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REGULAR HOMOTOPY CLASSES OF IMMERSED SURFACES

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§1. INTRODUCTION

In this paper we are concerned with the problem of classifying compact surfaces immersed in $\mathbb{R}^n$ up to regular homotopy†. This subject started in 1958 when Smale classified the immersions of the 2-sphere [17]. For $n \geq 4$ the problem was then completely solved by Hirsch ([8], theorems 8.2 and 8.4): if M^2 is a compact surface then for $n \geq 5$ any two immersions $f, g: M^2 \to \mathbb{R}^n$ are regularly homotopic, while the immersions $f: M^2 \to \mathbb{R}^4$ are completely classified by their normal class.

Concerning immersed surfaces in $\mathbb{R}^3$ two results are known:

(1) If M^2 is a compact surface, $h = \dim H_1(M^2, \mathbb{Z}_2)$ then the space $I(M^2, \mathbb{R}^3)$ of immersions $f: M^2 \to \mathbb{R}^3$ has 2^h connected components (James and Thomas [9], see also [16]).‡

(2) The cobordism group of compact surfaces immersed in $\mathbb{R}^3$ is $\mathbb{Z}_8$ (Wells [18]).

From the viewpoint of geometry the classification 1) is too fine in some sense, because it often happens that for given immersions $f, g: M^2 \to \mathbb{R}^3$ f is not regularly homotopic to g, but f is regularly homotopic to $g \circ \varphi$ where $\varphi: M^2 \to M^2$ is a diffeomorphism. From the geometric viewpoint however g and $g \circ \varphi$ are just two different parametrizations of "the same surface". On the other hand the cobordism classification (2) of immersed surfaces is too coarse in some sense, because a cobordism between immersed surfaces can change the topological type. These considerations may motivate the following definitions:

Let M^2 be a surface. An *immersed surface* in $\mathbb{R}^n$ of type M^2 is defined as an equivalence class of immersions $f: M^2 \to \mathbb{R}^n$, where two immersions $f, g: M^2 \to \mathbb{R}^n$ are considered as equivalent if there is a diffeomorphism $\varphi: M^2 \to M^2$ such that $f = g \circ \varphi$. For any immersion $f: M^2 \to \mathbb{R}^n$ we will denote the corresponding immersed surface by $[f]$.

If $f, g: M^2 \to \mathbb{R}^n$ are two immersions the immersed surfaces $[f]$ and $[g]$ are said to be regularly homotopic§ if f is regularly homotopic to $g \circ \varphi$ for some diffeomorphism $\varphi: M^2 \to M^2$. It is easy to see that regular homotopy defines an equivalence relation on the set of all immersed surfaces in $\mathbb{R}^n$ of type M^2.

Our aim is to classify compact immersed surfaces in $\mathbb{R}^n$ up to regular homotopy. For $n \geq 4$ the results of Hirsch cited above imply that two immersed surfaces $[f]$, $[g]$ in $\mathbb{R}^n$ are regularly homotopic if and only if f and g are regularly homotopic, so in this case the problem is solved. Therefore from now on we will concentrate on the case $n = 3$.

Let M^2 be a compact surface. In section 2 we will associate to any immersion $f: M^2 \to \mathbb{R}^3$ a $\mathbb{Z}_4$-valued quadratic form

$$q_f: H_1(M^2, \mathbb{Z}_2) \to \mathbb{Z}_4. \tag{1}$$

To say that q_f is a quadratic form means that for all $x, y \in H_1(M^2, \mathbb{Z}_2)$ we have

$$q_f(x+y) = q_f(x) + q_f(y) + 2x \cdot y. \tag{2}$$

† Throughout the term "surface" stands for "connected smooth 2-manifold without boundary". A regular homotopy is a smooth homotopy that is an immersion at each stage.

‡ Recently Hass and Hughes [7] obtained generalizations of the result of James and Thomas for immersions of surfaces into arbitrary 3-manifold.

§ A similar notion (called there "image homotopy") has been used in [10] to classify immersions of bounded surfaces into the plane.

Here $x \cdot y \in \mathbb{Z}_2$ denotes the intersection product of x and y and $2x \cdot y$ is the image of $x \cdot y$ under the natural inclusion $\mathbb{Z}_2 \to \mathbb{Z}_4$. In the context of cobordism of immersed surfaces the usefulness of the above quadratic form was already suggested by Sullivan, as is reported in [4]. Our key result concerning the classification of immersed surfaces in $\mathbb{R}^3$ up to regular homotopy is

THEOREM 2. *Let M^2 be a compact surface, $f, g: M^2 \to \mathbb{R}^3$ two immersions. Then*

(a) *f and g are regularly homotopic if and only if $q_f = q_g$.*

(b) *For every quadratic form $q: H_1(M^2, \mathbb{Z}_2) \to \mathbb{Z}_4$ there exists an immersion $f: M^2 \to \mathbb{R}^3$ such that $q = q_f$.*

(c) *The immersed surfaces $[f]$ and $[g]$ are regularly homotopic if and only if there is a linear automorphism $\alpha: H_1(M^2, \mathbb{Z}_2) \to H_1(M^2, \mathbb{Z}_2)$ such that $q_f = q_g \circ \alpha$.*

Theorem 2 reduces the problem of classifying immersed surfaces in $\mathbb{R}^3$ up to regular homotopy to the purely algebraic problem of classifying the $\mathbb{Z}_4$-valued quadratic forms on the inner product space $H_1(M^2, \mathbb{Z}_2)$. This algebraic classification is carried out in section 3. It turns out that the equivalence class of a quadratic form $q: H_1(M^2, \mathbb{Z}_2) \to \mathbb{Z}_4$ is completely determined by the value of the Gaussian sum

$$Q = \frac{1}{\sqrt{2^h}} \sum_{x \in H_1(M^2, \mathbb{Z}_2)} e^{2\pi i q(x)/4}. \tag{3}$$

Here we have set $h = \dim H_1(M^2, \mathbb{Z}_2)$. Q is called the *Arf invariant* of the quadratic form q.

THEOREM 3. (a) *Q is always an eighth root of unity:*

$$Q = e^{2\pi i \sigma/8}, \quad \sigma \in \mathbb{Z}_8$$

(b) *If $q, \tilde{q}$ are two quadratic forms on $H_1(M^2, \mathbb{Z}_2)$ then there exists an automorphism $\alpha: H_1(M^2, \mathbb{Z}_2) \to H_1(M^2, \mathbb{Z}_2)$ such that $\tilde{q} = q \circ \alpha$ if and only if the corresponding Arf invariants Q and $\tilde{Q}$ are equal.*

In section 4 we translate the results of the above classification back into geometric terms. If F and G are regular homotopy classes of immersed surfaces then we define the connected sum $F \# G$. This turns the set of all regular homotopy classes of compact immersed surfaces in $\mathbb{R}^3$ into an abelian semigroup H. The zero element of H is represented by any immersed sphere in $\mathbb{R}^3$. A set of generators for H is shown in Fig. 1. S is of course a standardly embedded torus. T is an immersed torus obtained by rotating a plane lemniscate around a vertical axis while rotating it also in its plane. B is the famous Boy surface [1, 3] (we have removed a disk so that one can see the inside), $\bar{B}$ is the mirror image of B.

THEOREM 4. (a) *Every compact orientable immersed surface in $\mathbb{R}^3$ that is not a sphere is regularly homotopic to a connected sum of several copies of S and T.*

(b) *Every compact nonorientable immersed surface in $\mathbb{R}^3$ is regularly homotopic to a connected sum of several copies of B and $\bar{B}$.*

(c) *By (a) and (b) the semigroup H is generated by S, T, B and $\bar{B}$. Defining relations for H are*

$$\begin{aligned} S \# S &= T \# T \\ B \# B \# B \# B &= \bar{B} \# \bar{B} \# \bar{B} \# \bar{B} \\ S \# B &= B \# B \# \bar{B} \\ S \# \bar{B} &= B \# \bar{B} \# \bar{B} \\ T \# B &= \bar{B} \# \bar{B} \# \bar{B} \\ T \# \bar{B} &= B \# B \# B. \end{aligned}$$

In section 5 we consider embedded surfaces. In section 6 we relate our classification of immersed surfaces up to regular homotopy to the cobordism classification. The key result is the following theorem:

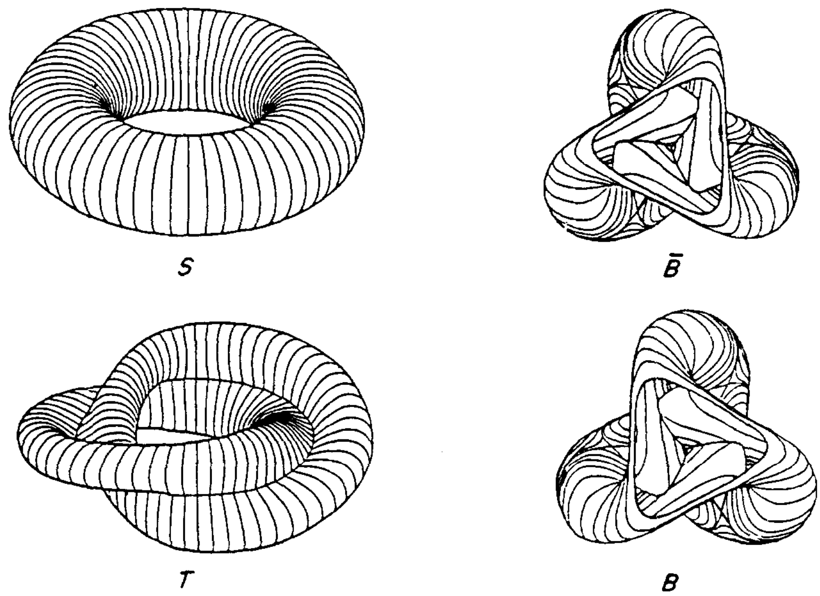

Fig. 1.

THEOREM 7. (a) *The Arf invariant* $Q_{[f]} \in \mathbb{Z}_8$ ($\mathbb{Z}_8$ = *multiplicative group of eighth roots of unity*) *of a compact surface immersed in* $\mathbb{R}^3$ *depends only on the cobordism class of* $[f]$.

(b) *The mapping of the cobordism group* $N_1(2)$ *of immersed surfaces in* $\mathbb{R}^3$ *onto* $\mathbb{Z}_8$ *defined by (a) is a group isomorphism.*

Since we do not use the results of [18] theorem 6 gives a new proof of the fact that $N_1(2)$ is isomorphic to $\mathbb{Z}_8$. Combining our theorems 2, 3 and 6 we obtain

THEOREM 8. *Two compact immersed surfaces in* $\mathbb{R}^3$ *of the same topological type are regularly homotopic if and only if they are cobordant.*

It is often much easier to find an explicit cobordism between two given immersed surfaces than to construct a regular homotopy (see the example at the end of section 6). Thus theorem 7 gives a convenient tool for determining in concrete situations the regular homotopy class of a given immersed surface in $\mathbb{R}^3$.

I would like to thank U. Abresch and W. Meyer for helpful discussions.

§2. THE QUADRATIC FORM OF AN IMMERSED SURFACE

Let M^2 be a compact surface, $f\colon M^2 \to \mathbb{R}^3$ an immersion. Our aim is to define a quadratic form $q_f\colon H_1(M^2, \mathbb{Z}_2) \to \mathbb{Z}_4$ that depends only on the regular homotopy class of f. As a first step we define a function $\tilde{q}_f\colon \Gamma \to \mathbb{Z}_4$ where Γ is the set of all smooth embeddings $\gamma\colon S^1 \to M^2$.

Let $\Gamma_f \subset \Gamma$ denote the set of all embeddings $\gamma\colon S^1 \to M^2$ such that $f \circ \gamma$ is also an embedding. For each $\gamma \in \Gamma_f$ there exists a tubular neighborhood N_γ of $\gamma(S^1)$ in M^2 such that $f|_{N_\gamma}$ is an embedding. Now we define $\tilde{q}_f(\gamma) \in \mathbb{Z}_4$ by

$$\tilde{q}_f(\gamma) \equiv \mathrm{lk}\,(f \circ \gamma, f(\partial N_\gamma)). \tag{4}$$

Here lk denotes linking number in $\mathbb{R}^3$, which is defined for any two disjoint 1-cycles in $\mathbb{R}^3$ (see [15], p. 132 for several equivalent definitions of lk). lk is well-defined because we are working throughout with a fixed orientation of $\mathbb{R}^3$. Also for (4) to make sense we must orient $f(\partial N_\gamma)$. This is done in accordance with the orientation on $\gamma(S^1)$ defined by γ (see Fig. 2).

LEMMA 1. *Let* $f, g\colon M^2 \to \mathbb{R}^3$ *be two regularly homotopic immersions,* $\gamma \in \Gamma_f \cap \Gamma_g$. *Then*

$$\tilde{q}_f(\gamma) = \tilde{q}_g(\gamma). \tag{5}$$

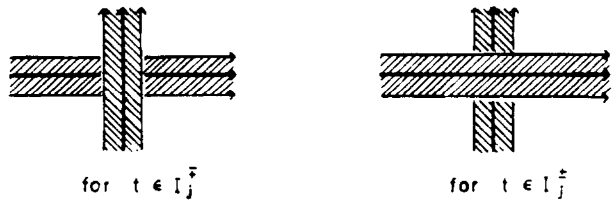

Fig. 2.

Proof. Let f_t: $M^2 \to \mathbb{R}^3$ for $0 \leq t \leq 1$ be a regular homotopy with $f_0 = f$ and $f_1 = g$. A transversality argument shows that by perturbing f_t slightly and choosing N_γ narrow enough we may assume the following behaviour of f_t:

(i) $f_t \circ \gamma$ fails to be an embedding only for a finite set $\{t_1, \ldots, t_k\}$ of values of t. For $1 \leq i \leq k$ the curve $f_{t_i} \circ \gamma$ has only one point $P_i \in \mathbb{R}^3$ of self-intersection.

(ii) There is an $\varepsilon > 0$ such that for $t \in [0, 1] - \cup (t_j - \varepsilon, t_j + \varepsilon)$ $f_t|_{N_\gamma}$ is an embedding and for $1 \leq j \leq k$ the homotopy $f_t|_{N_\gamma}$ behaves in a neighborhood of P_j as indicated in Fig. 2. We have set $I_j^- = [t_j - 2\varepsilon, t_j - \varepsilon]$, $I_j^+ = [t_j + \varepsilon, t_j + 2\varepsilon]$.

Using definition (3) given in [15] for the linking pairing it is clear that $\mathrm{lk}\,(f_{t_j-\varepsilon} \circ \gamma, f_{t_j-\varepsilon} f(\partial N_\gamma))$ and $\mathrm{lk}\,(f_{t_j+\varepsilon} \circ \gamma, f_{t_j+\varepsilon}(\partial N_\gamma))$ differ by ± 4. Because obviously the above linking number does not change at all on the intervals $[0, t_1 - \varepsilon]$, $[t_1 + \varepsilon, t_2 - \varepsilon], \ldots, [t_k + \varepsilon, 1]$ the lemma is proved. □

As a consequence of Lemma 1 we are now able to define $\tilde{q}_f$ on the whole of Γ, because for any $\gamma \in \Gamma$ we can achieve by a slight perturbation of f that $f \circ \gamma$: $S^1 \to \mathbb{R}^3$ is an embedding. Also by Lemma 1 $\tilde{q}_f: \Gamma \to \mathbb{Z}_4$ depends only on the regular homotopy class of f. Our next aim is to show that for $\gamma \in \Gamma$ $\tilde{q}_f(\gamma)$ depends only on $[\gamma]_{\mathbb{Z}_2}$ (for $R = \mathbb{Z}$ or $\mathbb{Z}_2$ $[\gamma]_R \in H_1(M^2, R)$ is defined as $\gamma_*[S^1]_R$ where $[S^1]_R$ is the canonical generator of $H_1(S^1, R)$).

Lemma 2. *Let M^2 be a compact surface, f: $M^2 \to \mathbb{R}^3$ a self-transversal immersion. Then there is a two dimensional submanifold V of M^2 with boundary such that*

(i) *$f|_V$ is an embedding*
(ii) *$M^2 - V$ is a disk.*

Proof. Choose a point $p \in M^2$ and smooth curves $\gamma_1, \ldots, \gamma_n$: $[0, 1] \to M^2$ such that $\gamma_j(0) = \gamma_j(1) = p$ and $\gamma_1, \ldots, \gamma_n$ generate the fundamental group of M^2. Furthermore one can assume that $\gamma_j|_{(0,1)}$ is an embedding for each j and that for $i \neq j$ we have $\gamma_i(0, 1) \cap \gamma_j(0, 1) = \emptyset$. Then we obtain V with the desired properties by smoothing the corners of a suitable tubular neighborhood of $\cup^j \gamma_j\,[0, 1]$. □

Let now M^2 be a compact surface, f: $M \to \mathbb{R}^3$ a self-transversal immersion, $V \subset M^2$ as in Lemma 2. Then according to Gordon and Litherland [6] we can define a symmetric bilinear form

$$G_V: H_1(V, \mathbb{Z}) \times H_1(V, \mathbb{Z}) \to \mathbb{Z} \tag{6}$$

as follows: Let $\hat{V}$ be the unit normal bundle of the embedding $f|_V$, π: $\hat{V} \to V$ the associated double covering of V. For every 1-simplex s in V let $\hat{s}$ denote the 1-chain $\hat{s} = \hat{s}_1 + \hat{s}_2$ in $\hat{V}$ where $\hat{s}_1$ and $\hat{s}_2$ are the two 1-simplices in $\hat{V}$ satisfying $s = \pi \circ \hat{s}_1$, $i = 1, 2$. It is easy to see that the correspondence $s \mapsto \hat{s}$ gives rise to a linear map τ: $H_1(V, \mathbb{Z}) \to H_1(\hat{V}, \mathbb{Z})$ (the "transfer map"). For $\delta > 0$ define $\hat{f}_\delta$: $\hat{V} \to \mathbb{R}^3$ by

$$\hat{f}_\delta(v) = f(\pi(v)) + \delta \cdot v. \tag{7}$$

Choose ε small enough such that the map

$$\begin{aligned} (0, \varepsilon] \times \hat{V} &\to \mathbb{R}^3 \\ (\delta, v) &\mapsto \hat{f}_\delta(v) \end{aligned} \tag{8}$$

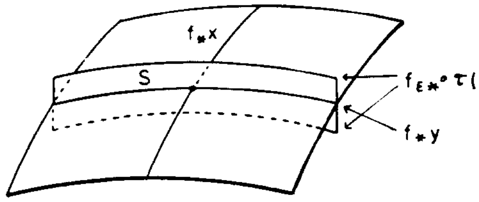

Fig. 3.

is an embedding. Then the *Goeritz form* G_V is defined by

$$G_V(x, y) = \mathrm{lk}\,(f_* x, \hat{f}_{\varepsilon *} \circ \tau(y)) \tag{9}$$

Here $\mathrm{lk}\colon H_1(f(V), \mathbb{Z}) \times H_1(\mathbb{R}^3 - f(V), \mathbb{Z}) \to \mathbb{Z}$ again denotes the linking pairing.

It is proved in [6] that G_V is symmetric. We need another property of G_V: For $x, y \in H_1(V, \mathbb{Z})$ let $x \cdot y \in \mathbb{Z}_2$ denote the mod 2 – intersection number of x and y.

LEMMA 3. $G(x, y) \equiv x \cdot y$ *mod* 2 *for all* $x, y \in H_1(V, \mathbb{Z})$.

Proof. Let $f_* x$ and $\hat{f}_{\varepsilon *} \circ \tau(y)$ be represented by cycles in $\mathbb{R}^3$ as indicated in Fig. 3.

Then the value of $G(x, y)$ modulo 2 can be determined by taking the mod 2 intersection product of $f_* x$ with any surface S spanning the link $\hat{f}_{\varepsilon *} \circ \tau(y)$, orientable or not. The assertion of the Lemma now becomes obvious if we choose S as indicated in Fig. 3. □

Thus $G(x, y)$ is determined modulo 2 by the images of x and y under the projection

$$p\colon H_1(V, \mathbb{Z}) \to H_1(V, \mathbb{Z}_2). \tag{10}$$

The value of $G(x, x)$ however is determined by $p(x)$ even modulo 4, as can be seen from the congruence

$$G(x + 2y, x + 2y) \equiv G(x, x) \bmod 4. \tag{11}$$

Thus there is a function $\tilde{G}_V\colon H_1(V, \mathbb{Z}_2) \to \mathbb{Z}_4$ making the following diagram commute:

$$\begin{tikzcd} H_1(V, \mathbb{Z}) \arrow[r, "x \mapsto G_V(x, x)"] \arrow[d, "p"'] & \mathbb{Z} \arrow[d, dashed] \\ H_1(V, \mathbb{Z}_2) \arrow[r, "\tilde{G}_V"] & \mathbb{Z}_4 \end{tikzcd} \tag{12}$$

As a consequence of Lemma 3 we have for all $x, y \in H_1(V, \mathbb{Z}_2)$

$$\tilde{G}_V(x + y) = \tilde{G}_V(x) + \tilde{G}_V(y) + 2x \cdot y, \tag{13}$$

that means $\tilde{G}_V$ is a quadratic form.

LEMMA 4. (a) *There is a unique map* $q_f\colon H_1(M^2, \mathbb{Z}_2) \to \mathbb{Z}_4$ *making the following diagram commute:*

$$\begin{tikzcd} \Gamma \arrow[r, "\tilde{q}_f"] \arrow[d] & \mathbb{Z}_4 \\ H_1(M^2, \mathbb{Z}_2) \arrow[ur, dashed, "q_f"] & H_1(V, \mathbb{Z}_2) \arrow[l, "i_*"'] \arrow[u, "\tilde{G}_V"'] \end{tikzcd} \tag{14}$$

(b) *In* (14) *the map* $\gamma \to [\gamma]_{\mathbb{Z}_2}$ *represented by the vertical arrow on the left is onto.*

(c) *The homomorphism* i_* *induced by the inclusion* $i\colon V \to M$ *is an isomorphism.*

Proof. The proof of (c) is left to the reader. (b) follows from known results concerning the representation of $\mathbb{Z}$-homology classes by simple closed curves [12]. Once we know by (c) that i_* is an isomorphism we can use the lower right triangle in (14) to define q_f. In order to see that with this definition the upper left triangle becomes commutative note that for every $\gamma \in \Gamma$ there exists a diffeotopy $\varphi_t: M^2 \to M^2$, $0 \le t \le 1$, $\varphi_0 = \mathrm{id}$ such that $\varphi_1(\gamma(S^1)) \subset V$. Then by Lemma 1 we have $\tilde{q}_f(\varphi_1 \circ \gamma) = \tilde{q}_f(\gamma)$. Also clearly $[\varphi_1 \circ \gamma]_{\mathbb{Z}_2} = [\gamma]_{\mathbb{Z}_2}$. Thus we have to show $q_f[\varphi_1 \circ \gamma]_{\mathbb{Z}_2} = \tilde{q}_f(\varphi_1 \circ \gamma)$, but this is clear from the definitions of $\tilde{q}_f$ and G_V. □

Lemma 5. (a) *q_f is independent of the choice of V and depends only on the regular homotopy class of f.*

(b) *For all $x, y \in H_1(M^2, \mathbb{Z}_2)$ we have*

$$q_f(x+y) = q_f(x) + q_f(y) + 2x \cdot y. \tag{15}$$

Proof. By Lemma 4(b) the map $\gamma \mapsto [\gamma]_{\mathbb{Z}_2}$ is onto, that means every $\mathbb{Z}_2$-homology class of M^2 can be represented by a simple closed curve. Thus q_f is completely determined by $\tilde{q}_f$. But we know by Lemma 1 that $\tilde{q}_f$ depends only on the regular homotopy class of f, hence the same is true for q_f. (b) follows from (13). □

Let $\hat{I}(M^2, \mathbb{R}^3)$ denote the set of all regular homotopy classes of immersions $f: M^2 \to \mathbb{R}^3$, $QF(M^2)$ the set of all quadratic forms $q: H_1(M^2, \mathbb{Z}_2) \to \mathbb{Z}_4$. By Lemma 5 the correspondence $f \mapsto q_f$ induces a map $\hat{q}: \hat{I}(M, \mathbb{R}^3) \mapsto QF(M^2)$.

Theorem 1. *Both $\hat{I}(M^2, \mathbb{R}^3)$ and $QF(M^2)$ are in a natural way affine spaces over the $\mathbb{Z}_2$-vector space $H^1(M^2, \mathbb{Z}_2)$.*

$$\hat{q}: \hat{I}(M^2, \mathbb{R}^3) \to QF(M^2) \tag{16}$$

is an affine isomorphism.

Proof, Step 1. We first describe the affine structure of $\hat{I}(M^2, \mathbb{R}^3)$. By Hirsch [8] there is a natural one-to-one correspondence between the elements of $\hat{I}(M^2, \mathbb{R}^3)$ and the homotopy classes of sections of the bundle $\mathrm{Inj}\,(TM^2, \mathbb{R}^3)$. Here $\mathrm{Inj}\,(M^2, \mathbb{R}^3)$ denotes the bundle over M^2 whose fibre at $p \in M$ consists of all linear injections of T_pM^2 into $\mathbb{R}^3$. The mentioned correspondence takes the regular homotopy class of an immersion $f: M^2 \to \mathbb{R}^3$ to the homotopy class of its differential $\mathrm{d}f$ (considered as a section of $\mathrm{Inj}\,(M^2, \mathbb{R}^3)$).

Now we fix a Riemannian metric on M^2 and denote by $\widetilde{\mathrm{Inj}}\,(TM^2, \mathbb{R}^3)$ the subbundle of $\mathrm{Inj}\,(TM^2, \mathbb{R}^3)$ whose fibre at $p \in M^2$ consists of all orthogonal injections of T_pM^2 into $\mathbb{R}^3$. We assert that the inclusion of $\widetilde{\mathrm{Inj}}\,(TM^2, \mathbb{R}^3)$ into $\mathrm{Inj}\,(TM^2, \mathbb{R}^3)$ is a fibre homotopy equivalence: Any injection $A: T_pM^2 \to \mathbb{R}^3$ can be decomposed uniquely as $A = \mathrm{Iso}\,(A) \circ \mathrm{Sym}\,(A)$ where $\mathrm{Iso}\,(A): T_pM^2 \to \mathbb{R}^3$ is an orthogonal injection and $\mathrm{Sym}\,(A): T_pM^2 \to T_pM^2$ is self-adjoint (set $\mathrm{Sym}\,(A) = (A^*A)^{1/2}$, $\mathrm{Iso}\,(A) = A(A^*A)^{-1/2}$). Then the fibre homotopy $\psi_t: \mathrm{Inj}\,(TM^2, \mathbb{R}^3) \to \mathrm{Inj}\,TM^2, \mathbb{R}^3)$

$$\psi_t(A) = \mathrm{Iso}\,(A) \circ [(1-t)\,\mathrm{Sym}\,(A) + t\,\mathrm{id}] \tag{17}$$

shows that $\mathrm{Iso}: \mathrm{Inj}\,(TM^2, \mathbb{R}^3) \to \widetilde{\mathrm{Inj}}\,(TM^2, \mathbb{R}^3)$ is a fibre homotopy inverse for the inclusion of $\widetilde{\mathrm{Inj}}\,(TM^2, \mathbb{R}^3)$ into $\mathrm{Inj}\,(TM^2, \mathbb{R}^3)$. Thus the homotopy classes of sections of $\widetilde{\mathrm{Inj}}\,(TM^2, \mathbb{R}^3)$ are in one-to-one correspondence with those of $\mathrm{Inj}\,(TM^2, \mathbb{R}^3)$, hence also with the elements of $\hat{I}(M^2, \mathbb{R}^3)$.

$\widetilde{\mathrm{Inj}}\,(TM^2, \mathbb{R}^3)$ is in a natural way a principal fibre bundle with group $SO(3)$: For $A \in \mathrm{Inj}\,(TM^2, \mathbb{R}^3)$, $g \in SO(3)$ define $g(A)$ as $g \circ A$. It is easy to check that with this $SO(3)$-action $\widetilde{\mathrm{Inj}}\,(TM^2, \mathbb{R}^3)$ becomes a principal bundle.

The set $[M^2, SO(3)]$ of homotopy classes of maps $h: M^2 \to SO(3)$ inherits a group structure from $SO(3)$. Furthermore the group $[M^2, SO(3)]$ acts in a natural way on the set of homotopy classes of sections of $\widetilde{\mathrm{Inj}}\,(TM^2, \mathbb{R}^3)$ and therefore (by the above correspondence) also on $\hat{I}(M^2, \mathbb{R}^3)$. It is not difficult to check that this group action is free and transitive.

Up to homotopy the maps $h: M^2 \to SO(3)$ are completely classified by the induced homomorphism

$$h_*: H_1(M^2, \mathbb{Z}_2) \to H_1(SO(3), \mathbb{Z}_2) \simeq \mathbb{Z}_2. \tag{18}$$

One can verify that the bijection of $[M^2, SO(3)]$ with $H^1(M^2, \mathbb{Z}_2)$ obtained in this way is actually a group isomorphism. Using this isomorphism we finally obtain a transitive free action of $H^1(M^2, \mathbb{Z}_2)$ on $\hat{I}(M^2, \mathbb{R}^3)$. With this action $\hat{I}(M^2, \mathbb{R}^3)$ becomes an affine space over $H^1(M^2, \mathbb{Z}_2)$. For $F, G \in \hat{I}(M^2, \mathbb{R}^3)$ we will denote by $\overrightarrow{FG}$ the unique element of $H^1(M^2, \mathbb{Z}_2)$ taking F to G.

Step 2. The structure of $QF(M^2)$ as an affine space over $H^1(M^2, \mathbb{R}_2)$ is obvious: The defining property (15) implies that the difference

$$q_2 - q_1 : H_1(M^2, \mathbb{Z}_2) \to \mathbb{Z}_4 \tag{19}$$

of any two quadratic forms $q_1, q_2 \in QF(M^2)$ is linear. Therefore it takes values in $2\mathbb{Z}_4 \simeq \mathbb{Z}_2$ and thus gives rise to a linear map

$$\overrightarrow{q_1 q_2} : H_1(M^2, \mathbb{Z}_2) \to \mathbb{Z}_2. \tag{20}$$

Step 3. To complete the proof of theorem 1 we only have to show that for any two regular homotopy classes $F, G \in \hat{I}(M^2, \mathbb{R}^3)$ we have

$$\overrightarrow{\hat{q}_F \hat{q}_G} = \overrightarrow{FG}\,. \tag{21}$$

(Every map $\hat{q}$ satisfying (21) is necessarily bijective).

Let $F, G \in \hat{I}(M^2, \mathbb{R}^3)$ be arbitrary and choose immersions $f, g \colon M^2 \to \mathbb{R}^3$ representing F and G respectively. Let $x \in H_1(M^2, \mathbb{Z}_2)$ be any homology class, $\gamma \in \Gamma$ an embedding of S^1 in M^2 with $[\gamma]_{\mathbb{Z}_2} = x$. By applying regular homotopies to f and to g we can assume that both $f \circ \gamma$ and $g \circ \gamma$ are given by

$$f \circ \gamma(e^{i\varphi}) = g \circ \gamma(e^{i\varphi}) = (\cos\varphi, \sin\varphi, 0). \tag{22}$$

To prove the last statement one can proceed in two steps:

(1) Any two immersions of S^1 in $\mathbb{R}^3$ are regularly homotopic, hence in particular the immersions $f \circ \gamma$ and $g \circ \gamma$ are regularly homotopic to the one given by (22).

(2) An obvious modification of the proof of Lemma 3.4 in [8] shows that the homotopies $S^1 \times I \to \mathbb{R}^3$ referred to in (1) can be written as $f_t \circ \gamma$ and $g_t \circ \gamma$ where $f_t, g_t \colon M^2 \to \mathbb{R}^3$, $t \in [0, 1]$ are regular homotopies.

Let $h \colon M^2 \to SO(3)$ be defined by the property

$$\mathrm{Iso}\,(dg_p) = h(p) \circ \mathrm{Iso}\,(df_p). \tag{23}$$

Then according to our definitions $\overrightarrow{FG}\,(x)$ depends on the homotopy class of $h \circ \gamma \colon S^1 \to SO(3)$:

$$\overrightarrow{FG}\,(x) = \begin{cases} 0 & \text{If } h \circ \gamma \text{ is homotopic to zero} \\ 1 & \text{otherwise.} \end{cases} \tag{24}$$

In order to describe $h \circ \gamma$ more explicitly we assume that for each $p \in \gamma(S^1)$ we have $df_p = \mathrm{Iso}\,(df_p)$. (This can be achieved by an application of [8] theorem 5.9). Then with the definition $\ell\,(e^{i\varphi}) = (-\sin\varphi, \cos\varphi, 0)$ for all $u \in S^1$ we have

$$(f \circ \gamma)'|_u = (g \circ \gamma)'|_u = \ell\,(u) \tag{25}$$

$$h \circ \gamma(u)\,\ell\,(u) = \ell\,(u)$$

and therefore for all $\mathscr{Y} \in \mathbb{R}^3$

$$h \circ \gamma(u)\,\mathscr{Y} = \cos\psi(u)\,\mathscr{Y} + \sin\psi(u)\,\ell\,(u) \times \mathscr{Y}. \tag{26}$$

Here $\psi \colon S^1 \to \mathbb{R}$ is a (possibly discontinuous) map such that $u \mapsto e^{i\psi(u)}$ is a smooth map $S^1 \to S^1$ of degree n, say. Now it is not difficult to see that we have

$$\overrightarrow{FG}\,(x) = \begin{cases} 0 & \text{if } n \equiv 0 \bmod 2 \\ 1 & \text{otherwise.} \end{cases} \tag{27}$$

On the other hand it follows from the definition (4) of q_f and q_g that we also have

$$\overrightarrow{q_f q_g}(x) = \begin{cases} 0 & \text{if } n \equiv 0 \bmod 2 \\ 1 & \text{otherwise.} \end{cases} \tag{28}$$

Combining (24) and (28) we now obtain (21). □

As an immediate consequence of theorem 1 we recover the result of James and Thomas [9] that $\hat{I}(M^2, \mathbb{R}^3)$ has exactly 2^h elements, where $h = \dim H_1(M^2, \mathbb{Z}_2)$. Also the parts (a) and (b) of theorem 2 (stated in the introduction) follow easily from theorem 1.

Proof of theorem 2(*c*): If $f: M^2 \to \mathbb{R}^3$ is an immersion, $\varphi: M^2 \to M^2$ a diffeomorphism then clearly $q_{f\circ\varphi} = q_f \circ \varphi_*$. Conversely let $f, g: M^2 \to \mathbb{R}^3$ be two immersions, $\alpha: H_1(M^2, \mathbb{Z}_2) \to H_1(M^2, \mathbb{Z}_2)$ a linear map such that $q_f = q_g \circ \alpha$. Then (15) implies that α preserves the intersection form "." on $H_1(M^2, \mathbb{Z}_2)$. By Lemma 6 below there is a diffeomorphism $\varphi: M^2 \to M^2$ such that $\varphi_* = \alpha$. Then $q_{g\circ\varphi} = q_f$ and by part (a) of theorem 2 f and g are regularly homotopic. □

LEMMA 6. *Let M^2 be a compact surface, $\alpha: H_1(M^2, \mathbb{Z}_2) \to H_1(M^2, \mathbb{Z}_2)$ a linear map preserving the intersection form. Then α is induced by a diffeomorphism $\varphi: M^2 \to M^2$.*

The proof is omitted here because it is similar to the proof of theorem 2 in [11]. For orientable M^2 Lemma 6 is actually a consequence of that theorem.

§3. $\mathbb{Z}_4$-VALUED QUADRATIC FORMS

Let V be a finite dimensional $\mathbb{Z}_2$-vector space equipped with a nondegenerate symmetric bilinear form ".": $V \times V \to \mathbb{Z}_2$. It is known that V decomposes as an orthogonal direct sum in one of the following ways:

$$V = H \oplus \ldots \oplus H \quad \text{or} \quad V = P \oplus \ldots \oplus P. \tag{29}$$

Here P is one dimensional with generating vector e, $e \cdot e = 1$ whereas H is two dimensional with basis e_1, e_2 and

$$(e_i \cdot e_j) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{30}$$

A quadratic form on V is a map $q: V \to \mathbb{Z}_4$ satisfying

$$q(x+y) = q(x) + q(y) + 2x \cdot y \tag{31}$$

for all $x, y \in V$. Setting $x = 0$ we see that (31) implies $q(0) = 0$.

Two quadratic form spaces (V, q) and $(\tilde{V}, \tilde{q})$ are said to be isomorphic if there is a vector space isomorphism $\alpha: V \to \tilde{V}$ such that $q = \tilde{q} \circ \alpha$. Note that by (31) α necessarily is an isometry with respect to the inner products on V and $\tilde{V}$. If (V_1, q_1) and (V_2, q_2) are quadratic form spaces then there is a unique quadratic form $q: V_1 \oplus V_2 \to \mathbb{Z}_4$ such that for $i = 1, 2$ we have $q|_{V_i} = q_i$, where $V_i \subset V_1 \oplus V_2$ are the canonical embeddings.

LEMMA 7. (a) *There are two isomorphism classes of quadratic forms on H, represented by q_S and q_T respectively as follows:*

$$q_S: \quad \begin{aligned} e_1 &\mapsto 0 \\ e_2 &\mapsto 0 \\ e_1 + e_2 &\mapsto 2 \end{aligned} \tag{32}$$

$$q_T: \quad \begin{aligned} e_1 &\mapsto 2 \\ e_2 &\mapsto 2 \\ e_1 + e_2 &\mapsto 2 \end{aligned} \tag{33}$$

(b) *There are two isomorphism classes of quadratic forms on* P, *represented by* q_B *and* $q_{\bar{B}}$ *respectively, where* $q_B(e) = 1$ *and* $q_{\bar{B}}(e) = -1$.

(c) *Writing* $(H, q_S) = H_S$ *etc. we have the following isomorphisms*:

$$\begin{gathered} H_S \oplus H_S \simeq H_T \oplus H_T \\ P_B \oplus P_B \oplus P_B \oplus P_B \simeq P_{\bar{B}} \oplus P_{\bar{B}} \oplus P_{\bar{B}} \oplus P_{\bar{B}} \\ H_S \oplus P_B \simeq P_B \oplus P_B \oplus P_{\bar{B}} \\ H_S \oplus P_{\bar{B}} \simeq P_B \oplus P_{\bar{B}} \oplus P_{\bar{B}} \\ H_T \oplus P_B \simeq P_{\bar{B}} \oplus P_{\bar{B}} \oplus P_{\bar{B}} \\ H_T \oplus P_{\bar{B}} \simeq P_B \oplus P_B \oplus P_B . \end{gathered} \tag{34}$$

Proof. The proof of (a) and (b) is left to the reader. To prove the first equation (34) let e_1, e_2 be a basis as in (32) for the first copy of H_S, $\tilde{e}_1, \tilde{e}_2$ similarly a basis for the second copy. Then evaluating q and "." on the basis

$$\begin{gathered} e_1 + \tilde{e}_1 + \tilde{e}_2, \quad e_2 + \tilde{e}_1 + \tilde{e}_2 \\ \tilde{e}_1 + e_1 + e_2, \quad \tilde{e}_2 + e_1 + e_2 \end{gathered} \tag{35}$$

we see that $H_S \oplus H_S$ is isomorphic to $H_T \oplus H_T$.

Concerning the second equation (34) let $e_1, \ldots e_4$ be generating elements of the four copies of P_B. Then the new basis

$$\begin{gathered} \hat{e}_1 = e_2 + e_3 + e_4 \quad \hat{e}_2 = e_3 + e_4 + e_1 \\ \hat{e}_3 = e_4 + e_1 + e_2 \quad \hat{e}_4 = e_1 + e_2 + e_3 \end{gathered} \tag{36}$$

satisfies $\hat{e}_i \cdot \hat{e}_j = \delta_{ij}$ and $q(e_i) = -1$ for all i. This establishes the asserted isomorphism. The other equations are proved similarly. □

Now for any quadratic form space (V, q) with $\dim V = h$ define the *Arf invariant* $Q_{(V,q)}$ by

$$Q_{(V,q)} = \frac{1}{\sqrt{2^h}} \sum_{x \in V} e^{2\pi i q(x)/4}. \tag{37}$$

The parts (a), (b), (c) of the following lemma are also proved in [4].

LEMMA 8. (a) $Q_{(V,q) \oplus (\tilde{V}, \tilde{q})} = Q_{(V,q)} \cdot Q_{(\tilde{V}, \tilde{q})}$.

(b) $Q_{(V,q)}$ *is always an eighth root of unity.*

(c) $Q^4_{(V,q)} = (-1)^{\dim V}$

(d) *Two quadratic forms* q_1, q_2 *on the same inner product space* $(V, \text{"}\cdot\text{"})$ *are isomorphic if and only if the corresponding Arf invariants are equal.*

Proof. (a) is a straightforward calculation. To prove (b) it is sufficient to consider the cases $(V, q) = H_S, H_T, P_B$ or $P_{\bar{B}}$. This is because an arbitrary quadratic form space (V, q) is an orthogonal direct sum of several copies of these spaces and by (a) Q behaves multiplicatively under orthogonal direct sum. In the mentioned special cases we obtain

$$\begin{gathered} Q_{H_S} = 1 \qquad Q_{H_T} = -1 \\ Q_{P_B} = \frac{1+i}{\sqrt{2}} \qquad Q_{P_{\bar{B}}} = \frac{1-i}{\sqrt{2}}. \end{gathered} \tag{38}$$

Part (c) of the Lemma is also a consequence of (38). We now prove (d). Using (29) and Lemma 7 (c) we see that every quadratic form space (V, q) can be represented in one of the following ways:

$$\begin{gathered} H_S \oplus H_S \oplus \ldots \oplus H_S \\ H_T \oplus H_S \oplus \ldots \oplus H_S \end{gathered} \tag{39}$$

$$\underbrace{P_{\bar{B}} \oplus \ldots \oplus P_{\bar{B}}}_{p \text{ times}} \oplus \underbrace{P_B \oplus \ldots \oplus P_B}_{q \text{ times}} \tag{40}$$

where $p \leq 3, p+q = \dim V$. But in the first case $Q_{(V,q)} = 1$, in the second $Q_{(V,q)} = -1$. Hence in the case of an inner product space $V = H \oplus \ldots \oplus H$ $Q_{(V,q)}$ is a complete invariant for the quadratic forms on V. Similarly if $V = P \oplus \ldots \oplus P$ and (V, q) is given by (40) it follows from (38) and (a) that here we have

$$Q_{(V,q)} = e^{2\pi i(q-p)/8} \tag{41}$$

Thus also in this case the invariant $Q_{(V,q)}$ is sufficient to distinguish the four cases $p = 0, 1, 2, 3$ in (40). □

Theorem 3 (stated in the introduction) follows immediately from Lemma 8.

If V happens to be a direct sum of copies of H (i.e. the inner product on V is symplectic) then (31) implies that q takes values in $2\mathbb{Z}_4 \simeq \mathbb{Z}_2$ and thus can be considered also as a quadratic form in the usual sense. The usual definition of the Arf invariant of q then amounts to assigning 0 to H_S and 1 to H_T, in contrast to (38). The only difference between the usual notation and ours is of course that we use in this case a multiplicative notation (instead of an additive one) for the cyclic group of two elements.

§4. IMMERSED SURFACE IN $\mathbb{R}^3$

We now use theorem 2 and the results of the last section to classify compact immersed surfaces in $\mathbb{R}^3$. We adopt the following terminology: Let $f: M^2 \to \mathbb{R}^3$ be an immersion, $\gamma: S^1 \to M^2$ an embedding. Then a tubular neighborhood of γ is called a left-handed Möbius band, an untwisted annulus, a right handed Möbius band or a twisted annulus depending on the value of $\tilde{q}_f(\gamma)$ being $-1, 0, 1$ or 2 respectively. Two such annuli or Möbius bands are said to be homologically *independent*, homologically *trivial* etc. if the corresponding homology classes $[\gamma]_{\mathbb{Z}_2}$ have the respective properties.

LEMMA 9. *Let M be an immersed torus in $\mathbb{R}^3$. Then*

(a) *M is regularly homotopic either to the standardly embedded torus S or to the torus T depicted in Fig. 1.*

(b) *If there is a homologically nontrivial untwisted annulus on an immersed torus M then M is regularly homotopic to S.*

(c) *If there are two homologically independent twisted annuli on an immersed torus M then M is regularly homotopic to T.*

Proof. All assertions follow almost immediately from theorem 2(c) and Lemma 7 (a). Note that in our case the inner product space $H_1(M^2, \mathbb{Z}_2)$ is of type H. One only has to find a homologically nontrivial untwisted annulus on S (easy) and two homologically independent twisted annuli on T. One such annulus on T is provided by a neighborhood of some embedded curve close to the lemniscate that serves as a "meridian" of T. Another twisted annulus, which is homologically independent from the first one, is given by a tubular neighborhood of one of the two preimages of the line of self-intersection. □

As an example we apply Lemma 9 to a torus of revolution with a lemniscate as meridian (Fig. 4a). Because clearly there is a homologically nontrivial untwisted annulus on this torus by Lemma 9(b) it must be regularly homotopic to the standard torus. An explicit description of such a homotopy is given in [13]. A similar argument as in Lemma 9 yields

LEMMA 10. *Every immersed projective plane P in $\mathbb{R}^3$ is regularly homotopic either to the "left-handed Boy surface" $\bar{B}$ or to the "right-handed Boy surface" B depicted in Fig. 1. If there is a right-handed Möbius band on P then P is regularly homotopic to B.*

Let M^2 and N^2 be compact surfaces, $f: M^2 \to \mathbb{R}$ and $g: N^2 \to \mathbb{R}^3$ immersions, $c: (-\varepsilon, 1+\varepsilon) \to \mathbb{R}^3$ an embedding such that $c(-\varepsilon, 1+\varepsilon)$ cuts $f(M^2)$ transversally at $c(0)$ and cuts

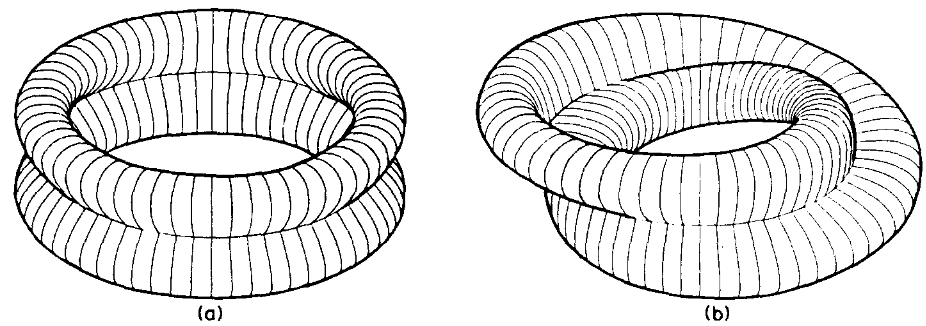

Fig. 4.

$g(N^2)$ transversally at $c(1)$. Then it is clear how to use a tubular neighborhood of $c(-\varepsilon, 1+\varepsilon)$ and a suitable smoothing process to define the connected sum $f \# g \colon M^2 \# N^2 \to \mathbb{R}^3$.

LEMMA 11. (a) *The regular homotopy class of $f \# g$ depends only on the regular homotopy classes of f and g (in particular not on the choice of c).*

(b) *The canonical isomorphism*

$$H_1(M \# N^2, \mathbb{Z}_2) \simeq H_1(M^2, \mathbb{Z}_2) \oplus H_1(N^2, \mathbb{Z}_2)$$

induces an isomorphism of quadratic form spaces

$$(H_1(M^2 \# N^2), q_{f \# g})$$
$$\simeq (H_1(M^2, \mathbb{Z}_2), q_f) \oplus (H_1(N^2, \mathbb{Z}_2), q_g).$$

Proof. The proof of (b) is left to the reader. (a) follows from (b) by theorem 2(a). □

Lemma 11 implies that the set of regular homotopy classes of compact immersed surfaces in $\mathbb{R}^3$ is an abelian semigroup H with respect to connected sum. Also by theorem 2(c) and Lemma 11 H is canonically isomorphic to the semigroup $\tilde{H}$ of isomorphism classes of quadratic form spaces over $\mathbb{Z}_2$. In Lemma 9 and Lemma 10 we made this correspondence explicit for a set of generators of $\tilde{H}$. Also by Lemma 8 the relations (34) are defining relations for $\tilde{H}$. Putting these pieces of information together we arrive at theorem 4 (stated in the introduction).

The two types of orientable surfaces of a given genus > 0 are visualized best by the "normal forms"

$$\begin{gathered} S \# S \# \ldots \# S \\ T \# S \# \ldots \# S. \end{gathered} \tag{42}$$

For nonorientable surfaces one has the representation

$$\underbrace{\bar{B} \# \ldots \# \bar{B}}_{p \text{ times}} \# B \# \ldots \# B \tag{43}$$

with $p \leq 3$, but it is often useful to have a less complicated picture: it follows from theorem 4 that every nonorientable surface is regularly homotopic to a connected sum of several copies of S and one of the following eight surfaces:

$$K_0, B, K_+, K_+ \# B, K_0 \# T, K_- \# \bar{B}, K_-, \bar{B}. \tag{44}$$

Under a reflection of $\mathbb{R}^3$ B is mapped onto $\bar{B}$, hence the usual picture of a Klein bottle (having a plane of symmetry) must represent the class $K_0 = B \# \bar{B}$. On the other hand it is not difficult to locate on the immersed Klein bottle shown in Fig. 4b two homologically independent right-handed Möbius bands. Hence this surface is of type $K_+ = B \# B$. The

mirror image of Fig. 4b then represents the class $K_- = \bar{B} \# \bar{B}$. Thus we have shown that all three types of immersed Klein bottles can be realized without triple points. Summarizing we obtain

THEOREM 5. (a) *Every immersed surface in $\mathbb{R}^3$ with even Euler characteristic is regularly homotopic to one without triple points.*

(b) *Every immersed surface in $\mathbb{R}^3$ with odd Euler characteristic is regularly homotopic to one with only a single triple point.*

It is known [2] that the number of triple points of a self-transverse immersed surface in $\mathbb{R}^3$ is always congruent modulo 2 to the Euler characteristic of the surface. Rather complete information about the double point set of an immersed surface can be gained from §7 of [5] combined with our section 6. In particular it follows that the *twisting invariant* of the double point set defined in [7] is *equal* to the Arf invariant.

§5. EMBEDDED SURFACES

If $f: M^2 \to \mathbb{R}^3$ is an embedding then the immersed surface $[f]$ is called an *embedded surface.*

THEOREM 6. *Any two compact embedded surfaces of the same topological type M^2 are regularly homotopic.*

Proof. Clearly M^2 has to be orientable. But for any compact orientable immersed surface the surface V constructed in Lemma 2 is a Seifert surface for the knot ∂V, and the Arf invariant of G_V is known to be an invariant of this knot [14]. If the compact immersed surface in question is embedded then of course ∂V is unknotted and therefore the Arf invariant (in our notation) is $+1$. But by theorem 2(b) and theorem 3(b) any two immersed surfaces with the same topological type and the same Arf invariant are regularly homotopic. □

Note that theorem 6 becomes false if one replaces "compact embedded surfaces of the same topological type" by "embeddings of a compact surface".

§6. COBORDISM OF IMMERSED SURFACES

We now want to relate our classification of immersed surfaces in $\mathbb{R}^3$ to the cobordism classification of immersions [18]. If M^2, N^2 are compact surfaces, $f: M^2 \to \mathbb{R}^3$, $g: N^2 \to \mathbb{R}^3$ two immersions then f is said to be cobordant to g if there are

(i) a 3-manifold X having as boundary the disjoint union $\partial X = M^2 \amalg N^2$.

(ii) an immersion $h: X \to \mathbb{R}^3 \times [0, 1]$ such that h is transversal to $\mathbb{R}^3 \times \{0, 1\}$ and

$$f \times \{0\} = h|_{M^2}, \quad g \times \{1\} = h|_{N^2}.$$

It is clear that the cobordism class of an immersion $f: M^2 \to \mathbb{R}^3$ depends only on the immersed surface $[f]$ and that regularly homotopic immersions are cobordant. Thus cobordism can also be considered as an equivalence relation on the set of regular homotopy classes of immersed surfaces. Theorem 7 (stated in the introduction) is equivalent to saying that the cobordism class of an immersed surface $[f]$ is fully described by its Arf invariant $Q_{[f]}$.

Proof of theorem 7. We first prove (a). Let $f: M^2 \to \mathbb{R}^3$, $g: N^2 \to \mathbb{R}^3$ be two cobordant immersions and let X and h be given as above. Let $t: \mathbb{R}^3 \times [0, 1] \to \mathbb{R}$ denote the coordinate function corresponding to the factor $[0, 1]$. Then we may assume without loss of generality that $t \circ h$ is a Morse function on X. Watching then the behaviour of the slices $\mathbb{R}^3 \times \{t\} \cap h(X)$ as t runs from 0 to 1 we see a regular homotopy for all but a finite number $\{t_1, \ldots, t_k\}$ of parameter-values, where one of the following modifications occurs:

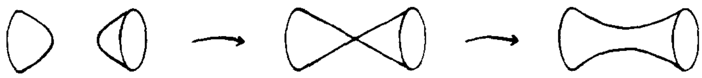

Fig. 5.

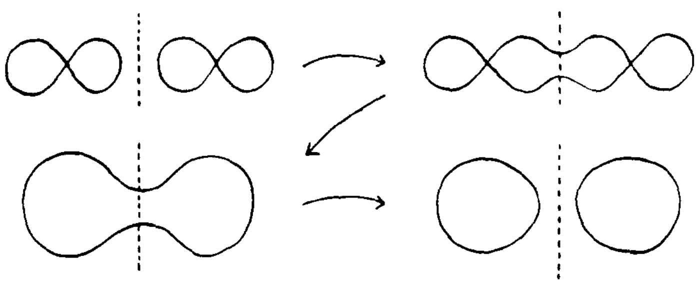

Fig. 6.

(a) a small sphere appears in the picture or vanishes (local minimum or maximum of $t \circ h$)

(b) the modification indicated in Fig. 5 or the reverse of it (critical point of $t \circ h$ of index 1 or 2).

Even if M^2 and N^2 are connected the intermediate stages of the above deformation may have several components. It is useful in this context to define the Arf invariant of a disconnected immersed surface as the product of the Arf invariants of its components.

It is clear that the modifications of type (a) do not change the Arf invariant. Concerning the modification indicated in Fig. 5 we distinguish two cases: (1) The two sheets on the left of Fig. 5 belong to different components of the surface. Then Fig. 5 amounts to replacing these two components by their connected sum. By Lemma 10 this does not change the Arf invariant. (2) The two sheets on the left of Fig. 5 belong to the same component of the surface. Then one can show that the effect of the modification in Fig. 5 is to attach to this component either a torus S or a Klein bottle $B \# \bar{B}$. Again this does not change the Arf invariant. This proves (a).

To prove (b) we have to show that any two compact immersed surfaces with the same Arf invariant are cobordant. We do this by showing that every compact immersed surface in $\mathbb{R}^3$ is cobordant to one of the eight surfaces (44). The latter cobordism is constructed in two steps: (1) Use the remark preceeding (44) to construct a regular homotopy ending in a connected sum of several copies of S with T or one of the surfaces (44). (2) Eliminate the tori of type S using modifications obtained from the one in Fig. 5 by reversing the arrows. $\square$

It is clear from the proof of theorem 7 that a cobordism between two immersed surfaces can be visualized as a deformation which fails to be a regular homotopy only for a finite number of critical stages, where modifications of type (a) or (b) occur. For example Fig. 6 indicates a cobordism between two tori of revolution by giving a sequence of meridian curves.

REFERENCES

1. F. APERY: La surface de Boy, thesis Strasbourg 1984.
2. T. F. BANCHOFF: Triple points and surgery of immersed surfaces. *Proc. Am. Math. Soc.* **46** (1974), 407–413.
3. W. BOY: Über die Curvatura integra und die Topologie geschlossener Flächen. *Math. Ann.* **57** (1903), 151–184.
4. E. H. BROWN: Generalizations of the Kervaire invariant, *Ann. Math.* **95** (1972), 368–383.
5. P. J. ECCLES: Multiple points of codimension one immersions. *Topology Symposium Siegen* 1979, Springer, LNM 788.
6. C. MCA. GORDON and R. A. LITHERLAND: On the signature of a link. *Invent. Math.* **47** (1978), 53–69.
7. J. HASS and J. HUGHES: Immersions of surfaces in 3-manifolds, *Topology* **24** (1985), 97–112.
8. M. W. HIRSCH: Immersions of manifolds. *Trans. Am. Math. Soc.* **93** (1959), 242–276.
9. I. JAMES and E. TOMAS: Note on the classification of cross-sections. *Topology* **4** (1966), 351–359.
10. L. H. KAUFFMAN and T. F. BANCHOFF: Immersions and mod-2 quadratic forms. *Am. Math. Mon.* **84** (1977), 168—185.

11. W. H. MEEKS III and J. PATRUSKY: Representing homology classes by embedded circles on a compact surface. *Ill. J. Math.* **22** (1978), 262–269.
12. W. H. MEEKS III: Representing codimension-one homology classes on closed nonorientable manifolds by submanifolds. *Ill. J. Math.* **23** (1979), 199–210.
13. J. P. PETIT: le retournement non trivial du tore. *C. R. Acad. Sci. Paris Ser.* A **287** (1978), 927–930.
14. R. ROBERTELLO: An invariant of knot cobordism. *Comm. Pure Appl. Math.* **18** (1965), 543–555.
15. D. ROLFSEN: *Knots and Links*, Publish or Perish 1976.
16. K. SCHLICHTING: Reguläre Homotopie geschlossener Flächen, Diplomarbeit TU Berlin 1977.
17. S. SMALE: A classification of immersions of the two sphere, *Trans. Am. Math. Soc.* **90** (1958), 281–290.
18. R. WELLS: Cobordism groups of immersion, *Topology* **5** (1966), 281–294.

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