Theorem 3. On a nonsingular algebraic variety, $c^2_k$ is the dimension of the $Q$-submodule of $H^4(M, Q)$ generated by cohomology classes dual to $(2n - d)$-dimensional rational cycles formed by intersecting $(2n + 2k - d)$-dimensional rational cycles with $(n - k)$-dimensional algebraic subvarieties of $M_n$.

Thus the index $c_{n-k}^{2n-d}$ in a sense gives the dimension of the set of $d$-dimensional cycles lying on $k$-dimensional algebraic subvarieties of $M_n$. The proof of this result follows from a theorem of Lefschetz, which asserts that a $(2n - 2)$-dimensional cycle is represented by a divisor (effective or not) if and only if its dual cohomology class is represented by a differential form of type $(1, 1)$, and from a theorem of Severi, which asserts that an irreducible algebraic subvariety $V_k \subset M_n$ is the complete intersection of $r - k$ divisors (effective or not) on $M_n$.

3. This corresponds to the Riemann conditions on $\Omega$.
4. F. Severi, Serie, sistemi d'equivalenza e corrispondenza algebriche sulla varietà algebriche (Rome, 1942).

ON DEHN'S LEMMA AND THE ASPHERICITY OF KNOTS

BY C. D. PAPAKYRIAKOPOULOS

INSTITUTE FOR ADVANCED STUDY

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Everything in this note will be considered from the semilinear point of view; i.e., any 3-manifold will be considered with a fixed triangulation (this is permissible according to Moise's work), any curve or line will be considered as polygonal, any surface as polyhedral, and so on.

The following theorem was first considered by Dehn, but it was pointed out by Kneser that Dehn's proof contains a gap.

Dehn's Lemma. Let $M$ be a 3-manifold, compact or not, with boundary which may be empty, and in $M$ let $D$ be a 2-cell with self-intersections (singularities), having as boundary the simple closed polygonal curve $C$ and such that there exists a closed neighborhood of $C$ in $D$ which is an annulus (i.e., no point of $C$ is singular). Then there exists a 2-cell $D_0$ with boundary $C$, semilinearly imbedded in $M$.

Johansson proved that, if Dehn's lemma holds for all orientable 3-manifolds, it also holds for all nonorientable ones. We prove that Dehn's lemma holds for all orientable 3-manifolds, and by a modification of our method we also prove the following theorem.

Sphere Theorem. Let $M$ be an orientable 3-manifold, compact or not, with boundary which may be empty, such that $\pi_1(M) \neq 0$, and which can be topologically imbedded in a 3-manifold $N$, having the following property: The first homology group of any nontrivial (but not necessarily proper) subgroup of $\pi_1(N)$, has an element of infinite order (note in particular that this holds if $\pi_1(N) = 1$). Then there exists a 2-sphere $S$ semilinearly imbedded in $M$, such that $S$ is not homotopic to zero in $M$. 
From these two theorems follow others, which may be stated easily if we introduce the following notions: Let \( F \) be a nonempty proper closed subset in \( S^3 \). We say that \( F \) is geometrically splittable if there is a 2-sphere \( S^2 \subset S^3 - F \) such that both components of \( S^3 - S^2 \) contain points of \( F \). We say that \( F \) is algebraically splittable if \( \pi_1(S^3 - F) \) is the free product of two groups, each of which is nontrivial.

**Theorem 1.** Let \( U \) be a nonempty proper open connected subset of the 3-sphere \( S^3 \). Then \( U \) is aspherical if and only if \( S^3 - U \) is not geometrically splittable.

The above theorem provides us with a solution of a problem of Whitehead.\(^5\)

**Corollary 1.** If \( F \) is a nonempty proper closed connected subset of \( S^3 \), then each component of \( S^3 - F \) is aspherical.

The above corollary provides us with a solution of a problem of Eilenberg.\(^6\) An immediate consequence of Corollary 1 is the following:

**Corollary 2.** If \( F \) is a connected graph or knot, then \( S^3 - F \) is aspherical.

The following theorem solves completely a problem initiated by Higman.\(^7\)

**Theorem 2.** Let \( K \) be a link in \( S^3 \). The following three statements are equivalent:

(i) \( S^3 - K \) is not aspherical.

(ii) \( K \) is geometrically splittable.

(iii) \( K \) is algebraically splittable.

Let \( K \) be a knot in \( S^3 \). According to Dehn (op. cit., Satz 2, p. 158), Specker,\(^8\) Papakyriakopoulos,\(^9\) and the above Corollary 2, the following theorem holds:

**Theorem 3.** (i) \( K \) is unknotted if and only if \( \pi_1(S^3 - K) \) is free cyclic. (ii) The number of ends of \( \pi_1(S^3 - K) \) is either 1 or 2. (iii) \( \pi_1(S^3 - K) \) has 1 end if \( K \) is knotted and 2 ends if \( K \) is unknotted.

The following statement is known as Hopf's conjecture.

**Theorem 4.** If \( U \) is an open connected subset of the 3-sphere, then \( \pi_1(U) \) has no element of finite order.

The proof is based on the sphere theorem and the following simple consequence of a theorem due to P. A. Smith:\(^10\) The fundamental group of an aspherical polyhedron (of finite dimension) has no element of finite order.

A Sketch of the Proof of Dehn's Lemma.—By a Dehn disk we mean a 2-cell, which may have singularities, but not on its boundary. We suppose that \( M \) is orientable and that \( D \) has no branch points (see Whitehead\(^{11}\)). Let \( G \) be the inverse image of \( D \) under \( f \), where \( G \) is a 2-cell. There is a finite set \( J \) of triples \((J', J'', \psi)\), where \( J', J'' \) are closed curves on \( G \), called the \( J \)-curves, and \( \psi: J' \to J'' \) is a "nice map" such that \( f(r) = f(\psi(r)) \), for any point \( r \in J' \), i.e., \( f(J') = f(J'') \) is a double line\(^{11}\) of \( D \). We emphasize that, according to Johansson,\(^{12}\) \( J' \neq J'' \) because \( M \) is orientable. So we have a map \( (G, J) \to D \) called a realized diagram. We denote by \( l(D) \) and \( d(D) \) the number of triple points\(^{11}\) and double lines of \( D \), respectively. The couple \((l(D), d(D))\) is called the complexity of \( D \).

Let \( V \subset M \) be a 3-manifold with boundary such that int \( V \) is a neighborhood of \( D - C \), \( C \subset \text{bdry} V \), and \( V \) is very small and "very nice," so that each face of \( \text{bdry} V \) is "parallel" to a face of \( D \). Such a \( V \) is called a prismatic neighborhood of \( D \) in \( M \). We observe that \( D \) is a deformation retract of \( V \). Let \( p_*: M_* \to V \) be the universal covering of \( V \), let \( D_* \) be a Dehn disk covering \( D \) just once, and let
Finally, let $V_*$ be a prismatic neighborhood of $D_*$ in $M_*$. So we have the following diagram:

\[
\begin{array}{c}
M_* \supset V_* \supset D_* \xrightarrow{f_*} (G, J_*) \\
p_* \downarrow \quad \quad \downarrow q_* \\
M \supset V \supset D \xrightarrow{f} (G, J)
\end{array}
\]

called an \textit{elementary tower} over $D \subset V \subset M$, where $f_*$ is the lifted map, and $f_*: (G, J_*) \rightarrow D_*$ is a realized diagram.

If $V$ is \textit{not} simply connected, then there is a covering translation $\tau$ of $p_*: M_* \rightarrow V$, such that $D_* \cap \tau^{-1}(D_*)$ is not empty, and so it consists of a finite number of closed curves $T_{*i}, i = s(\tau, 1), \ldots, s(\tau, d(\tau))$. These curves for all possible $\tau$’s are called the $T_{*i}$-curves. Then

\[d(D) = d(D_*) + \frac{1}{2} \sum_{\tau} d(\tau),\]

where the sum runs over all covering translations $\tau \neq 1$ (note that if $D_* \cap \tau^{-1}(D_*) = \emptyset$, then naturally $d(\tau) = 0$). Hence $d(D_*) < d(D)$. We emphasize that, in the special case where $D_*$ is a 2-cell, $q_*: (D_*, T_*) \rightarrow D$ is a realized diagram, where $T_*$ is now the set of all triples $(T', T'', \tau)$ such that $T' \in D_* \cap \tau^{-1}(D_*)$ and $T'' = \tau(T') \in \tau(D_*) \cap D_*$. If $V$ is \textit{simply connected}, then each of the components of bdry $V$ is a 2-sphere, according to a result due to Seifert. In this case $d(D_*) = d(D)$.

The diagram

\[
\begin{array}{c}
M_m \supset V_m \supset D_m \xrightarrow{f_m} (G, J_m) \\
p_m \downarrow \quad \quad \downarrow q_m \\
M_{m-1} \supset V_{m-1} \supset D_{m-1} \xrightarrow{f_{m-1}} (G, J_{m-1})
\end{array}
\]

is called a \textit{tower} over $D \subset M$ and is defined as follows: The diagram (m) is an elementary tower over $D_{m-1} \subset V_{m-1} \subset M_{m-1}$, for $m = 1, 2, \ldots$, where $M_0 = M$, $V_0 = V$, $D_0 = D$, $f_0 = f$, $J_0 = J$. Then

\[d(D_0) \geq d(D_1) \geq d(D_2) \geq \ldots \geq 0,
\]

and there exists a number $n \geq 0$, such that $d(D_i) > d(D_{i+1})$, for $i < n$, and $d(D_j) = d(D_{i+1})$ for $j \geq n$, i.e., $V_n$ is \textit{simply connected} but $V_i$ is \textit{not}. This number $n$ is called
the height of the tower. The following two cases are possible: (1) \( d(D_n) > 0 \), i.e., \( D_n \) is not a 2-cell; (2) \( d(D_n) = 0 \), i.e., \( D_n \) is a 2-cell.

In case (1) \( V_n \) is simply connected, and so \( \text{bdry} \, V_n \) is composed of 2-spheres, according to the result of Seifert mentioned above. Let \( D_n' \) be one of the 2-cells bounded by \( C_n \), on the component of \( \text{bdry} \, V_n \) containing the boundary \( C_n \) of \( D_n \). Then \( D' = p_1 \ldots p_n(D_n') \subset M \) is a Dehn disk with boundary \( C \) and "roughly speaking" having the following property: Either \( t(D') < t(D) \), or \( t(D') = t(D) \) and \( d(D') < d(D) \), i.e., we may say that complexity of \( D' \) < complexity of \( D \).

In case (2) we prove that there exists a triple \((J', J'', \psi)\) of \( f: (G, J) \to D \) such that \( J' \) and \( J'' \) are disjoint simple closed curves. The proof of this is rather algebraic and makes use of the fact that \( q_n: (D_n, T_n) \to D_n-1 \) is a realized diagram, as we have observed above, and that each element of \( T_n \) is a triple \((T', T'', \tau)\), where \( \tau \) is a covering translation of \( p_n: M_n \to V_n-1 \). Then by a cut ("Umschaltung";14 note that this is the only case in which we can apply Dehn's process without any danger [see Dehn, op. cit. p. 150, B, and Kneser, op. cit., p. 260]) of \( D \) along \( J = f(J') = f(J'') \) we obtain a new Dehn disk \( D' \subset M \), with boundary \( C \) and such that either \( t(D') < t(D) \) or \( t(D') = t(D) \) and \( d(D') < d(D) \), i.e., we may say that complexity of \( D' \) < complexity of \( D \).

In the same way we obtain from \( D' \) a new Dehn disk \( D'' \subset M \) with boundary \( C \) such that complexity of \( D'' \) < complexity of \( D' \), and so on. Thus, after a finite number of repetitions of the above process, we finally obtain a Dehn disk in \( M \) with boundary \( C \) and complexity \((0,0)\), i.e., we obtain a 2-cell in \( M \) with boundary \( C \).

As far as the proof of the sphere theorem is concerned, we restrict ourselves to the remark that the method of proof makes use of the above process, standard Hurewicz theorems, and the Poincaré duality theorem.