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Poincaré Spaces, Their Normal Fibrations and Surgery

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Let X be a Poincaré space of dimension m, i.e. there is an element $[X] \in H_m(X)$ such that $[X] \cap : H^q(X) \to H_{m-q}(X)$ is an isomorphism for all q. An oriented spherical fibre space ξ, $E_0 \xrightarrow{\pi} X$ with fibre F homotopy equivalent to S^{k-1}, is called a Spivak normal fibre space for X if there is $\alpha \in \pi_{m+k}(T(\xi))$ of degree 1, i.e. such that $h(\alpha) \cap U_\xi = [X]$, where $T(\xi) = X \cup_\pi c(E_0)$, $U_\xi \in H^k(T(\xi))$ is the Thom class; h: $\pi_* \to H_*$ the Hurewicz homomorphism. Let us always assume that X is the homotopy type of a CW complex.

Differential topology and surgery theory in particular has had great success in studying the problem of finding manifolds of the same homotopy type as a given Poincaré space, and classifying them (see [2] for example). In this theory, one finds weaker structures for Poincaré spaces analogous to well known structures on manifolds, and then studies the obstructions to lifting the structure to the strong type found on a manifold. For example, the Spivak normal fibre space is the analog of the stable normal bundle to the embedding of a manifold in the sphere, and the problem of making this spherical fibre space into a linear bundle (up to fibre homotopy type) is central to the theory of surgery.

In his thesis Spivak [5] showed that if $\pi_1 X = 0$ then a Poincaré space X has a Spivak normal fibre space for $k > \dim X$, unique up to fibre homotopy equivalence (see also [2, Ch. I, § 4]). His proof extends to non-simply connected Poincaré spaces X, provided that Poincaré duality holds with local coefficients and X is dominated by a finite complex [6]. However, the proof of uniqueness, due essentially to Atiyah [1] makes use of only ordinary homology, and Spanier-Whitehead S-theory. Hence it is quite plausible to think that the Spivak normal fibre space exists also, without any assumptions but Poincaré duality with ordinary integer coefficients. We shall show that this is in fact the case.

Theorem A. *Any Poincaré space has a Spivak normal fibre space.*

A relative version for Poincaré pairs is also true, and follows as an easy corollary as we shall see later (see § 1, Theorem A′).

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Suppose that X satisfies Poincaré duality with local coefficients, i.e.

$$[X]\cap\colon H^q(X;M)\to H_{m-q}(X;M)$$

is an isomorphism for local coefficients in any $\pi_1 X$ module M. We call such an X an (oriented) *local* Poincaré space.

Corollary 1. *If X is a local Poincaré space then $\pi_1(X)$ is finitely generated.*

This is of course not true without local Poincaré duality, since one could take the wedge of a Poincaré space with any number of acyclic complexes without changing the homology (and hence preserving Poincaré duality), while increasing the size of the fundamental group indefinitely.

I have not been able to prove that $\pi_1 X$ is finitely presented. However, we have the following:

Corollary 2. *If X is a local Poincaré space and if $\pi_1 X$ is finitely presented, then X is dominated by a finite complex.*

The two corollaries will follow by using the theorem to construct a map of degree 1 of a compact manifold into a Spivak normal fibre space Y for X with $\pi_1 Y=\pi_1 X$, and Y dominating X.

The proof of the theorem proceeds by first noticing that one needs only prove it for sufficiently high dimensions, and that if ζ is the Spivak normal fibre space of a Poincaré space Z, and if X is embedded (in a Poincaré space sense) in Z with a normal fibre space η, then $\zeta|X+\eta$ is the Spivak normal fibre space of X. Then we show that for a Poincaré space X of dimension $m\geqq 4$, one can always embed (in this sense) X in a 1-connected Poincaré space Z, so that the theorem now follows from the 1-connected version of Spivak.

The latter step is carried out by showing that one can "kill" $\pi_1 X$ by "surgery", i.e. one can find an analogous process to the well known smooth surgery, to find a Poincaré bordism Z of X, (i.e., $(Z, X\cup X')$ is a Poincaré pair (defined later)) so that Z and X' are 1-connected.

Now suppose X is a Poincaré space in a weaker sense, for example only with certain coefficient groups:

Let X be a CW complex and suppose $[X]\in H_m(X)$ so that

$$[X]\cap\colon H^q(X;\Lambda)\to H_{m-q}(X;\Lambda)$$

is an isomorphism for a fixed coefficient ring Λ. We call X a Λ-Poincaré space. The Λ-Spivak normal fibre space of X is defined analogously, i.e. a fibre space ξ over X with 1-connected fibre F, such that $H_*(F;\Lambda)\cong H_*(S^{k-1};\Lambda)$, and with $\alpha\in\pi_{m+k}(T(\xi))$ such that $h(\alpha)\cap U_\xi=[X]$.

This raises the question:

Is there an existence or uniqueness theorem for Spivak normal fibre spaces mod Λ for certain Λ?

It turns out that for $Z_{(p)}$ Poincaré spaces, where $Z_{(p)}$ is the integers localized at a prime p, a simple argument due to Frank Quinn yields the result as a consequence of our technique, together with the localization theory developed by Sullivan.

While all our results are proved for oriented Poincaré spaces one can easily deduce them for unoriented Poincaré spaces X, i.e. with an orientation class in $H_m(X; Z^t)$ where Z^t is the twisted integer coefficient system associated with some non-trivial action of $\pi_1 X$ on Z.

For X is Poincaré embedded (see § 1) in the non-trivial real line bundle Y over X corresponding to Z^t and $(Y, \partial Y)$ is an oriented Poincaré pair, so that the result follows from the oriented version for Poincaré pairs.

In § 1, we carry out the preliminaries of Poincaré embedding, etc., and deduce the two corollaries from the theorem. In § 2, we carry out the process of "surgery" on Poincaré spaces to "kill" π_1, which is the heart of the proof.

§ 1. Poincaré Embedding

Let X be a Poincaré space of dimension m. Let η be a $(k-1)$-spherical fibre space over X, π: $E_0 \to X$ the total space and projection map of η. A Poincaré *embedding* of X in Z with normal fibre space η, consists of a map f: $E_0 \to Y$ for some space Y with $H_{m+k-1}(Y)=0$, and a homotopy equivalence h: $Z \to X \cup_\pi (E_0 \times [0, 1)) \cup_f Y$ (i.e. the double mapping cylinder of π, f). We denote this by $X \subset_p Z$.

Note first the following (c.f. [3])

(1.1) **Proposition.** *The Spivak normal fibre space of X is (stably) the normal fibre space for a Poincaré embedding of $X \subset_p S^{m+k}$.*

Hence in fact the existence of the Spivak normal fibre space is the analog of Whitney's theorem on the embeddability of manifolds in Euclidean space.

Proof. If $X \subset_p S^{n+k}$ with normal fibre space η^k, then, the composition

$$S^{m+k} \xrightarrow{h} X \cup_\pi (E_0 \times [0, 1]) \cup_f Y \to (X \cup_\pi (E_0 \times [0, 1]) \cup_f Y)/Y = T(\eta)$$

is easily seen to have degree 1.

Conversely, suppose that $\alpha \in \pi_{m+k}(T(\eta^k))$, $k>1$, such that

$$h(\alpha) \cap U_\xi = [X].$$

First note that $T(\eta) \cong E_0(\eta+\varepsilon^1)/\rho X$, where ε^1 is the trivial S^0-bundle, $\rho\colon X \to E_0(\eta+\varepsilon^1)$ the canonical cross-section. Let $Y = T(\eta) \cup_\alpha e^{m+k+1}$, $f\colon E_0(\eta+\varepsilon^1) \to T(\eta) \to Y$ be the natural map. Then it follows easily from calculating homology that $Z = X \cup_\pi E_0(\eta+\varepsilon^1) \cup_f Y$ has the homology of the sphere S^{m+k+1}, and since $\pi_1(Y)=0$ and $\pi\colon E_0(\eta+\varepsilon^1) \to X$ induces isomorphism on π_1 (if $k>2$), it follows from van Kampen's theorem that Z is 1-connected and hence homotopy equivalent to S^{m+k+1} by a standard argument of obstruction theory.

We can now deduce the two corollaries.

Since X has a Spivak normal fibre space by Theorem A we have a homotopy equivalence $h\colon S^{m+k} \to X \cup_\pi (E_0 \times [0,1]) \cup_f Y = Z$. Now Z is the double mapping cylinder of π and f so that $E_0 \times (-1,1) \subset Z$ as an open set. Using the real coordinate in $(-1,1)$ we can make h transverse regular to E_0, so that $N^{m+k-1} = h^{-1}(E_0)$ and $h|N\colon N \to E_0$ is a map of degree 1 (using the Mayer-Vietoris sequence, and the fact that $h\colon S^{m+k} \to Z$ has degree 1). Then we use the following, which is proved by Pettus [4]:

(1.2) **Proposition.** *Let X, Y be local Poincaré spaces, and let $f\colon X \to Y$ be of degree 1, i.e. $f_*[X]=[Y]$. Then $f_*\colon \pi_1 X \to \pi_1 Y$ is onto.*

This is the generalization of the well known theorem of Hopf on manifolds. One shows that if Y is a local Poincaré space and $p\colon \tilde{Y} \to Y$ is a finite (l-fold) cover, then $\tilde{Y}$ is a Poincaré space with $t[Y]$ as orientation in $H_m(\tilde{Y})$ where $t\colon H_m(Y) \to H_m(\tilde{Y})$ is the transfer map. Then $p_* t[Y] = l[Y]$ so p_* is of degree l. Hence if $f\colon X \to Y$ factors through $p\colon \tilde{Y} \to Y$, its degree must be divisible by l, so that a map of degree 1 does not factor through any finite cover.

If $p\colon \tilde{Y} \to Y$ is an infinite cover, then $H_m(\tilde{Y}) \cong H_m(Y; A)$ where A is the left $\pi_1(Y)$ module $Z(C)$, where C is the right coset space of $\pi_1(Y)/\pi_1(\tilde{Y})$. By Poincaré duality $H_m(Y;A) \cong H^0(Y;A)$ and $H^0(Y;A)$ = elements of A invariant under the action of $\pi_1(Y)$. But since $A = Z(C)$ consists of finite linear sums of elements in C and $C = \pi_1(Y)/\pi_1(\tilde{Y})$ is infinite, no element of A is invariant, so $H^0(Y;A)=0$. Hence $H_m(\tilde{Y})=0$ for $p\colon \tilde{Y} \to Y$ an infinite cover, so a map of non-zero degree cannot factor through an infinite cover. It follows that $f_*\colon \pi_1 X \to \pi_1 Y$ is onto.

Now $h|N\colon N \to E_0$ has degree 1, and if X is a local Poincaré space, then so is E_0. Hence $(h|N)_*\colon \pi_1(N) \to \pi_1(E_0)$ is onto. Since N is a compact manifold $\pi_1 N$ is finitely generated, while $\pi_1(E_0) \cong \pi_1(X)$ if $k>2$. So Corollary 1 follows.

In fact we can easily deduce the more general version.

Definition. $(X, \partial X)$ is a local Poincaré pair if there is a class

$$[X] \in H_m(X, \partial X)$$

such that the Poincaré duality isomorphisms hold:

$$[X]\cap:\ H^q(X,\partial X;M)\to H_{m-q}(X;M)$$
$$\partial[X]\cap:\ H^q(\partial X;M_0)\ \to H_{m-q-1}(\partial X;M_0)$$

for any $\pi_1 X$-module M, $\pi_1(\partial X)$-module M_0.

Corollary 1'. *If $(X,\partial X)$ is a local Poincaré pair, then $\pi_1 X$ is finitely generated.*

Proof. It follows easily that the double $X_+\cup_{\partial X} X_-$ is a local Poincaré space, so that $\pi_1(X_+\cup_{\partial X} X_-)$ is finitely generated by Corollary 1. But X is a retract of $X_+\cup_{\partial X} X_-$, so $\pi_1(X)$ is finitely generated.

Note ∂X is a local Poincaré space, so that $\pi_1(\partial X)$ is finitely generated.

To deduce Corollary 2, we consider the map $h|N$: $N\to E_0$, and since we are now assuming $\pi_1 X\cong\pi_1 E_0$ is finitely presented, it follows that we can do surgery on N and $h|N$ to get h': $N'\to E_0$ of degree 1 so that h'_*: $\pi_1 N'\to\pi_1 E_0$ is an isomorphism (see [2; proof of (IV.1.13)]). Now h' is a map of degree 1 and hence h'_*: $H_*(N';M)\to H_*(E_0;M)$ is onto by Poincaré duality, for any $\pi_1 E_0$ module M. It then follows easily that we can do surgery on h': $N'\to E_0$, to get h'': $N''\to E_0$ with h'' $[n/2]$-connected where $n=m+k-1=\dim E_0$ (see again [2; (IV.1.13)]). We may assume (by adding ε^1 to η if necessary) that $n=2j$, so that by Poincaré duality h''_*: $H_i(N'';M)\to H_i(E_0;M)$ is an isomorphism for $i\neq j$, and onto for $i=j$, for $M=$the group ring of $\pi_1(E_0)$. It then follows from Wall [7, Theorem 8 (iii) and (i)] that E_0 is dominated by a finite complex. But E_0 dominates X, so that X is dominated by a finite complex.

Similarly to Corollary 1', we can easily deduce:

Corollary 2'. *Let $(X,\partial X)$ be a local Poincaré pair such that $\pi_1 X$ is finitely presented. Then X is dominated by a finite complex.*

Now we proceed to the preliminaries in the proof of the theorem.

(1.3) **Lemma.** *Let $X\subset_p Z$ be a Poincaré embedding of the Poincaré space X of dimension m, in a Poincaré space Z of dimension $m+k$, with normal fibre space η. If ζ is the Spivak normal fibre space of Z, then $\zeta|X+\eta$ is the Spivak normal fibre space of X.*

Proof. $Z=X\cup_\pi(E_0\times[0,1])\cup_f Y$ so that we have a natural map

$$T(\zeta)\to T(\zeta)/T(\zeta|Y),$$

and $T(\zeta)/T(\zeta|Y)$ is easily seen to be homotopy equivalent to $T(\zeta|X+\eta)$. It follows easily that $S^{m+k+l}\to T(\zeta^l)\to T(\zeta^l|X+\eta^k)$ is of degree 1, which proves the lemma.

In fact we have the following more general fact:

(1.3′) **Lemma.** *Let X, Z, W be Poincaré spaces of dimensions m, $m+k$ and $m+k+l$ respectively. Suppose $X \subset_p Z$ with normal fibre space η^k and $Z \subset_p W$ with normal fibre space ζ^l. Then $X \subset_p W$ with normal fibre space $\zeta|X+\eta$.*

Proof. We may suppose that $Z = X \cup_\pi (E_0(\eta) \times [0,1]) \cup_f Y$, and $W = Z \cup_p (E_0(\zeta) \times [0,1]) \cup_g U$, where π: $E_0(\eta) \times 0 \to X$, p: $E_0(\zeta) \times 0 \to Z$ are the projections of the fibre spaces, f: $E_0(\eta) \times 1 \to Y$, g: $E_0(\zeta) \times 1 \to U$. Set $V = Y \cup_{p'} (E_0'' \times [0,1]) \cup_{g'} U$, where $E_0'' = E_0(\zeta|Y)$ and p', g' are the restrictions of p, g. Let $E_0' = E_0(\eta + \zeta|X)$, and note that

$$E_0' \cong E_0(\eta) \cup_{p_1} (\bar{E}_0 \times [0,1]) \cup_{p_2} E_0(\zeta|X)$$

where $\bar{E}_0 = E_0(\pi^*(\zeta|X))$ is the induced fibre space over $E_0(\eta)$,

$$p_1\colon\ \bar{E}_0 \times 0 \to E_0(\eta)$$

its projection, p_2: $\bar{E}_0 \times 1 \to E_0(\zeta|X)$ the natural map of the fibre spaces covering π.

Define h: $E_0' \times 1 \to V$ by

$$h|E_0(\eta) = f\colon\ E_0(\eta) \to Y \subset V$$

$$h|E_0(\xi|X) = g|E_0(\xi|X)\colon\ E_0(\xi|X) \to U \subset V$$

$$h|\bar{E}_0 \times t = p_2\colon\ \bar{E}_0 \times t = E_0(\pi^*(\xi|X))$$

$$\to E_0(\xi|X) \times t \subset E_0(\xi|Y) \times t \subset V,$$

and it is easy to see that $W \cong X \cup_q (E_0' \times [0,1]) \cup_h V$, where q: $E_0' \times 0 \to X$ is the projection of the fibre space.

(1.4) **Lemma.** *It suffices to prove Theorem A for $m = \dim X$ sufficiently large.*

For, if $X \times S^r$ has a Spivak normal fibre space, then so does X, by (1.3), since $X \subset_p X \times S^r$.

In § 2, we will show that if $m = \dim X \geqq 4$, there is a Poincaré pair $(Z, X \cup X')$ with X', Z 1-connected. Then, if $W = Z_+ \cup_X Z_-$, it follows that $(W, X' \cup X')$ is a Poincaré pair with W 1-connected and $\partial W = X' \cup X'$ the union of 1-connected components. It follows from Spivak's theorem (see [3, (I.4.4)]) that there is a Spivak normal fibre space for W, i..e. a $(k-1)$-spherical fibre space ξ over W and an element $\alpha \in \pi_{m+k+1}(T(\xi), T(\xi|\partial W))$ such that $h(\alpha) \cap U_\xi = [W] \in H_{m+1}(W, \partial W)$.

(1.5) **Lemma.** *If $(W, \partial W)$ is a Poincaré pair with Spivak normal fibre space ξ^k, then $p^*(\xi)$ is the Spivak normal fibre space for $W \times [0,1]$, and hence $p^*(\xi)|D$ is the Spivak normal fibre space for the double $D = W \times 0 \cup (\partial W \times [0,1]) \cup W \times 1$, where p: $W \times [0,1] \to W$ is the natural projection, $D = \partial(W \times [0,1])$.*

It is easy to check that $T(p^*(\xi))/T(p^*(\xi)|\partial(W\times[0,1]))\cong\Sigma(T(\xi)/T(\xi|\partial W))$ and the natural map $T(\xi)/T(\xi|\partial W)\to\Sigma\,T(\xi|\partial W)$ is of degree 1. Then (1.5) follows easily.

Hence, $p^*(\xi)$ is the Spivak normal fibre space for $W\cup_{\partial W}W$, $X\subset_p W\cup_{\partial W}W$, with trivial normal fibre space, so by (1.3), $p^*(\xi)|X$ is the Spivak normal fibre space of X. This completes the proof of Theorem A except for the construction of Z which will be done in § 2.

We note that it is easy to extend Theorem A to Poincaré pairs:

Theorem A′. *Let $(X,\partial X)$ be a Poincaré pair. Then X has a Spivak normal fibre space ξ^k, i.e. a $(k-1)$-spherical fibre space over X and an element $\alpha\in\pi_{m+k}(T(\xi),T(\xi|\partial X))$ such that $h(\alpha)\cap U_\xi=[X]\in H_m(X,\partial X)$.*

Proof. The double $D=X_+\cup_{\partial X}X_-$ has a Spivak normal fibre space by Theorem A, i.e. ξ' over D such that $\alpha'\in\pi_{m+k}(T(\xi'))$ with $h(\alpha')\cap U_{\xi'}=[D]\in H_m(D)$. Then the composite map α

$$S^{m+k}\to T(\xi')\to T(\xi')/T(\xi'|X_-)=T(\xi'|X_+)/T(\xi'|\partial X)$$

and $\xi=\xi'|X_+$ fulfill the conditions of the Spivak normal fibre space.

§ 2. Surgery on Poincaré Spaces

In this section we show how to do surgery on Poincaré spaces to "kill" π_1. Recall the process that one uses with a smooth oriented manifold M^m, $m\geqq 4$. Take a set of generators $\gamma_1,\dots,\gamma_k\in\pi_1(M)$ and represent γ_i by a smooth embedding $g_i\colon S^1\to M$, $i=1,\dots,k$. Then each $g_i(S^1)$ has a trivial normal bundle so we have $\bar g_i\colon S^1\times D^{m-1}\to M$. Let $M_0=M-\operatorname{int}\left(\bigcup_i \bar g_i(S^1\times D^{m-1})\right)$. Then $\partial M_0=\bigcup_i S^1\times S^{m-2}$, and let $M'=M_0\cup\bigcup_i D^2\times S^{m-2}$. Then $M\cup M'=\partial W$, where $W=M\times[0,1]\cup\bigcup_i D_i^2\times D^{m-1}$ with $\bar g_i(S^1\times D^{m-1})\times 1$ identified with $S_i^1\times D^{m-1}\subset\partial(D_i^2\times D^{m-1})$. It follows easily, since $m\geqq 4$, that $\pi_1(M')\cong\pi_1(W)\cong\pi_1(M)/(\gamma_1,\dots,\gamma_k)=0$. We shall try to imitate this construction in the category of Poincaré spaces to prove:

(2.1) **Theorem.** *Let X be a connected Poincaré space of dimension $m\geqq 4$. Then there is a Poincaré pair $(Y,X\cup X')$, (i.e. X is one component of ∂Y), with Y, X' 1-connected, $H_i(Y,X)=0$ for $i\neq 2$, and $H_2(Y,X)$ free abelian.*

(2.2) **Lemma.** *If X is a Poincaré space, then $H_*(X)$ is finitely generated.*

Proof. By replacing X by its singular complex without changing its homological properties we may assume that X is a (infinite) CW complex. Let $i\colon K\subset X$ be a finite subcomplex such that $[X]\in\operatorname{im} i_*$, $[X]=i_*\mu$, $\mu\in H_n(K)$. By naturality of cap product $[X]\cap x=(i_*\mu)\cap x=i_*(\mu\cap i^*x)\in i_*H_*(K)$ and i_* is onto. Since K is a finite complex $H_*(K)$ is finitely generated, so $i_*H_*(K)=H_*(X)$ is finitely generated.

Let $\alpha_1, \ldots, \alpha_k \in \pi_1 X$ be a finite number of elements such that $h(\alpha_1), \ldots, h(\alpha_k)$ generate $H_1(X)$. Set $Y' = X \cup \bigcup D_i^2$, attaching D_i^2 by α_i. Then $H_1(Y') = H_1(X)/(h(\alpha_1), \ldots, h(\alpha_k)) = 0$.

(2.3) **Lemma.** *Suppose $H_1 K' = 0$. Then there is a space K and a map $f\colon K' \to K$, such that $\pi_1 K = 0$ and $f_*\colon H_*(K') \to H_*(K)$ is an isomorphism.*

Proof. Attach 2-cells to K' to get $\bar{K} = K' \cup \bigcup D^2$ so that $\pi_1(\bar{K}) = 0$. By the Hurewicz theorem $\pi_2(\bar{K}) = H_2(\bar{K})$, and we have the exact homology sequence

$$\to H_2(K') \to H_2(\bar{K}) \to H_2(\bar{K}, K') \to 0$$

since $H_1(K') = 0$, and we have by excision that

$$H_2(\bar{K}, K') \cong H_2(\bigcup D^2, \bigcup \partial D^2),$$

which is a free abelian group. Let B be a set of free generators of $H_2(\bar{K}, K')$ and choose $B' \subset \pi_2(\bar{K})$ so that B' maps bijectively to B. Attach a 3-cell to $\bar{K}$ for each element of B' to get $K = \bar{K} \cup \bigcup D^3$, so that $H_*(K) = H_*(\bar{K})/(B') \cong H_*(K')$.

Let Y be constructed from Y' by Lemma 2.3. We shall construct $X' \to Y$ such that $(Y, X \cup X')$ is a Poincaré pair, with X' 1-connected, which will complete the proof.

Note that in the case of a manifold described above, we have $\bigcup D_i^2 \times D^{m-1} \subset W$ and if we take the Thom-Pontrjagin construction we get $f\colon W \to \vee S_i^{m-1}$ and $f(M') = *$. It is easy to check that if $i\colon M \subset W$, then $(fi)^*(H^{m-1}(\vee S_i^{m-1}))$ is Poincaré dual to $\partial H_2(W, M) \subset H_1(M)$.

Hence let us take a map $f\colon Y \to \bigvee_k S^{m-1}$ such that

$$\begin{array}{ccccc} H^{m-1}(\vee S^{m-1}) & \xrightarrow{f^*} & H^{m-1}(Y) & \xrightarrow{i^*} & H^{m-1}(X) \\ \downarrow{\scriptstyle\varphi} & & & & \downarrow{\scriptstyle[X]\cap} \\ H_2(Y, X) & & \xrightarrow{\partial} & & H_1(X) \end{array}$$

commutes, where φ is some isomorphism of these two free abelian groups of rank k. This can be done using obstruction theory, since $H^i(Y) \cong H^i(X) = 0$ for $i > m$, so any homomorphism of $H^{m-1}(\vee S^{m-1})$ into $H^{m-1}(Y)$ is realizable as f^* for some $f\colon Y \to \vee S^{m-1}$. Replace f by a fibre map with fibre F. We shall try to find X' by suitably modifying F. It is clear that much modification is necessary since it may be that $H_i(F) \neq 0$ in arbitrarily high dimensions. We will first show that F has the necessary properties in dimensions $\leqq m$.

(2.4) **Lemma.** *Let $f\colon Y \to \Sigma K$ be a fibre map with fibre F, where $H_i(K) = 0$ for $i \neq k-1$, and $H_{k-1}(K)$ is free, and suppose for some n,*

$j_*: H_{n-k+1}(Y; G) \to H_{n-k+1}(Y, F; G)$ maps onto. Then for an element $x \in H_n(Y; G)$, $x \cap f^*(H^k(\Sigma K)) = 0$ if and only if $x \in i_* H_n(F; G)$, where $i: F \to Y$ is the inclusion.

Proof. It is clear that $i_* y \cap f^* H^k(\Sigma K) = 0$. In the other direction consider the exact sequence with G coefficients of the pair (Y, F):

$$\cdots \longrightarrow H_n(F; G) \xrightarrow{i_*} H_n(Y; G) \xrightarrow{j_*} H_n(Y, F; G) \xrightarrow{\partial} \cdots.$$

We wish to show that $x \in \operatorname{im} i_*$ which is equivalent to $j_* x = 0$ by exactness.

Using the technique of [8] we note the isomorphism

(2.5) $$H_*(Y, F; G) \cong H_*(F \times (cK, K); G)$$

where cK is the cone on K. For if we let $\Sigma K = c_+ K \cup c_- K$, and $Y_\pm = f^{-1}(c_\pm K)$, $Y_0 = f^{-1}(K)$, then the inclusion $(Y, F) \subset (Y, Y_-)$ is a homotopy equivalence, and $(Y_+, Y_0) \subset (Y, Y_-)$ is an excision. Thus $H_*(Y, F) \cong H_*(Y, Y_-) \cong H_*(Y_+, Y_0)$. Now Y_+ is a fibre space over $c_+ K$ which is contractible, so $(Y_+, Y_0) \cong F \times (c_+ K, K)$ which proves (2.5). The exact sequence of (Y, F) with the isomorphism (2.5) will be called the Wang sequence.

Since $H_{i-1}(K) \cong H_i(c_+ K, K) \cong \bar{H}_i(\Sigma K)$ is zero for $i \neq k$ and free for $i = k$, it follows from the Künneth formula that

$$H_n(Y, F; G) = H_{n-k}(F; G) \otimes H_k(cK, K).$$

Since $H_k(cK, K)$ is free, an element $z \in H_{n-k}(F; G) \otimes H_k(cK, K)$ is zero if and only if $z \cap (1 \otimes \bar{y}) = 0$ for each $\bar{y} \in H^k(cK, K)$. But we have the commutative diagram

(2.6) $$\begin{array}{ccccc} H_n(Y) & \xrightarrow{j_*} & H_n(Y, F) & \xrightarrow{\cong} & H_n(F \times (cK, K)) \\ \downarrow{\scriptstyle \cap f^* y} & & \downarrow{\scriptstyle \cap \bar{f}^*(\bar{y})} & & \downarrow{\scriptstyle \cap(1 \otimes \bar{y})} \\ H_{n-k}(Y) & \xleftarrow{i_*} & H_{n-k}(F) & \xrightarrow{1} & H_{n-k}(F) \end{array}$$

where $y \in H^k(\Sigma K) = H^k(\Sigma K, *)$, $\bar{y} = p^* y$, $p: (cK, K) \to (\Sigma K, *)$, $\bar{f}: (Y, F) \to (\Sigma K, *)$ is induced by f. It follows that $i_*(j_* x \cap \bar{f}^*(\bar{y})) = x \cap f^* y = 0$ so that $j_* x \cap \bar{f}^* \bar{y} \in \partial H_{n-k+1}(Y, F; G)$. Since j_* maps onto $H_{n-k+1}(Y, F; G)$ and $\partial j_* = 0$ it follows that $j_* x \cap \bar{f}^* \bar{y} = 0$. But from (2.6), it follows that $j_* x \cap \bar{f}^* \bar{y} = j_* x \cap (1 \otimes \bar{y})$ so that $j_* x = 0$, proving the lemma.

(2.7) We note that using the Wang sequence, i.e. the exact sequence of (Y, F) and (2.5) one may show that $H_{n-k+1}(Y)$ maps onto $H_{n-k+1}(Y, F) \cong H_{n-2k+1}(F)$ under any of the following circumstances

(1) $n < 2k - 1$ (so that $n - 2k + 1 < 0$),

(2) $n = 2k - 1$ and $f_*: H_k(Y) \to H_k(\Sigma K)$ is onto.

(3) $n = 2k$ and $H_1(F) = 0$.

(2.8) **Proposition.** *Let X be a Poincaré space of dimension n, $j\colon X\to Y$ such that $H_i(Y,X)=0$ for $i\neq p+1$, $H_{p+1}(Y,X)$ is free on l generators, $p+q+1=n$, $p<q$. Suppose $f\colon Y\to \bigvee_l S^{q+1}$ is a fibre map with fibre F and suppose that the diagram*

$$\begin{array}{ccccc} H^{q+1}(\bigvee S^{q+1}) & \xrightarrow{f^*} & H^{q+1}(Y) & \xrightarrow{j^*} & H^{q+1}(X) \\ \downarrow{\scriptstyle\varphi} & & & & \downarrow{\scriptstyle [X]\cap} \\ H_{p+1}(Y,X) & & \xrightarrow{\partial} & & H_p(X) \end{array}$$

commutes, where φ is some isomorphism. Then there is an element $[Y]\in H_{n+1}(Y,X\cup F)$ such that $\partial[Y]=[X]-[F]$, for some $[F]\in H_n(F)$ and $([Y]\cap)=\varphi\colon H^{q+1}(Y,F)\to H_{p+1}(Y,X)$ is an isomorphism.

Proof. Since $p<q$, $n=p+q+1<2q+1=2(q+1)-1$. Now $0=j_*\partial\varphi(y)=j_*([X]\cap j^*f^*(y))=j_*[X]\cap f^*(y)$ for all $y\in H^{q+1}(\bigvee S^{q+1})$, so that Lemma 2.4 and (2.7)(1) imply that $j_*[X]=i_*z$ for some $z\in H_n(F)$, $i\colon F\subset Y$. Hence there is a $w\in H_{n+1}(Y,X\cup F)$ with $\partial w=[X]-z$. It remains to show that w,z can be chosen so that $w\cap\colon H^{q+1}(Y,F)\to H_{p+1}(Y,X)$ is an isomorphism.

Using the Wang sequence situation as before with $K=\bigvee S^q$, recall that we have the commutative diagram

$$\begin{array}{ccc} F\times(c\vee S^q,\ \bigvee S^q) & \xrightarrow{e} & (Y,F) \\ \downarrow{\scriptstyle \text{proj.}} & & \downarrow{\scriptstyle \bar f} \\ (c\vee S^q,\ \bigvee S^q) & \xrightarrow{d} & (\bigvee S^{q+1},*) \end{array}$$

where the maps e and d induce homology isomorphisms.

(2.9) **Lemma.** *For $x\in H_{n+1}(Y,F)$ we have the commutative diagram*

$$\begin{array}{ccccc} H^{q+1}(F\times(c\vee S^q,\ \bigvee S^q)) & \xleftarrow[e^*]{\cong} & H^{q+1}(Y,F) & \xrightarrow{1} & H^{q+1}(Y,F) \\ \downarrow{\scriptstyle (e_*^{-1}x)\cap} & & \downarrow{\scriptstyle x\cap} & & \downarrow{\scriptstyle k_*x\cap} \\ H_{p+1}(F) & \xrightarrow{i_*} & H_{p+1}(Y) & \xrightarrow{h_*} & H_{p+1}(Y,X) \end{array}$$

where $k\colon (Y,F)\to(Y,F\cup X)$, $h\colon Y\to(Y,X)$ are inclusions.

This is just the usual commutativity of the cap product with inclusions, where $H_*(F)$ is identified with $H_*(F\times c\vee S^q)$.

From (2.9) we may deduce

(2.10) **Lemma.** *The map*

$$\beta\colon H_{n+1}(Y,F)\to \operatorname{Hom}(H^{q+1}(Y,F),H_{p+1}(Y))$$

given by $(\beta(x))(y)=x\cap y$, is onto, where $x\in H_{n+1}(Y,F)$, $y\in H^{q+1}(Y,F)$.

Proof. First we note that $e^*\colon H^{q+1}(Y, F) \xrightarrow{\cong} H^{q+1}(c \vee S^q, \vee S^q)$ which is free. Let $b_1, \dots, b_r$ be a basis for $H^{q+1}(Y, F)$. To prove (2.10) it suffices to show that for any element $z \in H_{p+1}(Y)$ and any s, $1 \leqq s \leqq r$, there is an element $x \in H_{n+1}(Y, F)$ such that $x \cap b_s = z$ and $x \cap b_j = 0$, $j \neq s$, since any homomorphism is the sum of these simple ones.

Now $i_*\colon H_{p+1}(F) \to H_{p+1}(Y)$ is onto since F is the fibre of $f\colon Y \to \vee S^{q+1}$ and $p < q$. Let $z' \in H_{p+1}(F)$, such that $i_* z' = z$. Let $g_1, \dots, g_r$ be the basis of $H^{q+1}(c \vee S^q, \vee S^q)$ corresponding to $b_1, \dots, b_r$, $e^* b_i = g_i$, and let $\bar{g}_1, \dots, \bar{g}_r$ be the dual basis of $H_{q+1}(c \vee S^q, \vee S^q)$ so that $\bar{g}_i \cap g_j = \delta_{ij}$. Set

$$x' = z' \otimes \bar{g}_s \in H_{p+1}(F) \otimes H_{q+1}(c \vee S^q, \vee S^q) = H_{n+1}\big(F \times (c \vee S^q, \vee S^q)\big).$$

Then $x' \cap g_i = (z' \otimes \bar{g}_s) \cap g_i = \delta_{is} z'$. If $x = e_* x'$, then

$$x \cap b_i = (e_* x') \cap b_i = i_*(x' \cap e^* b_i) = i_*(x' \cap g_i) = i_*(\delta_{is} z') = \delta_{is} z$$

which completes the proof of (2.10).

Now we may complete the proof of (2.8). For we have $w \in H_{n+1}(Y, X \cup F)$ such that $\partial w = [X] - z$, $z \in H_n(F)$. Hence the diagram

$$\text{(2.11)} \qquad \begin{array}{ccccccccc}
\cdots \to & H^q(X) & \longrightarrow & H^{q+1}(Y, X \cup F) & \longrightarrow & H^{q+1}(Y, F) & \xrightarrow{j^*} & H^{q+1}(X) & \to \cdots \\
& \downarrow{\scriptstyle [X]\cap} & & \downarrow{\scriptstyle w\cap} & & \downarrow{\scriptstyle w\cap} & & \downarrow{\scriptstyle [X]\cap} & \\
\cdots \to & H_{p+1}(X) & \longrightarrow & H_{p+1}(Y) & \xrightarrow{h_*} & H_{p+1}(Y, X) & \xrightarrow{\partial} & H_p(X) & \to \cdots
\end{array}$$

commutes, where the upper sequence is the exact sequence of the triple $(Y, X \cup F, F)$ with $H^*(X \cup F, F)$ identified with $H^*(X)$ and the lower sequence is the exact sequence of the pair (Y, X). If $\bar{f}\colon (Y, F) \to (\vee S^{q+1}, *)$ is the map of pairs induced by f, then $\bar{f}^*\colon H^{q+1}(\vee S^{q+1}, *) \to H^{q+1}(Y, F)$ is an isomorphism, and $\varphi \bar{f}^{*^{-1}}$ is an isomorphism so that

$$\begin{array}{ccc}
H^{q+1}(Y, F) & \xrightarrow{j^*} & H^{q+1}(X) \\
\downarrow{\scriptstyle \varphi \bar{f}^{*^{-1}}} & & \downarrow{\scriptstyle [X]\cap} \\
H_{p+1}(Y, X) & \xrightarrow{\partial} & H_p(X)
\end{array}$$

commutes. Then $\partial(\varphi \bar{f}^{*^{-1}} - w\cap) = 0$, and $H^{q+1}(Y, F)$ is free abelian, so that $(\varphi \bar{f}^{*^{-1}} - w\cap)$ factors through $h_*\colon H_{p+1}(Y) \to H_{p+1}(Y, X)$, $(\varphi \bar{f}^{*^{-1}} - w\cap) = h_* \alpha$, $\alpha\colon H^{q+1}(Y, F) \to H_{p+1}(Y)$. By (2.10), there is an element $x \in H_{n+1}(Y, F)$ such that $x\cap = \alpha$, so that $\varphi \bar{f}^{*^{-1}} - w\cap = h_* \circ (x\cap)$. But $h_* \circ (x\cap) = (k_* x)\cap$ from (2.9) so that $(w + k_* x)\cap = \varphi \bar{f}^{*^{-1}}$ is an isomorphism which completes the proof of (2.8).

Let us apply (2.8) in our original situation with $p = 1$, $q + 1 = m - 1$, $n = m$, where $m \geqq 4$, so $p = 1 < q = m - 2$. Since $[X]\cap$ is an isomorphism and $H^i(Y, F)=0$ for $i \neq m-1$, $i<m+1$ (using the Wang sequence and the fact that $\pi_1(F)=0$), it follows from (2.11) and the Five Lemma that for $y=(w+k_* x)$

$$y \cap: H^i(Y, F \cup X) \to H_{m+1-i}(Y)$$

is an isomorphism for $0<i<m+1$.

Let $F^{(j)}$ be the j-skeleton of F (or of its singular complex). Then we have that $H_m(F^{(m)}, F^{(m-1)}) \cong \pi_m(F^{(m)}, F^{(m-1)})$ is the free group on the m-cells of F, i.e. the group of m-chains, and

$$Z_m(F) \to H_m(F) \to 0$$

is exact, where $Z_m(F) \subset \pi_m(F^{(m)}, F^{(m-1)})$ is the subgroup of cycles. Let $\alpha \in Z_m(F) \subset \pi_m(F^{(m)}, F^{(m-1)})$ be such that $\{\alpha\}=\partial_2 y$ in $H_m(F)$, where $\partial_2: H_{m+1}(Y, X \cup F) \to H_m(F)$ is associated with the triple $(Y, X \cup F, X)$. From the exact sequence

$$0 \to Z_m(F) \to C_m(F) \xrightarrow{\partial} \partial C_m(F) \to 0$$

we can find a direct summand $B \subset \pi_m(F^{(m)}, F^{(m-1)})$ such that $\partial | B: B \to \partial C_m(F)$ is an isomorphism, since $\partial C_m(F)$ is free. Let $X'=F^{(m-1)} \cup \bigcup_{b \in \bar{B}} e_b^m \cup_{\partial\alpha} e^m$, where $\bar{B}$ is a free basis for B, $\partial: \pi_m(F^{(m)}, F^{(m-1)}) \to \pi_{m-1}(F^{(m-1)})$. If $l: X' \to F$ is the inclusion, $l_*: H_i(X') \to H_i(F)$ is an isomorphism for $i<m$, $H_m(X')=Z$ with generator $[X']$ and $l_*[X']=\partial_2 y$. It follows that $[X]-[X']=\partial[Y]$, $[Y] \in H_{m+1}(Y, X \cup X')$ and $l_*[Y]=y$, where l now denotes the map of pairs $(Y, X \cup X') \to (Y, X \cup F)$.

Then $H^i(Y, X')=H^i(Y, F)$ for $i<m+1$, $H^i(Y, X')=0$ for $i \geqq m+1$, so that it follows from (2.11) (with X' replacing F) and (2.8) that $(Y, X \cup X')$ is a Poincaré pair, which completes the proof of (2.1).

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