Russian Math. Surveys 31:2 (1976), 71–138
From Uspekhi Mat. Nauk 31:2 (1976), 69–134

HERMITIAN K-THEORY. THE THEORY OF CHARACTERISTIC CLASSES AND METHODS OF FUNCTIONAL ANALYSIS

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This paper gives a survey of results on Hermitian K-theory over the last ten years. The main emphasis is on the computation of the numerical invariants of Hermitian forms with the help of the representation theory of discrete groups and by signature formulae on smooth multiply-connected manifolds.

In the first chapter we introduce the basic concepts of Hermitian K-theory. In particular, we discuss the periodicity property, the Bass–Novikov projections and new aids to the study of K-theory by means of representation spaces. In the second chapter we discuss the representation theory method for finding invariants of Hermitian forms. In §5 we examine a new class of infinite-dimensional Fredholm representations of discrete groups. The third chapter is concerned with signature formulae on smooth manifolds and with various problems of differential topology in which the signature formulae find application.

Contents

Introduction

One of the classical problems of algebra is that of classifying Hermitian (skew-Hermitian) quadratic forms and their automorphisms. Hermitian K-theory, the basic concepts of which were first worked out in [14], studies unimodular quadratic (Hermitian and skew-Hermitian) forms and their automorphisms over an arbitrary ring Λ with involution up to "stable" equivalence, which is somewhat weaker than ordinary isomorphism (see §§1, 2). Unimodularity means that the matrix of the quadratic (Hermitian or skew-Hermitian) form has an inverse. Λ-modules are supposed to be free or projective. The stable equivalence classes of forms over a given ring Λ with involution form a group with respect to direct sum, which is denoted by $K_0^h(\Lambda)$ in the Hermitian case and by $K_0^{sh}(\Lambda)$ in the skew-Hermitian case. In classical number theory one considers integral or rational forms, with $\Lambda = \mathbf{Z}$ or $\mathbf{Q}$, forms over the ring of integral elements of some algebraic number field or over such a field itself. The groups of equivalence classes of forms over rings of numbers were first studied by Witt and are known as Witt groups.

For a long time now, many authors have used number-theoretic results on integral quadratic forms in the theory of smooth simply-connected manifolds (Milnor [48], Rokhlin [79] and others). In 1966, in papers by Novikov and Wall ([52], [13]) it was shown that the problems of modifying even-dimensional multiply-connected manifolds with fundamental group π (technically very important in the method of classifying manifolds due to Novikov and Browder ([51])) leads to the need to compute the groups $K_0^h(\Lambda)$ and $K_0^{sh}(\Lambda)$, the analogues of the Witt group for the group ring $\Lambda = \mathbf{Z}[\pi]$, whose involution has the form $g \to \sigma(g)g^{-1}$, where $\sigma\colon \pi \to \{\pm 1\}$ is the "orientation automorphism". In its most general form, this result was obtained by Wall [4]. Novikov [52] has examined the fundamental case of the free Abelian group $\pi = \mathbf{Z}^s$, which arose in the proof of the topological invariance of the Pontryagin in classes; he also pointed to the deep connection between an algebraic problem concerning Hermitian Λ-forms and the problem of classifying all homotopically invariant expressions in the Pontryagin–Hirzebruch classes (see also the survey [78]; its first version was a preprint for the International Congress of Mathematicians at Moscow in 1966; it was published later, with some additions, in 1970). In 1968 Wall showed that the problem of modifying odd-dimensional manifolds required the study of classes of "stably equivalent" automorphisms of Hermitian (skew-Hermitian) forms over the group ring $\Lambda = \mathbf{Z}[\pi]$ of the fundamental group π of the original manifold (for precise definitions, see §1). This work was published in detail in 1970 ([78]). The classes of stable equivalence of automorphisms of Hermitian (skew-Hermitian) forms constitute commutative groups with respect to direct sums, which are denoted by $K_1^h(\Lambda)$ and $K_1^{sh}(\Lambda)$. Wall in his papers calls all these objects

"L-groups", where $L_0(\pi) = K_0^h(\Lambda)$, $L_1(\pi) = K_1^h(\Lambda)$, $L_2(\pi) = K_0^{sh}(\Lambda)$, $L_3(\pi) = K_1^{sh}(\Lambda)$. For group rings $\Lambda = \mathbf{Z}[\pi]$ these groups have been given the name of the "Wall groups" of π. In 1968–69 Shaneson in [5], developing an idea of Browder [57], computed from purely geometric considerations these Wall groups for the case, very important in topology, of the free Abelian group $\pi = \mathbf{Z}^s$ (for the idea, see §10). However, Shaneson's result was non-effective: it remained completely unclear how one could find by algebraic operations from a given Hermitian form the complete system of its stable invariants. Finally, in 1969 in [13], the present author and Gel'fand used transition to the character group to study Hermitian forms over the group ring $\mathbf{Z}[\pi]$, where $\pi = \mathbf{Z}^n$, reducing their study to that of quadratic forms over the ring $\mathbf{C}^*(T^n)$ of functions on the torus.[1] In this paper, it was shown that the Hermitian K_0^h-groups over the ring $\Lambda = C(X)$ of complex-valued function coincide with the ordinary groups $K^0(X)$ in homotopy theory of the classes of stable equivalence of vector bundles (see §4). The connection between Hermitian forms and ordinary K-theory, constructed from vector bundles, for the ring of functions $\mathbf{C}^*(X)$, and also differential-topological arguments for the group ring $\mathbf{Z}[\pi]$, led to the idea that it was necessary to establish for all rings Λ with involution a single homological "Hermitian K-theory" $K_n^h(\Lambda)$, possessing the property of "Bott periodicity" if 2-torsion is neglected. From the point of view of Hermitian K-theory, periodicity means that the skew-Hermitian (Hermitian) groups $K_0^{sh}(\Lambda)$ should coincide, in reverse order, with the Hermitian (skew-Hermitian) groups $K_2^h(\Lambda)$. The problem of constructing such a theory was first solved by Novikov in 1970, using ideas of the Hamiltonian formalism (see [14]), for the group $K^h \otimes \mathbf{Z}[\frac{1}{2}]$ (see also §2 of this survey). Novikov proved homology properties of Hermitian K-theory and established a link between K_0^h and K_2^{sh}, clearly constructing analogues of the so-called Bass projections, which are similar to the suspension homomorphism in homotopy theory.

In particular, for the group $\pi = \mathbf{Z}^s$, these Bass–Novikov projections enable us to describe in an algebraically effective way the invariants of Hermitian (skew-Hermitian) forms and automorphisms. A precise construction of Hermitian K-theory, with the 2-torsion taken into account, based on the ideas of [14], was given by Ranicki in [65] and [66]. Sharpe [80] developed Milnor's approach to define and study the group $K_2^h(\Lambda)$. Using ideas of Quillen [67], Karoubi later developed in [73], [74], [75] a homotopy approach to the construction and proof of the 4-periodicity of the Hermitian $K^h \otimes \mathbf{Z}[\frac{1}{2}]$-theory (see §3).

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[1] Quadratic forms over the ring $\Lambda = \mathbf{Z}[\pi]$, where π is a free Abelian group (a lattice in $\mathbf{R}^n$) arise in the theory of a solid body as the square matrix of interacting separate parts (atoms) of the lattice, $n = 1, 2, 3$. In this theory Λ-forms always lead to forms over the ring of functions $\mathbf{C}^*(T^n)$, using characters; but there is interest in the classification relative to orthogonal transformations or in eigenvalues as functions of a point of the torus (phonon spectra).

We have already mentioned above the connection between the Hermitian K-groups for the group ring $\Lambda = \mathbf{Z}[\pi]$ of an infinite group π and integrals of the rational Pontryagin–Hirzebruch classes over certain cycles. A connection between the ordinary signature $\tau(M^{4k})$ of the integral quadratic form of the intersection of cycles on $H_{2k}(M^{4k})$ and characteristic classes was found by Rokhlin, Thom, and Hirzebruch back in the early '50's (see [36]–[38]). This link is called the "Hirzebruch formula" and takes the form $\langle L_k(M^n), [M^n]\rangle = \tau(M^r)$, where L_k is the Hirzebruch polynomial of the Pontryagin classes. Thanks to this formula, for example, were discovered the smooth structures on spheres (Milnor [49]). In the middle of the 60's analogues of this formula were discovered in the theory of multiply-connected manifolds, when the homotopy invariance of the integrals of the Pontryagin–Hirzebruch classes along special cycles (intersections of cycles) of codimension 1) were established by Novikov in a number of special cases in 1965 ([51], [52]) and were then completely proved in papers by Rokhlin, Kasparov, Hsiang and Farrell ([53]–[55]). By 1970 it was completely clear that the problem of finding all "multiply connected analogues of the Hirzebruch formula" or all homotopically invariant relations on the rational Pontryagin classes of multiply-connected manifolds reduce completely to the problem of Hermitian K-theory. The author (see [46] and §8 of this survey) found a new homotopy invariant of multiply-connected manifolds, which generalizes the ordinary signature: to each closed manifold M^n there corresponds an element of Hermitian $K_*\otimes \mathbf{Z}[\frac{1}{2}]$-theory, which is invariant also with respect to the bordisms $\Omega_* K(\pi, 1)$ that preserve the fundamental group. Thus, there arises a homomorphism ϕ: $\Omega_*(K(\pi, 1)) \to K^h_*(\Lambda)\otimes \mathbf{Z}[\frac{1}{2}]$. All homotopically invariant expressions in the Pontryagin classes reduce to purely algebraic invariants on the Hermitian K-groups $K^h_*(\Lambda)$, where $\Lambda = \mathbf{Z}[\pi]$. If $[M^n] \in \Omega_n(K(\pi, 1))$ is the bordism class of the manifold in question, then according to a conjecture of Novikov ([52], [78]) there must exist a purely algebraic homomorphism ("Chern character") σ: $K^h_*(\Lambda)\otimes \mathbf{Z}[\frac{1}{2}] \to H_*(\pi; Q)$, $\Lambda = \mathbf{Z}[\pi]$, and a "generalized Hirzebruch formula" $\langle \sigma\circ\varphi[M^n], x\rangle = \langle L_k(M^n), D\psi^*(x)\rangle$, where $x \in H^*(\pi; Q)$, $\pi = \pi_1(M)$, ψ: $M^n \to K(\pi, 1)$ is the natural map, L_k the Hirzebruch polynomial, and D the Poincaré duality operator.

However, for non-Abelian infinite groups π there were no such methods for finding invariants of Hermitian Λ-forms or indeed methods of defining the homomorphism σ. In an important paper [26], Lusztig, starting from the methods of Atiyah–Singer and using a continuous analogue of a construction of the author [46], developed a new analytical approach to these problems: he worked with the finite-dimensional representations of π and the families of elliptic operators (or elliptic complexes) associated with them. In particular, he gave a new proof of the theorem already mentioned of the homotopy invariance of the integrals of the rational Pontryagin–Hirzebruch classes along intersections of cycles of codimension 1 and

obtained some particular results for individual cycles, dual to the cohomology cycles of a non-commutative fundamental group π, provided that it can be regarded as a discrete subgroup of a semisimple Lie group (§7).

The author, in [16] and [17] has applied methods of functional analysis and the theory of infinite-dimensional representations to the problem of computing the Hermitian K-groups. In particular, the "Fredholm representations" studied in [17] enable us to find a number of new invariants of Hermitian forms over the group ring (see §6) and for a wide class of groups to give a complete classification of homotopically invariant expressions in the rational characteristic classes (see §9). To the class of groups studied in [17] there belong the fundamental groups of all manifolds of non-positive curvature. For this class of groups π the following theorem is completely proved: 1) The scalar product $\langle L_k(M^n)x, [M^n]\rangle$ is homotopically invariant for every cohomology class x that is obtained from the image of the map $M^n \to K(\pi, 1)$; here $L_k(M^n)$ is the Hirzebruch polynomial of the Pontryagin classes and M^n is a closed manifold. 2) For a manifold with fundamental group π there are no other homotopically invariant expressions in the rational Pontryagin classes.

Note that the Fredholm representations of the ring of functions $\mathbf{C}^*(X)$ were used by Kasparov, realizing an idea of Atiyah [18], for an explicit construction of the K-theory of homologies (see [20] and §5).

Very recently Solov'ev has studied discrete groups of algebraic groups over a locally compact local field. An example of such a field is the field of p-adic numbers. For these algebraic groups, Tits and Bruhat have proposed the construction of a contractible complex, the so-called Bruhat–Tits structure, on which the algebraic group acts. The factor space by the action of a discrete subgroup without elements of finite order is an Eilenberg–MacLane space and has a number of remarkable properties. In particular, for such complexes we can construct analogues of the de Rham complex of exterior differential forms and produce an ample set of Fredholm representations, which allow us to establish the homotopy invariance of expressions in the Pontryagin–Hirzebruch classes, provided that the fundamental groups are isomorphic to discrete subgroups of algebraic groups.

In conclusion we note that for integral group rings of a finite group π a number of strong results have recently been obtained [81], [82] concerning the computation of Hermitian K-groups, which we do not discuss in the survey.

CHAPTER 1

GENERAL CONCEPTS OF ALGEBRAIC AND HERMITIAN K-THEORY

§1. Elementary concepts of K-theory

There are many monographs on stable algebra, studying the topic from a

purely algebraic point of view. We mention here the book by Bass [1] and the lectures of Swan [2].

1. Modules over a ring Λ. The basic objects of our study will be modules over some associative ring with unity. As a rule, we shall denote the ring by Λ. In most cases the base ring Λ will be furnished with a supplementary structure – an involutory antiautomorphism, that is, a linear map $*: \Lambda \to \Lambda$ such that for any $\lambda, \lambda_1, \lambda_2 \in \Lambda$,

$$(\lambda_1\lambda_2)^* = \lambda_2^*\lambda_1^*, \quad (\lambda^*)^* = \lambda. \tag{1.1}$$

A ring with involution arises in a natural way when one studies the homotopy properties of multiply-connected topological spaces, in particular, smooth or piecewise linear manifolds, topological or homology manifolds. If X is a topological space and $\pi = \pi_1(X)$ its fundamental group, then in many problems one has to study the homology of X with coefficients in the group ring $\Lambda = \mathbf{Z}[\pi]$, taken as a local system of coefficients. The group ring $\Lambda = \mathbf{Z}[\pi]$ of π has a natural involution satisfying (1.1) and uniquely determined by the condition $g^* = g^{-1} (g \in \pi)$. Thus, let Λ be an arbitrary associative ring with involution, and M a finitely-generated left Λ-module. We denote by M^* the group of Λ-homomorphisms of the Λ-module M to Λ: $M^* = \mathrm{Hom}_\Lambda(M, \Lambda)$. The group M^* is equipped with the structure of a left Λ-module by setting $(\lambda\alpha)(m) = \alpha(m)\lambda^*$ ($\alpha \in M^*$, $m \in M$, $\lambda \in \Lambda$). If $\beta: M \to N$ is a Λ-homomorphism of Λ-modules, then we denote by β^* the homomorphism $\beta^*: N^* \to M^*$, given by $\beta^*(\alpha)(m) = \alpha(\beta(m))$ ($m \in M$, $\alpha \in N^*$). This β^* is a Λ-homomorphism of Λ-modules.

Although in the present section we do not consider systematically graded modules, we give here all the same the relevant definitions for graded objects. By a graded Λ-module M we mean a left Λ-module M, together with a fixed decomposition of M as a direct sum of Λ-modules $M = \oplus_k M_k$. If $m \in M_k$, then m is said to be homogeneous of degree k, $\deg m = k$. Let N be a second graded Λ-module, $N = \oplus_k N_k$ and $\alpha: M \to N$ a Λ-homomorphism. Then α is said to be homogeneous of degree t if $\alpha(M_k) \subset N_{k+t}$. In that case we write $\deg \alpha = t$. In particular, if M is a graded Λ-module, then M^* splits into the direct sum $M^* = \oplus_k M_k^*$, where M_k^* consists of all the homogeneous homomorphisms of degree $(-k)$ from M to the trivially graded ring Λ, that is, those homomorphisms $\alpha: M \to \Lambda$ such that $\alpha(M_i) = 0$ for $i \neq k$. Thus, the Λ-module M^* is turned in a natural way into a graded Λ-module $M^* = \oplus_j (M_j^*)$, where $(M^*)_j = (M_{-j})^*$.

For graded Λ-modules the concept of dual homomorphism has to be adjusted. In fact, if $\beta: M \to N$ is a homogeneous homomorphism of graded Λ-modules, then by β^* we denote the homomorphism $\beta^*: N^* \to M^*$, given by

$$\beta^*(\alpha)(m) = (-1)^{\deg\alpha\cdot\deg\beta}\alpha(\beta(m)) \quad (m\in M,\ \alpha\in(N^*)_{\deg\alpha}).$$

This β^* is a homogeneous homomorphism, and

$$\deg\beta^* = \deg\beta.$$

The operation of duality of Λ-modules is not involutory for arbitrary Λ-modules, but there is a natural homomorphism of Λ-modules ε_M: $M \to (M^*)^*$, which is defined on homogeneous elements by the formula

$$(\varepsilon_M(m))(\alpha) = (-1)^{\deg m\cdot\deg\alpha}\alpha(m) \quad (\alpha\in M^*,\ m\in M).$$

The homomorphism ε_M is homogeneous and $\deg\varepsilon_M = 0$.

If M is a projective graded Λ-module, then ε_M is an isomorphism.

2. **The algebraic K-functor.** We turn now to the description of concepts relating to algebraic and Hermitian K-theory. Suppose, as before, that Λ is an associative ring, and let $\mathcal{M}$ be the set of all finitely generated projective Λ-modules. By $K_0(\Lambda)$ we denote the Grothendieck group generated by the set $\mathcal{M}$ with the operation of direct sum of modules, in which the relation $[M] = 0$ is introduced for any free Λ-module $M\in\mathcal{M}$.

Suppose next that Γ is the set of automorphisms of free finitely generated Λ-modules. We define the group $K_1(\Lambda)$ to be the factor group of the free Abelian group generated by Γ factored by the following relations:

a) if (M_1, γ_1) and (M_2, γ_2) are two automorphisms of Γ, then

$$[M_1, \gamma_1] + [M_2, \gamma_2] - [(M_1\oplus M_2), (\gamma_1\oplus\gamma_2)] = 0, \tag{1.2}$$

b) if (M, γ_1) and (M, γ_2) are two automorphisms of Γ for one and the same module M, then

$$[M, \gamma_1] + [M, \gamma_2] - [M, \gamma_1\gamma_2] = 0. \tag{1.3}$$

An automorphism γ is said to be simple if $[\gamma] = 0$ in $K_1(\Lambda)$.

The groups $K_0(\Lambda)$ and $K_1(\Lambda)$ defined above are one of the basic objects of study of algebraic K-theory. On the other hand, these groups turn up in various problems of algebraic topology, as the domain of values of important homotopic and topological invariants of spaces and manifolds. The most important example of such an invariant is the Whitehead torsion for the homotopic equivalence of two chain complexes.

In our exposition the group $K_0(\Lambda)$ and $K_1(\Lambda)$ will play an auxiliary role in the development of Hermitian K-theory, and we now turn to the study of its concepts.

3. **The Hermitian K-functor.** Suppose again that we are given an associative ring Λ with involution. We consider a free finitely generated Λ-module M. Let α: $M^* \to M$ be an isomorphism of Λ-modules satisfying the relation

$$\varepsilon_M^{-1}\alpha^* = (-1)^k\alpha \tag{1.4}$$

(as we have defined the homomorphism ε_M for graded modules, we suppose here that M has the null grading).

When K is even, we say that we are given a Hermitian form α and when k is odd, a skew-Hermitian form α. Thus, a (skew-)Hermitian form over Λ is a pair (M, α) consisting of a free Λ-module M and an isomorphism α satisfying (1.4). For brevity we speak henceforth of Hermitian forms, distinguishing Hermitian and skew-Hermitian forms only when confusion would otherwise arise. Two Hermitian forms (M_1, α_1) and (M_2, α_2) are said to be equivalent if there is an isomorphism β: $M_1 \to M_2$ such that $\alpha_2 = \beta\alpha_1\beta^*$, that is, such that the following diagram is commutative:

$$\begin{array}{ccc} M_1 & \xrightarrow{\beta} & M_2 \\ {\scriptstyle\alpha_1}\uparrow & & \uparrow{\scriptstyle\alpha_2} \\ M_1^* & \xleftarrow{\beta^*} & M_2^* \end{array} \tag{1.5}$$

With each Hermitian form (M, α) we can associate a matrix with entries in Λ. To do this we fix in M a free basis $(m_1, \ldots, m_s)$ over Λ. Let $(m_1^*, \ldots, m_s^*)$ be the dual basis for M^*, so that $m_i^*(m_j) = \delta_{ij}$, where $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$. Then the homomorphism α: $M^* \to M$ determines the matrix A consisting of elements of Λ whose rows are the coordinate elements of $\alpha(m_i^*)$ in the basis $(m_1, \ldots, m_s)$:

$$\alpha(m_i^*) = \sum_{j=1}^{s} \lambda_{ij} m_j. \tag{1.6}$$

The condition (1.4) in terms of $A = \| \lambda_{ij} \|$ then takes the following form:

$$\lambda_{ij}^* = (-1)^k \lambda_{ji}. \tag{1.7}$$

If (M_1, α_1) and (M_2, α_2) are two Hermitian forms and A_1 and A_2 their matrices with respect to the bases, then the equivalence condition (1.5) can be stated in the following way: there exists an invertible matrix $B = \| \mu_{ij} \|$ with coefficients in Λ such that

$$A_1 = B^* A_2 B, \tag{1.8}$$

when by $B^* = \| \nu_{ij} \|$ we mean the matrix for which $\nu_{ij} = \mu_{ji}^*$.

The set of equivalence classes of Hermitian forms over Λ will be denoted by $\mathcal{M}_k$. This set splits into disjoint subsets according to the number of generators of the free module on which the Hermitian form is defined, or, what comes to the same thing, the order of the matrix corresponding to the Hermitian form. We introduce on $\mathcal{M}_k$ the operation of the direct sum of two Hermitian forms (M_1, α_1) and (M_2, α_2): we set

$$M = M_1 \oplus M_2, \qquad \alpha = \alpha_1 \oplus \alpha_2. \tag{1.9}$$

Thus, the direct sum of the two Hermitian forms (denoted by $(M_1, \alpha_1) \oplus (M_2, \alpha_2)$) is the pair (M, α) determined by (1.9). To equivalent Hermitian forms there correspond, according to (1.9), equivalent direct sums, so that

the operation of direct sum is well-defined on $\mathscr{M}_h$.

Now we are in the position to define the group $K^h_{2k}(\Lambda)$, which we call the Hermitian K-functor of Λ. By definition, $K^h_{2k}(\Lambda)$ is the factor group of the free Abelian group generated by the set $\mathscr{M}_h$ of classes of equivalent Hermitian forms, with the following relations:

$$[M_1, \alpha_1] + [M_2, \alpha_2] - [(M_1, \alpha_1) \oplus (M_2, \alpha_2)] = 0, \tag{1.10}$$

$$[M, \alpha] = 0 \tag{1.11}$$

for the Hermitian form (M, α) whose matrix A is

$$A = \begin{pmatrix} 0 & 1 \\ (-1)^k & 0 \end{pmatrix}. \tag{1.12}$$

To define the groups $K^h_{2k+1}(\Lambda)$ we proceed as in algebraic K-theory. Before defining $K^h_{2k+1}(\Lambda)$ we note that according to (1.11) the Hermitian form (1.12) and also the direct sum of any number of copies of the form (1.12) determine the trivial element in $K^h_{2k}(\Lambda)$. We call the Hermitian form (M, α) Hamiltonian if M has a basis $(m_1, \ldots, m_s, h_1, \ldots, h_s)$ in which the matrix A of α has the form

$$A = \begin{pmatrix} 0 & E \\ (-1)^k E & 0 \end{pmatrix}, \tag{1.13}$$

where E is the unit matrix of order s.

Let (M, α) be a Hamiltonian form. We consider an isomorphism $\gamma\colon M \to M$ preserving the form α, that is, such that

$$\alpha = \gamma\alpha\gamma^*. \tag{1.14}$$

Isomorphisms satisfying (1.14) are called Hamiltonian transformations of (M, α). Thus, as in the case of algebraic K-theory, we can define the direct sum of Hamiltonian transformations $(M_1, \alpha_1, \gamma_1)$ and $(M_2, \alpha_2, \gamma_2)$ by setting

$$(M_1, \alpha_1, \gamma_1) \oplus (M_2, \alpha_2, \gamma_2) = ((M_1 \oplus M_2), (\alpha_1 \oplus \alpha_2), (\gamma_1 \oplus \gamma_2)). \tag{1.15}$$

Let Γ_h be the set of Hamiltonian transformations of Hamiltonian forms. The group $K^h_{2k+1}(\Lambda)$ is defined as the factor group of the free Abelian group generated by Γ_h, with the following relations:

a) if $(M_1, \alpha_1, \gamma_1)$, $(M_2, \alpha_2, \gamma_2)$ are two Hamiltonian transformations, then

$$[M_1, \alpha_1, \gamma_1] + [M_2, \alpha_2, \gamma_2] - [(M_1, \alpha_1, \gamma_1) \oplus (M_2, \alpha_2, \gamma_2)] = 0, \tag{1.16}$$

b) if (M, α, γ_1), (M, α, γ_2) are two Hamiltonian transformations of one and the same Hamiltonian form (M, α), then

$$[M, \alpha, \gamma_1] + [M, \alpha, \gamma_2] - [M, \alpha, \gamma_1\gamma_2] = 0, \tag{1.17}$$

c) $\left[\begin{pmatrix} A & 0 \\ 0 & A^{*-1} \end{pmatrix}\right] = 0.$

4. Other versions of Hermitian K-theory. We have introduced the simplest

version of K-theory related to Hermitian forms over a ring Λ. In fact, however, in various problems we have to consider several modifications of the groups $K_k^h(\Lambda)$ that are better adapted to one problem or another.

We introduce in this subsection the most important versions of Hermitian K-theory, describing their domain of applicability on the one hand and their interconnections on the other.

a) Historically, the first groups of Hermitian K-theory appeared in differential topology in the description in algebraically invariant terms of the obstructions to the possibility of some modifications or other of smooth manifolds. The corresponding groups were denoted by $L_k(\Lambda)$ and called the Wall groups (after the author who first studied them – see [3], or, for a detailed exposition, the book [4]).

A Hermitian form (M, α) on a free Λ-module M can be regarded as a bilinear function

$$\beta(m_1, m_2) = \alpha^{-1}(m_1)(m_2) \qquad (m_1, m_2 \in M). \tag{1.18}$$

The function β takes its values in Λ. The condition (1.17) for α to be (skew-) Hermitian implies for β that

$$\beta(m_1, m_2) = (-1)^k(\beta(m_2, m_1))^* \qquad (m_1, m_2 \in M). \tag{1.19}$$

Such bilinear functions arise in smooth multiply-connected manifolds as intersection indices of cycles realized by embedded submanifolds. The intersection indices of cycles are closely connected with the indices of self-intersection of embedded submanifolds, and the algebraic expression of this connection is embodied in the definition of the Wall groups.

For let Λ' denote the group

$$\Lambda' = \Lambda/\{\nu - (-1)^k\nu^* \colon \nu \in \Lambda\}. \tag{1.20}$$

We consider a function $\gamma\colon M \to \Lambda'$ satisfying the following conditions:

$$\gamma(m_1 + m_2) = \gamma(m_1) + \gamma(m_2) + \beta(m_1, m_2), \tag{1.21}$$

$$\beta(m, m) = \gamma(m) + (-1)^k(\gamma(m))^*, \tag{1.22}$$

$$\gamma(\lambda m) = \lambda\gamma(m)\lambda^* \qquad (\lambda \in \Lambda). \tag{1.23}$$

All these equations are to be understood in the following way: we have to choose representatives in the coset $\gamma(m)$, carry out all operations on the representative in Λ, and then associate with it the corresponding coset.

A collection of functions (β, γ) satisfying (1.19) and (1.21)–(1.23), and such that the homomorphism α in (1.18) is a Hermitian form, is called a quadratic form over Λ. Quadratic forms also admit the operation of direct sums. For if (M_1, β_1, γ_1) and (M_2, β_2, γ_2) are two quadratic forms, then we set $M = M_1 \oplus M_2$, $\beta = \beta_1 \oplus \beta_2$ and we define γ by the relation (1.21), where we suppose that γ coincides with γ_i $(i = 1, 2)$ on M_i.

A Hamiltonian quadratic form is a direct sum of forms (M, β, γ) of the following kind: the Λ-module M has two generators, the matrix A of the

bilinear form β is

$$A = \begin{pmatrix} 0 & 1 \\ (-1)^k & 0 \end{pmatrix}$$

with respect to the basis (m_1, m_2), and γ is determined with respect to this basis by the formulae $\gamma(m_1) = \gamma(m_2) = 0$. Thus, we define the group $L_{2k}(\Lambda)$ on the set of quadratic forms as the factor group of the free Abelian group generated by the quadratic forms, factored by relations analogous to those for $K^h_{2k}(\Lambda)$.

The groups $L_{2k+1}(\Lambda)$ are defined with the help of automorphisms of free Λ-modules preserving the pair of functions (β, γ), which specify a Hamiltonian quadratic form.

b) The second version of the Wall group is bound up with the study of obstructions to the construction of maps of smooth manifolds up to simple homotopy equivalence. In this case it is supposed that every free Λ-module is furnished with a class of equivalent bases, a pair of bases for M being said to be equivalent if the transition from one to the other is determined by a simple automorphism α: $M \to M$, that is, $[\alpha] = 0$ in the group $K_1(\Lambda)$.

Then we distinguish the class of those quadratic forms (M, β, γ) for which the bilinear form β is determined by a simple isomorphism α: $M^* \to M$ in any of the equivalent bases of M and the corresponding dual basis for M^*. Correspondingly, two quadratic forms (M_1, β_1, γ_1) and (M_2, β_2, γ_2) are taken to be equivalent if there is a simple isomorphism δ: $M_1 \to M_2$ sending one quadratic form into the other.

The class of automorphisms preserving Hamiltonian quadratic forms can also be restricted to the subclass consisting of only simple automorphisms.

The corresponding analogues of Hermitian K-theory are also called the Wall groups and denoted by $L^s_k(\Lambda)$.

c) In these three versions of Hermitian K-theory we have observed a number of restrictions in the definitions. In the first place, we have fixed the class of Λ-modules on which the forms are considered. Up to now this class has been either the free Λ-modules or the free Λ-modules with a basis. Secondly, we have fixed the class of forms to be considered. We have the cases of Hermitian forms, quadratic forms, and simple quadratic forms. Thirdly, and finally, we have fixed an equivalence relation between forms, that is, we have defined which pairs of forms are to be regarded as equivalent and which forms as equivalent to zero.

Analogous restrictions have been observed in defining the classes of automorphisms preserving a Hamiltonian form.

Other versions of Hermitian K-theory have also been treated in the literature. We mention the most important of these, indicating the class of modules and of forms and the equivalence relation in each of these versions. A useful class of Λ-modules to consider in a number of problems (see §2) is the class of projective Λ-modules. The forms are then the so-called (simple)

Hermitian forms, that is, those bilinear functions β on M of the kind (1.18) for which there exists a function γ satisfying (1.21)–(1.23). However, the function γ itself is not fixed and is not considered. We denote the corresponding groups by $\widetilde{K}_k^h(\Lambda)$ ($\widetilde{K}_k^s(\Lambda)$). Finally, we can take as trivial the forms (M, α), where $M = N \oplus N^*$ and the morphism

$$\begin{pmatrix} 1 & 0 \\ 0 & \varepsilon_N \end{pmatrix} \alpha\colon\ (N^* \oplus N) \to (N \oplus N^*)$$

is written in matrix form as

$$\begin{pmatrix} 0 & E \\ (-1)^k E & 0 \end{pmatrix}.$$

Such forms are called projective-Hamiltonian.

5. Relations among the different versions of Hermitian K-theory. Despite the abundance of versions of Hermitian K-theory, the differences between them are not very important. Roughly speaking, if we ignore in Hermitian K-theory elements of order two, then it seems that all the versions are isomorphic to each other. As an illustration we consider the previously defined K-functors $K_k^h(\Lambda)$, $\widetilde{K}_k^h(\Lambda)$, $\widetilde{K}_k^s(\Lambda)$, $L_k(\Lambda)$ and $L_k^s(\Lambda)$. It is quite clear that the groups $L_k(\Lambda)$ and $K_k^h(\Lambda)$ can be obtained from the form defining $L_k^s(\Lambda)$ by "forgetting" the fine structure and going over to a coarser structure.

Thus, we have natural homomorphisms

$$\begin{array}{ccccc} L_k^s(\Lambda) & \to & L_k(\Lambda) & \to & \widetilde{K}_k^h(\Lambda) \\ & \searrow & & \nearrow & \downarrow \\ & & \widetilde{K}_k^s(\Lambda) & \to & K_k^h(\Lambda) \end{array}$$

These homomorphisms connecting all these groups, with the exception, perhaps, of $K_k^h(\Lambda)$, have a common property: the kernel and cokernel of each of the homomorphism consists of elements of order a power of 2. One can give more precise estimates on the orders of elements in the kernel and cokernel. In this connection see, for example, [5], [6], [7]. Thus, the map $L_k^s(\Lambda) \to L_k(\Lambda)$ can be included ([5]) in an exact sequence of groups

$$\ldots \to W_k(\Lambda) \to L_k^s(\Lambda) \to L_k(\Lambda) \to W_{k-1}(\Lambda) \to \ldots$$

The groups $W_k(\Lambda)$ are effectively described in terms of $K_1(\Lambda)$ and consist of elements of order 2.

We make now a remark that is important later on. If Λ contains the number $\frac{1}{2}$, then the groups $L_k(\Lambda)$, $\widetilde{K}_k^h(\Lambda)$, and $K_k^h(\Lambda)$ simply coincide. In fact, these three groups differ from one another in that we consider in place of the Hermitian form α the bilinear form β, with a function γ satisfying (1.21)–(1.23) added. We show that if $\frac{1}{2} \in \Lambda$, then γ is uniquely determined by the properties (1.21)–(1.23). For, let β be an arbitrary bilinear form satisfying (1.19). Then we set

$$\gamma(m) = \frac{1}{2}\beta(m, m) \qquad (m \in M). \tag{1.24}$$

It is not difficult to verify that $(\gamma(m))^* = (-1)^k\gamma(m)$ and that (1.21)–(1.23) are satisfied. To see that γ is uniquely determined by (1.21)–(1.23), it is sufficient to note that the ring splits into the direct sum of two subgroups $\Lambda = \Lambda^+ \oplus \Lambda^-$, consisting of the self-conjugate and skew-conjugate elements, respectively, while Λ' (1.20) is isomorphic to Λ^+ or Λ^-, according to the parity of k.

If γ is a function satisfying (1.21)–(1.23), then, for example, for even k we get

$$\gamma(m) \in \Lambda^+, \quad \beta(m, m) = 2\gamma(m), \tag{1.25}$$

that is, γ is uniquely determined.

Thus, when $\frac{1}{2} \in \Lambda$, the set of Hermitian forms $\mathscr{M}_h$ coincides with the set of quadratic forms, that is, the groups $L_k(\Lambda)$, $\widetilde{K}_k^h(\Lambda)$, and $K_k^h(\Lambda)$ are isomorphic.

The functional methods to which the present article is dedicated are insensitive to adjoining to Λ the element $\frac{1}{2}$, and therefore, from the point of view of these methods we cannot distinguish the different versions of Hermitian K-theory $L_k(\Lambda)$, $\widetilde{K}_k^h(\Lambda)$, $K_k^h(\Lambda)$ and $L_k^s(\Lambda)$, $\widetilde{K}_k^s(\Lambda)$, respectively. Later we shall see that from the point of view of functional methods all these versions of Hermitian K-theory are altogether indistinguishable.

§2. Periodicity in Hermitian K-theory

In §1 we have defined the groups $K_k^h(\Lambda)$ so that automatically for $k \equiv s \pmod 4$ the groups $K_k^h(\Lambda)$ and $K_s^h(\Lambda)$ are isomorphic. This fictitious periodicity has nowhere been justified. As a matter of fact, we are concerned here with four different groups $K_k^h(\Lambda)$ ($k = 0, 1, 2, 3$), and the question of the interpretation of these groups as a finite segment of some 4-periodic sequence of groups remains open. The incentive for considering algebraic K-theory as a 4-periodic theory is founded on the remarkable properties of topological K-theory discovered by Bott.

Let X be a topological space and $K(X)$ the group generated by the locally trivial vector bundles on X. If the fibre of the bundle is a finite-dimensional complex space and the structure group is the group of unitary transformations, then we denote it by $K_U(X)$. In the case of a real vector bundle and the group of orthogonal transformations as structure group, we denote it by $K_O(X)$. There is also a version of topological K-theory the group of symplectic transformations of a vector space over the field of quaternions as structure group. In this case the group of bundles is denoted by $K_{Sp}(X)$. In all three versions Bott has established a formula which is now known as Bott periodicity.

If (X, x_0) is a space with base point, then by $K^0(X, x_0)$ we denote the subgroup of $K(X)$ generated by the zero-dimensional elements of $K(X)$. We

denote by $S^p(X)$ the p-fold suspension of the space with base point (X, x_0)

$$S^pX = (S^p \times X)/(X \vee S^p).$$

Here S^p is the p-dimensional sphere with base point and the symbol $\vee$ denotes the connected sum of two topological spaces with base points. Then Bott periodicity can be written in the form of the following equalities:

$$K_U^0(S^2X) = K_U^0(X, x_0), \tag{2.1}$$

$$K_O^0(S^8X) = K_O^0(X, x_0), \tag{2.2}$$

$$K_{Sp}^0(S^8X) = K_{Sp}^0(X, x_0). \tag{2.3}$$

Now we introduce an argument to show that in algebraic K-theory we can also speak of periodicity in the sense of (2.1)–(2.3).

It turns out that the groups $K(X)$ of topological K-theory have a purely algebraic description. To see this we consider the ring $\Lambda = \mathbf{C}(X)$ of all continuous complex-valued functions on X. Then we can associate with each complex vector bundle ξ over X the Λ-module $M(\xi)$ of all continuous sections of ξ. If X is compact, then $M(X)$ is a finitely-generated projective Λ-module. Moreover, the correspondence $\xi \to M(\xi)$ establishes an equivalence between the category of complex vector bundles and that of finitely generated Λ-modules and, consequently, an isomorphism of the groups $K_U(X)$ and $K_0(\Lambda)$. Generalizing the above argument to the rings of real-valued functions $\mathbf{R}(X)$ and of quaternions $\mathbf{Q}(X)$, we get natural isomorphisms

$$K_U(X) = K_0(\mathbf{C}(X)),$$
$$K_O(X) = K_0(\mathbf{R}(X)),$$
$$K_{Sp}(X) = K_0(\mathbf{Q}(X)).$$

Therefore, we can interpret Bott periodicity solely in algebraic terms. For this it is more convenient to replace (2.1), (2.2), and (2.3) by relations among the K-groups for the spaces X and $X \times T^p$, where T^p is the p-dimensional torus, that is,

$$T^p = \underbrace{S^1 \times \ldots \times S^1}_{p \text{ times}}.$$

In the case, for example, of K_U, Bott periodicity takes the following form. We set $K_U^{-i}(X, x_0) = K_U^0(S^iX)$. Then (2.1) is equivalent to the exactness of the sequences:

$$0 \to K_U^{-1}(X) \to K_U^0(X \times S^1) \to K_U^0(X) \to 0, \tag{2.4}$$

$$0 \to K_U^0(X) \to K_U^{-1}(X \times S^1) \to K_U^{-1}(X) \to 0. \tag{2.5}$$

In truth, the sequence (2.4) is exact almost by definition, while (2.5) would be exact if we could replace $K_U^0(X)$ by $K_U^{-1}(X)$. The fact that the last two groups can change places is equivalent to Bott periodicity.

The sequences (2.4) and (2.5) can acquire a real meaning in algebraic

K-theory. In fact, the ring of continuous functions $\mathbf{C}(X \times S^1)$ contains the subring of Laurent polynomials $\mathbf{C}(X)[z, z^{-1}] \subset \mathbf{C}(X \times S^1)$. To see this, we have to represent S^1 as the set of complex numbers of norm 1. Then $\mathbf{C}(X \times S^1)$ is the completion of $\mathbf{C}(X)[z, z^{-1}]$ with respect to a suitable norm. The method of going from an arbitrary ring Λ to the ring of Laurent polynomials $\Lambda[z, z^{-1}]$ contains no topological ingredient at all and can serve as an analogue for the passage from X to $X \times S^1$. Therefore, in algebraic K-theory one of the essential problems is that of establishing the connection between the K-groups for Λ and its Laurent extension $\Lambda[z, z^{-1}]$.

The analogous problem in algebraic K-theory was solved by Bass ([63], [64], see also [1], Ch. 12; [2], 226). Roughly speaking, if Λ_z denotes the Laurent extension of Λ, then $K_1(\Lambda_z)$ splits into the direct sum of the groups $K_0(\Lambda)$ and $K_1(\Lambda)$ and yet another summand Nil (Λ), consisting of the elements of the form $(1 + \nu z^{\pm 1})$, where ν is a nilpotent homomorphism over Λ. In many important examples the group Nil (Λ) is trivial. The decomposition of $K_1(\Lambda_z)$ into a direct sum is effected with the help of the Bass projections

$$K_1(\Lambda_z) \to K_1(\Lambda) \to K_1(\Lambda_z),$$
$$K_1(\Lambda_z) \to K_0(\Lambda) \to K_1(\Lambda_z).$$

It turns out that there is a similar situation in the case of Hermitian K-theory. The connection between the K-groups for the rings Λ and Λ_z was established by Novikov [14]. He constructed projections that decompose the Hermitian K-groups over Λ_z into the direct sum of Hermitian K-groups over Λ. However, all the constructions were carried out in the 2-adic localization of the ring and the K-groups. Later Ranicki ([65], [66]) sharpened Novikov's theorem by taking account of the 2-torsion in the rings and the K-groups and constructed projections for "twisted" Laurent bundles. Note that parallel (and rather earlier in time) similar questions were solved by purely geometric methods (see §10).

The basic idea enabling us to construct projections for Hermitian K-theory appears to lie in a new approach to the question of defining the K-groups. If we wish to construct K_0 and K_1 in any category with direct sums, and with zero, then we must have:

a) a class of objects containing 0 and closed with respect to direct sums,

b) a concept of equivalence between two objects of the class,

c) a concept of a means (or process in the terminology of [14]) of establishing equivalence. In the case of algebraic K-theory, these objects are the projective Λ-modules, equivalent objects are equivalent projective Λ-modules, and equivalence is established by means of isomorphisms. Then the group $K_0(\Lambda)$ is constructed as the Grothendieck groups of the category of projective Λ-modules factored by the free Λ-modules. The group $K_1(\Lambda)$ is constructed with the help of all isomorphisms of a free Λ-module. There

is also a concept of equivalence of isomorphisms (just as, for K_0, there is the concept of equivalence of projective modules). In exactly the same way, if we wanted to define the next group $K_2(\Lambda)$, then we should have means of establishing the equivalence of isomorphisms, and also a concept of the equivalence of such means.

In the case of Hermitian K-theory the objects of the basic category are projective modules together with Hermitian or skew-Hermitian non-degenerate forms on a module.

Two Hermitian forms (M, α) and (M', α') are taken to be equivalent if there is an isomorphism:

$$\beta\colon M \to M', \tag{2.6}$$

such that $\alpha' = \beta\alpha\beta^*$. The trivial objects are the projective-Hamiltonian Hermitian forms. The classes of equivalent Hermitian forms generate the group $K^h_{2k}(\Lambda)$. To obtain the group $K^h_{2k+1}(\Lambda)$ we choose as objects the isomorphisms of the projective-Hamiltonian form (2.6). For these isomorphisms we have to define an equivalence relation. Let $M = M_p \oplus M_x$ be a direct sum decomposition of the Λ-module M, in which the Hamiltonian Hermitian form α has the matrix $A = \begin{pmatrix} 0 & 1 \\ (-1)^k & 0 \end{pmatrix}$. The submodules M_p and M_x are such that α is identically zero on each of them, while the composite map

$$M_p \to M \xrightarrow{\alpha^{-1}} M^* \to M_p^*$$

is an isomorphism. Such submodules are called Lagrangian planes. Every Hamiltonian transformation (2.6) sends a Lagrangian plane into another one. In particular, if β is a Hamiltonian transformation, then the image $\beta(M_p)$ of M_p also is a Lagrangian plane. Moreover, each Lagrangian plane $L \subset M_p \oplus M_x$ is the image of some Hamiltonian transformation β:

$$L = \beta(M_p). \tag{2.7}$$

The transformation β in (2.7) is determined not uniquely, but up to composition with a Hamiltonian transformation γ leaving M_p invariant. By analogy with the Hamiltonian formalism in dynamical systems we say that there is an elementary transfer process from β to $\beta\gamma$. Another type of elementary transfer process from β to a transformation β' consists in adjoining to $M = M_p \oplus M_x$ two additional coordinates

$$M \to M \oplus \Lambda(p_{s+1}) \oplus \Lambda(x_{s+1})$$

and then changing coordinates by

$$\begin{cases} p'_{s+1} = p_{s+1} - \sum \gamma_k x_k - \varphi x_{s+1}, \\ p'_k = p_k + \gamma_k^*, \ x'_k = x_k, \ x'_{s+1} = x_{s+1}. \end{cases} \tag{2.8}$$

This operation also has an analogue in the Hamiltonian formalism if the new coordinate x_{s+1} is regarded as the time coordinate. One can also iterate this, regarding x_{s+1} as a many-dimensional vector.

Then two Hamiltonian transformations (or, what comes to the same thing, two Lagrangian planes) are said to be equivalent if there is a

composition of elementary processes sending the one into the other. The equivalence classes of transformations form the group $K^h_{2k+1}(\Lambda)$. The next stage consists in choosing as fundamental objects processes taking one fixed Lagrangian plane (say, $M_p \subset M$) to another fixed Lagrangian plane $L \subset M$. Here L can be chosen so that it projects along M_x onto a direct summand. It turns out that the composition of elementary operations sending M_p to L is completely determined by the matrix φ in (2.8), which in this case is a non-degenerate matrix of the other sign of symmetry. Consequently, we can introduce an equivalence relation between two compositions of elementary operations if the corresponding skew-Hermitian forms are equivalent. In this way we succeed in interpreting the groups $K^h_{4k+2}(\Lambda)$ from the point of view of algebraic K-theory for Hermitian forms as distinguishing them in the category of equivalences of elements of $K^h_{4k+1}(\Lambda)$.

The approach we have given has a direct analogue in homotopy theory. For let (X, Y) be the set of continuous maps of a space X to a space Y respecting base point. Two maps φ and ψ are regarded as equivalent if they are homotopic: $\varphi \sim \psi$. We denote the classes of homotopic maps by $[X, Y]_0$. Then homotopies between two fixed maps form a new category in which we can introduce an equivalence relation (namely, homotopy). We denote the classes of equivalent homotopies by $[X, Y]_1$. Carrying on, in this way, we can construct sets $[X, Y]_k$ for each $k \geqslant 0$. In topology, one can make the remarkable observation that the sets $[X, Y]_k$ can be included in homology theory (under certain natural restrictions) and this means that one can apply the apparatus of homological algebra for computing these sets.

As a matter of fact, Hermitian K-theory has similar properties, which enable us to carry out computations comparing the Hermitian K-groups for the ring Λ and its Laurent extension Λ_z. If we ignore torsion, that is, if we tensor all groups with the ring of rational numbers, then we have the following decomposition:

$$K^h_n(\Lambda_z) = K^h_n(\Lambda) \oplus K^h_{n-1}(\Lambda). \tag{2.9}$$

The precise decompositions are more intricate and require that we consider in parallel the three versions of Hermitian K-theory associated with Hermitian forms on projective, projective-free, or free modules over Λ ([65]).

§3. Algebraic and Hermitian K-theory from the point of view of homology theory

1. **Classifying spaces.** The method of defining Hermitian K-theory in §2 points to the possibility of describing the K-groups as homotopy invariants of certain classifying spaces. The first such description was proposed by Quillen ([67]). We consider the space $BGL(\Lambda)$. Since the derived group of $GL(\Lambda)$ is equal to its second derived group, there is a map

f: $BGL(\Lambda) \to BGL(\Lambda)^+$ under which the fundamental group $\pi_1(BGL(\Lambda)) = GL(\Lambda)$ is mapped onto its Abelianization, and the integral cohomologies are isomorphic. Moreover, $BGL(\Lambda)^+$ is universal with respect to these properties. Then the space $BGL(\Lambda)^+$ is uniquely determined by the ring Λ and has functional properties.

Quillen has proposed defining the K-groups in the following way:

$$K_i\Lambda = \pi_i(BGL(\Lambda)^+). \tag{3.1}$$

The previously known groups $K_0\Lambda$, $K_1\Lambda$, $K_2\Lambda$, agree with the definition (3.1). This definition is convenient from the homotopy point of view. However, whether it is mathematically natural remains unclear.

At the same time Volodin proposed another definition, closer in spirit to the point of view of the processes explained in §2 ([68], [70]). With the help of elements of $GL(\Lambda)$ one can construct a simplicial complex. Its vertices are the elements (matrices) of $GL(\Lambda)$. The one-dimensional simplexes are the elementary matrices, and so on. The homotopy groups of the resulting space (after shifting the dimension by 1) turn out to be the groups $K_i(\Lambda)$. The merit of the space introduced by Volodin lies in the fact that its homotopy properties clearly reflect the homotopy structure of the group of pseudo-isotopies of a multiply-connected manifold. For example, the group $\pi_0(PISO(M))$ is expressed in terms of $K_2(\Lambda)$, $\pi_1(PISO(M))$ in terms of $K_3(\Lambda)$, and so on ([72]).

Later the homotopy equivalence of the spaces defined by Quillen and by Volodin was established ([69]).

There is still another definition of the algebraic K-groups, based precisely on the analogy of the Laurent extension of Λ in algebra and multiplication by the circle in topology ([71]). In the case of regular rings, at least for $i \leqslant 2$, the groups $K_i(\Lambda)$ agree with the other definitions.

2. **Periodicity in Hermitian K-theory.** For Hermitian K-theory Karoubi has also considered Quillen's classifying space ([73]). In the case of a regular ring Λ, he constructs versions of Hermitian K-theory that are needed for the passage to the Laurent extension of the rings, $U^n(\Lambda)$, $V^n(\Lambda)$. Relying on ideas of Novikov, he establishes 4-periodicity for the homotopy groups of the classifying spaces ([74], [75]).

Moreover, from the point of view of the homotopy groups of classifying spaces one can clarify the deviation from 4-periodicity by taking account of the elements of order a power of 2. Just as in topological K-theory, the periodicity of the K-groups is established by multiplying a generator u_4 into the group $K_0^h(\Lambda)$. It turns out that actually multiplication by u_4 is not an isomorphism. There is a generator $u_{-4} \in K_{-4}^h(\Lambda)$ such that $u_4 \cdot u_{-4} = 4 \in K_0^h(\Lambda)$. Consequently, after 2-adic localization, u_4 becomes invertible. For comparison we may point out that in real topological K-theory there is an analogous picture for studying the 4-periodicity of the theory (see [76]).

CHAPTER 2

HERMITIAN K-THEORY AND REPRESENTATION THEORY

§4. Finite-dimensional representations

We turn now to the discussion of functional methods for studying the invariants of Hermitian K-theory. Our main problem is as follows.

Let Λ be an associative ring with involution and (M, α) a Hermitian form over Λ. We have to find effective methods of finding invariants of (M, α) that distinguish inequivalent forms. In §3, as a matter of fact, such a complete set of invariants was given for Hermitian forms for one special class of rings, namely, group rings of free Abelian groups. The invariants of (M, α) are the signatures of the numerical quadratic forms that are the images of some chain of Bass–Novikov projections.

1. **Symmetric representations.** We consider another way of finding numerical invariants of the Hermitian form (M, α) over Λ, based on the study of the representations of Λ. Let $\rho\colon \Lambda \to \mathrm{Mat}(l, \mathbf{C})$ be a symmetric representation of Λ in the ring $\mathrm{Mat}(l, \mathbf{C})$ of complex matrices of order l (see [8], 221). In $\mathrm{Mat}(l, \mathbf{C})$ we fix the involution of ordinary complex conjugation of a matrix and then

$$\rho(\lambda^*) = (\rho(\lambda))^* \qquad (\lambda \in \Lambda). \tag{4.1}$$

By means of a symmetric representation ρ of Λ we can associate with each Hermitian form (M, α) a well-defined numerical Hermitian form on a complex vector space. Namely, let V be the complex l-dimensional vector space on which the ring $\mathrm{Mat}(l, \mathbf{C})$ acts. This action can be specified in terms of a basis $v_1 \ldots v_l$ for V, If V is furnished with a Hermitian metric in which the vectors $(v_1 \ldots v_l)$ are orthonormal, then the involution on $\mathrm{Mat}(l, \mathbf{C})$ corresponds to the operation of duality for the linear operators on V. Using ρ, we turn V into a left Λ-module. To do this we set

$$\lambda v = \rho(\lambda)v \qquad (\lambda \in \Lambda,\ v \in V). \tag{4.2}$$

Let M be a free Λ-module with a finite number of generators $(m_1 \ldots m_s)$. We consider the tensor product of the Λ-modules M and V. Since it is necessary for the tensor product that one of the modules has a left- and the other a right-module structure, we arrange that a right-module structure is assigned to V by the formula

$$v\lambda = \rho(\lambda^*)(v) \qquad (\lambda \in \Lambda,\ v \in V). \tag{4.3}$$

It is not difficult to verify that the right action of Λ on V satisfies all the properties of right action of a ring. Thus, we set

$$M_\rho = M \otimes_\Lambda V, \tag{4.4}$$

that is, M_ρ is the factor group of the tensor product $M \otimes V$ by the relations:

$$\lambda m \otimes v = m \otimes \rho(\lambda^*)v \qquad (m \in M,\ v \in V,\ \lambda \in \Lambda). \tag{4.5}$$

Notice that the tensor product M_ρ has lost the structure of a Λ-module. On the other hand, M_ρ admits the structure of an action by any ring whose action commutes with the action of Λ on one of its factors. Such a ring, for example, is the field of complex numbers $\mathbf{C}$ whose action on V commutes with the action of $\mathrm{Mat}(l, \mathbf{C})$, by definition. Consequently, M_ρ is a complex vector space, of dimension $l \cdot s$.

Moreover, if β: $M \to \Lambda$ is a module homomorphism, then we get a homomorphism

$$\beta \otimes 1\colon\ M_\rho \to \Lambda_\rho = V. \tag{4.6}$$

Thus, there is a natural homomorphism

$$\delta\colon\ M^* \to \mathrm{Hom}_{\mathbf{C}}(M_\rho, V). \tag{4.7}$$

By a direct computation it can be verified that δ is a left Λ-module homomorphism. In this way, a left Λ-module structure is inherited by $\mathrm{Hom}_{\mathbf{C}}(M_l, V)$ from the left Λ-module structure on V. On the other hand, the Λ-module $\mathrm{Hom}_{\mathbf{C}}(M_l, V)$ is isomorphic to $(M_l)^* \otimes_{\mathbf{C}} V$. Therefore, we get a natural Λ-module homomorphism:

$$\bar{\delta}\colon\ M^* \to (M_\rho)^* \otimes V. \tag{4.8}$$

Next, by analogy with (4.6) we have the homomorphism

$$\bar{\delta} \otimes 1\colon\ (M^*)_\rho \to (M_\rho^* \otimes_{\mathbf{C}} V) \otimes_\Lambda V. \tag{4.9}$$

Now we study the space $(M_\rho^* \otimes_{\mathbf{C}} V) \otimes_\Lambda V = (M_\rho)^* \otimes_{\mathbf{C}} (V \otimes_\Lambda V)$. If the image of $\rho(\Lambda)$ is the whole matrix algebra $\mathrm{Mat}(l, \mathbf{C})$, then $(V \otimes_\Lambda V)$ is isomorphic to the one-dimensional space $\mathbf{C}$. In general, we can at least assert that the Hermitian metric on the complex space V well-defines a complex-valued linear function defined on $V \otimes_\Lambda V$ by the formula

$$\gamma(v_1 \otimes v_2) = \langle v_1, v_2 \rangle. \tag{4.10}$$

Thus, we can construct the composite of homomorphisms

$$\bar{\bar{\delta}}\colon\ (M^*)_\rho \to (M_\rho)^*, \tag{4.11}$$

$$\bar{\bar{\delta}}\colon\ (1 \otimes \gamma)(\bar{\delta} \otimes 1). \tag{4.12}$$

If M is a free Λ-module, then $\bar{\delta}$ is an isomorphism. We now return to the Hermitian form (M, α). Here α: $M^* \to M$ is a Λ-module isomorphism, and by definition (1.7), $\varepsilon_M^{-1}\alpha^* = (-1)^k \alpha$. We multiply α tensorially by the Λ-module V and obtain the homomorphism

$$\bar{\alpha}\colon\ (\alpha \otimes 1)\colon\ (M^*)_\rho \to M_\rho, \tag{4.13}$$

We then set

$$\alpha_\rho\colon\ (M_\rho)^* \to M_\rho, \tag{4.14}$$

$$\alpha_\rho = \bar{\alpha} \circ (\bar{\bar{\delta}})^{-1}. \tag{4.15}$$

The homomorphism (4.14) satisfies a relation analogous to (1.17),

(4.16) $$\varepsilon_{M_\rho}^{-1}\alpha_\rho^* = (-1)^k \alpha_\rho,$$

that is, it is a (skew-) Hermitian form over $\mathbf{C}$ with complex conjugation as involution.

Our definition of the numerical Hermitian form (M_ρ, α_ρ), constructed from (M, α) and the symmetric representation ρ of Λ, does not refer to any basis in M or V. If in M we fix a free basis $(m_1 \ldots m_s)$ and in V a basis $(v_1 \ldots v_l)$, we can distinguish in their tensor product M_ρ the basis $\{(m_i \otimes v_j),\ 1 \leqslant i \leqslant s,\ 1 \leqslant j \leqslant l\}$. Then in this basis the matrix of (M_ρ, α_ρ) takes the following simple form. Let

(4.17) $$A = \| \lambda_{ij} \|, \qquad \lambda_{ij} \in \Lambda_\varepsilon$$

be the matrix of (M, α) in the basis $(m_1 \ldots m_s)$. We consider the matrix A_ρ of order $l \cdot s$, split into square blocks of order l each (altogether $s \times s$ blocks), each block being equal, respectively, to $\rho(\lambda_{ij}) \in \mathrm{Mat}(l, \mathbf{C})$:

(4.18) $$A_\rho = \| \rho(\lambda_{ij}) \|.$$

This A_ρ is the matrix of (M_ρ, α_ρ) in the basis $\{(m_i \otimes v_j),\ 1 \leqslant i \leqslant s,\ 1 \leqslant j \leqslant l\}$. The verification that A_ρ is Hermitian and non-degenerate is trivial.

2. **Signatures of Hermitian forms.** The invariant definition of the numerical Hermitian form (M_ρ, α_ρ) from (M, α) and ρ shows that if two forms (M_1, α_1) and (M_2, α_2) are equivalent, then so are the forms $(M_{1\rho}, \alpha_{1\rho})$ and $(M_{2\rho}, \alpha_{2\rho})$. Moreover, if (M, α) is Hamiltonian, then so is (M_ρ, α_ρ). In the same way, if ρ_1 and ρ_2 are equivalent representations of Λ and (M, α) is a Hermitian form, then the forms $(M_{\rho_1}, \alpha_{\rho_1})$ and $(M_{\rho_2}, \alpha_{\rho_2})$ are equivalent. Therefore, ρ induces a homomorphism of Hermitian K-theories:

(4.19) $$\mathrm{sign}_\rho\colon K^h_{2k}(\Lambda) \to K^h_{2k}(\mathbf{C}).$$

Moreover, for equivalent representations ρ_1 and ρ_2 the homomorphisms (4.19) coincide:

(4.20) $$\mathrm{sign}_{\rho_1} = \mathrm{sign}_{\rho_2}.$$

The notation sign_ρ is justified by the fact that the group $K^h_{2k}(\mathbf{C})$ of numerical Hermitian forms is isomorphic to the group of integers and the unique invariant of a numerical Hermitian form is its signature. We call $\mathrm{sign}_\rho(M, \alpha)$ the signature of (M, α) with respect to ρ.

Thus, every symmetric representation of Λ gives us a way of obtaining a numerical invariant for Hermitian forms over Λ. The larger the stock of symmetric representation of Λ we have available, larger will be a priori the number of numerical invariants of Hermitian forms we can get. Indeed, in some cases, by running through the whole set of finite-dimensional symmetric representations of Λ we obtain a complete set of numerical invariants for Hermitian forms over Λ, that is, with the help of the invariants (4.19) we can distinguish any pair of inequivalent Hermitian forms over Λ.

Before considering the class of rings Λ for which the finite-dimensional signatures are a complete set of invariants of Hermitian forms we make a number of remarks.

Let Λ be a ring with involution and $\rho\colon \Lambda \to \mathrm{Mat}(l, \mathbf{C})$ a symmetric representation. Then ρ induces a representation of the tensor product $\Lambda \otimes_{\mathbf{Z}} \mathbf{C}$ of Λ with the field of complex numbers. Thus, with the help of signatures with respect to symmetric representations we can construct invariants, as a matter of fact, not of Hermitian forms over Λ but of their images in $K^h_{2k}(\Lambda \otimes_{\mathbf{Z}} \mathbf{C})$ under the homomorphism

$$K^h_{2k}(\Lambda) \to K^h_{2k}(\Lambda \otimes_{\mathbf{Z}} \mathbf{C}), \tag{4.21}$$

induced by the change of rings

$$\Lambda \to \Lambda \otimes_{\mathbf{Z}} \mathbf{C}. \tag{4.22}$$

Consequently, we can from the very outset suppose that our ring Λ is an algebra over $\mathbf{C}$.

We suppose next that Λ splits into the direct sum of two rings $\Lambda = \Lambda_1 \oplus \Lambda_2$, that is, Λ has a self-conjugate projection μ in the centre of Λ, $\mu(\Lambda) = \Lambda_1$, $(1 - \mu)(\Lambda) = \Lambda_2$. The projection μ is the identity on Λ_1 and $(1 - \mu)$ the identity on Λ_2. Then $K^h_{2k}(\Lambda)$ is isomorphic to the direct sum of groups $K^h_{2k}(\Lambda_1) \oplus K^h_{2k}(\Lambda_2)$. For if $A = \| \lambda_{ij} \|$ is the matrix of a Hermitian form over Λ, then the matrices

$$A_1 = \| \mu\lambda_{ij} \| \qquad (\mu\lambda_{ij} \in A_1),$$
$$A_2 = \| (1 - \mu)\lambda_{ij} \|, \qquad (1 - \mu)\lambda_{ij} \in \Lambda_2,$$

define Hermitian forms over Λ_1 and Λ_2 respectively. Conversely, if A_1 and A_2 are the matrices of Hermitian forms over Λ_1 and Λ_2, respectively, then $A = A_1 + A_2$ is the matrix of a Hermitian form over Λ.

Thus, a complete set of invariants of Hermitian forms over $\Lambda = \Lambda_1 \oplus \Lambda_2$ is the union of complete sets of invariants of Hermitian forms over Λ_1 and Λ_2.

Finally, a third remark concerns a normed ring Λ with involution. This (see [8]) is a ring Λ with involution which is at the same time a normed space, such that the following relations hold:

$$| 1 | = 1, \qquad | \lambda\mu | \leqslant | \lambda | \cdot | \mu | \qquad (\lambda, \mu \in \Lambda), \tag{4.23}$$

$$| \lambda^* | = | \lambda | \qquad (\lambda \in \Lambda). \tag{4.24}$$

A complete normed ring Λ is called a Banach ring (or Banach algebra).

For normed rings Λ it is natural to limit oneself to consider only bounded symmetric representations, that is, homomorphisms

$$\rho\colon \Lambda \to \mathrm{Mat}(l, \mathbf{C}), \tag{4.25}$$

such that

$$\| \rho(\lambda) \| \leqslant C \, | \lambda | \qquad (\lambda \in \Lambda), \tag{4.26}$$

for some constant C. Let $\mathscr{R}$ denote the set of all representations (4.25) satisfying (4.26) with one and the same constant C. In this case, in fact, we

obtain invariants not of Λ itself, but of its completion $\hat{\Lambda}$, which is a Banach ring. Moreover, our results are completely unchanged if we change the norm on Λ preserving the continuity of all the representations in $\mathscr{R}$. Among all these norms there is a greatest one. We set

$$\| \lambda \| = \sup_{\rho \in \mathscr{R}} \| \rho(\lambda) \| \qquad (\lambda \in \Lambda). \tag{4.27}$$

Then

$$\| \lambda \| \leqslant C \, | \lambda | \qquad (\lambda \in \Lambda). \tag{4.28}$$

Consequently, $\| \lambda \|$ is the greatest norm among all ring norms for which the representations of the set $\mathscr{R}$ are bounded.On the other hand, with respect to the norm (4.27), Λ satisfies the relation

$$\| \lambda\lambda^* \| = \| \lambda \|^2 \qquad (\lambda \in \Lambda), \tag{4.29}$$

that is, the completion of Λ in the norm (4.27) is a C^*-algebra (see [10]).

In the case of the group ring $\Lambda = \mathbf{C}[\pi]$ of a discrete group π, the representations provide us with a certain absolute norm relative to which each symmetric representation ρ of Λ is continuous. For if the existence of a supremum (4.27) for arbitrary normed rings is guaranteed by (4.26), then in the case of a group ring Λ we obtain the existence of a supremum from other properties.

Thus, for a group ring Λ we set:

$$\| \lambda \| = \sup_{\rho} \| \rho(\lambda) \| \qquad (\lambda \in \Lambda), \tag{4.30}$$

where the supremum is taken over all symmetric representations of Λ. If $f \in \pi$ is any element, then according to (1.3)

$$\rho(f)^* = \rho(f^*) = \rho(f^{-1}) = \rho(f)^{-1}. \tag{4.31}$$

Consequently,

$$\| \rho(f) \| \leqslant 1 \tag{4.32}$$

for any symmetric representation ρ. Then if $\lambda = \sum_i x_i f_i$ ($f_i \in \pi$, $x_i \in \mathbf{C}$), we have the inequality

$$\| \rho(\lambda) \| \leqslant \sum_i | x_i | \tag{4.33}$$

for all symmetric representations ρ of the ring. Hence the supremum (4.30) exists and

$$\| \lambda \| \leqslant \sum_i | x_i |. \tag{4.34}$$

Summarising what has been said above we can state the following conclusion.

In the study of the invariants of Hermitian forms with the help of representation theory we may suppose without loss of generality that the base ring Λ is a C^*-algebra.

3. **Two examples of group rings.** In conclusion we quote two examples

of group rings Λ for which we can obtain, with the help of the theory of finite-dimensional representations, a full set of invariants of Hermitian forms.

The first example is that of the group ring $\Lambda = \mathbf{C}[\pi]$ of a finite group π.

This ring splits into the direct sum of matrix algebras

$$\begin{cases} \Lambda = \bigoplus_{i=1}^{s} \Lambda_i, \\ \Lambda_i = \mathrm{Mat}\,(l_i, \mathbf{C}), \end{cases} \tag{4.35}$$

and the projections

$$\rho_i\colon\ \Lambda \to \Lambda_i \tag{4.36}$$

form a complete set of pairwise inequivalent irreducible symmetric representation of Λ. Then

$$K^h_{2r}(\Lambda) = \bigoplus_{i=1}^{s} K^h_{2k}(\Lambda_i). \tag{4.37}$$

It only remains for us to exhibit the structure of Hermitian forms over the full matrix algebra Mat(l, $\mathbf{C}$). Let A be the matrix corresponding to a Hermitian form (M, α) over Mat(l, $\mathbf{C}$):

$$A = \| \lambda_{ij} \|, \quad \lambda_{ij} \in \mathrm{Mat}\,(l, \mathbf{C}),$$

$$\lambda_{ij} = (-1)^k \lambda^*_{ji} \qquad (1 \leqslant i, j \leqslant r).$$

We can regard A as a numerical matrix $\widetilde{A}$ of order lr, split into blocks λ_{ij}. Then $\widetilde{A}$ is non-degenerate and Hermitian: $\widetilde{A}^* = (-1)^k \widetilde{A}$. Conversely, if $\widetilde{A}$ is a non-degenerate Hermitian numerical matrix of order lr, then by splitting it into blocks each of order l, we obtain a matrix A of order s whose elements belong to Mat(l, $\mathbf{C}$). Similarly, if A_1 and A_2 are two matrices over Mat(l, $\mathbf{C}$) determining equivalent Hermitian forms, then their numerical representatives $\widetilde{A}_1$ and $\widetilde{A}_2$ are equivalent matrices of order lr. Conversely, if $\widetilde{A}_1$ and $\widetilde{A}_2$ are two numerical non-degenerate matrices and $\widetilde{B}^* \widetilde{A}_1 \widetilde{B} = \widetilde{A}_2$, then by splitting $\widetilde{A}_1$, $\widetilde{A}_2$, and $\widetilde{B}$ into blocks of order l, we obtain matrices A_1, A_2, and B with elements in Mat(l, $\mathbf{C}$) for which $B^* A_1 B = A_2$.

From the arguments above we see that

$$K^h_{2k}(\mathrm{Mat}\,(l, \mathbf{C})) = K^h_{2k}(\mathbf{C}) = \mathbf{Z}, \tag{4.38}$$

that is, for Hermitian forms over Mat(l, $\mathbf{C}$) a complete set of invariants consists of a single number, the signature of the numerical representation of the Hermitian form.

Thus, we can state the following proposition:

THEOREM 4.1. *Let $\Lambda = \mathbf{C}[\pi]$ be the group ring of a finite group π with the natural involution, and let $\{\rho_i\}$ $(1 \leqslant i \leqslant s)$ be a complete system of irreducible unitary representations of π. Then*

$$K^h_{2k}(\Lambda) = \mathbf{Z}^s, \tag{4.39}$$

and the projection on the ith component in the free Abelian group $\mathbf{Z}^s$ *can be identified with the signature relative to the representation* ρ_i:

$$\operatorname{sign}_{\rho_i}\colon K^h_{2k}(\Lambda) \to \mathbf{Z}.$$

As a second example we consider the group ring $\Lambda = \mathbf{C}[\mathbf{Z}^l]$ of the free Abelian group $\mathbf{Z}^l$ of rank l. In the preceding section we have explained that in place of Λ we should consider its completion with respect to a certain norm induced by the system of symmetric representations of the ring.

Instead of considering the set of all symmetric representations, we limit ourselves for a while to the class of one-dimensional unitary representations, that is, to the characters of $\mathbf{Z}^l$. Obviously, the set of characters of the free Abelian group $\mathbf{Z}^l$ is isomorphic to the Cartesian product of l copies of the circle, that is, to the torus T^l of dimension l. As coordinates on T^l we can take the sets of complex numbers $(z_1 \ldots z_l)$ of modulus 1. Let $\{a_1 \ldots a_l\}$ be a free basis for $\mathbf{Z}^l$, and $a \in \mathbf{Z}^l$ any element,

$$a = \coprod_{i=1}^{l} a_i^{n_i}. \tag{4.40}$$

Then from a we can construct a function on T^l:

$$\bar{a}(z_1, \ldots, z_n) = z(a) = \prod_{i=1}^{l} z_i^{n_i}. \tag{4.41}$$

The formula (4.41) gives us a symmetric homomorphism of $\Lambda = \mathbf{C}[\mathbf{Z}^l]$ into the ring $\mathbf{C}(T^l)$ of continuous functions on T^l:

$$\chi\colon \Lambda \to \mathbf{C}(T^l) \tag{4.42}$$

(the involution on $\mathbf{C}(T^l)$ is defined as complex conjugation on the values of a function). Then the norm on Λ, given by (4.27) on the set of characters of $\mathbf{Z}^l$, is the same as the uniform norm for continuous functions on T^l. Consequently the completion $\hat{\Lambda}$ of Λ in this norm coincides with the ring $\mathbf{C}[T^l]$ or, more precisely, the homomorphism (4.42) extends to a ring isomorphism

$$\hat{\chi}\colon \hat{\Lambda} \to \mathbf{C}[T^l]. \tag{4.43}$$

The norm of Λ induced by the uniform norm on $\mathbf{C}(T^l)$ need not a priori be the greatest norm in which any symmetric representation of Λ is continuous, since in (4.27) the supremum is taken not with respect to all representations of Λ, but only with respect to the characters of $\mathbf{Z}^l$. But nonetheless it can be established that the norm in Λ defined by (4.30) with respect to all representations of Λ is equivalent to the uniform norm of $\mathbf{C}(T^l)$. For if $\rho\colon \Lambda \to \operatorname{Mat}(s, \mathbf{C})$ is a symmetric representation, then the matrices $\rho(a_1) \ldots \rho(a_l)$ are unitary and commute pairwise. Consequently, in some other basis all these matrices reduce to diagonal form, that is, ρ is the direct sum of one-dimensional representations. Therefore, the norm (4.30) can be estimated in terms of the norm (4.27) constructed from the one-dimensional representations.

Thus, the study of the invariants of Hermitian forms over $\Lambda = \mathbf{C}[\mathbf{Z}^l]$ reduces with the help of representation theory to the study of the invariants of the ring of continuous functions $\mathbf{C}(T^l)$.

We can generalize the problem and consider the class of continuous functions $\mathbf{C}(X)$ over an arbitrary topological (compact) space ([13], and [14], 271–273). In this case a Hermitian form (M, α) is represented by a Hermitian matrix

$$A = \| \lambda_{ij} \|, \quad \lambda_{ij} = \dot{\lambda}^*_{ij} \qquad (1 \leqslant i, j \leqslant s), \tag{4.44}$$

whose elements are complex-valued functions λ_{ij} on X. We can interpret the matrix-valued function (4.44) as a continuous family of non-degenerate Hermitian forms on the trivial vector bundle $X \times \mathbf{C}^s$. We fix a point $x \in X$. Then on the fibre $x \times \mathbf{C}^s$ we obtain an individual non-degenerate Hermitian form. We split $x \times \mathbf{C}^s$ into the orthogonal sum of two subspaces V_x^+ and V_x^- on which our Hermitian form is positive definite and negative definite, respectively. Then the union of the subspaces V_x^+ for all $x \in X$ forms a subbundle

$$\xi_+ = \bigcup_{x \in X} V_x^+ \tag{4.45}$$

of the trivial bundle $X \times \mathbf{C}^s$. Similarly, the union of the subspaces V_x^- forms a subbundle

$$\xi_- = \bigcup_{x \in X} V_x^-, \tag{4.46}$$

and

$$\xi_+ \oplus \xi_- = X \times \mathbf{C}^s. \tag{4.47}$$

Thus, with the help of (4.45) and (4.46)) we associate with each Hermitian form (M, α) two vector bundles $\xi_+(M, \alpha) = \xi_+$ and $\xi_-(M, \alpha) = \xi_-$ over X. It is quite clear that the construction of ξ_+ and ξ_- does not depend on the choice of a free basis in M (that it, does not depend on the choice of A in the class of equivalent Hermitian matrices).

This construction admits a converse, that is, if we are given two finite-dimensional bundles ξ_+ and ξ_- over our spaces and a unitary isomorphism

$$\varphi\colon \xi_+ \oplus \xi_- \to X \times \mathbf{C}^s, \tag{4.48}$$

then we can construct a Hermitian form (M, α) over $\mathbf{C}(X)$. To do this we assign to each fibre $x \times \mathbf{C}^s$ the Hermitian form that is equal to the Hermitian metric on the fibre $\varphi(\xi_+)_x$ of the subbundle $\varphi(\xi_+)$, while on the fibre $\varphi(\xi_-)_x$ of the subbundle $\varphi(\xi_-)$ it is equal to the Hermitian metric with the opposite sign and leaves the two subspaces $\varphi(\xi_+)_x$ and $\varphi(\xi_-)_x$ orthogonal. As a result we have a continuous family of Hermitian forms on the bundle $X \times \mathbf{C}^s$ or, what comes to the same thing, a non-degenerate Hermitian matrix over $\mathbf{C}(X)$.

If we replace the isomorphism (4.48) by another unitary isomorphism φ', then by our construction we obtain a Hermitian matrix over $\mathbf{C}(X)$, which is equivalent to the original one.

So we have, in fact, established the following theorem:

THEOREM 4.2. *Let X be a compact topological space and $\mathbf{C}(X)$ the ring of continuous complex-valued functions on X, with complex conjugation as involution. The map*

$$(M, \alpha) \to \xi_+(M, \alpha) - \dim \xi_-(M, \alpha) \tag{4.49}$$

induces an isomorphism

$$\tau\colon K^h_{2k}(\mathbf{C}(X)) \to K(X) \tag{4.50}$$

of the group $K^h_{2k}(\mathbf{C}(X))$ with the topological K-functor of complex vector bundles over X.

4. Real representations. We make a few remarks concerning applications of real representations. It stands to reason that just as in the case of complex representations from a real representation we can construct without any change signatures of Hermitian forms. A natural change of Λ in this case would be to replace it by $\Lambda \otimes_{\mathbf{Z}} \mathbf{R}$ and to complete it in to the norm of type (4.30) constructed from the set of all the real representations. However, in the case of the free Abelian group $\mathbf{Z}^l$ we cannot interpret the completion of $\Lambda = \mathbf{R}[\mathbf{Z}^l]$ as the ring of continuous functions of some topological space. It seems more natural in this case to consider in the character group T^l the involution of complex conjugation and the anticomplex involution on all the bundles considered over T^l. This approach of reducing all objects over the real field to analogous objects over the complex field can be found for bundles in [15] and for Hermitian forms in [13].

§5. Infinite-dimensional Fredholm representations

1. Definition of Fredholm representations. In §4 we have considered the construction of invariants of Hermitian forms with the help of finite-dimensional symmetric representations of a ring Λ in the full matrix ring $\mathrm{Mat}(l, \mathbf{C})$. We have shown that in the case of the group ring $\Lambda = \mathbf{C}[\pi]$ of a finite group π we can construct in this way a complete set of invariants of Hermitian forms. In fact, the finite-dimensional rings are the only class of rings for which finite-dimensional signatures form a complete set of invariants of Hermitian forms.

For example, even in the case of the group ring of a free Abelian group the finite-dimensional representations do not give us new invariants of Hermitian forms compared with the ordinary signature of numerical Hermitian forms. For if $\pi = \mathbf{Z}^l$ and $\rho\colon \mathbf{Z}^l \to U(n)$ is a unitary representation, then there is a continuous family of homomorphism $\rho_t\colon \mathbf{Z}^l \to U(n)$ $(0 \leqslant t \leqslant 1)$ with $\rho_0 = \rho$, $\rho_1 = E$. Since the signature of a Hermitian matrix does not change for nearby Hermitian forms, for an arbitrary Hermitian form (M, α) over $\Lambda = \mathbf{C}[\mathbf{Z}^l]$ the function $\mathrm{sign}_{\rho_t}(M, \alpha)$ is constant. Consequently, the signature $\mathrm{sign}_{\rho}(M, \alpha)$ is the same as $\mathrm{sign}_{\rho_1}(M, \alpha)$ relative to the homomorphism of $\mathbf{Z}^l$ into the trivial group.

Thus, the problem arises naturally of searching for a class of representations of Λ with the help of which we might be able to construct a more complete set of invariants of Hermitian forms.

In the present section we consider such a class of infinite-dimensional representations, which furnish us with new invariants of Hermitian forms (see [16], [17]). We call them Fredholm representations of the ring. With the help of Fredholm representations we can construct invariants not only of Hermitian forms, but also of other objects, such as classifying spaces for discrete groups and elliptic operators on compact manifolds. We give these examples at the end of the section despite the fact that at a first glance they do not appear to be related to the matter being studied.

We now turn to the precise definition of Fredholm representations.

We fix two Hilbert spaces H_1 and H_2. Let ρ_1 and ρ_2 be two symmetric representations of Λ in the rings of bounded operators on H_1 and H_2, respectively. Next, let F: $H_1 \to H_2$ be a Fredholm operator satisfying the following condition:

(5.1) For each $\lambda \in \Lambda$ the operator $F\rho_1(\lambda) - \rho_2(\lambda)F$ is compact.

The triple $\rho = (\rho_1, F, \rho_2)$ consisting of ρ_1, ρ_2, and F and satisfying (5.1) is called a Fredholm representation.

Although from the point of view of representation theory it is more natural to call $\rho = (\rho_1, F, \rho_2)$ a wreath product of the two representations ρ_1 and ρ_2, all the same we call it a Fredholm representation. The thought behind this nomenclature is that in all future applications the pair of representations ρ_1 and ρ_2 related by the Fredholm operator F embodies the formal difference of ρ_1 and ρ_2, that is, the virtual representation $\rho_1 - \rho_2$.

We generalize the definition of Fredholm representations to the case of Fredholm complexes. Let $H_i (1 \leqslant i \leqslant s)$ be Hilbert spaces and $F_i (1 \leqslant i \leqslant s-1)$ be bounded operators

$$H_1 \xrightarrow{F_1} H_2 \xrightarrow{F_2} \ldots \xrightarrow{F_{s-1}} H_s, \tag{5.2}$$

such that the $F_i F_{i-1}$ are compact. Then (5.2) is said to be a Fredholm complex if there are also operators $G_i (2 \leqslant i \leqslant s)$:

$$H_1 \xleftarrow{G_2} H_2 \xleftarrow{G_3} \ldots \xleftarrow{G_s} H_s, \tag{5.3}$$

such that the operators

$$F_{i-1}G_i + G_{i+1}F_i - 1 \tag{5.4}$$

are compact. Now just as for Fredholm operators, so also for Fredholm complexes there is a well-defined index of a Fredholm complex, not depending on homotopies of the operators F_i in the class of Fredholm complexes, nor on varying the F_i by a compact component. Moreover, for the F_i forming a Fredholm complex we can choose compact summands K_i such that the new operators $F_i' = F_i + K_i$ form a genuine complex, that is, $F_i' F_{i+1}' = 0$. Here the homology of the newly formed complex

$$H_1 \xrightarrow{F_1'} H_2 \xrightarrow{F_2'} \dots \xrightarrow{F_{s-1}'} H_s \tag{5.5}$$

is finite-dimensional, and the index of the Fredholm complex (5.2) is the alternating sum of the dimensions of the homology groups of the complex (5.5).

Suppose now that $\rho_i (1 \leqslant i \leqslant s)$ are symmetric representations of Λ in the rings of bounded operators on the spaces $H_i (1 \leqslant i \leqslant s)$.

We say that we are given a Fredholm complex of representations of Λ if the following condition is satisfied:

(5.6) For each $\lambda \in \Lambda$ the operators $F_i \rho_i(\lambda) - \rho_{i+1}(F_i)$ are compact.

2. Signatures of Hermitian forms. It turns out that with the help of Fredholm representations (or of Fredholm complexes of representations) of a ring Λ we can construct numerical invariants of Hermitian forms of the type of signatures of finite-dimensional representations.

We turn at once to the construction of these invariants. We consider for simplicity the case of a Fredholm representation $\rho = (\rho_1, F, \rho_2)$.

Let (M, α) be an arbitrary Hermitian form over Λ, and A the matrix of (M, α) with respect to some free basis for the Λ-module M,

$$A = \| \lambda_{ij} \|, \quad \lambda_{ij} = \lambda_{ji}^* \qquad (1 \leqslant i, j \leqslant s).$$

By analogy with §3 we consider the matrix

$$A_{\rho_l} = \| \rho_l (\lambda_{ij}) \| \qquad (l = 1, 2). \tag{5.7}$$

The elements of the matrix (5.7) are operators acting on the Hilbert space H_l. Therefore, (5.7) can be interpreted as the expression of a bounded operator acting on the direct sum of s copies of H_l. We denote this operator by

$$\bar{A}_{\rho_l}\colon (H_l)^s \to (H_l)^s \qquad (l = 1, 2). \tag{5.8}$$

Moreover, we set

$$\bar{F} = \| F \delta_{ij} \| \qquad (1 \leqslant i, j \leqslant s). \tag{5.9}$$

Then we obtain the following diagram:

$$\begin{array}{ccc} (H_1)^s & \xrightarrow{\bar{F}} & (H_2)^s \\ {\scriptstyle \bar{A}_{\rho_1}}\uparrow & & \uparrow{\scriptstyle \bar{A}_{\rho_2}} \\ (H_1)^s & \xrightarrow{\bar{F}} & (H_2)^s \end{array} \tag{5.10}$$

in which the operator $\bar{F}\bar{A}_{\rho_1} - \bar{A}_{\rho_2}\bar{F}$ is compact. It is quite clear from the Hermitian nature of A and the symmetry of ρ_l that the operator (5.8) is Hermitian, invertible, and bounded. From spectral theory it follows that the Hilbert space $(H_l)^s$ splits uniquely into the orthogonal sum of two sub-

spaces by means of projections commuting with $\bar{A}_{\rho_l}$,

$$(H_l)^s = H_l^+ \oplus H_l^- , \tag{5.11}$$

and $\bar{A}_{\rho_l}$ is represented in the form of a matrix corresponding to the decomposition (5.11),

$$\bar{A}_{\rho_l} = \begin{pmatrix} A_l^+ & 0 \\ 0 & -A_l^- \end{pmatrix} \tag{5.12}$$

then the operators A_l^+ and A_l^- are positive.

We represent $\bar{F}$ also in matrix form corresponding to the decomposition (5.11)

$$\bar{F} = \begin{pmatrix} F_1 & F_2 \\ F_3 & F_4 \end{pmatrix}. \tag{5.13}$$

The condition for the diagram (5.10) to commutate up to compact operators can be expressed as follows:

$$F_1 A_1^+ - A_2^+ F_1 \in \mathscr{K}, \tag{5.14}$$

$$F_2 A_1^- + A_2^+ F_2 \in \mathscr{K}, \tag{5.15}$$

$$F_3 A_1^+ + A_2^- F_3 \in \mathscr{K}, \tag{5.16}$$

$$F_4 A_1^- - A_2^- F_4 \in \mathscr{K}, \tag{5.17}$$

where $\mathscr{K}$ denotes the space of compact operators.

From (5.15) and (5.16) it follows that F_2 and F_3 are compact. Thus, when we change the operator (5.13) by a compact component, we find that the operator represented by the matrix

$$\begin{pmatrix} F_1 & 0 \\ 0 & F_4 \end{pmatrix},$$

is Fredholm, or, what comes to the same thing, that F_1 and F_4 are Fredholm operators. We then get an invariant of the Hermitian form (M, α) from the equality

$$\operatorname{sign}_\rho(M, \alpha) = \operatorname{index} F_1 - \operatorname{index} F_4. \tag{5.18}$$

Now (5.18) does not depend on the choice of a basis in M, and when (M, α) is Hamiltonian, this invariant vanishes:

$$\operatorname{sign}_\rho(M, \alpha) = 0. \tag{5.19}$$

Next, for a direct sum of Hermitian forms we have

$$\operatorname{sign}_\rho((M_1, \alpha_1) \oplus (M_2, \alpha_2)) = \operatorname{sign}_\rho(M_1, \alpha_1) + \operatorname{sign}_\rho(M_2, \alpha_2), \tag{5.20}$$

which means that, with (5.19), we have defined a homomorphism

$$\operatorname{sign}_\rho : K_{4k}^h(\Lambda) \to \mathbf{Z}. \tag{5.21}$$

In the case of skew-Hermitian forms the construction of the invariant $\operatorname{sign}_\rho(M, \alpha)$ is completely analogous. The only difference is the decomposition of the Hilbert space (5.11) for which the operators are represented in the form

$$\bar{A}_{\rho_l} = i \begin{pmatrix} A_l^+ & 0 \\ 0 & -A_l^- \end{pmatrix}, \tag{5.22}$$

where A_l^+ and A_l^- are positive operators.

Thus, for each Fredholm representation ρ we have constructed a homomorphism

$$\text{sign}_\rho\colon\ K_{2k}^h(\Lambda) \to \mathbf{Z}. \tag{5.23}$$

So far we have considered the construction of the signature of Hermitian forms for a Fredholm representation of the ring Λ. If we are now given a Fredholm complex ρ of representations of Λ of the form (5.5), then with its help we can associate with each Hermitian form (M, α) a numerical invariant by a process analogous to (5.8)–(5.18). We denote this invariant by $\text{sign}_\rho(M, \alpha)$.

We make two remarks concerning the behaviour of the signature of a Fredholm representation $\rho = (\rho_1, F, \rho_2)$ of Λ. Let K be a compact operator and $F' = F + K$. Then the triple $\rho' = (\rho_1, F', \rho_2)$ is also a Fredholm representation, and

$$\text{sign}_\rho(M, \alpha) = \text{sign}_{\rho'}(M, \alpha). \tag{5.24}$$

Thus, from the point of view of signatures of Hermitian forms we can regard ρ and ρ' as equivalent.

The second remark concerns sufficient conditions for $\text{sign}_\rho(M, \alpha) = 0$ to hold for any Hermitian form (M, α). Suppose that the Fredholm representation $\rho = (\rho_1, F_1, \rho_2)$ is such that F is invertible and the operators $F\rho_1(\lambda) - \rho_2(\lambda)F$ are small in norm in comparison with F^{-1}. Then $\text{sign}_\rho(M, \alpha) = 0$. The precise formulation of the sufficient condition is the following: Let $\rho_t = (\rho_{1t}, F_t, \rho_{2t})$ be a continuous family of Fredholm representations, parameterized by a numerical parameter t. Suppose that the operators F_t are invertible and that for any $\lambda \in \Lambda$,

$$\lim_{t\to\infty} \| F_t^{-1}\rho_2(\lambda^{-1}) F_t \rho_1(\lambda) \| = 0. \tag{5.25}$$

Then $\text{sign}_\rho(M, \alpha) = 0$ for any Hermitian form (M, α).

The condition we have given is not invariant, but it is useful in a priori estimates of signatures.

3. **Connection with finite-dimensional representations.** Suppose that a Fredholm representation $\rho = (\rho_1, F, \rho_2)$ satisfies a rather stronger condition than (5.1), namely, for any $\lambda \in \Lambda$

$$F\rho_1(\lambda) - \rho_2(\lambda)F = 0. \tag{5.26}$$

Then the kernel and cokernel of F are ρ_1- and ρ_2-invariant finite-dimensional spaces. We denote the representations of Λ on the kernel and cokernel of F by ρ_1' and ρ_2', respectively. Then

$$\text{sign}_\rho(M, \alpha) = \text{sign}_{\rho_1'}(M, \alpha) - \text{sign}_{\rho_2'}(M, \alpha). \tag{5.27}$$

The formula (5.27) justifies the point of view that when the Fredholm operator ρ satisfies (5.26), then it should be identified with the formal difference of ρ_1' and ρ_2'.

In particular, in the case of the group ring $\Lambda = \mathbf{C}[\pi]$ of a finite group π each Fredholm representation is equivalent to the formal difference of finite-dimensional representations in the sense of the remark in §5.2. Namely, if $\rho = (\rho_1, F, \rho_2)$ is a Fredholm representation, then there is a compact operator K such that by setting $F' = F + K$, $\rho' = (\rho_1, F', \rho_2)$, we obtain a Fredholm representation satisfying (5.26). To see this, we set

$$K = \frac{1}{|\pi|} \sum_{g \in \pi} \rho_2(g^{-1}) F \rho_1(g) - F.$$

Then

$$F' = \frac{1}{|\pi|} \sum_{g \in \pi} \rho_2(g^{-1}) F \rho_1(g), \qquad \rho_2(h^{-1}) F' \rho_1(h) = F'.$$

Thus, in the case of the group ring of a finite group, the Fredholm representations, as was to be expected (see §4.3, Theorem 4.1), do not provide us with signatures of invariant Hermitian forms that are new in comparison with those derived from finite-dimensional representations.

4. Elliptic operators for Fredholm representations. Apparently, Fredholm representations were first introduced by Atiyah [18] in the study of problems concerning the index of elliptic operators on a compact closed smooth manifold.

We consider a smooth compact closed manifold X, the Hilbert space of square-summable functions $L_2(X)$ on X, and a pseudo-differential operator D: $L_2(X) \to L_2(X)$ of order zero. The ring of continuous functions $\Lambda = \mathbf{C}(X)$ acts on $L_2(X)$, and the commutator $D\varphi - \varphi D$ ($\varphi \in \Lambda$) is a pseudo-differential operator of order (-1) [19] and, consequently, compact. In the case of an elliptic operator D we obtain in this way a Fredholm representation ρ of Λ on the Hilbert space $L_2(X)$, $\rho = (\rho_1, D, \rho_1)$, where ρ_1 is a representation of $\Lambda = \mathbf{C}(X)$ on $H = L_2(X)$ for which $\rho_1(\varphi)\psi = \varphi\psi$.

Consequently, in describing the homotopy invariants of an elliptic operator D and, in particular, its index, we can raise the problem of finding invariants of a Fredholm representation of the ring of continuous functions $\Lambda = \mathbf{C}(X)$ on X.

It turns out that with every Fredholm representation ρ of $\Lambda = \mathbf{C}(X)$ we can associate an element $\varkappa(\rho) \in K_0(X)$, where $K_0(X)$ is the group of the generalized homology theory, dual to K-theory, constructed on the basis of complex vector bundles. The elements x of $K_0(X)$ can be identified with the natural transformations of the further $K^0(X \times *)$ to the function $K^0(*)$, that is, with the homomorphisms $x(Y)$: $K^0(X \times Y) \to K^0(Y)$, such that for any continuous map φ: $Y_1 \to Y_2$ the following diagram commutes:

$$\begin{array}{ccc} K^0(X \times Y_1) & \xrightarrow{x(Y_1)} & K^0(Y_1) \\ {\scriptstyle (1\times\varphi)^*}\uparrow & & \uparrow{\scriptstyle \varphi^*} \\ K^0(X \times Y_2) & \xrightarrow{x(Y_2)} & K^0(Y_2) \end{array}$$

Thus, to determine an element $\varkappa(\rho) \in K_0(X)$ it is sufficient to associate with the Fredholm representation ρ of $\Lambda = \mathbf{C}(X)$ the homomorphisms

$\varkappa(\rho)(Y)$: $K^0(X \times Y) \to K^0(Y)$. Let $\xi \in K^0(X \times Y)$ be a finite-dimensional bundle over the space $X \times Y$. We denote by $\Gamma_y(\xi)$ the space of continuous sections of ξ over the subspace $X \times \{y\}$ $(y \in Y)$. The space $\Gamma_y(\xi)$ is a finitely generated projective module over $\Lambda = \mathbf{C}(X)$. Then D induces the Fredholm operator

$$F_y = 1 \otimes D\colon\ \Gamma_y(\xi) \otimes_\Lambda H \to \Gamma_y(\xi) \otimes_\Lambda H \tag{5.28}$$

up to a compact component. So we obtain a continuous family of Fredholm operators F_y, parametrized by the points of Y, and operating fibrewise on the bundle $\mathscr{H}$ with the Hilbert fibre $\Gamma_y(\xi) \otimes_\Lambda H$ over Y. The family of Fredholm operators F_y determines (see [15]) an element of $K^0(Y)$. So we have completely determined an element $\varkappa(\rho) \in K_0(X)$ from the Fredholm representation ρ of $\Lambda = \mathbf{C}(X)$.

Kasparov [20] has found conditions under which two Fredholm representations ρ_1 and ρ_2 of $\Lambda = \mathbf{C}(X)$ lead to one and the same element of $K_0(X)$. If $\varkappa(\rho) = 0$, then ρ can be obtained by means of the following operations:

a) homotopies in the class of Fredholm representations,

b) the addition of direct summands $\rho' = (\rho_1', F, \rho_2')$ in which F is invertible and satisfies (5.26).

With the help of the invariant $\varkappa(\rho)$ of a Fredholm representation ρ of $\Lambda = \mathbf{C}(X)$ one can give a more transparent description of the Atiyah–Singer formula [21] for the index of an elliptic operator. Let $\rho = (\rho_1, D, \rho_2)$ be a Fredholm representation of Λ on the Hilbert space $H = L_2(X)$, constructed from the elliptic pseudo-differential operator D of order zero. Let $\sigma(D) \in K_c^0(T^*X)$ be the element of $K_c^0(T^*X)$ determined by the symbol of D. Since the cotangent bundle T^*X is a quasicomplex manifold, there is Poincaré duality, that is, an isomorphism $\partial\colon K_c^0(T^*X) \to K_0(T^*X) = K_0(X)$. Here K_c^0 denotes the "compact K-functor", that is, the relative groups for the one-point compactification of the base. Then from the Atiyah–Singer formula we get the following relation

$$\partial\sigma(D) = \varkappa(\rho_1, D, \rho_2). \tag{5.29}$$

The Atiyah–Singer formula itself for the index of an elliptic operator can be obtained as an analogue of the direct image for elliptic operators. If $\varphi\colon X_1 \to X_2$ is a continuous map of compact manifolds and $\rho = (\rho_1, F_1, \rho_2)$ a Fredholm representation of $\Lambda_1 = \mathbf{C}(X_1)$, then by applying the change of rings $\varphi^*\colon \Lambda_2 = \mathbf{C}(X_2) \to \Lambda_1 = \mathbf{C}(X_2)$ we get a Fredholm representation $\varphi_*(\rho) = (\varphi^*\rho_1, F, \varphi^*\rho_2)$ of the ring Λ_2. Here we have the relation

$$\varphi_*(\varkappa(\rho)) = \varkappa(\varphi_*(\rho)). \tag{5.30}$$

In particular, if the manifold X_2 consists of a single point, $X_2 = \mathrm{pt}$, then

$$\varkappa(\varphi_*(\rho)) = \operatorname{index} F. \tag{5.31}$$

Consequently, applying (5.29) we come to the formula for the index of D:

$$\text{index } D = \varphi_*(\partial\sigma(D)) = \varphi_!(\sigma(D)), \tag{5.32}$$

where

$$\varphi_!\colon K_c^0(T^*X) \to K^0(\text{pt}) = \mathbf{Z}$$

is the direct image in K-theory ([22]). The derivation of the traditional Atiyah–Singer formula in terms of the characteristic classes of the elements of $\sigma(D)$ is by now standard routine in K-theory.[1]

§6. Fredholm representations and bundles over classifying spaces

In §5 we have described how with the help of the Fredholm representations of a ring Λ we can find signature invariants of Hermitian forms over Λ and, in case $\Lambda = C(X)$ is the ring of continuous functions on a manifold X, invariants of elliptic operators.

If Λ is the group ring of a discrete group π, $\Lambda = C[\pi]$, then with the help of Fredholm representations we can describe the homotopy invariants of the classifying space $B\pi$ of π, namely, vector bundles over $B\pi$.

1. Finite-dimensional representations and bundles. The connection between vector bundles over a classifying space $B\pi$ and finite-dimensional representations of π was considered earlier in papers of Atiyah and Hirzebruch [22], [23]. With each finite-dimensional representation $\rho\colon \pi \to \text{Mat}(n, C)$ of π we can associate a finite-dimensional complex bundle ξ_ρ over $B\pi$. To do this we consider the universal covering space over $B\pi$:

$$\varphi\colon E\pi \to B\pi.$$

The group π acts freely on $E\pi$, and if $B\pi$ is a simplicial complex, then $E\pi$ also has a simplicial structure, the projection φ is a simplicial map, and the action of π is also simplicial.

The representation ρ provides a linear action of π on the Euclidean space $V = C^n$. We consider the direct product $E\pi \times V$ with the diagonal action, that is, $g(x, y) = (g(x), \rho(g)y)$ $(x \in E\pi, y \in V)$. We can construct the following commutative diagram of maps:

$$\begin{array}{ccc} E\pi \times V & \xrightarrow{\tilde{\varphi}} & (E\pi \times V)/\pi \\ \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \tilde{\psi}} \\ E\pi & \xrightarrow{\varphi} & B\pi \end{array}$$

Here $(E\pi \times V)/\pi$ denotes the space of orbits of the action of π on $E\pi \times V$, ψ is the projection onto the first factor, $\psi(x, y) = x$, φ and $\tilde{\varphi}$ are the natural projections onto the space of orbits, and $\tilde{\psi}$ is the map of the orbit space induced by ψ. Thus $\tilde{\psi}$ is a locally trivial bundle with fibre V. For if $x \in B\pi$ is any point, then there is a neighbourhood U of x such that the inverse image $\varphi^{-1}(U)$ splits into the disjoint union of neighbourhoods

[1] It is equivalent to a proof of the formula (5.29).

$\{gW,\ g \in \pi\}$, $\varphi^{-1}U = \bigcup_{g\in\pi} gW$, and φ homeomorphically maps each component gW onto U. Then $\widetilde{\varphi}$ homeomorphically maps $\psi^{-1}(gW) = gW \times V$ onto the set $\widetilde{\varphi}^{-1}(U)$.

Consequently, $\widetilde{\psi}$ is a locally trivial bundle. Since the action of π on V is linear, the transition functions of $\widetilde{\psi}$ also are linear, hence $\widetilde{\psi}$ is a vector bundle. We denote by ξ_ρ the bundle $\widetilde{\psi}$: $(E\pi \times V)/\pi \to B\pi$. We also denote by ξ_ρ the induced element of $K(B\pi)$.

It is not difficult to verify that

$$\xi_{(\rho_1 \oplus \rho_2)} = \xi_{\rho_1} \oplus \xi_{\rho_2}, \qquad \xi_{\rho_1 \otimes \rho_2)} = \xi_{\rho_1} \otimes \xi_{\rho_2}.$$

Thus, the correspondence $\rho \to \xi_\rho$ induces a homomorphism of the ring $\mathcal{R}(\pi)$ of virtual representations of π to the ring $K(B\pi)$:

$$\xi\colon\ \mathcal{R}(\pi) \to K(B\pi). \tag{6.1}$$

It is natural to investigate the question in what cases ξ is an isomorphism or to describe the image and kernel of ξ.

When π is a compact Lie group, Atiyah and Hirzebruch ([22], [23]) have proved that $\ker \xi = 0$, while the image $\operatorname{Im} \xi$ is dense in the ring $K(B\pi)$, equipped with the topology induced by the finite-dimensional skeletons of $B\pi$ (see also [24]).

2. Fredholm representations and bundles. Just as in the case of the signature invariants of Hermitian forms, so finite-dimensional representations of infinite discrete groups provide a small stock of bundles from the group $K(B\pi)$. There are only individual results, describing the image of (6.1) for discrete subgroups of the real symplectic group $\mathrm{Sp}(2n, \mathbf{R})$. For example, it is shown in [25] that if $\pi \subset \mathrm{Sp}(2n, \mathbf{R})$ is a discrete torsion-free group and if the homogeneous space $\mathrm{Sp}(2n, \mathbf{R})/\pi$ is compact, then the inclusion homomorphism $H^p(B\ \mathrm{Sp}(2n, \mathbf{R})) \to H^p(B\pi)$ is an epimorphism for $p < (n+2)/4$. For an application of this algebraic result to obtain signature invariants of Hermitian forms, see Lusztig [26].

With the help of Fredholm representations it becomes possible to obtain a wide class of vector bundles over $B\pi$.

We describe the corresponding geometrical construction of the bundle $\xi_\rho \in K(B\pi)$ for a Fredholm representation $\rho = (\rho_1, F, \rho_2)$ of a discrete group π. In this case it is convenient to represent the elements of $K(B\pi)$ not as linear combinations of vector bundles, but as continuous families $\{F_x\}$ of Fredholm operators, parametrized by the points x of $B\pi$ (see [15]). If we are given a continuous family $\{F_x\}$ $(x \in X)$ acting from one Hilbert space H_1 to another H_2 (where X is some topological spaces), and if the dimension of the kernel $\dim \operatorname{Ker} F_x$ of F_x is a locally constant function, then the element of $K(X)$ corresponding to $\{F_x\}$ is the difference of the two finite-dimensional bundles $\bigcup_{x\in X} \operatorname{Ker} F_x$ and $\bigcup_{x\in X} \operatorname{Coker} F_x$. Here, $\bigcup_{x\in X} \operatorname{Ker} F_x$ is to be understood as a subspace of the direct product

$X \times H_1$, and $\bigcup_{x \in X} \operatorname{Coker} F_x$ as the factor space $X \times H_2$. Thus, it will be convenient for us to interpret the family of Fredholm operators as a homomorphism of trivial bundles with Hilbert fibres

$$X \times H_1 \xrightarrow{F} X \times H_l \tag{6.2}$$
$$\searrow \quad \swarrow$$
$$X$$

The diagram (6.2) is commutative, and over each $x \in X$ the map F is a Fredholm operator of Hilbert spaces. It is quite clear that if in place of trivial bundles with Hilbert fibre we consider arbitrary bundles $\mathscr{H}_1$ and $\mathscr{H}_2$ with Hilbert fibres over X and a bundle homomorphism $F\colon \mathscr{H}_1 \to \mathscr{H}_2$, which is a Fredholm operator on each fibre, then we equally well come to an element of $K(X)$, since each locally trivial bundle with Hilbert fibre is isomorphic to a trivial bundle (see, for example, [15]).

We return now to the case when the base X is a classifying space $B\pi$. Let $\mathscr{H}_1$ and $\mathscr{H}_2$ be two bundles with Hilbert fibres H_1 and H_2, respectively, and let $F\colon \mathscr{H}_1 \to \mathscr{H}_2$ be a Fredholm bundle homomorphism on each fibre. We denote by $\tilde{\mathscr{H}}_1$ and $\tilde{\mathscr{H}}_2$ the inverse images of $\mathscr{H}_1$ and $\mathscr{H}_2$ under the projection $\varphi\colon E\pi \to B\pi$ of the universal covering of $B\pi$. Then π acts freely both on $E\pi$ and on each of the bundles $\tilde{\mathscr{H}}_1$ and $\tilde{\mathscr{H}}_2$. Also, the homomorphism F induces a homomorphism $\tilde{F}\colon \tilde{\mathscr{H}}_1 \to \tilde{\mathscr{H}}_2$, that is equivariant under the action of π. Conversely, if $G\colon \tilde{\mathscr{H}}_1 \to \tilde{\mathscr{H}}_2$, is an arbitrary equivariant homomorphism then it is always induced by some homomorphism $F\colon \mathscr{H}_1 \to \mathscr{H}_2$.

Therefore, to specify an element of $K(B\pi)$, it is enough to give an equivariant homomorphism $G\colon \tilde{\mathscr{H}}_1 \to \tilde{\mathscr{H}}_2$ that is a Fredholm operator on each fibre, for some pair of bundles $\tilde{\mathscr{H}}_1$ and $\tilde{\mathscr{H}}_2$ with Hilbert fibre and free action by π on the base space.

Let ρ be a Fredholm representation of π, that is, a triple $\rho = (\rho_1, F, \rho_2)$ consisting of two unitary representations on Hilbert spaces H_1 and H_2, and a Fredholm operator $F\colon H_1 \to H_2$ such that for each $g \in \pi$ the operator $\rho_2(g)F - F\rho_1(g)$ is compact. We have to associate with ρ an element ξ_ρ of $K(B\pi)$. For this purpose we construct over $E\pi$ two equivalent bundles $\tilde{\mathscr{H}}_1$ and $\tilde{\mathscr{H}}_2$ with Hilbert fibre and an equivariant homomorphism $G\colon \tilde{\mathscr{H}}_1 \to \tilde{\mathscr{H}}_2$ that is a Fredholm operator on each fibre.

For the bundles $\tilde{\mathscr{H}}_1$ and $\tilde{\mathscr{H}}_2$ we take the trivial bundles $\tilde{\mathscr{H}}_1 = E\pi \times H_1$, and $\tilde{\mathscr{H}}_2 = E\pi \times H_2$, with diagonal action of π, that is, $g(x, y) = (gx, \rho_k(g)y)$ $(x \in E\pi,\ y \in H_k)$. Then the map G can be regarded as a continuous, equivariant family of Fredholm operators $\{G_x\}$ $(x \in E\pi)$. Equivariance of $\{G_x\}$ means that

$$\rho_2(g)G_x = G_{gx}\rho_1(g). \tag{6.3}$$

We look for an equivariant family $\{G_x\}$ such that for any point $x \in E\pi$

(6.4) the difference $F - G_x$ is a compact operator.

If G'_x is another family satisfying (6.3) and (6.4), then by setting $G^t_x = G_x + tG'_x$ we get a homotopy between $\{G_x\}$ and $\{G'_x\}$ in the class of families satisfying (6.3) and (6.4). Consequently, both $\{G_x\}$ and $\{G'_x\}$ define one and the same element of $K(B\pi)$, which we denote by ξ_ρ.

It only remains for us to establish the existence of at least one family $\{G_x\}$, satisfying (6.3) and (6.4). To do this we represent $B\pi$ as a simplicial complex and $E\pi$ as a simplicial complex with simplicial action of π. We construct $\{G_x\}$ by an inductive process on the dimension of the skeletons of $E\pi$. The zero-dimensional skeleton $(E\pi)^0$ consists of a discrete set of vertices and splits into the union of pairwise disjoint orbits of the action of π. Let $A \subset (E\pi)^0$ be some orbit, $A = \bigcup_{g\in\pi} ga$, where $a \in A$ is one of the vertices of A. If at $a \in A$ we are given an operator G_a, then at the other points $b \in A$ the operator G_b is uniquely determined by the condition (6.3) of equivariance of the family by the formula

$$G_b = \rho_2(g)G_a\rho_1(g^{-1}), \tag{6.5}$$

where $g \in \pi$ is an element of π such that $b = ga$. Thus, to determine a family $G_x (x \in (E\pi)^0)$ satisfying (6.3) and (6.4) it is enough to choose a point a in each orbit A and to set $G_a = F$. At all other points of the zero-dimensional skeleton the family is defined by (6.5).

Then (6.3) is automatically fulfilled and (6.4) follows from the definition of a Fredholm representation.

Suppose now as an inductive hypothesis that we are given a family $\{G_x\}$ on the k-dimensional skeleton $(E\pi)^k$, satisfying (6.2) and (6.3). We consider the set of all $(k+1)$-dimensional simplexes of $E\pi$. Now π, acting freely on $E\pi$, permutes its simplexes leaving none of them invariant. Therefore, the set of all $(k+1)$-dimensional simplexes can be split into the disjoint union of "orbits" and in each orbit all the simplexes can be obtained as images of a given simplex of the "orbit" with the help of the action of π. Let Δ^{k+1} be one of the $(k+1)$-dimensional simplexes of $E\pi$. If the family $\{G_x\}$ is given for each $x \in \Delta^{k+1}$, then to get an equivariant family on the whole orbit we have to use (6.5). If the point $x \in \Delta^{k+1}$ belongs to the boundary $\partial\Delta^{k+1}$ of Δ^{k+1}, then (6.5) gives us an identity, by the inductive hypothesis. Therefore, it is sufficient to extend the family of operators $\{G_x\}$, which is defined by the inductive hypothesis on the boundary $\partial\Delta^{k+1}$, to a family defined on the whole simplex Δ^{k+1} with (6.3) preserved. Since (6.3) holds on $\partial\Delta^{k+1}$ and the space of compact operators is contractible (as a linear space), the required extension of $\{G_x\}$ to Δ^{k+1}, and hence to its whole orbit, always exists.

This completes the construction of the element $\xi_\rho \in K(B\pi)$ for a Fredholm representation.

NOTE. The reader should be warned against trying to prove that ξ_ρ is trivial, on the basis of (6.3). The fact is that (6.3) is not invariant under

a change of coordinates in the fibres of the bundles $\mathscr{H}_1$ and $\mathscr{H}_2$. Therefore a trivialization of $\mathscr{H}_1$ and $\mathscr{H}_2$ over $B\pi$ induces over $E\pi$ a trivialization of $\tilde{\mathscr{H}}_1$ and $\tilde{\mathscr{H}}_2$ in which, generally speaking, (6.3) is not satisfied.

3. Modification of the construction of a bundle. The construction described above can be generalized to the case of continuous families of Fredholm representations. Let $\rho = \{\rho_x\} = \{\rho_{1x}, F_x, \rho_{2x}\}$ be a continuous family of Fredholm representations, parametrized by the points of some topological space X. Then by applying the construction of subsection 2 we get a continuous family of Fredholm operators $\{G_{x,y}\}$, parametrized by the points of the direct product $X \times B\pi$, that is, an element $\xi_\rho \in K(X \times B\pi)$. If we ignore the torsion in $K(X \times B\pi)$, this group can be identified with the zero grading of the tensor product $K(X \times B\pi) \approx [K^*(X) \otimes K^*(B\pi)]_0$, that is, ξ_ρ splits into the sum

$$\xi_\rho = \sum_k \alpha_k \otimes \beta_k,$$

where the $\{\alpha_k\}$ form a basis for $K^*(X)$, and $\beta_k \in K^*(B\pi)$. Consequently, the family ρ of Fredholm representations gives us a whole set of homotopy invariants β_k, lying in $K^*(B\pi)$. Moreover, if we can guarantee that for some point x of a closed subspace Y of X the family $\{G_{x,y}\}$ consists of invertible operators, then we can define ξ_ρ as an element of the relative group $K(X \times B\pi, Y \times B\pi)$.

As a condition for the invertibility of a family of operators $\{G_{x,y}\}$ we can take the following:

(6.6) $\| F^{-1}\rho_2(g) F\rho_1(g^{-1}) - 1 \| < 1$, for any $g \in \pi$.

Then in the construction of $\{G_{x,y}\}$ we can add to (6.3) the condition

$$\| F^{-1}G_{x,y} - 1 \| < 1, \tag{6.7}$$

guaranteeing the invertibility of operators of $\{G_{x,y}\}$.

Next, we can start not from a Fredholm representation $\rho = (\rho_1, F, \rho_2)$ of π, but from a Fredholm complex (5.2) of representations of π. In this case we have to construct a continuous equivariant family of Fredholm complexes $\{G_{i,x}\}$ $(1 \leqslant i \leqslant s-1, x \in E\pi\}$, such that

(6.8) The operators $F_i - G_{i,x}$ are compact for each point $x \in E\pi$.

In the case of a continuous family $\rho = \{\rho_x\}(x \in X)$ of Fredholm representations we can repeat almost word for word the construction of $\xi_\rho \in K(X \times B\pi)$ and also write down a condition on ρ for ξ_ρ to be defined as an element of the relative group $K(X \times B\pi, Y \times B\pi)$.

Finally, if $\rho = (\rho_1, F_1, \rho_2)$ and $\rho' = (\rho_1, F', \rho_2)$ are two Fredholm representations of π such that $F' - F$ is compact, then $\xi_\rho = \xi_{\rho'}$.

4. The construction of Fredholm representations for a particular class of groups. As we have indicated in subsection 2 above, we are interested in the question how large the class of bundles over $B\pi$ is that we can construct with the help of Fredholm representations. A priori, this class may be larger than that of bundles obtained from finite-dimensional bundles.

We notice to begin with that, by analogy with the signatures of Hermitian forms, in the case of a Fredholm representation $\rho = (\rho_1, F, \rho_2)$ satisfying (5.26) rather than (5.1), the element $\xi_\rho \in K(B\pi)$ coincides with the difference $\xi_{\rho_1'} - \xi_{\rho_2'}$, where ρ_1' and ρ_2' are representations of π on the kernel and cokernel of F, respectively. Taking into account that for finite groups π each Fredholm representation can be changed into one satisfying (5.26) by adding a compact operator to the representation, we find that for a finite group π the classes of bundles over $B\pi$ constructed with the help of Fredholm representations and of finite-dimensional representations are identical.

In any case, if we wish to describe the class of bundles over $B\pi$ representable in the form ξ_ρ for some Fredholm representation ρ of π, then we must present an explicit construction of a Fredholm representation.

There is an important class of infinite groups π for which such a geometrical construction of Fredholm representations is possible. This class of groups is defined by the following property:

(6.9) The space $B\pi$ is homotopically equivalent to a compact Riemannian manifold with a metric of non-positive curvature in every two-dimensional direction.

In particular, this class of groups includes finitely generated free Abelian groups, the fundamental groups of compact two-dimensional surfaces, and discrete groups of motions of homogeneous spaces of semisimple non-compact Lie groups.

Let X be a complete Riemannian manifold with metric of non-positive curvature, $\pi_1(X) = \pi$, homotopically equivalent to $B\pi$. Then its universal covering $\widetilde{X}$ is diffeomorphic to Euclidean space, and moreover, there is on $\widetilde{X}$ an equivariant metric of non-positive curvature. We consider the space T^*X of the cotangent bundle of X, which we denote by Y. In this space the metric on X induces a natural metric such that in the universal cover $\widetilde{Y}$, the group π preserves the metric.

Thus, we have the commutative diagram

$$\begin{array}{ccc} \widetilde{Y} & \xrightarrow{\widetilde{\psi}} & Y \\ {\scriptstyle\widetilde{\varphi}}\downarrow & & \downarrow{\scriptstyle\varphi} \\ \widetilde{X} & \longrightarrow & X \end{array}$$

Next, we consider on X the exterior power $\Lambda_k(cT^*X)$ of the complexification of the cotangent bundle. We set $\xi_k = \widetilde{\psi}^*\varphi^*(\Lambda_k(cT^*X))$. The points of $\widetilde{Y}$ are pairs $(\widetilde{x}, y)$, $(\widetilde{x} \in \widetilde{X})$, where y is a cotangent vector to $\widetilde{X}$ at $\widetilde{x}$.

The vectors of the bundle ξ_k split into linear combinations of exterior products of the form $y_1 \wedge \ldots \wedge y_k \in \xi_k$, where $y_1, \ldots, y_k$ are cotangent vectors to $\widetilde{X}$.

We consider the complex of bundles

$$\xi_0 \xrightarrow{a_0} \xi_1 \xrightarrow{a_1} \ldots \xrightarrow{a_{n-1}} \xi_n, \tag{6.10}$$

where the homomorphisms $a_0, a_1, \ldots, a_{n-1}$ are the operators of exterior multiplication by covectors of the form $(y + i\omega(\widetilde{x}))$ at $(\widetilde{x}, y) \in \widetilde{Y}$. Here $\omega(\widetilde{x})$ is a section of the cotangent bundle (that is, a differential form) of $\widetilde{X}$. Note that π acts freely on $\widetilde{X}$ and $\widetilde{Y}$, and also on the bundles $\xi_0, \ldots, \xi_n$. Moreover, from the point of view of this action the operator of exterior multiplication on the covector y at $(\widetilde{x}, y)$ is equivariant, and the commutator of a_j and the action of $g \in \pi$ is the operator of exterior multiplication by the covector $(\omega(g\widetilde{x}) - g^*\omega(\widetilde{x}))$.

If the covector $(y + i\omega(\widetilde{x}))$ at $(\widetilde{x}, y) \in \widetilde{Y}$ is different from zero, then the complex (6.10) is exact, that is, has trivial homology.

We denote by $\widetilde{\psi}_!(\xi_j)$ the "direct image" of ξ_j, that is, the bundle with Hilbert fibre whose fibre is constructed at each point $y \in Y$ as the orthogonal sum of the finite-dimensional fibres of ξ_j at all the inverse images of y. A unitary action of π (which also respects fibres) is induced in the infinite-dimensional bundle $\widetilde{\psi}_!(\xi_j)$, and the homomorphisms $a_0, \ldots, a_{n-1}$ induce homomorphism of the "direct images", so that we obtain a new complex

$$\widetilde{\psi}_!(\xi_0) \xrightarrow{A_0} \widetilde{\psi}_!(\xi_1) \xrightarrow{A_1} \ldots \xrightarrow{A_{n-1}} \widetilde{\psi}_!(\xi_n). \tag{6.11}$$

The complex (6.11) is the required family of Fredholm complexes of representations, provided that by a suitable choice of $\omega(\widetilde{x})$ we can guarantee that the conditions in the definition of Fredholm representations hold.

The commutator of A_j and the action of $g \in \pi$ can be described by a diagonal matrix in which each term is the operator of exterior multiplication by the covector $(\omega(g\widetilde{x}) - g^*\omega(\widetilde{x}))$, and $\widetilde{x} \in \widetilde{X}$ runs through the orbit of the action of π. Therefore, for the commutator to be compact it is sufficient that

$$\lim_{\widetilde{x}\to\infty} \|\omega(g\widetilde{x}) - g^*\omega(\widetilde{x})\| = 0. \tag{6.12}$$

For the complex (6.11) to be Fredholm at each $y \in Y$ it is sufficient that on each orbit the homology groups of the complex (6.10) are trivial everywhere, with the exception of finitely many points of each orbit, that is, that $\omega(\widetilde{x})$ vanishes only at finitely many points of each orbit, and that the norm $\|\omega(\widetilde{x})\|$ is bounded below at infinity.

We construct such a function $\omega(\widetilde{x})$. By the duality induced by the Riemannian metric it is sufficient to produce a vector field with similar properties. For this purpose we fix an initial point $\widetilde{x}_0$ and consider the (unique) geodesic $\gamma(\widetilde{x})$ beginning at $\widetilde{x}_0$ and ending at a point $\widetilde{x}$. Let $\omega(\widetilde{x})$ be the tangent vector to $\gamma(\widetilde{x})$ at $\widetilde{x}$, whose length is equal to $l/(1 + l)$, where l is the length of $\gamma(\widetilde{x})$. Then $\omega(\widetilde{x}) = 0$ only at the point $\widetilde{x} = \widetilde{x}_0$, $\lim_{x\to\infty} \|\omega(x)\| = 1$, and since the curvature is non-positive condition (6.12) is satisfied.

So we have produced a family $\rho = \{\rho_y\}$ $(y \in Y)$ of Fredholm complexes of representations of π. Moreover, there is a compact subspace $Z \subset Y$ such that at the points $y \in Y \setminus Z$, ρ_y satisfies an additional condition of type (6.7) so that ξ_ρ can be defined as an element of the group $K_c(Y \times X)$ of bundles with compact support.

The Chern character of ξ_ρ (see [17] can be computed by the following formula:

$$\text{ch}\,\xi_\rho = \sum \sigma a_i \otimes b_i. \tag{6.13}$$

where $a_i \in H^*(H; \mathbf{Q})$ is a basis for the cohomology groups, $b_i \in H^*(X, \mathbf{Q})$ is the dual basis by Poincaré duality, and $\sigma a_i \in H_c^*(Y; \mathbf{Q})$ are the images in the cohomology groups under the Thom isomorphism.

Thus, with the help of a single family of Fredholm representations ρ we obtain for groups π satisfying (6.9) invariants b_i, which form a basis for the cohomology groups of $B\pi$.

5. The construction of individual Fredholm representations. In the preceding subsection we have presented for one class of groups π a family of Fredholm representations, and with the help of it we have been able to describe the cohomology of $B\pi$ (or, modulo torsion, the bundles over $B\pi$).

However, the question remains open what bundles over $B\pi$ can be described with help of Fredholm representations, not indirectly, but directly as bundles of the type ξ_ρ. It turns out that, starting from a family of Fredholm representations, we can construct new individual representations, whose invariants generate the invariants of the original family. It is appropriate to state a precise proposition.

THEOREM 6.1 [27]. *Let $\rho = \{\rho_x\}$ $(x \in X)$ be a family of Fredholm representations of a group π, $\xi_\rho \in K(X \times B\pi)$ the bundle constructed in subsection 2, and $\text{ch}\xi_\rho = \sum_k a_k \otimes b_k$, where the elements $a_k \in H^*(X; \mathbf{Q})$ form a basis for the cohomogy groups and $b_k \in H^*(B\pi; \mathbf{Q})$. Then we can find Fredholm representations ρ_k of π such that*

$$\text{ch}\xi_{\rho_k} = \lambda_k b_k \qquad (\lambda_k \neq 0). \tag{6.14}$$

Theorem 6.1 explains the idea behind studying families of Fredholm representations. For example, from this theorem and the construction of subsection 4 it follows that, in the case of a group π satisfying (6.9), "almost all" the bundles can be constructed with the help of Fredholm representations of π. Here the term "almost all" means that the bundles of type ξ_ρ generate a subgroup of finite index in $K(B\pi)$. In one particular case, $\pi = \mathbf{Z} \times \mathbf{Z}$, this corollary to Theorem 6.1 was first established by Solov'ev [29].

To prove Theorem 6.1 we need an additional geometrical construction, which is precisely followed through in the cases of families of finite-dimensional representations of π.

We consider a topological space X and a finite-dimensional bundle η over X on which π acts linearly. The action of π can be regarded as a family $\rho = \{\rho_x\}$ $(x \in X)$ of finite-dimensional representations of π on the fibre of η. This family ρ generates a (finite-dimensional) bundle $\xi_\rho \in K(X \times B\pi)$. Suppose that X is a compact closed manifold. Then π acts linearly on the space of sections $\Gamma(\eta \otimes \xi)$ of the tensor product of η with an arbitrary bundle ζ and also on the completions $H^s(\eta \otimes \zeta)$ of $\Gamma(\eta \otimes \zeta)$ in the Sobolev norm. We consider two bundles ζ_1 and ζ_2 and the elliptic pseudodifferential operator

$$D\colon \Gamma(\eta \otimes \zeta_1) \to \Gamma(\eta \otimes \zeta_2), \tag{6.15}$$

which is a Fredholm operator on the Sobolev spaces

$$D\colon H^s(\eta \otimes \zeta_1) \to H^s(\eta \otimes \zeta_2). \tag{6.16}$$

The commutator of D and the action of $g \in \pi$ is an operator of smaller degree, and so is a compact operator on the Sobolev spaces.

Thus, if we denote by ρ_1 and ρ_2 two representations of π acting on $H^s(\eta \otimes \zeta_1)$ and $H^s(\eta \otimes \zeta_2)$, then the triple $\bar{\rho} = (\rho_1, D, \rho_2)$ is a Fredholm representation.

The computation of $\xi_{\bar{\rho}} \in K(B\pi)$ can be made with the help of the Atiyah–Singer theorem on the index of an elliptic operator, or more precisely, its generalization to the case of families [28]. In particular, if the symbol of D has the form $1 \otimes \sigma\colon \varphi^*(\eta \otimes \zeta_1) \to \varphi^*(\eta \otimes \zeta_2)$, where $\varphi\colon T^*X \to X$ is the projection, then

$$\xi_{\bar{\rho}} = \psi_!(\xi_\rho \otimes \bar{\sigma}). \tag{6.17}$$

Here $\psi\colon T^*X \times B\pi \to B\pi$ is the projection, $\bar{\sigma} \in K(T^*X)$ the element defined by the symbol σ and $\psi_!$ the "direct image" in K-theory. Choosing D suitably, we can easily prove Theorem 6.1 by using theorems on realizing rational homology classes of X with the help of singular manifolds.

In the case of infinite-dimensional Fredholm representations, the situation is technically somewhat more complicated, but nevertheless remains analogous.

CHAPTER 3

CHARACTERISTIC CLASSES OF SMOOTH MANIFOLDS AND HERMITIAN K-THEORY

The theory of Hermitian forms and Hermitian K-theory have deep applications in the study of the invariants of smooth of piecewise-linear manifolds. It would be difficult to give a historical survey of all these applications. We note only the chief aspects of the profound connection between Hermitian K-theory and the theory of smooth manifolds. This is above all the problem of classifying smooth structures of a given homotopy type, and the homotopy invariant characteristic classes of smooth manifolds. In this chapter we give a survey of the problems concerned

with relations for characteristic classes of smooth manifolds that arise in the application of representation theory to the computation of characteristic classes. These relations are known as Hirzebruch formulae.

§7. Hirzebruch formulae for finite-dimensional representations

1. **Simply-connected manifolds.** The classical Hirzebruch formula is concerned with a compact closed $4k$-dimensional oriented manifold, say X.

We consider the cohomology group $H^{2k}(X; \mathbf{R})$ with real coefficients. It is a finite-dimensional vector space on which there is defined a non-degenerate symmetric bilinear form $\langle x, y\rangle = (xy, [X])$, $x, y \in H^{2k}(X; \mathbf{R})$. Here $[X]$ denotes the fundamental class of X, $[X] \in H_{4k}(X; \mathbf{Z})$. The signature of this form is called the signature of X and is denoted by $\tau(X)$. So we obtain an integral-valued function $\tau(X)$, defined on the set of all $4k$-dimensional oriented compact closed manifolds.

It is not difficult to see that for a disjoint sum $X = X_1 \cup X_2$ signatures are additive: $\tau(X) = \tau(X_1) + \tau(X_2)$, and if the manifold is the boundary of some $(4k + 1)$-dimensional oriented manifold W, $X = \partial W$, then the signature $\tau(X)$ of X is trivial. This means that $\tau(X)$ determines a linear function on the group Ω_{4k} of oriented bordisms. In bordism theory it is known that every linear function on the group of bordisms is a characteristic number, that is, there exists a characteristic class (say, L_{4k}) such that

$$\tau(X) = 2^{2k}\langle L_{4k}(X), [X]\rangle. \tag{7.1}$$

The equality (7.1) is usually called the Hirzebruch formula with an explicit description of the class L_{4k}. To define the characteristic class L_{4k} we denote by L the sum

$$L = \sum_{k=0}^{\infty} L_{4k}. \tag{7.2}$$

Then in terms of the *Wu* generators ([30]–[32]) the characteristic class (7.2) is

$$L = \prod_i \frac{x_i/2}{\operatorname{th}(x_i/2)}. \tag{7.3}$$

The characteristic class (7.2) or (7.3) is called the Hirzebruch class.

There are at least two distinct proofs of the Hirzebruch formula (7.1). One of these is based on the computation of the ring of oriented cobordisms and the verification of the Hirzebruch formula on multiplicative generators. For if $X = X_1 \times X_2$, then $\tau(X) = \tau(X_1)\,\tau(X_2)$. On the other hand, it follows from (7.3) that $L(X) = L(X_1)\,L(X_2)$. Thus, if the formula is true for two manifolds X_1 and X_2, then it is also true for $X = X_1 \times X_2$.

The second proof is based on the representation of the signature $\tau(X)$ of X as the index of an elliptic operator on X and the calculation of this index by means of the Atiyah–Singer formula ([31]). To represent $\tau(X)$

as the index of an elliptic operator it should be observed that according to de Rham theory the real cohomology of X is isomorphic to the de Rham homology, that is, the cohomology of the complex of exterior differential forms of X. Let $\Omega^j(X)$ denote the space of exterior differential forms of degree j on X, and d: $\Omega^j(X) \to \Omega^{j+1}(X)$ the operator of exterior differentiation. Then we obtain the de Rham complex

$$\Omega^0(X) \xrightarrow{d} \Omega^1(X) \xrightarrow{d} \ldots \xrightarrow{d} \Omega^n(X)$$

($n = \dim X$, $d^2 = 0$). On the space $\Omega(X) = \bigoplus_j \Omega^j(X)$ of all exterior differential forms there is the operation of exterior multiplication, which induces a ring structure on the de Rham homology groups.

In particular, to determine the signature $\tau(X)$ of X ($\dim X = 4k$) we consider the bilinear form on the cohomology of middle dimension $H^{2k}(X; \mathbf{R})$ induced by the equality

$$\langle \alpha, \beta \rangle = \int \alpha \wedge \beta, \quad \alpha, \beta \in \Omega^{2k}(X). \tag{7.4}$$

Suppose that a Riemannian metric is given on X. It induces on each of the spaces $\Omega^j(X)$ a scalar product

$$(\alpha, \beta) \qquad (\alpha, \beta \in \Omega^j(X)). \tag{7.5}$$

Then the bilinear form (7.4) can be reduced to (7.5) by means of some operator*

$$\langle \alpha, \beta \rangle = (\alpha, * \beta). \tag{7.6}$$

Let $\tau(\alpha) = i^{j(j-1)+2k} * (\alpha)$ ($\alpha \in \Omega^j(X)$). It is easily verified that $\tau^2 = 1$. Consequently, the space $\Omega(X)$ of all forms splits into the direct sum of two eigenspaces $\Omega(X) = \Omega^+ \oplus \Omega^-$ for the involution τ. Let δ be the operator formally adjoint to the operator of exterior differentiation with respect to the scalar product (7.5), and let $D = d + \delta$.

It is easy to verify that $D\tau = -\tau D$, that is, D maps Ω^+ into Ω^- and Ω^- into Ω^+. After this it remains to observe that

$$\operatorname{index} D^+ = \tau(X). \tag{7.7}$$

2. Multiply-connected manifolds. For a multiply-connected manifold X, $\dim 4k$, $\pi_1(X) = \pi$, we can define a whole series of signature invariants using the finite-dimensional unitary representations of the fundamental group. We consider for this purpose a unitary representation ρ of π on a finite-dimensional space V. We denote by $H^j(X; V)$ the cohomology groups of X with coefficients in the local system V. Although we cannot introduce a ring structure on the groups $H^j(X; V)$, we can define a bilinear form which, so to speak, can be constructed from X with the help of the cohomology product and from V with the help of the scalar product φ: $V \otimes V \to \mathbf{R}$ on V:

$$\langle \alpha, \beta \rangle = \langle \varphi(\alpha \otimes \beta), [X] \rangle \qquad (\alpha, \beta \in H^*(X; V)).$$

Then on the group of middle dimension $H^{2k}(X; V)$ there is a non-degenerate symmetric bilinear form whose signature we denote by $\tau_\rho(X)$. In the case of the trivial representation of π on $\mathbf{R}$ the signature $\tau_\rho(X)$ obviously coincides with the ordinary signature of the manifold defined in subsection 1. Just as for the ordinary signature we can prove that the numbers are invariant under the special bordisms in which the fundamental group of the "film" is isomorphic to the fundamental group of the boundary.

In the language of bordism theory, this property can be formulated in the following way. Let X be a smooth compact closed manifold and $\pi_1(X) = \pi$. Then (see [33]) there exists a continuous map $\varphi_X: X \to B\pi$ of X to the classifying space $B\pi$, unique up to homotopy, inducing an isomorphism of fundamental groups $(\varphi_X)_*: \pi_1(X) \to \pi_1(B\pi) = \pi$. If W is the film bounding X, $\partial W = X$, and if $\pi_1(W)$ is naturally isomorphic to $\pi_1(X)$ then φ_X extends to a continuous map $\varphi_W: W \to B\pi$. Thus, $\tau_\rho(X)$ is defined by the integral-valued function

$$\tau_\rho: \Omega_*(B\pi) \to \mathbf{Z} \tag{7.8}$$

on the group of oriented bordisms of the classifying space $B\pi$. (For the definition of bordism see [34]).

Each numerical function on the group of bordisms can be described by means of characteristic classes ([35]). In particular, for the functions (7.8) there are characteristic classes a_l of oriented manifolds and cohomology classes $b_l \in H^*(B\pi)$ such that

$$\tau_\rho(X) = \sum_l \langle a_l(X)\, \varphi_X^*(b_l), [X] \rangle. \tag{7.9}$$

The signature $\tau_\rho(X)$ has a number of algebraic properties which enable us to obtain a priori information about the right-hand side of (7.9). Firstly, $\tau_\rho(X)$ is an additive function on $\Omega_*(B\pi)$, that is,

$$\tau_\rho(X_1) + \tau_\rho(X_2) = \tau_\rho(X_1 \cup X_2). \tag{7.10}$$

Next, if Y is a simply-connected oriented manifold, then

$$\tau_\rho(X \times Y) = \tau_\rho(X) \cdot \tau(Y). \tag{7.11}$$

From (7.10) and (7.11) it follows that the right-hand side of (7.9) has only one component and that $a(X) = L(X)$. For to determine the classes $a_i(X)$ and b_i it is enough to check the formula on some additive basis for the bordism group $\Omega_*(B\pi)$. Moreover, since the left-hand side of (7.9) also is an integer, it is enough to check the same formula on an additive basis of the group $\Omega_*(B\pi) \otimes \mathbf{Q}$, where $\mathbf{Q}$ is the field of rational numbers. Such a basis can be chosen in the following way. Let Ω_*^{fr} be the group of framed bordisms ([34]). Then there is the following isomorphism:

$$\Omega_*(B\pi) \otimes \mathbf{Q} \approx \Omega_*^{fr}(B\pi) \otimes \Omega_* \otimes \mathbf{Q}. \tag{7.12}$$

Let $\{(Y_j, \varphi_{Y_j})\}$ be a basis for $\Omega_*^{fr}(B\pi)$. Then every bordism $(X, \varphi_X) \in \Omega_*(B\pi)$ can be represented as a linear combination

$$(X, \varphi_X) = \sum (Z_j \times Y_j, \varphi_{Y_j}), \tag{7.13}$$

where the $Z_j \in \Omega_*$ are simply-connected manifolds. The left-hand side of (7.9) takes the form

$$\tau_\rho(X) = \sum_j \tau(Z_j)\, \tau_\rho(Y_j, \varphi_{Y_j}). \tag{7.14}$$

On the right-hand side of (7.9) we may suppose without loss of generality that the classes $\{b_l\}$ form a basis for the cohomology $H^*(B\pi)$, dual to the basis $\{(Y_j, \varphi_{Y_j})\}$. Then

$$\tau_\rho(X) = \sum_j \langle a_j(Z_j \times Y_j)\, \varphi_{Y_j}^*(b_j), [Z_j \times Y_j]\rangle. \tag{7.15}$$

Since the tangent bundle to Y_j is trivial, $a_j(Z_j \times Y_j) = a_j(Z_j)$. Consequently,

$$\tau_\rho(X) = \sum_j \langle a_j(Z_j), [Z_j]\rangle. \tag{7.16}$$

Setting $Z_j = 0$ for $j \neq l$ we get

$$\tau(Z_l)\, \tau_\rho(Y_l, \varphi_{Y_l}) = \langle a_l(Z_l), [Z_l]\rangle. \tag{7.17}$$

Taking into account the classical Hirzebruch formula (7.1) we obtain for any simply-connected manifold Z_l, and therefore, in general, for any manifold Z_l, the relation

$$a_l(Z_l) = 2^{(\dim Z_l)/2}\, \tau_\rho(Y_l, \varphi_{Y_l})\, L(Z_l). \tag{7.18}$$

Turning to (7.15) we come to the simpler relation

$$\begin{cases} \tau_\rho(X) = 2^{2k} \sum_l \langle L(Z_l \times Y_l)\, \tau_\rho(Y_l, \varphi_{Y_l})\, \varphi_{Y_l}^*(b_l), [Z_l \times Y_l]\rangle = \\ \qquad\qquad = 2^{2k} \langle L(X)\, \varphi_X^*(b), [X]\rangle, \end{cases} \tag{7.19}$$

where $b = \sum_l \tau_\rho(Y_l, \varphi_{Y_l}) b_l$.

It remains only to establish the value of the cohomology class $b \in H^*(B\pi)$, which depends only on ρ.

Unfortunately, the bordism method fails here. Therefore, we have to use the other method described in subsection 1 to compute the classical signature of manifolds with the help of a suitable choice of elliptic operator. The first such calculation was carried out (in a somewhat more general setting) by Lusztig [26].

In contrast to the classical signature, we have to consider, in place of the de Rham complex, exterior differential forms with values in a finite-dimensional bundle $\varphi_X^*(\xi_\rho)$ (the dimension of ξ_ρ is the same as that of the vector space V on which π acts). The only difficulty that has to be overcome in constructing the elliptic Hirzebruch operator in our case consists in verifying that we can construct the operator of exterior differentiation of exterior differential forms with values in $\varphi_X^*(\xi_\rho)$. For this purpose it is

enough to remark that in a coordinate representation of $\varphi_X^*(\xi_\rho)$ we can choose the transition functions on the intersections of the local charts so that they are locally constant matrix functions. All the remaining constructions remain unchanged. It turns out that

$$b = \mathrm{ch}\rho = \mathrm{ch}\xi_\rho. \tag{7.20}$$

Thus, we obtain the multiply-connected Hirzebruch formula

$$\tau_\rho(X) = 2^{2k}\langle L(X)\,\varphi_X^*\,\mathrm{ch}\,\xi_\rho, [X]\rangle \tag{7.21}$$

for a finite-dimensional unitary representation ρ of the fundamental group of X.

The class of formulae (7.21) can be increased [26] by considering representations of π in the group of automorphisms $O(m, n)$ of the vector space V^{m+n}, preserving a bilinear Hermitian form of type (m, n). With the help of such a representation ρ we can define a signature $\tau_\rho(X)$ of cohomology with coefficients in the local system of coefficients V^{m+n}, and also an element $\eta_\rho \in K(B\pi)$. This element η_ρ is constructed as follows. Let ξ_ρ be the finite-dimensional bundle constructed as in §5, from the representation $\rho\colon \pi \to O(m, n) \subset GL(m+n)$ of π in the group of all linear automorphism of V^{m+n}. Then in each fibre of ξ_ρ we have a bilinear form of type (m, n), which induces a decomposition of ξ_ρ into the direct sum $\xi_\rho = \xi_\rho^+ \oplus \xi_\rho^-$ of two subbundles on one of which (ξ_ρ^+) the bilinear form is positive definite, and on the other (ξ_ρ^-) negative definite. Then by definition

$$\eta_\rho = \xi_\rho^+ - \xi_\rho^-. \tag{7.22}$$

As a result we obtain the Hirzebruch formula

$$\tau_\rho(X) = 2^{2k}\langle L(X)\,\mathrm{ch}\,\varphi_X^*\eta_\rho, [X]\rangle. \tag{7.23}$$

§8. Algebraic Poincaré complexes

1. The problem of classifying smooth structures and the homology invariants of a manifold. The Hirzebruch formulae obtained in §7 for multiply-connected manifolds with the help of finite-dimensional representations of the fundamental group are, apart from the purely mathematical beauty of the relations, of deep value for a number of problems in the theory of smooth manifolds.

We mention, first of all, that the definition of the signature $\tau_\rho(X)$ of X for a representation ρ only depends on the homotopy properties of the cohomology ring $H^*(X; V)$, that is, it is a homotopy invariant of X. This means that if $\varphi\colon X_1 \to X_2$ is a continuous map of a manifold X_1 to a manifold X_2 and a homotopy equivalence or, what comes to the same thing, if φ induces an isomorphism of all homotopy groups $\varphi_*^{\pi k}\colon \pi_k(X_1) \to \pi_k(X_2)$ $(k \geqslant 1)$, then the signature of X and X_2 are equal,

$\tau_\rho(X_1) = \tau_\rho(X_2)$, for each finite-dimensional representation ρ of $\pi = \pi_1(X_1) = \pi_1(X_2)$.

Consequently, the right-hand sides of the Hirzebruch formulae, which are characteristic numbers of the manifolds and hence do not a priori depend only on the homotopy type of the manifold but also on the smooth structure introduced on it, are in fact homotopically invariant.

Research on the homotopy invariance of characteristic classes (and also of the characteristic numbers derived from them) began to be developed about 25 years ago. The first results in this direction are due to Rokhlin [36] and Thom [37], who proved the homotopy invariance of the (unique) Pontryagin class on a four-dimensional oriented manifold. The proof was essentially based on the formula they found for the signature (7.1) for four-dimensional manifolds. In 1956 Hirzebruch [38] established, properly, the general formula (7.1) for arbitrary oriented $4k$-dimensional manifolds and so proved the homotopy invariance of the Pontryagin–Hirzebruch classes $L_{4k}(X)$, $\dim X = 4k$.

Further research showed that the Pontryagin classes, generally speaking, are not homotopy invariants (G. Whitehead, Dold, Thom) and for simply-connected manifolds the highest Hirzebruch class is the only homotopically invariant Pontryagin class. The last assertion was proved by Browder [39] and S. P. Novikov [40] on the basis of a full study of the problem of classifying smooth structures of given homotopy type of simply-connected manifolds. In fact, they established a stronger assertion: in each homotopy type of simply-connected manifolds the Pontryagin class is a homotopy invariant if and only if it is proportional to the highest Hirzebruch class.

In the case of multiply-connected manifolds the answer to the problem of the homotopy invariance of the characteristic classes is more complicated, but in one way or other it has been directly tied up with progress in the solution of the other problem: that of classifying smooth structures of given homotopy type of a multiply-connected manifold.

We turn now to a precise description of the problem of classifying smooth structures of given homotopy type. Let X_0 be a finite CW-complex. A smooth structure of homotopy type X_0 consists of a smooth closed manifold X and a homotopy equivalence $\varphi\colon X \to X_0$. Two smooth structures (X_1, φ_1) and (X_2, φ_2) of the homotopy type of X_0 are said to be equivalent if there is a smooth homeomorphism

$$\psi\colon X_1 \to X_2,$$

such that the diagram

$$\begin{array}{ccc} X_1 & \xrightarrow{\varphi_1} & X_0 \\ {\scriptstyle\psi}\searrow & & \nearrow{\scriptstyle\varphi_2} \\ & X_2 & \end{array}$$

is homotopically commutative, that is, the maps φ_1 and $\varphi_2\psi$ are homotopic. Then the set of all smooth structures of homotopy type X_0 splits

into classes of pairwise equivalent structures. We denote the set of these classes by $\mathcal{S}(X_0)$.

Then the problem of classifying smooth structures of given homotopy type X_0 consists in describing the set $\mathcal{S}(X_0)$ with the help of the homotopy invariants of X_0. In particular, one of the questions consists in finding methods of distinguishing smooth structures that are effective from the topological point of view.

We mention to begin with that for the set $\mathcal{S}(X_0)$ to be non-empty it is necessary that X_0 has certain properties, namely that X_0 is a Poincaré complex of some formal dimension n. This means that there is a cycle $[X_0] \in H_n(X_0; Z)$ such that the intersection homomorphism

$$\cap [X_0]\colon H^*(X_0; \Lambda) \to H_*(X_0; \Lambda)$$

is an isomorphism, where $\Lambda = Z[\pi]$ is the group ring of the fundamental group $\pi = \pi_1(X_0)$ of X_0.

One of the invariants of a smooth structure (X, φ) of given homotopy type X_0 is the "normal bundle of the smooth structure", that is, the element $\xi = (\varphi^{-1})^*(\nu(X)) \in K_O(X_0)$. It is clear that if the normal bundles of two smooth structures are distinct, then so are the smooth structures themselves. But on the one hand, not every element of $K_O(X_0)$ can be realized as the "normal bundle of a smooth structure", while on the other hand, different smooth structures may a priori have the same "normal bundle".

For an element $\xi \in K_O(X_0)$ to be realized as a "normal bundle of a smooth structure" a number of conditions have to be satisfied. One such relation is easily written in geometric terms. Let ξ be a finite-dimensional (dim $\xi = N$) bundle over the complex X_0, and X_0^ξ the Thom complex of ξ (that is, the one-point compactification of the space of ξ). Suppose now that:

There exists a map of the $(N + n)$-dimensional sphere:

$$\chi\colon S^{N+n} \to X_0^\xi \text{ of degree } 1. \tag{8.1}$$

Condition (8.1) is one of the requirements that have to be imposed on ξ for it to be the "normal bundle of a smooth structure". For, if (X, φ) is a smooth structure, then the map induces a bundle homomorphism

$$\bar{\varphi}\colon \nu(X) \to \xi. \tag{8.2}$$

Recalling that the space of the normal bundle (for sufficiently large $N = \dim \nu(X)$) can be regarded as a domain of the sphere S^{N+n}, we can extend the map (8.2) to a map χ of the sphere S^{N+n} to the one-point compactification X_0^ξ of the space of ξ. The map χ is transverse-regular along the null section $X_0 \subset X_0^\xi$, and $\chi^{-1}(X_0) = X$.

(8.1) has another equivalent formulation when X_0 is a smooth manifold, in terms of the so-called J-functor and the Adams cohomology operations

(see [41]–[45]). It is useful for us to note only one fact: the set of elements $\xi \in K_O(X_0)$ satisfying (8.1) is a coset of a subgroup of finite index in $K_O(X_0)$.

However, (8.1) is by no means a sufficient condition for the existence of a smooth structure with a given normal bundle. The only thing that can be asserted is this: there is a manifold X and a map φ: $X \to X_0$ of degree 1 so that the inverse image $\varphi^*(\xi)$ is isomorphic to the normal bundle $\nu(X)$. But φ need not be a homotopy equivalence. Therefore, it is reasonable to ask with what modifications of X and of φ we end up with a homotopy equivalence.

Now at this point the famous technique of modifying smooth manifolds or surgery enters the stage. The manifold X, the map φ, and the isomorphism ψ: $\nu(X) \to \varphi^*(\xi)$ in the previous paragraph form a triple (X, φ, ψ), and among them we can establish a natural bordism relation. The triple (X, φ, ψ) is called the normal map of the manifold X to the pair (X_0, ξ). Using surgery we can answer the following question.

Let (X, φ, ψ) be a given normal map. When is there a triple (X', φ', ψ') bordant to it in which φ' is a homotopy equivalence? The answer was completely worked out by Novikov for simply-connected manifolds [40] and by Wall for multiply-connected manifolds [3], [4]. Namely, let $L_n(\pi)$ be the Wall group of π, defined in §1.4, and let $\pi = \pi_1(X_0)$. Then with each triple we can associate an element $\theta(X, \varphi, \psi) \in L_n(\pi)$ satisfying the following conditions:

a) $\theta(X, \varphi, \psi)$ depends only on the bordism class of the triple,

b) $\theta(X, \varphi, \psi) = 0$ if and only if there is another triple (X', φ', ψ') bordant to (X, φ, ψ) in which φ' is a homotopy equivalence.

The element $\theta(X, \varphi, \psi)$ is called the obstruction to modifying the map φ to become a homotopy equivalence. The first condition means that $\theta(X, \varphi, \psi)$ must be completely described at least up to elements of finite order by means of the characteristic classes of the map φ: $X \to X_0$. In fact, in the case of simply-connected manifolds and $n = 4k$, the group $L_{4k}(1)$ is isomorphic to the group of integers and is described by a single integral-valued parameter. Then

$$\theta(X, \varphi, \psi) = \tau(X) - \tau(X_0), \tag{8.3}$$

where $\tau(X)$ is the signature of the manifold.

Formula (8.3) and property b) of the surgery obstruction gives in the simply-connected case complete information on the homotopy invariance of the rational Pontryagin classes of the manifold. For, by the Hirzebruch formula, $\tau(X) = \langle L_{4k}(X), [X] \rangle$. Consequently, if φ: $X \to X_0$ is a homotopy equivalence of two manifolds, then $\theta(X, \varphi, \psi) = \tau(X) - \tau(X_0) = 0$. Therefore, $L_{4k}(X) = L_{4k}(X_0)$. To establish that every other Pontryagin class is not a homotopy invariant we must produce two manifolds X and X_0 and a homotopy equivalence φ: $X \to X_0$ such that $p(X) \neq \varphi^* p(X_0)$. Consequently,

in terms of the problem concerning surgery it is sufficient to produce a bundle ξ satisfying (8.1) and also such that $p(\xi) \neq p(X_0)$ and $\theta(X, \varphi, \psi) = 0$, that is, $L_{4k}(\xi) = L_{4k}(X_0)$. The latter is a problem concerning the algebraic properties of $K_O(X_0)$ and has been solved positively.

2. **Generalized signatures of multiply-connected manifolds.** We have come to the problem of describing the obstruction to surgery $\theta(X, \varphi, \psi) \in L_n(\pi)$ with the help of the characteristic classes of φ. But before we give this description, we turn our attention to the fact that in the case of simply-connected manifolds the obstruction to surgery $\theta(X, \varphi, \psi)$ can be expressed in terms of the signatures of X and Y_0,

$$\theta(X, \varphi, \psi) = \tau(X) - \tau(X_0), \tag{8.4}$$

that is, by the homotopy invariants of the two manifolds. That the signature $\tau(X)$ of X is the only homotopically invariant characteristic number follows from (8.4).

Thus, also in the multiply-connected case we must try to express the surgery obstruction $\theta(X, \varphi, \psi)$ in terms of the homotopy invariants of X and X_0. For simplicity we look for these invariants as elements of $K_n^h(\Lambda)$ for the group ring $\Lambda = \mathbf{Z}[1/2][\pi]$ (see [46]). For the changes that would have to be made for $\mathbf{Z}[\pi]$, see [7].

Thus, we consider a ring Λ with involution containing 1/2. Let (M, d) be a free chain complex over Λ, that is a free graded Λ-module M (see §1) and a homogeneous homomorphism d of degree (-1)

$$d\colon M \to M, \quad d^2 = 0.$$

Then the dual complex (M^*, d^*) also is a chain complex. Let $\beta\colon M^* \to M$ be a homogeneous homomorphism of degree n and suppose that the following conditions hold:

(8.5) a) $\beta^* = \beta$,

(8.6) b) $d\beta + \beta d^* = 0$,

c) the homomorphism β induces an isomorphism of the homology groups

$$H(\beta)\colon H^*(M^*, d^*) \to H(M, d). \tag{8.7}$$

The equality (8.5) must be understood as an identification of the modules M and $(M^*)^*$, that is, in the form $\varepsilon_M^{-1}\ \beta^* = \beta$. Then we call the triple (M, d, β) an algebraic Poincaré complex (APC) of formal dimension n.

Before we turn to the study of APC's, we mention that Hermitian forms and also automorphisms preserving a Hamiltonian form are particular cases of an APC or can be reduced to one. For let us consider a chain complex (M, d) such that $M_j = 0$ if $j \neq k$, $n = 2k$. Then the homomorphism β reduces to a map on the one term $\beta\colon M_k^* \to M_k$, while (8.7) means that β is an isomorphism. Then (8.5) on replacing the kth grading by the zero grading goes over into condition (1.17). Thus, the pair (M_k, β) is a (skew-) Hermitian form over Λ.

To interpret automorphisms in terms of APC's, we consider the APC (M, d, β), $n = 2k + 1$, such that $M_j = 0$ for $j \neq k, k+1$. Then β gives us the diagram:

$$\begin{array}{ccc} M_k & \xrightarrow{d} & M_{k+1} \\ {\scriptstyle \beta_k}\uparrow & & \uparrow{\scriptstyle \beta_{k+1}} \\ M^*_{k+1} & \xleftarrow{d^*} & M^*_k \end{array} \tag{8.8}$$

Let us forget the gradings on M. Then in place of d^* in (8.8) we must take $(-1)^k d^*$, according to (1.12). Thus, $d\beta + (-1)^k \beta d^* = 0$ in (8.8). Now (8.5) means that $(\beta_{k+1})^* = \beta_k$. If we introduce on the module $M^*_{k+1} \oplus M_{k+1}$ a (skew-) Hermitian Hamiltonian form by means of the matrix

$$\begin{pmatrix} 0 & \varepsilon \bar{M}^{-1}_{k+1} \\ (-1)^k & 0 \end{pmatrix}, \tag{8.9}$$

then the homomorphism

$$\begin{pmatrix} d^* \\ \beta_{k+1} \end{pmatrix} : M^*_k \to M^*_{k+1} \oplus M_{k+1}$$

determines a Lagrangian plane in $M^*_{k+1} \oplus M_{k+1}$ and hence a class of automorphisms preserving the form (8.9).

In the set of all APC's we can introduce a relation of bordism type. We consider a free chain complex (M, d) and a subcomplex (M_0, d) so that $M_0 \subset M$ is a direct summand. Let $\beta^* : M^* \to M$ be a homogeneous homomorphism of degree n such that

(8.10) a) $\beta^* = \beta$,

(8.11) b) $d\beta + \beta d^* \equiv 0 (\mathrm{mod}\, M_0)$,

c) the homomorphism β induces an isomorphism of the homology groups

(8.12) $H(\beta) : H(M^*, d^*) \to H(M/M_0, d)$.

Then we call $\alpha = (M, M_0, d, \beta)$ an algebraic Poincaré complex with boundary (APC with boundary), of formal dimension n.

From (8.10) and (8.11) it follows that the homomorphism $d\beta + \beta d^* : M^* \to M_0$ has a unique decomposition

$$\begin{array}{ccc} M^* & \xrightarrow{d\beta+\beta d^*} & M_0 \\ & {\scriptstyle j^*}\searrow \quad \nearrow{\scriptstyle \beta_0} & \\ & M^*_0 & \end{array}$$

where $j : M_0 \to M$ is the inclusion and the triple (M_0, d, β_0) is an APC of formal dimension $(n-1)$. From (8.12) we obtain the following diagram of exact homology sequences:

$$\begin{array}{ccccccc} H(M_0) & \to & H(M) & \to & H(M/M_0) & \to & H(M_0) \\ {\scriptstyle H(\beta_0)}\uparrow & & {\scriptstyle H(\beta)^*}\uparrow & & {\scriptstyle H(\beta)}\uparrow & & {\scriptstyle H(\beta_0)}\uparrow \\ H(M_0^*) & \to & H((M/M_0)^*) & \to & H(M^*) & \to & H(M_0^*) \end{array}$$

The triple (M_0, d, β_0) is called the boundary of the APC with boundary α and is denoted by $\partial\alpha$. If $\alpha_1 = (M_1, d_1, \beta_1)$ and $\alpha_2 = (M_2, d_2, \beta_2)$ are two APC's, then we call their sum the APC $(M_1 \oplus M_2, d_1 \oplus d_2, \beta_1 \oplus \beta_2)$. If $\alpha = (M, d, \beta)$ is an APC, then by $(-\alpha)$ we denote the APC $(M, d, -\beta)$. The operation $\alpha \to (-\alpha)$ is called change of orientation on the APC α.

Two APC's α_1 and α_2 are said to be bordant if there is an APC with boundary γ such that $\alpha_1 \oplus (-\alpha_2) = \partial\gamma$. Bordism is an equivalence relation.

The complete analogy with the theory of bordisms of smooth (or piecewise-linear) manifolds is not by chance. The fact is that with each closed oriented smooth simplicial manifold X with fundamental group $\pi = \pi_1(X)$ we can associate an APC $\sigma(X)$ over the ring $\Lambda = Z[1/2][\pi]$. To do this we take for the free chain complex (M, d) the complex of simplicial chains of X with local system of coefficients Λ, and for β the operator

$$\beta = \frac{1}{2}\{(\cap[X]) + (\cap[X])^*\}, \tag{8.13}$$

where $(\cap[X])$ is the operator of intersection with the fundamental cycle (see [47]).

The same construction can be made for a smooth oriented manifold X with boundary ∂X. Here we have

$$\partial\sigma(X) = \sigma(\partial X). \tag{8.14}$$

We must sound one note of caution. The operation $(\cap[X])$ depends, generally speaking, on the ordering of the zero-dimensional vertices of the simplicial decomposition of X. Therefore, strictly speaking, the APC $\sigma(X)$ is not uniquely determined. But it is not difficult to prove that for different orderings of the vertices of X we get bordant APC's, and (8.14) is true for coherent orderings on X and its boundary ∂X.

Thus, we find it convenient to introduce the following notation. We denote by $\Omega_n(\Lambda)$ the set of bordism classes of APC's. The sum operation for APC's induces a group structure on $\Omega_n(\Lambda)$.

Then all the above constructions and examples can be presented as several homomorphisms.

a) The group $K_n^h(\Lambda)$ is mapped homomorphically to $\Omega_n(\Lambda)$:

$$\psi\colon K_n^h(\Lambda) \to \Omega_n(\Lambda), \tag{8.15}$$

b) The groups of oriented bordisms $\Omega_n^{SO}(B\pi)$ of the classifying space $B\pi$ are mapped homomorphically to $\Omega_n(\Lambda)$:

$$\sigma\colon \Omega_n^{SO}(B\pi) \to \Omega_n(\Lambda). \tag{8.16}$$

To establish the existence of the homomorphism (8.15) corresponding to

the interpretation of Hermitian forms and automorphisms as APC's, we have to verify that to equivalent Hermitian forms and automorphisms there correspond bordant APC's.

We turn now to a description of those properties of ψ and σ that explain to us the usefulness of introducing APC's. First of all, the analogy between APC's and smooth manifolds can be taken further. In particular, the operation of elementary surgeries of smooth manifolds and of "glueing on handles" can be interpreted "on the level" of their chain complexes so that analogous elementary operations can be defined for APC's. The inclusion of a sphere in the smooth manifold is replaced by a homomorphism of a free Λ-module to the group of cycles of the chain complex. The absence of any obstruction to the construction of such homomorphisms (in contrast to the embedding of a sphere in the manifold) leads to the conclusion that Milnor's method of killing homotopy groups ([48], [49]) of smooth manifolds can be applied to "killing" the homology groups of an APC. Then every APC α is bordant to another APC α' whose homology groups are non-zero only in the middle dimension (for $n = 2k$) or the two middle dimensions (for $n = 2k + 1$).

Moreover, chain-homotopically equivalent APC's are bordant.

As a result we come to the following assertion.

THEOREM 8.1. *The homomorphism* $\psi\colon K_n^h(\Lambda) \to \Omega_\Lambda(\Lambda)$ *is an isomorphism.*

Thus, we cannot distinguish the groups $K_n^h(\Lambda)$ and $\Omega_n(\Lambda)$, that is, with each APC of even formal dimension we can associate a Hermitian form and with each APC of odd formal dimension an automorphism preserving a Hamiltonian form. We need not use Theorem 8.1 to do this, by the way, but can use the following simple argument. Let (M, d, β) be an APC, which we present in terms of the following diagram:

$$\begin{array}{ccccccc} M_0 & \xleftarrow{d} & M_1 & \xleftarrow{d} & \cdots & \xleftarrow{d} & M_n \\ \uparrow{\scriptstyle\beta} & & \uparrow{\scriptstyle\beta} & & & & \uparrow{\scriptstyle\beta} \\ M_n^* & \xleftarrow{d^*} & M_{n-1}^* & \xleftarrow{d^*} & \cdots & \xleftarrow{d^*} & M_0^* \end{array} \tag{8.17}$$

This can be regarded as a biregular complex. Then we can construct from (8.17) a new complex in which the grading is equal to the sum of the two gradings in (8.17):

$$M_0 \leftarrow M_1 \oplus M_n^* \leftarrow \cdots \leftarrow M_n \oplus M_1^* \leftarrow M_0^*, \tag{8.18}$$

and the differential is equal to the sum of the differentials (see [50]) in (8.17).

The chain complex (8.8) is "self-dual" and exact. Consequently, we can split it symmetrically with respect to its ends into the direct sum of three complexes (lopping off direct components from both ends) one of which has the form

$$0 \leftarrow N_k \leftarrow N_k^* \leftarrow 0 \tag{8.19}$$

when $n = 2k$, or the form

$$0 \leftarrow N_k \leftarrow M_{k+1} \oplus M_{k+1}^* \leftarrow N_k^* \leftarrow 0 \tag{8.20}$$

when $n = 2k + 1$. The complex (8.19) determines a Hermitian form and (8.20) a Lagrangian plane in Hamiltonian form.

Taking Theorem 8.1 into account, we may suppose that the homomorphism (8.16), in fact, maps the group of oriented bordisms $\Omega_n^{SO}(B\pi)$ to $K_n^h(\pi)$:

$$\sigma\colon \Omega_n^{SO}(B\pi) \to K_n^h(\pi). \tag{8.21}$$

We denote by $\gamma\colon L_n(\pi) \to K_n^h(\pi)$ the natural homomorphism of the Wall group $L_n(\pi)$ to $K_n^h(\pi)$. Let X_0 be a smooth manifold, $\pi = \pi_1(X_0)$, (X, φ, ψ) a normal map of X to the pair (X_0, ξ), that is, a map $\varphi\colon X \to X_0$ and a bundle isomorphism $\psi\colon \nu(X) \to \varphi^*(\xi)$. The obstruction $\theta(X, \varphi, \psi)$ to modifying the map by surgery to become a homotopy equivalence lies in $L_n(\pi)$.

THEOREM 8.2.

$$\gamma(\theta(X, \varphi, \psi)) = \sigma(X) - \sigma(X_0) \tag{8.22}$$

in $K_n^h(\pi)$.

Theorem 8.2 can be regarded as an analogue of (8.3) for simply-connected manifolds. Naturally, the element $\sigma(X)$ is called the generalized signature of the manifold. The generalized signature, just like the classical signature, is a homotopy invariant and a bordism invariant. Consequently, $\sigma(X)$ can be described with the help of the characteristic numbers of the bordisms of $B\pi$, that is, the numbers of the form $\langle p(X)\varphi_X^*(a), [X]\rangle$, where p is a Pontryagin class and $a \in H^*(B\pi)$ a cohomology class.

With the help of Theorem 8.2 we obtain an answer to the question what characteristic classes of multiply-connected manifolds are homotopy invariants.

THEOREM 8.3. *Let X be a multiply-connected manifold, $\pi = \pi_1(X)$. Then the only homotopically invariant characteristic numbers are those of the form*

$$\sigma_a(X) = \langle L(X)\varphi_X^*(a), [X]\rangle, \tag{8.23}$$

where $a \in H^*(B\pi; \mathbf{Q})$ and $L(X)$ is the Hirzebruch class of X.

The numbers (8.23) are called the higher signatures of X. For some classes of groups π it has been established that all the higher signatures are homotopy invariants. In 1965, S. P. Novikov ([51], [52]) established the homotopy invariance of the Hirzebruch class of codimension 1 and some partial results for other codimensions. There he conjectured[1] that if a

[1] See also [78].

cohomology class a is the direct product of 1-dimensional cohomology classes, then the corresponding higher signature $\sigma_a(X)$ is a homotopy invariant. In 1966, Rokhlin ([53]) proved this conjecture for two-dimensional classes, and later it was completely proved by Farrell and Hsiang [54] and by Kasparov [55].

The higher signatures $\sigma_a(X)$ are functions on bordisms

$$\sigma_a\colon\ \Omega_n^{SO}(B\pi) \to Q. \tag{8.24}$$

If $\sigma_a(X)$ for some class $a \in H^*(B\pi; \mathbf{Q})$ is a homotopy invariant, then the homomorphism (8.24) splits:

$$\begin{array}{ccc} \Omega_n^{SO}(B\pi) & \xrightarrow{\sigma_a} & Q \\ & {\scriptstyle\sigma}\searrow \quad \nearrow{\scriptstyle\delta_a} & \\ & K_n^h(\pi) & \end{array} \tag{8.25}$$

Conversely, if $\delta\colon K_n^h(\pi) \to \mathbf{Q}$ is a linear functional, then it determines a homotopically invariant characteristic number $\delta(\sigma(X))$ of X, which according to Theorem 8.3 is necessarily a higher signature $\sigma_a(X)$ for some cohomology class $a = a_\delta \in H^*(B\pi; \mathbf{Q})$. Consequently, running through all the linear functionals $\delta\colon K_n^h(\pi) \to \mathbf{Q}$, we obtain all the classes $a_\delta \in H^*(B\pi; Q)$ whose higher signatures are homotopically invariant.

§9. Hirzebruch formulae for infinite-dimensional Fredholm representations

The generalized signature $\sigma(X) \in K_n^h(\pi)$ of a multiply-connected manifold X gives a new interpretation of the multiply-connected Hirzebruch formulae (7.21). The left-hand side $\tau_\rho(X)$ of (7.21) is a homotopy invariant and, therefore, must depend only on the value of the generalized signature. A detailed analysis of the definition of $\tau_\rho(X)$ for a finite-dimensional representation ρ of the fundamental group $\pi = \pi_1(X)$ of a manifold X shows that this number can be expressed in the following way:

$$\tau_\rho(X) = \mathrm{sign}_\rho(\sigma(X)), \tag{9.1}$$

where sign_ρ is the signature of the Hermitian form defined in §4, (4.19).

Thus, the Hirzebruch formula can be written more naturally in the following form:

$$\mathrm{sign}_\rho(\sigma(X)) = 2^{2h}\langle L(X)\varphi_X^* \mathrm{ch}\,\xi_\rho,\ [X]\rangle. \tag{9.2}$$

This way of writing the Hirzebruch formula is useful to us because both the left- and the right-hand sides of it make sense for a wider class of representations: the infinite-dimensional Fredholm representations of π. For if ρ is a Fredholm representation of π, then the left-hand side of (9.2) is defined by (5.18), and the element ξ_ρ on the right-hand side is defined in

§6.2. Thus, we can state the following proposition [17]:

THEOREM 9.1. *Let X be a closed compact oriented manifold,* $\dim X = 4k$, $\pi_1(X) = \pi$, *and ρ a Fredholm representation of π. Then we have the Hirzebruch formula*

$$\operatorname{sign}_\rho(\sigma(X)) = 2^{2k}\langle L(X)\varphi_X^* \operatorname{ch}\xi_\rho, [X]\rangle, \tag{9.3}$$

where $\varphi_X: X \to B\pi$ is the classifying map for X.

If $\rho = \{\rho_Y\}$ $(y \in Y)$ is a family of Fredholm representations, then the left-hand side of (9.2) is no longer a number, but the bundle $\operatorname{sign}_\rho(\sigma(X)) \in K(Y)$, and the element ξ_ρ is a bundle with base space $B\pi \times Y$. By applying to the left-hand side the Chern character, we can rewrite the Hirzebruch formula as an equality of two cohomology classes of $H^*(Y; \mathbf{Q})$:

$$\operatorname{ch} \operatorname{sign}_\rho(\sigma(X)) = 2^{2k}\langle L(X)\varphi_X^* \operatorname{ch}\xi_\rho, [X]\rangle. \tag{9.4}$$

The proof of (9.3) for infinite-dimensional Fredholm representations does not repeat that of the Hirzebruch formula in the case of finite-dimensional representations. The latter is based on the observation that its left-hand side can be interpreted as the signature of certain cohomology groups and, consequently, can be reduced to the index of some elliptic operator. In the case of infinite-dimensional Fredholm representations this interpretation is not available to us, and we can only use the indirect definition of the Hermitian form $\sigma(X)$ for X.

We mention the highlights of the proof of (9.3) for infinite-dimensional Fredholm representations. Let X be a smooth simplicial oriented closed manifold, $\pi = \pi_1(X)$, and $\dim X = 4k$. To construct the Hermitian form $\sigma(X)$ we start from the APC of simplicial chains of X with the local system of coefficients $\Lambda = \mathbf{Z}[1/2][\pi]$. First of all, taking account of a remark in §4, we change the ring of scalars $\mathbf{Z}[1/2]$ to the field of complex numbers. Therefore, let $\Lambda = \mathbf{C}[\pi]$. Then $\sigma(X)$ is an APC of formal dimension $n = 4k$. To obtain a Hermitian form, we perform on $\sigma(X)$ a number of elementary operations of the type of Morse surgeries for smooth manifolds and factorizations by acyclic subcomplexes. The whole set of these operations can be organized as an APC α with boundary, of formal dimension $n + 1$, and the boundary $\partial\alpha$ is the direct sum of the APC $\sigma(X)$ and a Hermitian form $\sigma_0(X)$.

We consider now a Fredholm representation $\rho = (\rho_1, F, \rho_2)$ of π (and, hence, of Λ). The unitary representations ρ_1 and ρ_2 act on Hilbert spaces H_1 and H_2, respectively. By analogy with (4.4) we consider the tensor products of $\alpha(X)$, α, and $\sigma_0(X)$ with ρ_1 and ρ_2, respectively. In the end we obtain two infinite-dimensional "algebraic Poincaré complexes", say α_{ρ_1} and α_{ρ_2}, and the Fredholm operator F defines (up to compact operators) a Fredholm map $F_\alpha: \alpha_{\rho_1} \to \alpha_{\rho_2}$. Moreover, each elementary operation on APC's of the type of a Morse surgery or factorization by an

acyclic complex goes over, under the tensor product, to a pair of elementary operations compatible with the Fredholm map F_α. Therefore, it is not difficult to verify that $\operatorname{sign}_\rho(\sigma_0(X))$ can be computed with the help of APC's, by passing to the complexes $(\sigma(X))_{\rho_1}$ and $(\sigma(X))_{\rho_2}$ and using the constructions described by the diagrams (8.17)–(8.19).

The next stage consists in passing from the Hilbert "algebraic Poincaré complexes" $(\sigma(X))_{\rho_1}$ and $(\sigma(X))_{\rho_2}$ to some elliptic complex of differential operators.

Each of the unitary representations ρ_1 and ρ_2 induces infinite-dimensional bundles $\mathscr{H}_1$ and $\mathscr{H}_2$ on X, and F is a family of Fredholm operators $F_X : \mathscr{H}_1 \to \mathscr{H}_2$. Note that the bundles $\mathscr{H}_1$ and $\mathscr{H}_2$ are locally flat. We can, therefore, consider the spaces of exterior differential forms $\Omega^j(X; \mathscr{H}_1), \Omega^j(X; \mathscr{H}_2)$ with coefficients in $\mathscr{H}_1$ and $\mathscr{H}_2$.

Just as in the cases of finite-dimensional bundles, we construct involutions $\alpha_k : \Omega^j(X; \mathscr{H}_k) \to \Omega^{n-j}(X; \mathscr{H}_k)$ continuous in the appropriate Sobolev norms.

The de Rham complex of exterior differential forms $\{\Omega^j(X; \mathscr{H}_k), d\}$ can be mapped to the complex of simplicial cocycles $(\sigma(X))^*_{\rho_k}$ by integrating over simplexes. This homomorphism induces an isomorphism of cohomology groups. Then each elementary operation on $(\sigma(X))_{\rho_k}$ is matched by a corresponding operation on $\{\Omega^j(X; \mathscr{H}_k), d\}$. As the result of a finite number of such operations we obtain an isomorphism of two Hermitian forms. On the other hand, its signature can be computed as the index of an elliptic diagram, by splitting $\bigoplus_j \Omega^j(X; \mathscr{H}_k)$ into the direct sum $\Omega^+ \oplus \Omega^-$ of two eigensubspaces of α and constructing the Hirzebruch operator $(d + \delta)$: $\Omega^+ \to \Omega^-$ corresponding to this decomposition. The index of this diagram (taking into account the Fredholm map F_X) by the Atiyah–Singer formula is equal to $\langle L(X) \operatorname{ch}\xi_\rho, [X]\rangle$.

The formulae (9.3) and (9.4) provide us with plenty of homotopically invariant higher signatures, because the right-hand side of 9.3 is a higher signature for the cohomology class

$$a = \operatorname{ch}\xi_\rho. \tag{9.5}$$

In (9.4) we have on the right-hand side a whole family of higher signatures. For let $b_j \in H^*(Y; Q)$ be a basis for the cohomology groups and let $\operatorname{ch}\xi_\rho = \Sigma a_j \otimes b_j$. Then the right-hand side of (9.4) has the form

$$2^{2k} \sum_j \langle L(X)\, \varphi_X^*(a_j),\ [X]\rangle\, b_j = \sum_j \sigma_{a_j}(X)\, b_j. \tag{9.6}$$

Consequently, the higher signatures σ_{a_j} are homotopically invariant.

The Hirzebruch formulae, together with the image (6.1) of the homomorphism from the set of Fredholm representations $\mathscr{R}(\pi)$ into the group $K(B\pi)$ describe a set of higher signatures of varieties, which are homotopically

invariant.

For example, if π satisfies (5.9), then it follows from (6.13) that all the higher signatures of a manifold with fundamental group π are homotopically invariant.

Of course, the class of group π with homotopy invariance of the higher signatures for which we can obtain detailed information is much wider. For example, if π is a free product of two other groups, $\pi = \pi_1 * \pi_2$, then the classifying space $B\pi$ is homeomorphic to the connected sum of the classifying spaces, $B\pi = B\pi_1 \vee B\pi_2$. Consequently, the cohomology and the K-functor split into direct sums:

$$H^*(B\pi, *t) = H^*(B\pi_1, pt) \oplus H^*(B\pi_2, pt),$$
$$K^0(B\pi, *t) = K^0(B\pi_1, pt) \oplus K^0(B\pi_2, pt),$$

and the set of Fredholm representation $\mathscr{R}(\pi)$ contains the direct sum $\mathscr{R}(\pi_1) \oplus \mathscr{R}(\pi_2)$. Thus, the image of (6.1) for the group π contains the direct sum of the homomorphism (6.1) for π_1 and π_2. In particular, for a finitely generated free group π all the higher signatures are homotopically invariant.

There are $4k$-dimensional manifolds X involved in the Hirzebruch formulae. Therefore, the homotopy invariance of the higher signatures $\sigma_a(x)$ is also proved for $4k$-dimensional manifolds X and $4s$-dimensional cohomology classes $\in H^*(B\pi; Q)$. However, similar assertions on the homotopy invariance of the higher signatures can also be proved for other dimensions.

Suppose that π satisfies (6.9), and, for example, dim $X = 4k - 1$. We consider the direct product $X' = X \times S^1$ of X with the circle S^1. Then the group $\pi' = \pi_1(X \times S^1) = \pi \times Z$ also satisfies (6.9). Consequently, there is a family of Fredholm representations $\rho = \{\rho_y\}$ $(y \in Y)$ such that $\mathrm{ch}\xi_\rho = \Sigma\, a_i \otimes b_i$ and $a_i \in H^*(B\pi')$, $b_i \in H^*(Y)$, are bases for the cohomology groups. The space $B\pi'$ is homeomorphic to the direct product $B\pi' = B\pi \times S^1$. Therefore, the basis $\{a_i\}$ can be represented in the form $\{a_j', a_j' \otimes v\}$, $a_j' \in H^*(B\pi)$, $v \in H^1(S^1)$. The Hirzebruch formula gives us the homotopy invariance of the number $\sigma_{a_j' \otimes v}(X') = \langle L(X)\varphi_X^*(a_j' \otimes v), [X'] \rangle$. Since $L(X') = L(X)$, with $X' = X \times S^1$, it follows that

$$\sigma_{a_j'}(X) = \langle L(X)\, \varphi_X^*(a_j'),\ [X]\rangle = \langle L(X')\, \varphi_X^*(a_j' \otimes v),\ [X \times S']\rangle = \sigma_{a_j' \otimes v}(X').$$

Consequently, the higher signatures $\sigma_{a_j'}(X)$ for dim $a_j' = 4k - 1$ are homotopically invariant.

For the other dimensions the homotopy invariance of the higher signatures is established similarly.

§ 10. Other results on the application of Hermitian K-theory

1. The application of bordism theory to the Hirzebruch formulae. The Hirzebruch formulae both for finite-dimensional, and for infinite-dimensional

Fredholm representations have been proved on the basis of the Atiyah–Singer theorem on the index of elliptic operators.

In one particular case – for simply-connected manifolds – there is another way of proving the Hirzebruch formula on the basis of bordism theory. The difficulty in applying bordism theory in the case of the multiply-connected Hirzebruch formula lies in the choice and effective description of generators of the group of oriented bordisms $\Omega_*(B\pi)$ of the classifying space $B\pi$ of π. As was shown in §7, it follows from bordism theory that the right-hand side of the Hirzebruch formula always is of the form $\tau_\rho(X) = \langle L(X)\, a_\rho, [X]\rangle$, where $a_\rho \in H^*(B\pi)$ is some cohomology class depending on ρ. Consequently, the main difficulty is in proving that $a_\rho = \mathrm{ch}\xi_\rho$. In this direction there have been some definite advances ([56]), which also allow us to make clear the homotopy nature of the property of the higher signatures to be homotopy invariants of multiply-connected manifold.

By some analogy with the papers of Volodin ([68], [70]) we can construct for Hermitian K-theory a universal space $W(\Lambda)$, having a number of useful properties from the point of view of Hermitian K-theory. To do this, we consider a ring Λ with involution. In §8 we have defined an algebraic Poincaré complex over Λ and also an APC with boundary. We can go further in this direction and, by analogy with n-manifolds, define bundles of APC's over some simplicial complex K. The exact definition is as follows: let K be a simplicial complex, and denote by K also the category of all its subcomplexes and their inclusion morphisms. We consider the functor Π that associates with each subcomplex $K' \subset K$ the complex $\Pi(K')$ of free Λ-modules, with $\Pi(K' \cap K'') = \Pi(K') \cap \Pi(K'')$ and $\Pi(K' \cup K'') = \Pi(K') + \Pi(K'')$, and if $K' \subset K''$, then $\Pi(K') \subset \Pi(K'')$ as a direct summand. Next, suppose that with each oriented simplex $\sigma \subset K$ there is associated a homomorphism $\beta(\sigma)$: $\Pi(\sigma)^* \to \Pi(\sigma)$, satisfying the properties of an APC with respect to the boundary $\Pi(\partial\sigma) \subset \Pi(\sigma)$, of formal dimension $\dim \sigma$. Suppose, finally, that the homomorphisms are compatible in the following way:

$$d\beta(\sigma) + \beta(\sigma)\, d = \sum_{\substack{\sigma' \subset \sigma \\ \dim \sigma' = \dim \sigma - 1}} \beta(\sigma'), \tag{10.1}$$

where the summation is over all orientations of the boundary $\sigma' \subset \sigma$ compatible with that of σ. The functor Π and the homomorphism $\beta(\sigma)$ are called a sheaf of algebraic Poincaré complexes.

It is important to note the following assertion: let X be a simplicial smooth oriented manifold, and Π a sheaf of APC's over X. Then the pair $(\Pi(X), \beta_X)$, where

$$\beta_X = \sum_{\dim \sigma = \dim X} \beta(\sigma), \tag{10.2}$$

is an APC.

APC sheaves behave like bundles. In particular, it makes sense to speak of the inverse image of a sheaf of APC's under simplicial maps. Each simplicial complex K has a distinguished sheaf of APC's, that of its simplicial chains over the ring of integers, or if $\pi = \pi_1(K)$, also over the group ring $\Lambda = \mathbf{Z}[1/2][\pi]$ of π. Then $W(\Lambda)$ is the "universal" complex with universal sheaf Π_Λ of APC's over it.

If $f\colon X \to W(\Lambda)$ is a simplicial map of a smooth manifold X, then from the inverse image $f^*(\Pi_\Lambda)$ we construct the APC $\{f^*(\Pi_\Lambda)(X), \beta_X\}$ (see (10.2)) and according to Theorem 8.1, the element

$$\psi^{-1}(X, f) \in K_n^h(\Lambda), \qquad n = \dim X. \tag{10.3}$$

Now (10.3) depends only on the bordism class $(X, f) \in \Omega_n(W(\Lambda))$. Consequently, we obtain the homomorphism

$$\alpha\colon \Omega_n^{SO}(W(\Lambda)) \to K_n^h(\Lambda). \tag{10.4}$$

The elements of the homotopy group $\pi_n(W(\Lambda))$ determine in a natural way certain bordisms of $W(\Lambda)$, that is, there is a homomorphism $\omega\colon \pi_n(W(\Lambda)) \to \Omega_n^{SO}(W(\Lambda))$. It turns out that the composition

$$\alpha\omega\colon \pi_n(W(\Lambda)) \to K_n^h(\Lambda) \tag{10.5}$$

is an isomorphism.

As we have already shown, over the classifying spaces $B\pi$ there is the special sheaf of APC's of its simplicial chain with coefficients in the local system $\Lambda = \mathbf{Z}[1/2][\pi]$. Then we obtain the map

$$\zeta\colon B\pi \to W(\Lambda), \tag{10.6}$$

which induces a homomorphism $\zeta_*\colon \Omega_n^{SO}(B\pi) \to \Omega_n^{SO}(W(\Lambda))$. The composite map

$$\alpha\zeta_*\colon \Omega_n^{SO}(B\pi) \to K_n^h(\Lambda)$$

coincides with the generalized signature (8.21)

$$\sigma\colon \Omega_n^{SO}(B\pi) \to K_n^h(\pi).$$

All the homotopically invariant higher signatures can be written in the following way. Let $\gamma\colon K_n^h(\pi) \to \mathbf{Z}$ be a linear functional. The composite map $\alpha\gamma\colon \Omega_n^{SO}(W(\Lambda)) \to \mathbf{Z}$ is described by characteristic classes of the form

$$\alpha\gamma(X, f) = \langle L(X)f^*(a_\gamma), [X]\rangle,$$

where $a_\gamma \in H^*(W(\Lambda); \mathbf{Q})$ is some cohomology class. Then the higher signature $\alpha_b(X)$, $b = \zeta^*(a_\gamma) \in H^*(B\pi; \mathbf{Q})$ is homotopically invariant. Since the homotopy groups of $W(\Lambda)$ are periodic of period 4, then the cohomology ring $H^*(W(\Lambda); \mathbf{Q})$ is described by the generators of the cohomology group of the first four dimensions. It is convenient, in fact, to make at the outset

the change of rings $\Lambda \to \Lambda \otimes \mathbf{C}$. Then the period of the homotopy groups is two. Let $u_j \in H^1(W(\Lambda \otimes \mathbf{C}); \mathbf{Q})$ and $v_k \in H^2(W(\Lambda \otimes \mathbf{C}); \mathbf{Q})$ be bases for the cohomology groups. Then there are elements $v_{k,l} \in H^{2l}(W(\Lambda \otimes \mathbf{C}); \mathbf{Q})$ such that the cohomology ring $H^*(W(\Lambda \otimes \mathbf{C}); \mathbf{Q})$ is isomorphic to the ring of topological polynomials

$$H^*(W(\Lambda \otimes \mathbf{C}); \mathbf{Q}) = \Lambda(u_j) \otimes \mathbf{Q}[v_{k,l}].$$

From the generators $\{v_{k,l} \mid l \geqslant 1\}$ we form the Chern polynomials w_k, regarding the elements $v_{k,l}$ as elementary symmetric polynomials. Then the classes $w_k \in H^*(W(\Lambda \otimes \mathbf{C}); \mathbf{Q})$ are those whose inverse images $\zeta^*(w_k) \in H^*(B\pi; \mathbf{Q})$ give homotopically invariant higher signatures.

The Hirzebruch formulae have an elegant interpretation in terms of the bordisms of $W(\Lambda)$. Observe that under the change of rings $\Lambda \to \mathrm{Mat}(n, \Lambda) = \Lambda'$ the spaces $W(\Lambda)$ and $W(\Lambda')$ are homotopically equivalent. Therefore, a finite-dimensional representation $\rho\colon \Lambda \to \mathrm{Mat}(n, \mathbf{C})$ induces a map $\rho_*\colon W(\Lambda) \to W(\mathbf{C})$. So we obtain the following commutative diagram ($n = 4k$):

$$\begin{array}{ccc} \Omega_n(W(\Lambda)) & \xrightarrow{\alpha} & K_n^h(\Lambda) \\ \downarrow{\scriptstyle \rho_*} & & \downarrow{\scriptstyle \mathrm{sign}_\rho} \\ \Omega_n(W(\mathbf{C})) & \xrightarrow{\alpha} & K_n^h(\mathbf{C}) = \mathbf{Z} \end{array}$$

Then

$$2^{2k}\langle L(X)\, \varphi_X^* \operatorname{ch} \xi_\rho,\ [X]\rangle = \langle L(X)\, \varphi_X^* \zeta^* \rho^* a_\alpha,\ [X]\rangle,$$

that is,

$$\operatorname{ch} \xi_\rho = \zeta^* \rho^* a_\alpha, \qquad a_\alpha \in H^*(W(\mathbf{C}); Q).$$

The last equality raises the question of the existence of bundles over $W(\Lambda)$ whose Chern character determines the classes $\rho^* a_\alpha$. In fact, passing to Banach algebras, which from the point of view of representation theory, as we saw in §4, is not a restriction, we can introduce in $W(\Lambda)$ a weaker topology. To do this we consider all free Λ-modules furnished with free bases and we express all homomorphisms with the help of matrices with elements in the algebra Λ. Then we consider two APC's as being close if all their homomorphisms are close. We denote the weak topology of $W(\Lambda)$ by $\widetilde{W}(\Lambda)$. We obtain a continuous map

$$\varepsilon\colon W(\Lambda) \to \widetilde{W}(\Lambda). \tag{10.7}$$

It turns out that (10.7) is a weak homotopy equivalence. On the other hand, the zero-dimensional skeleton of $W(\Lambda)$ is no longer a discrete space in the weak topology, but is the space $F(\Lambda)$ of Hermitian forms over Λ. The inclusion

$$\varepsilon_0\colon F(\Lambda) \to \widetilde{W}(\Lambda) \tag{10.8}$$

induces an isomorphism of homotopy groups in the 2-adic localization

$$(\varepsilon_0)_*\colon\ \pi_k(F(\Lambda)) \otimes \mathbf{Z}[1/2] \xrightarrow{\approx} \pi_k(\widetilde{W}(\Lambda)) \otimes \mathbf{Z}[1/2].$$

Therefore, ignoring 2-torsion, we find that $W(\Lambda)$ is weakly homotopically equivalent to $F(\Lambda)$. The latter space can be interpreted as the classifying space $BGL(\Lambda)$ of $GL(\Lambda)$. We can also extract more precise information about the homotopy properties of (10.8). From this point of view, (10.6) is a map classifying bundles over $B\pi$ with Banach fibre Λ and with the same representation of π in its group ring Λ.

2. **Rational-homological manifolds.** The Hirzebruch formulae are true for a wider class of manifolds, although the methods of proof given here are suitable only for smooth manifolds (see [56]). Solov'ev has shown that the homomorphism (10.2) defines an APC when X is a simplicial rational-homological manifold. Consequently, all the assertions of §10.1, in particular, the Hirzebruch formulae, are true for rational-homological manifolds.

3. **Geometrical methods of computing the Wall groups.** In this direction the first work to be mentioned is apparently the paper of Browder and Levine ([57]), which is concerned with the problem of surgery of a submanifold of codimension 1. Later a series of papers by Farrell and Hsiang ([58], [59], [54]) and then by Shaneson ([5]) solved, as a matter of fact, the problem of computing the Wall groups for a free Abelian group.

The geometrical idea (as distinct from the algebraic methods considered in §§2 and 3) consists in the following: each element $\theta \in L_n(\pi)$ of the Wall group can be realized as an obstruction to surgery of a normal map to a homotopy equivalence. If π splits into the direct product $\pi = \pi' \times \mathbf{Z}$, we can take as a model manifold X with fundamental group π the direct product of X' with fundamental group π' and the circle S^1:

$$X = X' \times S^1.$$

Then each normal map

$$\text{(10.9)} \qquad \begin{cases} \varphi\colon\ Y \to X' \times S^1, \\ \psi\colon\ \nu(Y) \to \varphi^*(\xi) \end{cases}$$

can without loss of generality be regarded as being transversal-regular along $X' \subset X$. Thus, we obtain a new normal map

$$\text{(10.10)} \qquad \begin{cases} \overline{\varphi}\colon\ Y' \to X', \\ \overline{\psi}\colon\ \nu(Y') \to \overline{\varphi}^*(\xi), \\ Y' = \varphi^{-1}(X'),\ \ \overline{\varphi} = \varphi/Y'. \end{cases}$$

Realizing any element $\theta \in L_n(\pi)$ as an obstruction $\theta = \theta(Y, \varphi, \psi)$ to modifying the map (10.9) to a homotopy equivalence, we get the element

$$\text{(10.11)} \qquad q(\theta) = \theta(Y', \overline{\varphi}, \overline{\psi}) \in L_{n-1}(\pi').$$

(10.11) gives us the construction of a homomorphism (provided that it is well-defined) $q\colon\ L_n(\pi) \to L_{n-1}(\pi')$. The precise computation of the depend-

ence of $q(\theta)$ on the realization (10.9) shows that, in fact, by the given method we can map the Wall group $L_n^s(\pi)$ of obstruction to surgeries to a simple homotopy equivalence

$$q\colon L_n^s(\pi) \to L_{n-1}(\pi'). \tag{10.12}$$

Using the operation of slicing the manifold X along X' (that is, by passage to $X' \times I$), (10.12) fits into an exact sequence

$$0 \to L_n^s(\pi') \to L_n^s(\pi) \to L_{n-1}(\pi') \to 0. \tag{10.13}$$

With the help of (10.13) it has been possible to obtain full information about the Wall groups for free Abelian groups.

Further generalizations of (10.13) have been applied to various classes of amalgamated products of groups. (In this context, see the papers of Cappell [60], [61].)

Clearly, we have not been able to give a full survey of applications of K-theory, since at the present time this area is in a state of rapid development. The reader can find a more complete survey of the results of foreign authors on Hermitian K-theory up to 1972 in the collection of papers of the Seattle conference on algebraic K-theory [62].

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Received by the Editors 14 October 1974

Translated by I. R. Porteous