus, W_p is obtained from W_{p-1} by adding p-handles B_σ, $\dim\sigma = p$, and $M = H_0 \cup H_1 \cup \cdots \cup H_n$.

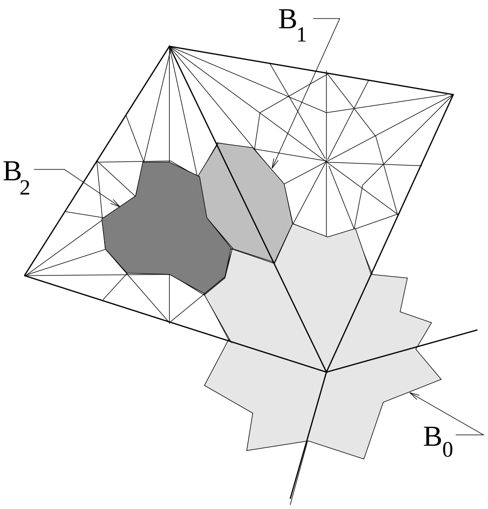


Fig. 6.1

A *handle decomposition* of a *PL* n-manifold M is a presentation $M = H_0 \cup H_1 \cup \cdots \cup H_n$, where H_0 is a disjoint union of n-balls, and $W_p = \bigcup_{i\le p} H_i$ is obtained from W_{p-1} by adding p-handles, $1 \le p \le n$. It may be that $W_p = W_{p-1}$, in which case the decomposition has no handles of index p. That is, we allow $H_p = \emptyset$.

If M is a *PL* n-manifold with boundary ∂M, let C be a regular neighborhood of ∂M in M, $(C, \partial M) \cong (\partial M \times I, \partial M \times \{0\})$. Assume that $C = N(\partial K'', K'')$ for some *PL* triangulation K of M. Construct the n-balls B_σ as above for $\sigma \notin \partial K$. Then $M = C \cup H_0 \cup \cdots H_n$ as before, and $W_p = C \cup \bigcup_{i\le p} H_i$ is obtained from W_{p-1} by adding p-handles. Notice that any attaching set that meets C meets it in $\dot{N}(\partial K'', K'')$. Any such presentation of M is a *handle decomposition of* M, rel ∂M.

Finally, we extend the idea of a handle decomposition to a cobordism between $(n-1)$-manifolds. A *cobordism* is a triple $(W; M, M')$, where W is a *PL* n-manifold and $\partial W = M \cup M'$, $M \cap M' = \emptyset$. A *handle decomposition* of W, rel M, is a presentation $W = C \cup H_0 \cup H_1 \cup \cdots H_n \cup C'$, where C and C' are regular neighborhoods of M and M' in W, respectively, and $W_p = C \cup \bigcup_{i\le p} H_i$ is obtained from W_{p-1} by adding p-handles.

Dual Handle Decompositions. Notice that if $W = C \cup H_0 \cup H_1 \cup \cdots H_n \cup C'$ is a handle decomposition of a cobordism $(W; M, M')$, and if $H^{(p)}$ is a p-handle with characteristic map h, then $H^{(p)} \cap (\bigcup_{i>p} H_i \cup C') = h(J^p \times \partial J^{n-p})$. That is, $H^{(p)}$ can be thought of as an $(n-p)$-handle added to $\bigcup_{i>p} H_i \cup C'$. With this point of view, we write $H^{(p)} = \tilde{H}^{(n-p)}$ and call $\tilde{H}^{(n-p)}$ the *dual* $(n-p)$*-handle* determined by $H^{(p)}$. Thus, we also get a handle decomposition $W = C' \cup \tilde{H}_0 \cup \tilde{H}_1 \cup \cdots \cup \tilde{H}_n \cup C$, where $\tilde{H}_p = H_{n-p}$. Dual handle structures are closely related to dual cell structures described in Section 2.

Handle decompositions arising from triangulations of a manifold are generally too large to be of much use, although they often provide a place to get started. The goal is to try to find the simplest possible handle decomposition. For example, if a cobordism $(W; M, M')$ has a handle decomposition with *no* handles, then $W = C \cup C'$ is a product: $(W; M, M') \cong (M \times I; M \times \{0\}, M \times \{1\})$. An obvious necessary condition for this to happen is that the inclusions $M_i \hookrightarrow W$, $i = 0, 1$, are homotopy equivalences. A cobordism $(W; M, M')$ satisfying this condition is called an *h-cobordism* between M and M', or simply an h-cobordism.

h-Cobordism Theorem 6.1. *Suppose $(W; M, M')$ is a compact h-cobordism, $\dim W \geq 6$, and W is simply connected. Then $(W; M, M') \cong (M \times I; M \times \{0\}, M \times \{1\})$.*

We shall outline a proof of the h-Cobordism Theorem in this section. Our treatment is taken from [50], where many of the omitted details may be found.

Simplifying handle decompositions. For the immediate discussion, we will let $(W; M, M')$ denote a compact cobordism with $\dim W = n$. Our first observation is that "sliding a handle" does not change the topology of the resulting manifold.

Lemma 6.2. *If $f, g\colon \partial I^p \times I^{n-p} \to \partial M'$ are ambient isotopic attaching maps, then $W \cup_f H^{(p)} \cong W \cup_g H^{(p)}$ by a homeomorphism that is fixed outside a regular neighborhood (collar) of M'.*

Proof. By Proposition 3.12, an isotopy of M' extends to W, fixing the complement of a collar on M'.

Lemma 6.3. *If $p \leq q$, then $(W \cup H^{(q)}) \cup H^{(p)} \cong (W \cup H^{(p)}) \cup H^{(q)}$, with $H^{(p)}$ and $H^{(q)}$ disjoint.*

Proof. Let S_a be the attaching sphere for $H^{(p)}$ and S_b the belt sphere for $H^{(q)}$. Then $\dim S_a + \dim S_b = (p-1) + (n-q-1) < n-1$ so that S_a can be general positioned to miss S_b in $\partial(W \cup H^{(q)})$. Use 3.14 and Lemma 6.2 to "squeeze" the p-handle so that $H^{(p)} \cap S_b = \emptyset$ as well. Let N be a regular neighborhood of the cocore of $H^{(q)}$ in $W \cup H^{(q)}$ such that $N \cap H^{(p)} = \emptyset$. Since $H^{(q)}$ is also a regular neighborhood, there is an isotopy of $W \cup H^{(q)}$ taking N to $H^{(q)}$. This isotopy slides $H^{(p)}$ off of $H^{(q)}$, so Lemma 6.2 applies to complete the proof.

As a consequence of Lemma 6.3, we can rearrange the addition of handles to a cobordism W so that the handles are added in nondecreasing order, thereby producing a handle decomposition of W. We now look at circumstances in which handles may be eliminated.

Suppose $W_1 = W \cup H^{(p)} \cup H^{(p+1)}$, $\dim W = n$. Then $H^{(p)}$ and $H^{(p+1)}$ are called *complementary handles* if the attaching sphere S_a of $H^{(p+1)}$ meets the belt sphere S_b of $H^{(p)}$ transversely in a single point. This means that near $x = S_a \cap S_b$, and after an ambient isotopy of the attaching map f for $H^{(p+1)}$, f matches up the product structure on $\partial H^{(p+1)}$ with that on $\partial H^{(p)}$.

Lemma 6.4. *Suppose that $W_1 = W \cup H^{(p)} \cup H^{(p+1)}$, where $H^{(p)}$ and $H^{(p+1)}$ are complementary handles. Then $W_1 \cong W$ by a PL homeomorphism that is the identity outside a collar on M'.*

Proof. Let $h\colon J^p \times J^{n-p} \to H^{(p)}$ be the characteristic map for $H^{(p)}$ and let $f\colon (\partial J^{p+1}) \times J^{n-p-1} \to \partial(W \cup H^{(p)})$ be the attaching map for $H^{(p+1)}$. Using the Regular Neighborhood Theorem and Lemma 6.2, we may assume that $h(J^p \times (\{1\} \times J^{n-p-1})) = f((J^p \times \{1\}) \times J^{n-p-1})$, and that $f^{-1} \circ h | J^p \times (\{1\} \times J^{n-p-1}) = \mathrm{id}$. (See Fig. 6.2.) Thus, we see that W_1 is obtained from W by attaching an n-ball $B = H^{(p)} \cup H^{(p+1)}$ to M' along an $(n-1)$-ball in ∂B. That is, W_1 shells to W.

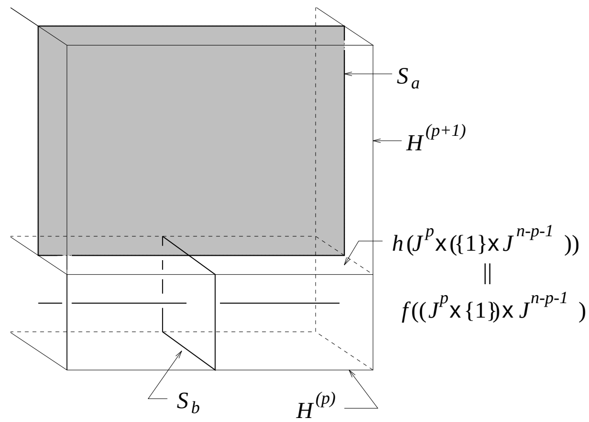


Fig. 6.2

The reverse of this process allows one to introduce a cancelling pair of handles to a cobordism.

In general, if $W_1 = W \cup H^{(p)} \cup H^{(p+1)}$, then we can define the *incidence number* $\varepsilon(H^{(p+1)}, H^{(p)})$ as follows. There is a strong deformation retraction of $W \cup H^{(p)}$ onto $W \cup h(I^p \times \{0\})$, where h is the characteristic map for $H^{(p)}$. The composition $W \cup H^{(p)} \to W \cup h(I^p \times \{0\}) \to (W \cup h(I^p \times \{0\})/W \cong S^p$ gives a mapping $g\colon W \cup H^{(p)} \to S^p$. If S_a is the attaching sphere for $H^{(p+1)}$, then the restriction $g|S_a$ gives a mapping of $S^p \cong S_a \to S^p$. Choose orientations for (each) S^p, and define $\varepsilon(H^{(p+1)}, H^{(p)})$ to be the degree of this map. Thus, $\varepsilon(H^{(p+1)}, H^{(p)})$ is an integer, which is well-defined up to sign. If S_a and the belt sphere S_b for $H^{(p)}$ are in general position in $\partial(W \cup H^{(p)})$, then they intersect transversely in a finite number of points. If $H^{(p)}$ and $H^{(p+1)}$ are given orientations, then $\varepsilon(H^{(p+1)}, H^{(p)})$ is the algebraic intersection number of S_a and S_b in $\partial(W \cup H^{(p)})$. If $H^{(p)}$ and $H^{(p+1)}$ are a complementary pair of handles, then, clearly, $\varepsilon(H^{(p+1)}, H^{(p)}) = \pm 1$. The next lemma gives conditions under which, up to an ambient isotopy of attaching maps, the converse is true. The proof uses another form of the Whitney Trick and may be found in [50], Ch. 6.

Lemma 6.5. (Handle Cancellation Lemma) *Suppose* $W_1 = (W \cup H^{(p)}) \cup_f H^{(p+1)}$, $2 \le p \le n-4$, $n \ge 6$, M' *is simply connected, and* $\varepsilon(H^{(p+1)}, H^{(p)}) = \pm 1$. *Then the attaching map* f *for* $H^{(p+1)}$ *is ambient isotopic to an attaching map* g *such that, in* $W_2 = (W \cup H^{(p)}) \cup_g H^{(p+1)}$, $H^{(p)}$ *and* $H^{(p+1)}$ *are complementary handles. Thus,* $W_1 \cong W_2 \cong W$.

Lemma 6.6. (Handle Addition Lemma) *Suppose* $W_1 = W \cup_{f_1} H_1^{(p)} \cup_{f_2} H_2^{(p)}$, *where* $H_1^{(p)} \cap H_2^{(p)} = \emptyset$, $2 \le p \le n-2$, *and* M' *is simply connected. Then* f_1 *is ambient isotopic to* f_3, *where* $[f_3] = [f_1] + [f_2]$ *in* $\pi_p(M')$.

Proof. If $2 \le p \le n-2$ and M' is connected, we can connect PL embedded $(p-1)$-spheres S_1 and S_2 with a PL "ribbon" $D = g(I \times I^{p-1})$ in M', where g is a PL embedding, $g(I \times I^{p-1}) \cap S_1 = g(\{0\} \times I^{p-1})$ and $g(I \times I^{p-1}) \cap S_2 = g(\{1\} \times I^{p-1})$. (See [50], Ch. 5.) In this way we can add the homotopy classes of $[f_1]$ and $[f_2]$ in $\pi_{p-1}(M')$ (this requires $2 \le p \le n-2$), and if M' is simply connected, the resulting class is independent of D. Inside the boundary of the p-handle $H_1 = H_1^{(p)}$ in W_1, there is a "parallel" copy of its core: $B = h(J^p \times \{x\})$, where $x \in \partial J^{n-p}$ and h is a characteristic map for H_1. Connect the boundary sphere S of B to the attaching sphere S_2 for the handle H_2 with a ribbon D in M'.

Now we have the collapses $B \cup D \cup S_2 \searrow S_2$ and $B \cup D \cup S_2 \searrow S_3 = (S_1 \cup D \cup S_2) - g(I \times \operatorname{int} I^{p-1})$ in $M_2 = \partial(W \cup H_1) - M$. Thus, in M_2, the Regular Neighborhood Theorem provides an ambient isotopy of the attaching map f_1 to $f_3\colon S^{p-1} \to M'$ with $[f_3] = [f_1] + [f_2]$ in $\pi_p(M')$. The new handle can be moved off of $H_1^{(p)} \cup H_2^{(p)}$.

Lemma 6.7. *Suppose that $(W; M, M')$ is a connected cobordism. Then W has a handle decomposition with no 0- or n-handles.*

Proof. If a 1-handle $H^{(1)}$ joins C to a 0-handle $H^{(0)}$, then $(H^{(0)}, H^{(1)})$ is a complementary pair. Proceed by induction: if there are any 0-handles, then there is one that is joined to C by a 1-handle. The n-handles in a handle decomposition of W are the 0-handles in the dual decomposition.

If M is simply connected, one can use a handle decomposition $(W; M, M') = C \cup H_1 \cup \cdots \cup H_{(n-1)} \cup C'$, where $H_p = H_1^{(p)} \cup \cdots \cup H_{n_p}^{(p)}$ is the (disjoint) union of handles of index p, to compute the homology of the pair $(W; M)$. Form the chain complex whose pth chain group, $\mathcal{C}_p$, is generated by the (oriented) p-handles. If $H_i^{(p)}$ is a p-handle, define $\partial(H_i^{(p)}) = \sum_j \varepsilon(H_i^{(p)}, H_j^{(p-1)}) H_j^{(p-1)}$. Observe that if $X \supseteq Y$ is a polyhedron with $\dim(X - Y) \le p$, and if $f\colon (X, Y) \to (W, M)$ is a mapping, then f is homotopic, rel Y, to a mapping $g\colon X \to C \cup H_1 \cup \cdots \cup H_p$. Just proceed inductively: use general position to get $f(X)$ disjoint from the cocores of higher dimensional handles and then use the handle structure to get $f(X)$ miss the handles themselves. A similar general position argument can also be used to prove the following useful fact.

Lemma 6.8. *Suppose that $W = C \cup H_0 \cup \cdots \cup H_n \cup C'$ is a handle decomposition, $W^{(p)} = C \cup H_1 \cup H_1 \cup \cdots \cup H_p$, and $M^{(p)} = \partial W^{(p)} - M$. Then*

(a) $\pi_i(W, W^{(p)}) = 0$ *for* $i \le p$, *and*
(b) $\pi_i(W, M^{(p)}) = 0$ *for* $i \le \min\{p, n-p-1\}$.

Lemma 6.9. *Suppose that W is connected, $n \ge 6$, M is simply connected, and $(W; M, M')$ has a handle decomposition with no handles of index $< p$, $1 \le p \le n-4$. If $H_p(W, M) = 0$, then there is another handle decomposition with no handles of index $\le p$ and with the same number of handles of index $> p+1$.*

Proof. Let $W = C \cup H_1 \cup \cdots \cup H_n \cup C')$ be a handle decomposition for W.

Case 1: $p = 1$. Let $H^{(1)}$ be a 1-handle in H_1; $H^{(1)}$ is attached to $M_0 = \partial C - M$. Let α be an arc in $\partial H^{(1)}$ parallel to its core, and let β be a *PL* arc in C joining the endpoints of α. Use general position to get α to miss all 2-handles and β to miss the cores of the 1- and 2-handles so that $\gamma = \alpha \cup \beta \subseteq \partial(C \cup H^{(1)}) - M$. By Lemma 6.8 and general position, γ bounds a *PL* disk D in $M_1 = \partial(C \cup H_1 \cup H_2) - M$. By 4.6 D is locally flat in M_1. Introduce a complementary pair of 2- and 3-handles $H^{(2)}$ and $H^{(3)}$ so that $H^{(2)}$ has attaching sphere ∂D and D lies in the attaching sphere of $H^{(3)}$. Then $H^{(1)}$ and $H^{(2)}$ are complementary handles and may be eliminated. After rearranging handles, we get a new decomposition with $H^{(1)}$ eliminated and a new $H^{(3)}$ introduced.

Case 2: $1 < p \leq n-4$. As $H_p(W, M) = 0$ and there are no handles of index $< p$, the boundary map $\partial\colon \mathcal{C}_{p+1} \to \mathcal{C}_p$ must be onto. If $H^{(p)}$ is a p-handle, then $\sum_i n_i \varepsilon(H_i^{(p+1)}, H^{(p)}) = 1$ for some chain $\sum_i n_i H_i^{(p+1)}$ in $\mathcal{C}_{p+1}$. Thus we can use handle addition to get $\varepsilon(H_i^{(p+1)}, H^{(p)}) = 1$ for some $(p+1)$-handle $H_i^{(p+1)}$ and $\varepsilon(H_j^{(p+1)}, H^{(p)}) = 0$ for $j \neq i$. We may then cancel $H^{(p)}$ and $H_i^{(p+1)}$.

Proof of the h-Cobordism Theorem. Let $(W; M, M')$ be a compact h-cobordism, $\dim W \geq 6$, with W is simply connected. Let $(W; M, M') = C \cup H_1 \cup \cdots \cup H_{n-1} \cup C'$ be a handle decomposition with no 0- or n-handles. Use Lemma 6.9 to eliminate handles of index $\leq n-4$, leaving only handles of index $(n-3)$, $(n-2)$, and $(n-1)$. Thus the dual handle decomposition only has handles of index 1, 2, and 3. Eliminate the dual 1-handles from the dual decomposition, leaving only dual 2- and 3-handles. Since we are working over the integers, the matrix of the boundary map from $\mathcal{C}_3$ to $\mathcal{C}_2$, which is invertible over $\mathbb{Z}$, may be diagonalized, and the elementary operations may be realized by handle additions. Hence, we arrive at a handle decomposition with only complementary handles in dimensions 2 and 3, which may be cancelled as well, leaving a decomposition with no handles. Hence, W is a product $M \times I$.

Corollary 6.10. (Strong Poincaré Conjecture) [3],[55] *If M is a k-connected, closed PL n-manifold, $n \geq 6$, $k = \lfloor n/2 \rfloor$, then M is PL homeomorphic to S^n.*

Proof. Remove the interiors of disjoint, *PL* n-balls from M. The result is an h-cobordism between boundary spheres, which is a product $S^{n-1} \times I$.

If W is not simply connected, one must define incidence numbers as elements of the group ring $\mathbb{Z}[\pi_1(M)]$, and the proof of the h-cobordism theorem may break down at the last step, when we eliminate the dual 2- and 3-handles. In this case the matrix of the boundary map is a non-singular matrix over $\mathbb{Z}[\pi_1(M)]$, and it can be diagonalized if, and only if, we are given the additional hypothesis that the inclusion $M \subseteq W$ is a simple homotopy equivalence. An h-cobordism $(W; M, M')$ with this property is called an *s-cobordism.*

Theorem 6.11. (s-Cobordism Theorem) *Suppose that $(W; M, M')$ is an h-cobordism, $n \geq 6$. Then $(W; M, M') \cong (M \times I; M \times \{0\}, M \times \{1\})$ if, and only if, $(W; M, M')$ is also an s-cobordism.*

There are also relative versions of the h- and s-cobordism theorems. There proofs proceed very much the same as in the absolute case. A *relative cobordism* between $(n-1)$-manifolds M and M' with boundaries is a triple $(W; M, M')$, where W is an n-manifold such that $\partial W = M \cup V \cup M'$, where $M \cap M' = \emptyset$, $\partial V = \partial M \cup \partial M'$, $V \cap M = \partial M$, and $V \cap M' = \partial M'$. A relative cobordism is an h-cobordism (s-cobordism) if all of the inclusions $M, M' \subseteq W$, $\partial M, \partial M' \subseteq V$ are homotopy equivalences (simple homotopy equivalences).

Theorem 6.12. (Relative s-Cobordism Theorem) *Suppose W is a compact relative h-cobordism, $n \geq 6$, such that $V \cong \partial M \times I$. Then $(W; M, M') \cong (M \times I; M \times \{0\}, M \times \{1\})$ if, and only if, $(W; M, M')$ is also an s-cobordism.*

7. Isotopies, Unknotting

In this section we explore further the question: Given polyhedra X and Y, when are homotopic embeddings $f, g\colon X \to Y$ ambient isotopic? The first result is Zeeman's unknotting theorem [65] .

Theorem 7.1. [65] *A PL sphere pair (S^n, S^q) or a proper PL ball pair (B^n, B^q), $n-q \geq 3$ is unknotted.*

Proof. [50] The proof is by induction (eventually) on n. The case $n = 3$ is trivial. The case $n = 4$ follows from Theorem 4.5(i), for sphere pairs, and Theorem 4.7, for ball pairs. Theorem 4.5(ii) gives the result for sphere pairs when $n = 5$, and the Regular Neighborhood Theorem for Pairs can, in turn, be used to show that ball pairs unknot.

Assume, then, that $n \geq 6$. Let (B^n, B^q) be a ball pair, $n-q \geq 3$. By induction, (B^n, B^q) is locally flat. Hence if N is a regular neighborhood of B^q in B^n, then, by 3.22, (N, B^q) is an unknotted ball pair. Let $W = \mathcal{C}\ell(B^n - N)$, $M = N \cap W$. Then W is a relative cobordism between $(M, \partial M)$ and $(M', \partial M')$, where $M' = \mathcal{C}\ell((\partial B^n \cap W) - N')$ and N' is a small collar of ∂M in $\partial B^n \cap W$. Since B^q is contractible and $B^n - B^q$ is simply connected, by general position, W is an h-cobordism. Thus $W \cong M \times I$ and so (B^n, B^q) is unknotted.

Suppose (S^n, S^q) is a sphere pair, $n \geq 6$, and $n - q \geq 3$. Let $K > L$ be a triangulation of $S^n \supseteq S^q$, and let v be a vertex of L. Then $(\mathrm{st}(v, K), \mathrm{st}(v, L))$ is an unknotted ball pair, as is the complementary pair. Thus (S^n, S^q) is obtained by gluing two unknotted ball pairs together.

Corollary 7.2 *Any proper (n, q)-manifold pair is locally flat, if $n - q \geq 3$.*

Since the Whitehead group of $\mathbb{Z}$ is trivial, any h-cobordism between manifolds with fundamental group $\mathbb{Z}$ is an s-cobordism. Thus the proof of Theorem 7.1 works for locally flat sphere pairs in codimension 2, provided the pair is "homotopically unknotted."

Theorem 7.3. *Suppose (S^n, S^{n-2}), $n \geq 6$, is a locally flat sphere pair. Then (S^n, S^{n-2}) is unknotted if, and only if, $S^n - S^{n-2}$ has the homotopy type of a circle.*

We consider next the weaker question: When are isotopic embeddings of X into M ambient isotopic? (See Section 3 for definitions.) The answer is yes whenever the isotopy is "locally extendable". Given an isotopy $F\colon X \times I \to M \times I$, (or a proper isotopy of pairs) and a point $(x, t) \in X \times I$, F is *locally extendable* at (x, t) if there are neighborhoods V of x in X, U of $F(x, t)$ in M, and J of t in I and a level preserving embedding $h\colon U \times J \to M \times J$ such that $h(y, t) = (y, t)$ for all $y \in U$ and $h(F(x, s), s) = F(x, s)$ for $(x, s) \in V \times J$. In other words, local collars of $F(X \times \{t\})$ in $F(X \times [t, 1])$ (if $t < 1$) and in $F(X \times [0, t])$ (if $t > 0$) are extendable (locally) to collars of $M \times \{t\}$ in $M \times [t, 1]$ and $M \times [0, t]$. By Theorem 3.19, a locally extendable isotopy is extendable at $X \times \{t\}$ for all $t \in I$, meaning we can choose $V = X$ and $U = M$. This fact together with a standard compactness argument proves the following extension theorem.

Theorem 7.4. *If X is a compact polyhedron and $F\colon X \to Y$ is a locally extendable isotopy, then F is ambient.*

If Q and M are PL q- and n-manifolds, respectively, with $n - q \geq 3$, then, by Zeeman's Unknotting Theorem, any isotopy of Q in M is locally extendable. This fact, together with Proposition 3.12 establishes the following corollary.

Corollary 7.5. [32]*Suppose Q and M are PL q- and n-manifolds, respectively, with $n-q \geq 3$, and Q is compact. Then any proper isotopy of $(Q, \partial Q)$ in $(M, \partial M)$ is ambient.*

If X and Y are polyhedra and X is compact, a proper PL embedding $f\colon ((v{*}X), X) \to (w{*}Y, Y)$ is *unknotted* if there is a PL homeomorphism $h\colon w{*}Y \to w * Y$ fixing Y such that hf is the cone on $f|X$. Lickorish's Cone Unknotting Theorem [38] is a polyhedral analogue of Theorem 7.1. We state it next without proof.

Theorem 7.6. [38] *Suppose that X is a compact $(q-1)$-dimensional polyhedron and $f\colon (v * X, X) \to (B^n, S^{n-1})$ is a proper embedding, $n - q \geq 3$. Then f is unknotted.*

Theorem 7.6 allows one to prove that a PL isotopy of a polyhedron in a PL manifold in codimension ≥ 3 is locally extendable. Thus, the line of argument above can be used to prove the following isotopy extension theorem due to Hudson [30].

Corollary 7.7. [30] *Suppose that X is a compact q-dimensional polyhedron, M is a PL n-manifold, $n - q \geq 3$. Then any isotopy of X in M is ambient.*

An embedding $F\colon X \times I \to Y \times I$ satisfying $F^{-1}(Y \times \{i\}) = X \times \{i\}$, $i = 0, 1$, is a *concordance* of X in Y (from F_0 to F_1). Thus, an isotopy of X in Y is a level-preserving concordance. Hudson [31] obtains the following improvement of Corollary 7.7. Rourke [47] gives a "handle straightening" argument for this result as well.

Theorem 7.8. [31] *Suppose that X is a compact q-dimensional polyhedron, M is a PL n-manifold, $n - q \geq 3$. Then concordant embeddings of X in M are ambient isotopic.*

Corollary 7.9. *Suppose that Q is a compact PL q-manifold, M is a PL n-manifold, $n - q \geq 3$, and $f\colon (Q, \partial Q) \to (M, \partial M)$ is a (proper) PL embedding such that f is $(2q - n + 1)$-connected. If $g\colon (Q, \partial Q) \to (M, \partial M)$ is a PL embedding that is homotopic to f rel ∂Q, then f and g are ambient isotopic.*

8. Approximations, Controlled Isotopies

Many of the results of Sections 5, 6, and 7 have "controlled" analogues. In this section we state without proof a few of the basic theorems of this type. The first result is Miller's Approximation Theorem [42].

Theorem 8.1. [42] *Suppose that Q is a PL q-manifold, M is a PL n-manifold, $n - q \geq 3$, and $f\colon (Q, \partial Q) \to (M, \partial M)$ is a topological embedding. Then for every $\epsilon\colon Q \to (0, \infty)$, there is a PL embedding $g\colon (Q, \partial Q) \to (M, \partial M)$ such that $d(f, g) < \epsilon$.*

(This result was initially announced by T. Homma [27] , but a problem was discovered in his proof by H. Berkowitz. A proof along the lines originally presented by Homma can be found in [9].) Using Miller's theorem for q-cells, Bryant [8] was able to extend Miller's theorem to embeddings of polyhedra.

Theorem 8.2. [8] *Suppose that X is a q-dimensional polyhedron and $f\colon X \to M$ is a topological embedding into a PL n-manifold. Then for every $\epsilon\colon Q \to (0,\infty)$, there is a PL embedding $g\colon X \to M$ such that $d(f,g) < \epsilon$.*

The next theorem is an ϵ version of the isotopy theorems of Section 7. This result is an amalgamation of results due primarily to Connelly [18] and Miller [41], with contributions and improvements due to Cobb [14] , Akin [1] , and Bryant-Seebeck [10].

Theorem 8.3. [18, 41] *Suppose that (X,Y) is a polyhedral pair, $\dim Y < \dim X = q$, M is a PL n-manifold, and $f\colon (X,Y) \to (M,\partial M)$ is a proper topological embedding, $n-q \geq 3$. Then for every $\epsilon\colon X \to (0,\infty)$ there is a $\delta\colon X \to (0,\infty)$ such that if $g_i\colon (X,Y) \to (M,\partial M)$ are PL embeddings, $i = 0,1$, within δ of f, then there is an ϵ-push H of M such that $Hg_0 = g_1$.*

There are useful variations on Theorem 8.3. For example, if $\dim(X-Y) \leq n-3$, $f|Y$ is a *PL* embedding, and $g_i|Y = f|Y$, $i = 0,1$, then one can get an ϵ-push H of M rel ∂M with $Hg_0 = g_1$.

There are a number of controlled versions of the engulfing theorem (Theorem 5.2), although mostly they have been replaced by Quinn's End Theorem [46]. Bing's article "Radial Engulfing" [5], contains a variety of such theorems the reader is encouraged to survey. We state one such result from [10]. It requires a definition: A subset Y of a space X is *1-LCC* (1-locally co-connected) in X if for each $x \in Y$ and each neighborhood U of x in X, there is a neighborhood V of x in X such that the inclusion $\pi_1(V-Y) \to \pi_1(U-Y)$ is trivial. An embedding $f\colon Y \to X$ is 1-LCC if $f(Y)$ is 1-LCC in X.

Theorem 8.4. [10] *Suppose $f\colon X \to M$ is a 1-LCC embedding of a q-dimensional polyhedron X into a PL n-manifold M, $n-q \geq 3$, $n \geq 5$. Then for every $\epsilon > 0$, there is a $\delta > 0$ such that if $g\colon X \to M$ is a PL embedding within δ of f and U is any neighborhood of $g(X)$, then there is an ϵ-push H of M such that $H_1(f(X)) \subseteq U$.*

If X is not compact then ϵ and δ are functions of X. The 1-LCC condition allows one to push $f(X)$ off of the 2-skeleton of a neighborhood of $f(X)$ and then close to the dual $(n-3)$-skeleton. One then engulfs the dual $(n-3)$-skeleton with U. These are ϵ versions of Stallings arguments in [57].

Using Theorems 8.2, 8.3, and 8.4, and an infinite process due to Homma [26] and Gluck [23] , one can deduce the following "taming" theorem of Bryant-Seebeck [10] .

Theorem 8.5. [10] *Suppose $f\colon X \to M$ is a 1-LCC embedding of a q-dimensional polyhedron X into a PL n-manifold M, $n-q \geq 3$, $n \geq 5$. Then for every $\epsilon > 0$, there is an ϵ-push H of M such that $H_1 f$ is PL.*

There is a 4-dimensional analogue of this result due to R. D. Edwards (unpublished), obtained from Casson-Freedman handle theory for 4-manifolds [21] .

9. Triangulations of Manifolds

We conclude this chapter with a discussion of the two most central issues in *PL* topology: existence and uniqueness of triangulations of topological manifolds. Classically, these questions dealt with *PL* triangulations of manifolds, although there are obvious related questions concerning triangulations in general. Existence and uniqueness of *PL* triangulations of manifolds of dimensions ≤ 3 have been known for some time, the case $n = 3$ being due to Moise [44] and Bing [4]. The uniqueness question (the Hauptvermutung) can be asked for polyhedra in general. Milnor was the first to construct examples of polyhedra that are homeomorphic, but not piecewise linearly homeomorphic [43]. Edwards was able to exhibit a non-combinatorial triangulation of S^n, $n \geq 5$, by showing that the k-fold suspension of a non-simply connected 3-manifold constructed by Mazur [40] is topologically homeomorphic to S^{k+3}. (See [19].) Edwards and Cannon [11] solved the "Double Suspension Problem" in general by proving the following theorem.

Theorem 9.1. [11] *Suppose that H^n is a PL n-manifold with the homology of S^n. Then, for each $k \geq 1$, the polyhedron $S^k * H$ is topologically homeomorphic to S^{k+n+1}.*

Whenever H is not simply connected, which can happen when $n \geq 3$, the polyhedron $S^k * H$ is not *PL* homeomorphic to the standard S^{k+n+1}. Thus, uniqueness of triangulations of topological manifolds fails if one does not require triangulations to be *PL*. The problem of uniqueness of *PL* triangulations is more subtle yet.

Suppose that M and N are *PL* n-manifolds and $h\colon M \to N$ is a topological homeomorphism. Sullivan's idea [58],[59] was to prove that M and N are *PL* homeomorphic by taking a handle decomposition of M and, inductively, "straightening" their images under h. This idea presents a *handle problem*, that is, a topological homeomorphism $h\colon B^k \times \mathbb{R}^{n-k} \to V^n$ onto a *PL* manifold V^n that is *PL* on a neighborhood of $S^{k-1} \times \mathbb{R}^{n-k}$. The handle can be *straightened* if there is an isotopy H of V^n, fixed on a neighborhood of $S^{k-1} \times \mathbb{R}^{n-k}$ and outside a compact set, such that $H_1 h$ is *PL* on $h\colon B^k \times B^{n-k}$. Sullivan showed that, for $n \geq 5$, there was a possible $\mathbb{Z}/2$ obstruction to straightening 3-handles [58]. In his solution to the annulus conjecture, Kirby [36] showed how to straighten 0-handles when $n \geq 5$. Kirby and Siebenmann (see [37]) proved that k-handles can be straightened provided $n \geq 5$ and $k \neq 3$. Whether or not 3-handles could be straightened depended upon the Hauptvermutung for the n-torus $T^n = S^1 \times \cdots S^1$. The following result of Hsiang and Shaneson [28] , Wall [61] and Casson classifies the *PL* structures on T^n, showing, in particular, that they are not all equivalent.

Theorem 9.2. *For $n \geq 5$, the set of PL equivalence classes of PL manifolds topologically homeomorphic to T^n, is in one-to-one correspondence with the set of orbits of $(\Lambda^{n-3}\mathbb{Z}^n) \otimes \mathbb{Z}/2$ under the natural action of $GL(n, \mathbb{Z})$. The standard torus corresponds to the zero element under this action.*

($\Lambda^k \mathbb{Z}^n$ denotes the kth exterior algebra on $\mathbb{Z}^n$.) In particular there are non-standard T^5's. The classification implies that if M is a non-standard, or fake, torus, then even covers of M will be standard, while odd covers are not. Kirby [36] used the first fact in his "torus trick" to prove the annulus conjecture, while the second fact is used to disprove the Hauptvermutung. (See [37].)

Theorem 9.3. [37] *Given a PL n-manifold M, $n \geq 6$ or $n \geq 5$ if $\partial M = \emptyset$, the isotopy classes of PL structures on M are in one-to-one correspondence with the elements of $H^3(M; \mathbb{Z}/2)$.*

The coefficient group $\mathbb{Z}/2$ appears as the homotopy group $\pi_3(\mathrm{TOP/PL})$, where TOP/PL is the fiber of the forgetful map $B_{\mathrm{TOP}} \to B_{\mathrm{PL}}$ of classifying spaces for topological and piecewise linear bundles. Further handle analysis leads to the following existence theorem of Kirby-Siebenmann.

Theorem 9.4. [37] *Given a topological n-manifold M, $n \geq 6$, or $n \geq 5$ if ∂M has a PL triangulation, there is a well-defined obstruction in $H^4(M; \mathbb{Z}/2)$ to triangulating M as a PL manifold extending the triangulation on ∂M.*

There are, in fact, topological 4-manifolds that do not admit *PL* triangulations. One such may be constructed using results of Kervaire and Freedman. Kervaire shows [35] that there is a homology 3-sphere H, the Poincaré 3-sphere, that bounds a parallelizable *PL* 4-manifold M with signature 8. Freedman shows [21] that H bounds a contractible topological 4-manifold M'. $V = M \cup_H M'$ is then a closed 4-manifold with signature 8. A result of Rohlin (see [34]) states that V cannot have a *PL* triangulation, for, if it did, its signature would be divisible by 16. (A manifold is *parallelizable* if its tangent bundle is a product bundle.)

It is clear that if M has a *PL* triangulation, then so does $M \times \mathbb{R}^k$ for $k \geq 1$. Kirby-Siebenmann prove a very strong converse to this fact for higher dimensional topological manifolds.

Theorem 9.5. (Product Structure Theorem) [37] *Suppose that M is a topological n-manifold and that $M \times \mathbb{R}^k$, $k \geq 1$, is triangulated as a PL manifold. If $n \geq 6$, or $n \geq 5$ and $\partial M = \emptyset$, then there is a PL triangulation of M inducing an equivalent triangulation on $M \times \mathbb{R}^k$.*

There are relative versions of the Product Structure Theorem, which the reader may find in [37] I, section 5. The following, rather obvious corollary has proved useful in applications.

Corollary 9.6. *Suppose that M is a topological n-manifold with boundary, $n \geq 6$, such that* int M *has a PL triangulation. Then M has a PL triangulation.*

In particular, any topological, n-dimensional submanifold (with boundary) of a *PL* n-manifold M, $n \geq 6$, has a *PL* structure. Likewise, if a compact subset C of a topological n-manifold, $n \geq 6$, has vanishing Čech cohomology in dimension 4, then naturality of the obstruction in Theorem 9.4 implies that sufficiently small manifold neighborhoods of C have *PL* structures.

The question as to whether a topological n-manifold has a triangulation (*PL* or not) has been investigated extensively by Galewski and Stern. (See, e.g., [22].) Let θ_3^H denote the group obtained from oriented *PL* homology 3-spheres, under the operation of connected sum, #, modulo those that bound acyclic *PL* 4-manifolds. There is a homomorphism $\mu \colon \theta_3^H \to \mathbb{Z}/2$ (the Kervaire-Milnor-Rohlin map) defined by $\mu(H) = [\sigma(W)/8]$, where $\sigma(W)$ denotes the signature of any parallelizable 4-manifold W with boundary H. If H is the Poincaré homology 3-sphere, then $\mu(H) = 1$, so that μ is surjective.

Theorem 9.7. [22] *Suppose M is a topological n-manifold, $n \geq 6$ or $n \geq 5$ if ∂M is triangulated. Then there is an element $t_M \in H^5(M, \partial M; \ker \mu)$ such that $t_M = 0$ iff there is a triangulation of M compatible with the given triangulation on ∂M. Moreover, the set of concordance classes of triangulations of M rel ∂M is in one-to-one correspondence with the elements of $H^4(M, \partial M; \ker \mu)$.*

Triangulations K_0 and K_1 of M are *concordant* if there is a triangulation K of $M \times I$ restricting to K_i on $M \times \{i\}$, $i = 0, 1$.

Galewski and Stern [22] and Matumoto [39] have shown that all compact topological n-manifolds ($n \geq 6$ or $n \geq 5$ if ∂M is triangulated) can be triangulated if there is a homology 3-sphere H such that $\mu(H) = 1$ and $H\#H$ bounds a parallelizable 4-manifold with signature 0. At the time of this writing it is unknown whether such a 3-manifold exists or whether every topological n-manifold, $n \geq 5$, can be triangulated. Casson, however, has found a topological 4-manifold that cannot be triangulated [13].

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