A procedure for killing homotopy groups of differentiable manifolds

by

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A "surgery" on a differentiable manifold $W$ of dimension $n = p + q + 1$ has the effect of removing an imbedded sphere of dimension $p$ from $W$, and replacing it by an imbedded sphere of dimension $q$ (see §1). This construction is closely related to cobordism theory (§2). The main objective of this paper (§§3-6) is to study the extent to which this construction can be used to simplify the homotopy groups of a given manifold. A typical result is the following:

**Corollary to Theorem 3.** Let $W$ be a triangulated differentiable manifold of dimension $2m$. Suppose that $W$ is compact and that its tangent bundle restricted to the $(m-1)$-skeleton is a trivial bundle. Then by performing a series of surgeries on $W$, one can obtain a manifold $W'$ which is $(m-1)$-connected.

**Conventions.** All manifolds are to be oriented and differentiable of class $C^\infty$. The letter $W$ is used for a manifold which may have boundaries. The letter $V$ is reserved for a manifold without boundaries.

I am grateful to R. Thom for describing "surgery" to me, and for pointing out the possibility of using it to kill homotopy groups.

*Added in proof.* Much of the material below has been obtained independently by A. H. Wallace [14].

1. **The construction.** Let $D^{m+1}$ denote the unit disk in the euclidean space $R^{m+1}$, with boundary $S^m$ and with center 0. The product manifold $S^m \times S^m$ can be considered either

   (1) as the boundary of $S^m \times D^{m+1},$

   or

   (2) as the boundary of $D^{m+1} \times S^m.$

   Given any imbedding of $S^m \times D^{m+1}$ in a manifold $W$ of dimension $n = p + q + 1$, a new manifold $W'$ can be formed by removing the interior of $S^m \times D^{m+1}$ and replacing it by the interior of $D^{m+1} \times S^m$. This procedure will be called surgery. To be more precise:

   **Definition.** Given a differentiable, orientation preserving imbedding

   $f : S^m \times D^{m+1} \to W$

   $1$ The author holds a Sloan fellowship.
with $p + q + 1 = n$, let $\chi(W, f)$ denote the quotient manifold obtained from the disjoint sum

$$ (W - f(S^r \times 0)) + (D^{r+1} \times S^q) $$

by identifying $f(u, \theta)$ with $(\theta u, \epsilon)$ for each $u \in S^p$, $v \in S^q$, $0 < \theta < 1$. Thus $\chi(W, f)$ is an oriented differentiable manifold. The boundary of $W$ (if any) is equal to the boundary of $\chi(W, f)$. If $W'$ denotes any manifold which is diffeomorphic to $\chi(W, f)$ under an orientation preserving diffeomorphism, then we will say that $W'$ can be obtained from $W$ by surgery of type $(p + 1, q + 1)$.

This construction clearly makes sense in the range $0 \leq p < n$ (that is the range $p, q \geq 0$). It will be convenient to extend it to the cases $p = -1$ or $p = n$ by defining

$$ D^0 = \{0\}; \quad S^{-1} = \text{the vacuous set}.$$ 

With these conventions, a surgery of type $(0, n + 1)$ replaces $W$ by the disjoint sum $W' + S^q$, while a surgery of type $(n + 1, 0)$ replaces $W + S^n$ by $W'$.

It is clear that $W$ and $W'$ play a symmetrical role in this construction. If $W' = \chi(W, f)$ is obtained from $W$ by a surgery of type $(p + 1, q + 1)$, then $W$ can be obtained from $W'$ by a surgery of type $(q + 1, p + 1)$.

Given a sequence $W_1, \cdots, W_s$ of manifolds such that each $W_i$ can be obtained from $W_{i-1}$ by a surgery we will say that $W_s$ is $\chi$-equivalent to $W_1$.

2. $\chi$-equivalence and cobordism. Consider manifolds $V$ without boundary which are compact and oriented.

**Theorem 1.** Two such manifolds are $\chi$-equivalent if and only if they belong to the same cobordism class.

**Proof.** Let $L$ denote the locus of points $(x, y)$ in $\mathbb{R}^{r+1} \times \mathbb{R}^{q+1}$ which satisfy the inequalities

$$ -1 \leq ||x||^2 - ||y||^2 \leq 1,$$

and

$$ ||x||^2 - ||y||^2 < \sinh(1) \cosh(1).$$

Thus $L$ is a differentiable manifold with two boundaries. The “upper” boundary,

$$ ||x||^2 - ||y||^2 = 1,$$

is diffeomorphic to $S^q \times (\text{Interior } D^{r+1})$ under the correspondence

$$ (u, \theta) \leftrightarrow (u \cosh \theta, v \sinh \theta), \quad 0 \leq \theta < 1.$$

The lower boundary,

$$ ||x||^2 - ||y||^2 = -1,$$

is diffeomorphic to $(\text{Interior } D^{r+1}) \times S^q$ under the correspondence

$$ (u, \theta) \leftrightarrow (u \sinh \theta, v \cosh \theta).$$

Consider the orthogonal trajectories of the surfaces $||x||^2 - ||y||^2 = \text{constant}$. 

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The trajectory which passes through the point $(x, y)$ can be parametrized in the form

$$ t \rightarrow (x, (1 + 1 - e^t) y).$$

[Thus the product $||x|| / ||y||$ is constant along any orthogonal trajectory.] If $x$ or $y$ is zero this trajectory is a straight line segment tending to the origin. For $x$ and $y$ different from zero it is a hyperbola which leads from some well defined point $(u \cosh \theta, v \sinh \theta)$ on the upper boundary of $L$ to the corresponding point $(u \sinh \theta, v \cosh \theta)$ on the lower boundary.

Now let $V$ be a differentiable manifold without boundary of dimension $n = p + q + 1$, and let $f: S^q \times D^{r+1} \rightarrow V$ be an embedding. Construct an $(n + 1)$-dimensional manifold $\omega(V, f)$ as follows. Start with the disjoint sum

$$ (V - f(S^q \times 0)) \times \mathbb{D}^1 + L.$$

For each $u \in S^p$, $v \in S^q$, $0 < \theta < 1$, and $c \in \mathbb{D}^1$ identify the point $((u, \theta) \times c)$ in the first summand with the unique point $(x, y) \in L$ such that

1. $||x||^2 - ||y||^2 = c$,
2. $(x, y)$ lies on the orthogonal trajectory which passes through the point $(u \cosh \theta, v \sinh \theta)$.

It is not difficult to see that this correspondence defines a diffeomorphism

$$ f(S^q \times (\text{Interior } D^{r+1} \times 0)) \times D^1 \leftrightarrow L \cap (D^{r+1} \times 0) \times (D^1 \times 0).$$

It follows from this that $\omega(V, f)$ is a well defined differentiable manifold.

This manifold $\omega(V, f)$ has two boundaries, corresponding to the values $c = ||x||^2 - ||y||^2 = +1$, and $-1$. The upper boundary, $c = +1$, can be identified with $V$, letting $x \in V$ correspond to

$$ ((x, 1) \times f(S^q \times 0)) \times D^1 \quad \text{for } x \times f(S^q \times 0),$$

$$ (u \cosh \theta, v \sinh \theta) \times L \quad \text{for } f(u, \theta) \times 0.$$

The lower boundary can be identified with $\chi(V, f)$; letting $x \in V - f(S^q \times 0)$ correspond to $(x, -1)$; and letting $(u, v) \times D^{r+1} \times S^q$ correspond to $(u \sinh \theta, v \cosh \theta)$.

It is clear that $\omega(V, f)$ is compact if and only if $V$ is compact. Since orientations can be chosen appropriately, this completes the proof that $\chi$-equivalent manifolds belong to the same cobordism class.

An immediate consequence is the following.

**Corollary ( Thom).** The Stiefel-Whitney numbers, Pontrjagin numbers, and the index of a compact manifold $V$ are invariant under surgery.

Now suppose that $V$ and $V'$ are compact manifolds which belong to the same cobordism class. Thus $V$ and $V'$ together bound a compact manifold $W$.

**Lemma 1.** There exists a differentiable map $g: W \rightarrow [0, 1]$ with the following three properties:

1. $g^0 = V'$,
2. $g^1 = V$.
(2) The gradient of $g$ vanishes only at isolated interior points $w_1, \ldots, w_k \in W$. Furthermore the matrix of second derivatives of $g$ at each $w_i$ is non-singular.

(3) The values $c_i = g(w_i)$ are distinct; say $0 < c_1 < \cdots < c_k < 1$.

Using a partition of unity, it is easy to construct a function $g : W \to [0, 1]$ which satisfies condition (1), and has non-zero gradient along the boundary of $W$. Hence $g_0$ has non-zero gradient along some compact neighborhood $K_0$ of the boundary.

Choose open sets $B_1, \ldots, B_k \subset W$ which are diffeomorphic to open subsets of euclidean space, and which cover the compact set $\text{Closure}(W - K_0)$. Then $g_0$ restricted to $B_i$ can be approximated by a function $g'_i$ which satisfies condition (2) throughout $B_i$. (See Morse [6, Theorem 10], or Whitney [13, Theorem 12A].)

Let $A_1, \ldots, A_k$ be an open covering of $\text{Closure}(W - K_0)$ such that $\text{Closure} A_i \subset B_i$. Let $\lambda_i : W \to [0, 1]$ be a differentiable function which takes the value of $1$ on $A_i$, and has carrier contained in $B_i$. Now define

$$g_i(w) = (1 - \lambda_i(w))g_0(w) + \lambda_i(w)g'_i(w) \quad \text{for} \quad w \in B_i,$$

$$g_i(w) = g_0(w) \quad \text{for} \quad w \notin B_i.$$ 

Then $g_i : W \to R$ will satisfy condition (2), at least throughout $\text{Closure} A_i$. Furthermore, if $g'_i$ is sufficiently close to $g_0$, then

(a) the modified function $g_i$ will still have values between $0$ and $1$; and

(b) $g_i$ restricted to $K_0$ will still satisfy condition (2).

In other words $g_i$ will satisfy condition (2) throughout the compact set $K_i = K_0 \cup \text{Closure} A_i$.

Now continue by induction, constructing functions $g_n : W \to [0, 1]$ which satisfy condition (1), and which satisfy (2) throughout the set $K_1 = K_0 \cup \text{Closure} A_1, \ldots, K_n = K_{n-1} \cup \text{Closure} A_n$.

Since $K_n$ is equal to $W$, it follows that $g_n$ satisfies condition (2) everywhere.

If $g$, on the other hand, is not a critical point of $g_i$ in a small neighborhood of one point, and is zero outside a larger neighborhood, we can separate the two values. This completes the proof of Lemma 1.

The proof of Theorem 1 continues as follows. Define $c_0 = 0, c_{k+1} = 1$. If both $c'$ and $c''$ lie between $c_i$ and $c_{i+1}$, note that the manifolds $g^{-1}(c')$ and $g^{-1}(c'')$ are diffeomorphic. This is proved by taking any Riemannian metric on $W$, and using the orthogonal trajectories of the surfaces $g^{-1}(c)$ to define the diffeomorphism. Call any one of these surfaces $V_i$. Thus $V_i$ is diffeomorphic to $V'$ and $V_i$ is diffeomorphic to $V$. We will prove that each $V_i$ can be obtained from $V$, by surgery.

Let $B(c)$ denote the open ball of radius $r$ in $R^n$. According to Morse [7], p. 172] there exists a neighborhood $U$ of $w$, in $W$, and a coordinate diffeomorphism $\phi : B(2r) \to U$ so that the function $\phi \phi^{-1}(0, \ldots, 0)$ takes the form

$$c_1 + z_2 + \cdots + z_k - z_{k+1} + \cdots + z_l,$$

for some $-1 \leq p \leq n$, and some $\varepsilon > 0$. We will abbreviate this as

$$\phi(x, y) - c_i = ||x||^2 - ||y||^2,$$

where $x \in R^p, y \in R^{n-p}, p + q = 1$. Note that $\phi(0, 0) = w_i$.

We may assume that $g^{-1}(c_i + \varepsilon) = V_i$ and that $g^{-1}(c_i - \varepsilon) = V_{i-1}$. Define an imbedding

$$f : S^i \times D^{n-i} \to V_i$$

by $f(x, y) = \phi(x \cosh \theta, y \sinh \theta)$. It will be shown that $V_{i-1}$ is diffeomorphic to $x(V_i)$. An imbedding $h : L \to W$ is defined by $h(x, y) = \phi(x \cosh \theta, y)$. Choose a Riemannian metric on $W$ which reduces to the euclidean metric

$$dx_i^2 + \cdots + dx_k^2 + dy_i^2 + \cdots + dy_l^2$$

through the closure of $h(L)$. The imbedding $h$ carries the surfaces $||x||^2 - ||y||^2 = \varepsilon$ into surfaces $g^{-1}(c_i)$. Hence it carries orthogonal trajectories into orthogonal trajectories.

Define a diffeomorphism $h : \chi(V_i, f) \to V_{i-1}$ as follows. Starting at any point $z$ of $V_i$, $f(S^i \times 0)$ the orthogonal trajectory is a non-singular curve which leads from $z$ to some well defined point $k(z)$ in $V_{i-1}$. In the case $z = f(u, 0) = h(u \cosh \theta, v \sinh \theta)$ this orthogonal trajectory leads from $z$ to the point $h(u \cosh \theta, v \cosh \theta)$ of $V_{i-1}$. Hence defining $h(\theta, 0) = \phi(u \cosh \theta, v \cosh \theta)$ for all $(\theta, 0) \in D^{n-i} \times S^i$, we obtain a well defined diffeomorphism from $\chi(V_i, f)$ to $V_{i-1}$. This completes the proof that cobordant manifolds are $x$-equivalent.

Remark. A similar proof shows that the region $g^{-1}(c_i - \varepsilon, c_i + \varepsilon)$ between $V_i$ and $V_{i-1}$ is diffeomorphic to $w(V_i, f)$.

3. Killing homotopy classes. Let $W$ be a manifold of dimension $n$ and let $\lambda : r(W)$ be a homotopy class which we wish to kill. To simplify the discussion, assume that $W$ is connected.

Definition. An imbedding $f : S^i \times D^{n-i} \to W$ represents the homotopy class $\lambda$ if $\lambda = f_*([1])$, where $[1]$ is a generator of the infinite cyclic group $\pi_1(S^i \times D^{n-i})$. [To be more precise one should specify base points for these homotopy groups, but this will be left to the reader.]

Suppose that $\lambda$ is represented by such an imbedding $f : S^i \times D^{n-i} \to W$. Suppose further that $n \geq 2p + 2$. Let $W' = \chi(W, f)$. 

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LEMMA 2. Under these conditions, the homotopy groups $\pi_*(W')$ are isomorphic to $\pi_*(W)$ for $i < p$; and $\pi_p(W')$ is isomorphic to $\pi_p(W)$ modulo a subgroup which contains $\lambda$.

Proof. As usual set $q = n - p - 1$. Note that $p < q$. Let $X$ denote the space which is formed from the topological sum $W = (D^m \times D^q) \cup (D^m \times D^q)$ by identifying $(u, y)$ with $(u, y)$ for each $(u, y) \in S \times D^q$. The subset $W' \cup (D^m \times 0)$ is clearly a deformation retract of $X$. This subset is formed from $W'$ by attaching a $(p + 1)$-cell using the map $u \rightarrow f(u, 0)$ as attaching map. It follows immediately that the inclusion homomorphism $\pi_i(W) \rightarrow \pi_i(X)$ is an isomorphism for $i < p$, and is onto for $i = p$. Furthermore the homotopy class $\lambda$ of the attaching map lies in the kernel of this homomorphism.

But $W'$ is also imbedded topologically in $X$, and a similar argument shows that the homomorphism $\pi_i(W') \rightarrow \pi_i(X)$ is an isomorphism for $i < q$. In particular it is an isomorphism for $i \leq p$. Together with the preceding paragraph, this completes the proof.

Remark. The manifold $\omega(W', f)$ could be used in place of the space $X$ in this argument.

The above argument can also be applied to the case $p = 0$. In this case the appropriate statement is:

LEMMA 2'. Let $W$ be a manifold of dimension $n \geq 2$ with $k$ components, $k \geq 2$. Let $f: S^r \times D^q \rightarrow W$ be an orientation preserving embedding which carries the two components of $S^r \times D^q$ into distinct components of $W$. Then $\chi(W, f)$ has only $k - 1$ components.

The proof will be left to the reader.*

In order to make use of Lemma 2 it is necessary to answer the following question. For which homotopy classes $\lambda \in \pi_*(W)$ does there exist an imbedding $S^r \times D^q \rightarrow W$ which represents $\lambda$?

Let $r^*$ denote the tangent bundle of $W$ and let $f^* : S^r \rightarrow W$ denote any map in the homotopy class of $\lambda$. Let $f^* r^*$ denote the induced bundle over $S^r$.

LEMMA 3. Assume that $n \geq 2p + 1$. Then there exists an imbedding $S^r \times D^q \rightarrow W$ which represents $\lambda$ if and only if the induced bundle $f^* r^*$ is trivial.

The proof of Lemma 3 will be based on the following.

LEMMA 4. Let $\xi$ be an $m$-dimensional vector space bundle over a complex $K$ of dimension $p < m$. Let $\sigma^*$ denote the trivial line bundle over $K$. Then $\xi$ is a trivial bundle if and only if the Whitney sum $\xi \oplus \sigma^*$ is trivial.

Proof. Let $B(SO_m)$ denote any classifying space for the rotation group. First observe that there exists a fibre bundle over $B(SO_m)$ with fibre $S^r$ and total space $B(SO_m)$. [This is seen as follows. The group $SO_m$ acts freely on the total space $E$ of its universal bundle, with orbit space $E/SO_m = B(SO_m)$. Hence the subgroup $SO_r$ also acts freely on $E$. Define $B(SO_r)$ as the orbit space $E/SO_r$. It follows easily that $B(SO_r)$ is a bundle over $B(SO_m)$ with fibre $SO_r/SO_m = S^r$.]

Let $j : K \rightarrow B(SO_m)$ be a classifying map for the bundle $\xi$. Then $j^* : K \rightarrow B(SO_m)$ is a classifying map for the Whitney sum $\xi \oplus \sigma^*$. If $j^*$ is null-homotopic then it follows from the covering homotopy theorem that $j$ is homotopic to a map of $K$ into the fibre $S^r$. Since $K$ has dimension less than $m$, this implies that $j$ is null-homotopic. 4

Proof of Lemma 3. If $n \geq 2p + 1$ then any homotopy class of maps from $S^r$ to $W$ contains an imbedding

$f^* : S^r \rightarrow \text{Interior } W$.

(See Whitney [11, Theorem 2].) Let $r^*$ denote the tangent bundle of $S^r$ and let $r^* \sigma$ denote its normal bundle. Then the Whitney sum $r^* \sigma^* \oplus r^* \sigma$ can be identified with $f^* r^* : \text{the tangent bundle of } W \text{ restricted to } S^r$.

If $f^* r^*$ is trivial then it follows that $r^* \sigma^*$ is trivial. To see this, let $\sigma^*$ denote the trivial $k$-dimensional vector space bundle over $S^r$. Then $\sigma^* \otimes r^*$ is known to be trivial. (Identify $\sigma^*$ with the normal bundle of $S^r$ in $B^r$.) Combining the relations

\[ \sigma^* \otimes r^* \cong r^* \sigma^* \quad \text{and} \quad \sigma^* \otimes r^* \cong \sigma^* \]

it follows that $r^* \sigma^* \cong r^* \sigma$. Together with Lemma 4 this implies that $r^* \sigma$ is trivial.

Now take a tubular neighborhood of $\partial S^r$ in $W$. This can be identified with the total space of the normal $D^r$-bundle. Hence this neighborhood is diffeomorphic to $S^r \times D^q$. The resulting imbedding

$S^r \times D^q \rightarrow W$

certainly represents the homotopy class of $f^*$. Conversely suppose that such an imbedding is given. Since $S^r \times D^q$ is parallelizable, it follows immediately that $f^* r^*$ is a trivial bundle. This completes the proof of Lemma 3.

This lemma can also be formulated as follows. Let $B(SO_r)$ denote a classifying space for the rotation group, and let

$T : W \rightarrow B(SO_r)$

be a classifying map for the tangent bundle of $W$. Then there exists an imbedding

$S^r \times D^q \rightarrow W$ which represents $\lambda$ if and only if the homomorphism

$T_* : \pi_*(W) \rightarrow \pi_*(B(SO_r))$

annihilates $\lambda$.

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* This proof is due to A. Dold.

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Combining this assertion with Lemmas 2 and 2' it is easy to prove the following.

**Theorem 2.** Let $W$ be a manifold of dimension $n \geq 2p + 1$. Then any homotopy class in $\pi_n(W)$ is represented by an imbedding $f : S^n \times D^{n-p} \to W$ such that the new manifold $\chi(W, f)$ is also a manifold.

Together with Lemmas 2 and 2' this clearly implies the following.

**Corollary.** Any compact manifold of dimension $n$ is $\chi$-equivalent to a manifold which is $[n/2 - 1]$-connected.

**Remark.** It is definitely not true that $\chi(W, f)$ is a manifold for any imbedding $f$. If this were true then any manifold in the trivial cobordism class would have to be a $\chi$-manifold. But a counter-example is provided by the disjoint sum $P_1(\mathbb{C}) + (-P_1(\mathbb{C}))$ where the minus sign stands for reversal of orientation.

**Lemma 5.** The $n + 1$ independent cross-sections $c_0, \ldots, c_n$ of $(r \oplus e)^n \to W$ can be chosen in such a way that, for each point $w = f(s) \in f(S^n)$, the first $p + 1$ of these vectors are given by

$$c_i(w) = f_*(e_i(s)), \quad i = 0, \ldots, p.$$
$c_0, \ldots, c_p$ of $S^{p+1}(\mathbb{D}^{p+1} \times 0)$. Thus the first $p + 1$ cross-sections can certainly be extended over $\mathbb{D}^{p+1} \times 0$.

But the remaining cross-sections $c_{p+1}, \ldots, c_n$ on $S^n \times 0$, are just the standard basis for the normal bundle of $\mathbb{D}^{p+1} \times 0$ in $L$. Hence these can be extended over $\mathbb{D}^{p+1} \times 0$. This extension can be carried out so that all $n + 1$ vectors remain independent.

Thus the tangent bundle of $(W, f)$, restricted to $W \cup (\mathbb{D}^{p+1} \times 0)$, is a trivial bundle. To complete the argument it is only necessary to observe that $W \cup (\mathbb{D}^{p+1} \times 0)$ is a deformation retract of $(W, f)$. This can be proved in two steps as follows.

**Step 1.** The manifold $(W, f)$ can be deformed into the subset $W \cup (\text{Closure} L)$ by a deformation which leaves $W \cup (\mathbb{D}^{p+1} \times 0)$ pointwise fixed. (The proof is not difficult.)

**Step 2.** The space $W \cup (\text{Closure} L)$ has $W \cup (\mathbb{D}^{p+1} \times 0)$ as deformation retract. In fact a retraction can be defined as follows, for $(x, y) \in \text{Closure} L$:

- $(x, y) \to (x, 0)$ if $|x| \leq 1$,
- $(x, y) \to (x, y(|x|^2 - 1)^{1/2}/|y|)$ otherwise.

This retraction is clearly homotopic to the identity.

This proves that $W \cup (\mathbb{D}^{p+1} \times 0)$ is a deformation retract of $(W, f)$, and therefore proves that $(W, f)$ is parallelizable.

To prove Theorem 2 observe that $\chi(W, f)$ is imbedded as a boundary of $\chi(W, f)$. Hence $(\mathbb{R}^2 \oplus \mathbb{S}^1)\chi(W, f)$ can be identified with the tangent bundle of $(W, f)$ restricted to $\chi(W, f)$. Since this is trivial it follows that $\chi(W, f)$ is a $\pi$-manifold. This completes the proof of Theorem 2, except for Lemma 5.

**Proof of Lemma 5.** Start with any $n + 1$ independent cross-sections $b_0, \ldots, b_n$ of $(\mathbb{R}^2 \oplus \mathbb{S}^1)\chi$.

For each $u \in S^n$ the required vectors $f_u : S^n \to f(u)$ can be expressed as linear combinations

$$f_u : S^n \to \sum_{i=0}^n z_i(u) \cdot b_i(u).$$

The correspondence $u \to (z_i(u))$ defines a map $z$ from $S^n$ to the Stiefel manifold consisting of all $(p + 1) \times (n + 1)$ matrices of rank $p + 1$. But this Stiefel manifold is $(n - p - 1)$-connected, hence $z$ is null-homotopic. Using the covering homotopy theorem it follows that $z$ can be lifted to a null-homotopic map $\tilde{z}$ which carries $S^n$ into the Stiefel manifold of all non-singular $(p + 1) \times (n + 1)$ matrices. Let $\tilde{z}_i(u)$ denote the $i$th component of $\tilde{z}(u)$. Then $\tilde{z}_i(u)$, for $0 \leq i \leq p$.

Now define new sections $c_0, \ldots, c_p$ as follows. Let $c_i(u) = \tilde{z}_i(u)$ outside of some neighborhood of $f_0(S^n)$, and let

$$c_0(u) = \sum \tilde{z}_i(u) \cdot b_i(u) \quad \text{for} \quad w = f_0(u).$$

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The null-homotopy of $f$ can be used to extend throughout the neighborhood. This completes the proof of Lemma 5 and of Theorem 2.

**5. k-parallelizable manifolds.** The next step will be to extend the above procedure so that it applies to something more general than a $\pi$-manifold.

**Definition.** A bundle $\xi$ over a space $B$ is $k$-trivial if for every compact $k$ of dimension $\leq k$ and every map $f : K \to B$, the induced bundle $f^*\xi$ is trivial. In case $B$ itself is a complex, this is equivalent to the requirement that the restriction of $\xi$ to the $k$-skeleton of $B$ should be trivial.

**Dissertation.** A manifold is $k$-parallelizable if its tangent bundle is $k$-trivial. Thus every (orientable) manifold is $1$-parallelizable. For large values of $k$, this condition implies that the manifold is actually parallelizable.

**Theorem 3.** Let $W$ be a compact manifold which is $k$-parallelizable, $1 \leq k < n$. Then $W$ is $\pi$-equivalent to a $k$-parallelizable manifold $W'$ which is $\pi$-equivalent where $\pi = \min (k, n/2 - 1)$.

**Proof.** Suppose by induction that $W$ is $(p - 1)$-connected with $1 \leq p \leq \min (k, n/2 - 1)$. Consider some triangulation of Interior $W$, together with its dual cell subdivision. Let $W'$ denote the $k$-skeleton of some of these cell subdivisions of $W$, the $k$-skeleton of the other. Then the skeleton $W'$ of $W'$ is a deformation retract of $(\text{Interior} W) - W_{n-k}$. Therefore the open manifold $(\text{Interior} W) - W_{n-k+1}$ is parallelizable, and hence is a $\pi$-manifold. Since $p \leq k$, any element of $\pi(p)$ is represented by a map of $S^n$ into the skeleton.

$$W' \subset (\text{Interior} W) - W_{n-k+1}.$$ 

Since $n \geq 2p + 1$ it follows from Theorem 2 that this homotopy class is represented by an imbedding

$$f : S^n \times D^{p+1} \to (\text{Interior} W) - W_{n-k+1}$$

such that the manifold $\chi((\text{Interior} W) - W_{n-k+1}, f)$ is again a $\pi$-manifold.

But $f$ can also be considered as an imbedding of $S^n \times D^{p+1}$ in $W$. We assert that the manifold $\chi(W, f)$ is $k$-parallelizable. The above argument shows that

$$\chi((\text{Interior} W) - W_{n-k+1}, f) = \chi((\text{Interior} W) - W_{n-k+1}, f)$$

is a $\pi$-manifold. But any map of a $k$-dimensional complex $K$ into $\chi(W, f)$ can be deformed into a map

$$g : K \to (\text{Interior} W, f) - W_{n-k+1},$$

since the set $W_{n-k+1}$ is a countable union of imbedded $k$-spheres, each having codimension $\geq k + 1$. Hence $\phi(\mathbb{R}^2 \oplus \mathbb{S}^1) = (\mathbb{R}^2 \oplus \mathbb{S}^1)$ is trivial. Using Lemma 4 and the assumption that $k < n$, this implies that $\phi(\mathbb{R}^2)$ is trivial. Therefore $\chi(W, f)$ is $k$-parallelizable.

Since $W$ is compact and $(p - 1)$-connected it follows that $\pi(k)$ is finitely
generated. Hence a finite number of iterations of this construction will kill the \( p \)th homotopy group. This completes the proof of Theorem 3.

**6. Killing the middle homotopy group.** Consider compact manifolds \( W \) of dimension \( 2m \). Using Theorem 2 or 3 it may be possible to kill the homotopy groups \( \pi_p(W) \) in dimensions \( p < m \). This section will study the possibility of killing \( \pi_m(W) \).

**Remark.** If one succeeds in obtaining a manifold \( W' \) which is \( m \)-connected, then the homotopy type of \( W' \) is almost determined by that of Boundary \( W \). The Poincaré duality theorem implies that \( H_i(W', \text{Boundary } W) = 0 \) for \( m < i < 2m - 1 \) and hence that \( H_i(W) \) is isomorphic to \( H_i(\text{Boundary } W) \) for \( m < i < 2m - 1 \). As an example, if Boundary \( W \) is vacuous, then it follows easily that \( W' \) has the homotopy type of a \( 2m \)-sphere.

In order to kill \( \pi_m(W) \), the first problem is to give a homotopy class by an imbedded sphere.

**Lemma 6.** If \( W \) is a simply connected manifold of dimension \( 2m > 4 \), then every element of \( \pi_m(W) \) is represented by an imbedding \( f_0 : S^m \to W \).

More generally, if \( V \) is a compact connected manifold of dimension \( m \), the proof will show that any homotopy class of maps \( V \to W \) is represented by an imbedding. For the special case \( W = \mathbb{R}^{2m} \) this result is due to Whitney [12]. The present proof makes use of Whitney's method. I do not know whether the assertion is true in the case \( m = 2 \).

First any map can be approximated by an immersion \( f : V \to W \) which has no self-intersections other than a finite number of double points:

\[
 f(a_0) = f(a_1), \quad f(a_2) = f(a_3), \ldots,
\]

where \( a_0, a_1, a_2, \ldots, a_n \) are distinct points of \( V \). (See Whitney, [11, Theorem 2].)

Since \( V \) is connected there exists an arc joining \( a_i \) to \( a_j \) in \( V \) which misses the points \( a_0, a_1, \ldots, a_{i-1} \). This arc projects to a simple closed curve \( C \) in \( W \) which is differentiable except for one angle.

Identify \( C \) with a simple closed curve in \( \mathbb{R}^2 \), having one angle. Since \( W \) is simply connected of dimension \( 2m \), the imbedding \( g \) of \( C \) in \( W \) can be extended to an imbedding \( g : \mathbb{R}^2 \to W \). Now a tubular neighborhood \( N \) of \( g(\mathbb{R}^2) \) in \( W \) will be diffeomorphic to \( \mathbb{R}^{2m} \).

Consider the immersion \( f^{-1}(N) \to N \). This has a double point, \( f(a_0) = f(a_1) \), where \( a_0 \) and \( a_1 \) belong to the same component of \( f^{-1}(N) \). Apply the construction described by Whitney [12, Theorems 3, 4] to this immersion. This has the effect of first adding one double point and then removing two double points. The map \( f \) is altered only in a compact subset of \( f^{-1}(N) \). Therefore the modified immersion \( f^{-1}(N) \to N \) gives rise to a new immersion \( f' : V \to W \) which has only \( k - 1 \) double points.

\footnote{Integer coefficients are to be understood.}

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Since the set \( N \subset W \) is homeomorphic to \( \mathbb{R}^{2m} \) it follows easily that \( f' \) is homotopic to \( f \). Iterating this construction \( k \) times we obtain an imbedding \( f_0 \) of \( V \) in \( W \), which completes the proof.

The next problem is to decide whether the normal bundle \( s^* \) of \( f_0(S^n) \) is trivial. This turns out to be a difficult question for \( m \) odd. Hence we will concentrate on the case \( m \) even.

**Lemma 7.** Let \( W \) be \( m \)-parallelizable of dimension \( 2m \) with \( m \) even. Let \( f_0 : S^m \to W \) be an imbedding which represents the homology class \( \mathbf{1} \in H_{m}(W) \). Then the normal bundle \( s^* \) of \( f_0(S^m) \) is trivial if and only if the intersection number \( (\beta, \beta) \) is zero.

[Here the notation \( (\alpha, \beta) \) stands for the intersection number of two homology classes \( \alpha \in H_i(W), \beta \in H_{m-i}(W) \). See Lefschetz [2].]

**Proof.** Let \( [\alpha] \in \pi_m(SO_m) \) denote the homology class which corresponds to the bundle \( s^* \). (See Steenrod [8, §18].) Just as in the proof of Lemma 3 it is seen that \( s^* \otimes s^{-1} \) is a trivial bundle and hence, by Lemma 4, that \( s^* \otimes s^{-1} \) is trivial. Therefore \( [\alpha] \) is annihilated by the homomorphism

\[
 \iota_* : \pi_{m-1}(SO_m) \to \pi_{m-1}(SO_{m+1}).
\]

Now consider the exact sequence

\[
 \pi_{m-1}(SO_{m+1}) \to \pi_{m-1}(SO_{m+1}) \to \pi_{m-1}(SO_m).
\]

The homomorphism \( \iota_* \) carries a generator of \( \pi_{m-1}(SO_{m+1}) \) to the class \( [\beta] \) which corresponds to the tangent bundle of \( S_{m+1} \). Hence \( [\beta] \) must be equal to some \( k[\mathbf{1}] \) where \( k \) is an integer.

The Euler class \( X(\xi) \) of an \( m \)-dimensional vector space bundle can be defined as the first obstruction to the existence of a non-zero cross-section. Note that the correspondence

\[
 [\xi] \to X(\xi)
\]

defines a homomorphism from \( \pi_{m-1}(SO_m) \) to the infinite cyclic group \( H^m(S^n) \).

**Proof.** Let \( X \in H^m(B(SO_m)) \) denote the universal Euler class and let \( g : S^m \to B(SO_m) \) denote a classifying map for \( \xi \). Thus \( X(\xi) = g^*X \). This is clearly an additive function of the homology class \( g \in \pi_m(B(SO_m)) \approx \pi_{m-1}(SO_m) \).

For the tangent bundle \( s^* \) of \( S^n \) the class \( X(s^*) \) is known to be twice a generator of \( H^m(S^n) \). (The hypothesis that \( m \) is even comes in here.) Since \( [\alpha] = k[\mathbf{1}] \) it follows that \( X(s^*) \) is equal to \( 2k \) times a generator.

This argument proves that the group \( \text{Image } \iota_* = \text{kernel } \iota_* \) generated by \( [\beta] \) is infinite cyclic. An element \( [\beta] \) in this group is zero if and only if the Euler class \( X(\xi) \) is zero.

But for the normal bundle \( s^* \), the class \( X(s^*) \) can be interpreted as the intersection number \( (\beta, \beta) \) multiplied by a generator of \( H^m(S^n) \). For given a normal vector field with only finitely many zeros, we can deform \( f_0(S^n) \) along these
vectors to obtain a new embedding which intersects $f_0(S^m)$ at only finitely many places. The multiplicity of each such intersection is equal to the index of the corresponding zero of the normal vector field.

Thus $e^*$ is trivial if and only if $(\beta, \beta)$ equals zero, which completes the proof.

Remark. This argument also proves that $(\beta, \beta)$ is equal to $2k$ and hence is always an even number. This fact will be important later.

Lemma 7 raises the following question: Does there exist a non-zero element $\beta$ of $H_\ast(W)$ such that the intersection number $(\beta, \beta)$ is zero?

Assume now that $W$ is compact and $(m-1)$-connected of dimension $2m$, and moreover assume that the boundary of $W$ has no homology in dimensions $m$, $m-1$. Then the Poincaré duality theorem implies that $H_\ast(W)$ is a free abelian group. (The group $H_\ast(W)$ is isomorphic to $H_\ast(W)$, Boundary $W$) and hence, by duality, to $H^{2m}(W)$. Since $H_{2m}(W) = 0$, the universal coefficient theorem implies that $H^{2m}(W)$ is free abelian.

Since $m$ is even the correspondence $\alpha \mapsto \langle \alpha, \alpha \rangle$ for $\alpha \in H_{2m}(W)$ defines a quadratic form with integer coefficients. The determinant of this quadratic form is $\pm 1$. (Compare Laudsota & Milnor [3, Lemma 1].) Our question now becomes: does the quadratic form of $W$ have a non-trivial zero?

**Lemma 8.** A quadratic form $\varphi$ with integer coefficients and with determinant $\pm 1$ has a non-trivial zero if and only if it is indefinite.

Both Lemma 8 and Lemma 9 (which follows) are immediate consequences of [3, Theorems 1, 2]. The more direct proofs given below are due to H. Sah. The basic reference for these proofs is B. Jones [4].

**Proof of Lemma 8.** If $\varphi$ is positive definite or negative definite then it clearly has only the trivial zero. Assume that $\varphi$ is indefinite of rank $r$. That is, if $r$ denotes the number of negative terms when $\varphi$ is diagonalized over the rational numbers, assume that $0 < r < r$.

Case 1. $r \geq 5$. According to [1, Corollary 27d], $\varphi$ has a non-trivial zero over the rational numbers (that is in $H_\ast(W; \mathbb{Q})$). Clearing denominators it follows that $\varphi$ has a non-trivial zero over the integers.

Case 2. $r = 3$ or 4. According to [1, Theorem 14], $\varphi$ has a non-trivial zero over the $p$-adic integers for $p$ odd. It clearly has a non-trivial zero over the real numbers. The Hasse symbol $c_\omega(\varphi)$ can be computed as on pages 38-39 of [1], and turns out to be $+1$ for $r = 1, 2$ and $-1$ for $r = 3$. The determinant of $\varphi$ is clearly equal to $(-1)^r$. Now using [1, Theorem 14] it follows that $\varphi$ has a non-trivial zero over the 2-adic numbers. Together with the Hasse-Minkowski theorem, [1, Theorem 27], this implies that $\varphi$ has a rational zero.

Case 3. $r = 2$. Then the determinant of $\varphi$ is $-1$, and the conclusion follows by an elementary argument. (See [1, Theorem 14a].) This completes the proof of Lemma 8.

The index of a quadratic form is defined as $r - 2k$. (For a diagonalized form, this is the number of positive terms minus the number of negative ones.) The index of the quadratic form $\alpha \mapsto \langle \alpha, \alpha \rangle$ of $W$ turns out to be invariant under the $x$-construction. (Compare the Corollary to Theorem 1.) Thus in order to have any hope of killing $H_\ast(W)$ we must assume that this index is zero.

Now suppose that $W$ is $m$-parallelizable. Then the quadratic form of $W$ takes on only even values. This follows from the remark after Lemma 7, together with Lemma 6 and the Hurewicz theorem. (For an alternative proof, compare [3, Lemma 3].)

The conditions which we have obtained for the quadratic form of $W$ turn out to be sufficient to characterize this form. Let $U$ denote the matrix

$$
\begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}
$$

**Lemma 9.** Consider a quadratic form over the integers with determinant $\pm 1$ and index 0, which takes on only even values. The matrix of this form, with respect to a suitable basis, is $\mathbb{Z}$-diagonal $(U, \cdots, U)$.

**Proof.** It will be convenient to use the notation $a \mapsto \langle a, a \rangle$, $a \in H$, for this quadratic form. According to Lemma 8 there exists a non-zero element $\beta$ of $H$ such that $(\beta, \beta) = 0$. We may assume that $\beta$ is indivisible. Since the determinant is $\pm 1$, it follows that there exists an element $a$ of $H$ with $(\beta, a) = 1$.

Define

$$
\beta_0 = a - \frac{1}{2}(a, a)\beta.
$$

It follows that $(\beta_0, \beta_0) = 1$ and $(\beta_0, a) = 0$. Thus the matrix $[\beta_n, \beta_n]$ for $i, j = 1, 2$ is equal to $U$.

Let $H'$ denote the set of $a$ in $H$ with $(\beta, a) = (\beta_0, a) = 0$. Since the determinant of $U$ is $\pm 1$ it follows that $H'$ splits into the direct sum of $H'$ and the free abelian group generated by $\beta$ and $\beta_0$. By induction on the rank we may choose a basis $\beta_1, \cdots, \beta$ for $H'$ which has the required form. This completes the proof of Lemma 9.

**Theorem 4.** Let $W$ be $m$-parallelizable and $(m-1)$-connected of dimension $2m$ where $m$ is even $\geq 2$. Suppose that the quadratic form of $W$ has index zero, and that the boundary of $W$ has no homology in dimensions $m, m-1$. Then $W$ is $x$-equivalent to an $m$-connected manifold.

**Proof.** According to Lemma 9 there exists a basis $\beta_0, \cdots, \beta$, for $H_\ast(W)$ so that the intersection matrix takes the form $\operatorname{diag}(U, \cdots, U)$. By Lemma 6 and the Hurewicz theorem, there exists an imbedding

$$
f_0 : S^m \hookrightarrow \text{Interior } W
$$

which represents the homology class $\beta_0$. By Lemma 7 the normal bundle of

---

* Here $\operatorname{diag}(U, \cdots, U)$ denotes the $r \times r$ matrix with $r/2$ copies of $U$ along the diagonal, and zeros elsewhere.
$f_0(S^n)$ is trivial. Hence $f_0$ can be extended to an embedding

$$f : S^n \times D^m \to W.$$  

Let $W' = x(W, f)$, $W_e = W - \text{Interior } f(S^n \times D^m)$. 

First consider the exact sequence

$$0 \to H_e(W) \to H_e(W) \to H_e(W \text{ mod } W_e) \to \cdots.$$  

By excision it is seen that $H_e(W \text{ mod } W_e)$ is infinite cyclic. A generator of this group has intersection number $\pm 1$ with the cycle $f_0(S^n)$. Thus for $a \in H_e(W)$ the image $f_0(a)$ is equal to the intersection number $\langle a, \beta_1 \rangle$ multiplied by a generator of $H_e(W \text{ mod } W_e)$. Hence $H_e(W)$ is isomorphic to the subgroup of $H_e(W)$ generated by the elements $\beta_1, \beta_2, \beta_3, \ldots, \beta_s$ with $\beta_0$ omitted. It is easily verified that $W_e$ is $(m - 1)$-connected.

Next consider the exact sequence

$$0 \to H_{m-1}(W \text{ mod } W_e) \to H_{m-1}(W) \to H_{m-1}(W \text{ mod } W_e) \to 0.$$  

In this case the group $H_{m-1}(W \text{ mod } W_e)$ is infinite cyclic. The homomorphism $u \to f_0(u, v)$

of $S^n$ in $W_e$. But this is clearly the homology class of $\beta_1$. Therefore $H_e(W)$ is a free abelian group with a basis corresponding to the elements $\beta_1, \beta_2, \ldots, \beta_s$. The manifold $W'$ is $(m - 1)$-connected.

The effect of this construction on $W$ is to replace the sphere $f_0(S^n)$ by a sphere of dimension $m - 1$. Any map of an $(m - 1)$-dimensional sphere into $W$ can be deformed so as to miss this $(m - 1)$-sphere, and hence can be deformed into the subspace $W_e$. This implies:

(1) that $W'$ is also $(m - 1)$-parallelizable and

(2) that the intersection number of any two elements of $H_e(W)$ is equal to the intersection number of two corresponding elements of $H_e(W)$. Thus the quadric form of $W'$ has matrix diag $(U, \ldots, U)$, with rank $r - 2$.

Now iterate this construction $r/2$ times. The resulting manifold $W''$ will still be $(m - 1)$-connected and the group $H_e(W'')$ will be zero. Together with the Hurwitz theorem, this completes the proof of Theorem 4.

Remark. If the hypothesis that $W$ has index zero is replaced by the weaker hypothesis that the quadric form of $W$ is indefinite, then it is at least possible to reduce the rank of $H_e(W)$. The above argument shows that the intersection matrix of $H_e(W)$ has the form diag $(U, X)$, for a suitable choice of basis. Applying surgery one obtains a manifold $W'$ with intersection matrix $X$.

In conclusion we consider the corresponding problem for $m$ odd. Then the quadric form of $W$ is replaced by a skew symmetric bilinear form. The following result is well known. (See Veblen [9, p. 133].) Let $U'$ denote the matrix

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$  

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Assertion. Every skew symmetric bilinear form with determinant $\pm 1$ has matrix diag $(U', \ldots, U')$, for a suitable choice of basis.

This takes the place of Lemma 9, and is easier to prove. Unfortunately, however, Lemma 7 is not so easy to replace, except in two special cases.

If $m$ is equal to $3$ or $7$ then the group $\pi_{m-1}(SO_n)$ is zero. Hence any $m$-sphere in a $2m$-dimensional manifold has trivial normal bundle. This makes it possible to prove the following analog of Theorem 4.

Theorem 4'. If $W$ is compact and $(m - 1)$-connected of dimension $2m$, with $m = 3$ or $7$, and if

$$H_{m-1}(\text{Boundary } W) = H_{m-1}(\text{Boundary } W) = 0,$$

then $W$ is $\pi$-equivalent to an $m$-connected manifold.

The proof goes just as before, with the assertion concerning skew symmetric forms in place of Lemma 9.

For $m$ odd but unequal to $1, 3, 7$ the situation is more difficult, and will be considered in a later paper.

References


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