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LINK GROUPS

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1. Summary

By a link homotopy is meant a deformation of one link onto another, during which each component of the link is allowed to cross itself, but such that no two components are allowed to intersect. The purpose of this paper is to study links under the relation of homotopy. The fundamental tool in this study will be the link group. The link group of a link is a factor group of the fundamental group of its complement, which is invariant under homotopy.

To each conjugate class of elements in the link group of a link with n components there corresponds a link with $n + 1$ components which is defined up to homotopy. A study of this group therefore, not only gives a method for distinguishing between links which are not homotopic, but also gives a procedure for obtaining a (possibly redundant) list of all homotopy classes of links with a given number of components. By means of the link group, an effective procedure is given for deciding whether or not a given link is trivial—that is homotopic to a collection of "unlinked" circles. A complete homotopy classification is given for links with three components in euclidean space, and for links such that every proper sublink is trivial.

I am indebted to R. H. Fox for assistance in the preparation of this paper.

2. The basic theorems

Let M be an open 3-dimensional manifold which possesses a regular triangulation. Let C be a circle. By an *n-link* L will be meant an ordered collection $(l_1, \cdots, l_n)$ of maps $l_i: C \to M$, where the images $l_1(C), \cdots, l_n(C)$ are to be disjoint. A link will be called *proper* if the maps $l_1, \cdots, l_n$ are all homeomorphisms.

Two links L and L' will be called *homotopic* if there exist homotopies h_{it} between the maps l_i and the maps l'_i so that the sets $h_{1t}(C), \cdots, h_{nt}(C)$ are disjoint for each value of t. Clearly the relation of homotopy is reflexive, symmetric and transitive.

Denote the image $l_1(C) \cup \cdots \cup l_n(C)$ of a link L by $| L |$. For each proper link L let $G(L)$ denote the fundamental group of the complement $M - | L |$. Let $L^i = (l_1, \cdots, l_{i-1}, l_{i+1}, \cdots, l_n)$ denote the $(n - 1)$-sublink obtained by removing the i^{th} component. To each such sublink there corresponds a natural inclusion homomorphism $G(L) \to G(L^i)$. Denote the kernel of this homomorphism by $A_i(L)$, and denote the commutator subgroup of A_i by $[A_i]$. Since the $[A_i]$ are normal subgroups of $G(L)$, their product $E = [A_1][A_2] \cdots [A_n]$ is also a normal subgroup. By the *link group* $\mathcal{G}(L)$ will be meant the factor group[1] $G(L)/E(L)$.

For example for a link with one component the subgroup E of G is just the commutator subgroup $[A]$ of the kernel of the natural homomorphism of G into the fundamental group of M. Thus in this case $\mathcal{G} = G/E = G/[A]$.

(REMARK. The following alternative definition of the link group is more intuitive, although less practical for computation. Consider the set of all closed loops in $M - | L |$ with base point x_0. A multiplication between loops is defined in the usual manner. Define two loops f and g to be equivalent, if the $(n + 1)$-link $(l_1, \cdots, l_n, fg^{-1})$ is homotopic to a link $(l'_1, \cdots, l'_n, x_0)$, for which the last component consists of a single point. The equivalence classes of loops now build a group. It will follow from Corollary 1 to Theorem 3 that this group is isomorphic to $\mathcal{G}$).

For the proof that $\mathcal{G}$ is invariant under homotopy, three lemmas will be needed. A homeomorphism of a one-dimensional complex C into M will be called *polygonal* if, for some subdivision of C, each simplex is mapped linearly into a simplex of M.

LEMMA 1. *Any map f of a one-dimensional complex C into M can be approximated arbitrarily closely by a polygonal homeomorphism of C into M, which is homotopic to f.*

Thus in particular, every link can be approximated arbitrarily closely by a proper link. The proof is easily given.

The next lemma is closely related to the Lefschetz duality theorem, and is proved in the same manner.

LEMMA 2. *Let U be an orientable n-dimensional manifold which possesses a regular triangulation; and let $U \cup \infty$ denote the one point compactification of U. Let X be a subcomplex of U, and let V be any closed subset which is disjoint from X. Then the singular homology group $H_r(U - V, X)$ is naturally isomorphic to the Čech-Dowker cohomology group $H^{n-r}(U - X \cup \infty, V \cup \infty)$.*

Let K denote an arbitrarily fine simplicial complex for U and let K^* denote the dual cell complex. Let V' be the smallest subcomplex of K^* which contains V, and let V'' be the open star neighborhood of V' with respect to K. Let X' be the open star neighborhood of X with respect to K^*. By a standard combinatorial argument it follows that $H_r(U - V'', X)$ is isomorphic to $H^{n-r}(U - X' \cup \infty, V' \cup \infty)$. Furthermore the pair $(U - X' \cup \infty, V' \cup \infty)$ is a deformation retract of $(U - X \cup \infty, V' \cup \infty)$.

Now pass to the limit, as the mesh of the complex K becomes arbitrarily fine. Since the neighborhoods V'' converge to the closed set V, it follows that the singular group $H_r(U - V, X)$ is the direct limit of the groups $H_r(U - V'', X)$. Thus in order to complete the proof it is only necessary to show that $H^{n-r}(U - X \cup \infty, V \cup \infty)$ is the direct limit of the corresponding groups for the neighborhoods V'. Since the $V' \cup \infty$ are compact spaces with intersection $V \cup \infty$, the Cech group $H^s(V \cup \infty)$ is the direct limit of the groups $H^s(V' \cup \infty)$.

[1] For the case $n = 1$, this factor group has been studied by R. H. Fox (see [2] in bibliography).

Now using the exact cohomology sequences of the pairs $(U - X \cup \infty, V \cup \infty)$ and $(U - X \cup \infty, V' \cup \infty)$, it follows that $H^{n-r}(U - X \cup \infty, V \cup \infty)$ is the direct limit of the groups $H^{n-r}(U - X \cup \infty, V' \cup \infty)$; which completes the proof.

LEMMA 3. *Let Y and Z be topological spaces, and let h_0 and h_1 be homeomorphisms of Y into Z. A homotopy between h_0 and h_1 induces an isomorphism between $H^r(Z, h_0(Y))$ and $H^r(Z, h_1(Y))$.*

The proof will be valid for any cohomology theory. Let Q_0 and Q_1 denote the mapping cylinders of h_0 and h_1. A natural isomorphism between $H^r(Z, h_i(Y))$ and $H^r(Q_i, Y)$ will be constructed for $i = 0, 1$. Since a homotopy between h_0 and h_1 induces a homotopy equivalence between the pairs (Q_0, Y) and (Q_1, Y), this will complete the proof.

The mapping cylinder Q_i is defined as the identification space of $Y \times [0, 1] \cup Z$ in which $(y, 1)$ is identified with $h_i(y)$ for each $y \in Y$. The spaces Y $(= Y \times [0])$ and Z can be considered as subspaces of Q_i. Let P_i denote the image of $Y \times [0, 1]$ in Q_i. The inclusion map $(Z, h_i(Y)) \rightarrow (Q_i, P_i)$ is clearly a homotopy equivalence. Since h_i is a homeomorphism, the inclusion map $Y \rightarrow P_i$ is also a homotopy equivalence. Using the exact sequences of the pairs (Q_i, Y) and (Q_i, P_i) it follows that the inclusion map $(Q_i, Y) \rightarrow (Q_i, P_i)$ induces isomorphisms of the cohomology groups. The maps $(Z, h_i(Y)) \rightarrow (Q_i, P_i) \leftarrow (Q_i, Y)$ now induce the required isomorphism between $H^r(Z, h_i(Y))$ and $H^r(Q_i, Y)$; which completes the proof.

THEOREM 1. *If two proper links are homotopic, then their link groups are isomorphic.*

It will be assumed that there is a fixed base point x_0 in M which is not on the path of the homotopy. Clearly any given homotopy can be approximated by one which permits such a point x_0.

CASE 1. Links with one component.

For a link L with one component, the subgroup $A/[A]$ of the link group $G/[A]$ may be described as follows. Let U denote the universal covering space of M, and let V denote the inverse image of $| L |$ in U. Then $U - V$ is a covering space of $M - | L |$, and its fundamental group equals the subgroup A of the fundamental group G of $M - | L |$. Therefore the abelianized group $A/[A]$ is isomorphic to the singular homology group $H_1(U - V)$ with integer coefficients.

The full link group $G/[A]$ may be described by a modification of this procedure. Let X denote the inverse image of x_0 in U, and let x_0' denote a base point in X. We will next define a homomorphism η of G onto a certain group associated with the singular homology group $H_1(U - V, X)$. Each element g of G is represented by a closed loop in $M - | L |$ with base point x_0. Such a loop is covered by a path in $U - V$ which starts at x_0' and ends at some point x of X. This path represents an element $\eta(g)$ of $H_1(U - V, X)$.

The image $\eta(G)$ clearly consists of all elements λ of $H_1(U - V, X)$ such that the boundary of λ in $H_0(X)$ has the form $x - x_0'$. A group operation in $\eta(G)$ is defined as follows. To each λ_1 with $\partial\lambda_1 = x_1 - x_0'$ there corresponds a unique covering transformation ϕ_1 of $U - V$ over $M - L$ which carried x_0' into x_1. Define the product of two such elements by

$$\lambda_1 \cdot \lambda_2 = \lambda_1 + \phi_1(\lambda_2).$$

Under this group operation the map η becomes a homomorphism, and it is easily verified that the kernel of η is $[A]$. Note that the group of all covering transformations ϕ is just the fundamental group F of M. Thus η defines an isomorphism of the link group $G/[A]$ onto a certain group $\eta(G)$ which is defined in terms of $H_1(U - V, X)$, together with the boundary homomorphism $H_1(U - V, X) \to H_0(X)$ and the operations of F on these two groups.

By Lemma 2, $H_1(U - V, X)$ is isomorphic to the Čech-Dowker group $H^2(U - X \cup \infty, V \cup \infty)$. By Lemma 3, this group is invariant under homotopies of $V \cup \infty$. But a homotopy of L induces a homotopy of $V \cup \infty$. Certainly the boundary homomorphism $H_1(U - V, X) \to H_0(X)$ and the operations of F on these two groups are also invariant under homotopies of L. Therefore $G/[A]$ is invariant under homotopies of L.

CASE 2. L has more than one component, but only the i^{th} component is moved by the homotopy.

Applying Case 1 to the link l_i in the manifold $M - | L^i |$ it follows that $G/[A_i]$ is invariant under homotopy of l_i. Furthermore it is easily verified that the kernel $A_j[A_i]/[A_i]$ of the natural homomorphism $G(L)/[A_i(L)] \to G(L^j)/[A_i(L^j)]$ is invariant, for each $j \neq i$. The commutator subgroup of this kernel is

$$[A_j[A_i]/[A_i]] = [A_j][A_i]/[A_i].$$

The product over all $j \neq i$ of the groups $[A_j][A_i]/[A_i]$ is just $E/[A_i]$. Therefore the factor group $\mathcal{G} = G/E$ is invariant under homotopy of l_i.

CASE 3. An arbitrary homotopy.

First suppose that, for some subdivision $0 = t_0 < t_1 < \cdots < t_k = 1$ of $[0, 1]$, the homotopy h_t satisfies the following two conditions:

(1) Only one component of the link is moved during each interval $[t_j, t_{j+1}]$.

(2) The links $h_{t_0}, h_{t_1}, \cdots, h_{t_k}$ are proper.

Then the desired isomorphism is obtained by repeated application of Case 2. But, with the use of Lemma 1, any homotopy can be approximated by one which satisfies these conditions. This completes the proof of Theorem 1.

REMARK. The isomorphism which is obtained does not actually depend on the approximation which is chosen. However the proof of this assertion is somewhat complicated, and will not be given here.

The meridians to a proper link L are elements of the link group of L, defined as follows. It will be assumed that M is an orientable manifold, and that fixed orientations have been chosen for M and C.

Let $p(t)$ be a path leading from the base point x_0 to a point $p(1)$ of $l_i(C)$, but not touching L for $t < 1$. Choose a small neighborhood N of $p(1)$; and form a closed loop in $M - | L |$ as follows. Traverse the path p from x_0 to a point in

N; then traverse a closed loop in N which has linking number $+1$ with $l_i(C) \cap N$ considered as a cycle of $(N, \dot{N})$; and return to x_0 along p. This defines an element α_i of $\mathcal{G}(L)$ which will be called the i^{th} *meridian* of L with respect to the path p. Making use of the isomorphism $A_i(L)/[A_i] \approx H_1(U - V)$ of Theorem 1, it is easy to see that α_i is unique and well-defined for all sufficiently small neighborhoods N.

Let $\mathfrak{A}_i$ denote the kernel of the natural homomorphism $\mathcal{G}(L) \rightarrow \mathcal{G}(L^i)$. An element $\bar{\beta}_i$ of $\mathcal{G}(L^i)$ is defined as follows. Traverse the path p from x_0 to $p(1)$; then traverse $l_i(C)$ in the positive direction; and return to x_0 along p. This element $\bar{\beta}_i$ corresponds to a coset $\beta_i\mathfrak{A}_i$ in $\mathcal{G}(L)$. This coset will be called the i^{th} *parallel* of L with respect to p.

If the path p is replaced by some other path, then the pair $(\alpha_i, \beta_i\mathfrak{A}_i)$ is clearly replaced by some conjugate pair $(\lambda\alpha_i\lambda^{-1}, \lambda\beta_i\lambda^{-1}\mathfrak{A}_i)$.

THEOREM 2. *The isomorphism of Theorem* 1 *preserves the pairs* $(\alpha_1, \beta_1\mathfrak{A}_1), \cdots, (\alpha_n, \beta_n\mathfrak{A}_n)$ *up to conjugations.*

More precisely: if the pair $(\alpha_i, \beta_i\mathfrak{A}_i(L))$ in $\mathcal{G}(L)$ corresponds to $(\alpha_i', \beta_i'\mathfrak{A}_i(L'))$ in $\mathcal{G}(L')$ under the isomorphism, and if $(\alpha_i'', \beta_i''\mathfrak{A}_i(L'))$ is a meridian and parallel pair in $\mathcal{G}(L')$, then $\alpha_i'' = \lambda\alpha_i'\lambda^{-1}$ and $\beta_i''\mathfrak{A}_i(L') = \lambda\beta_i'\lambda^{-1}\mathfrak{A}_i(L')$ for some λ in $\mathcal{G}(L')$. The proof is easily given: it is only necessary to check this assertion through each stage of the proof of Theorem 1.

Nearly all of the above considerations can still be carried through if the manifold M is non-orientable. The only change is that it is no longer possible to distinguish between α_i and α_i^{-1}.

If L is a polygonal link, then meridians can be defined not only in the link group, but also in the fundamental group $G(L)$. (A closed loop in $M - |L|$ is defined just as before. However in this case it represents a well-defined element of $G(L)$.)

LEMMA 4. *Let L be a polygonal link, and let $a_i \in G(L)$ be a meridian to the i^{th} component. Then the normal subgroup generated by a_i is A_i. For an arbitrary proper link L, the normal subgroup generated by a meridian $\alpha_i \in \mathcal{G}(L)$ is $\mathfrak{A}_i$.*

Let N be a smooth tubular neighborhood of $l_i(C)$. The group $G(L)$ can be persented as the free product of the fundamental groups of $N - l_i(C)$ and $M - N - |L|$, with relations corresponding to the boundary of N. If the component l_i is removed, the only change is that the relation $a_i = 1$ is added in the group of $N - l_i(C)$. Therefore the kernel of the homomorphism $G(L) \rightarrow G(L^i)$ is the normal subgroup generated by a_i. The corresponding assertion for the link group of an arbitrary link follows immediately, by passing to a polygonal approximation.

THEOREM 3. *Let L be a proper n-link, and let f and f' be closed loops in $M - |L|$ with base point x_0. If f and f' represent conjugate elements of the link group $\mathcal{G}(L)$, then the $(n + 1)$-links (L, f) and (L, f') are homotopic.*

It is sufficient to consider the special case in which f and f' represent the same element of $\mathcal{G}$. For if hfh^{-1} is any loop conjugate to f, then the links (L, f) and (L, hfh^{-1}) are clearly homotopic.

For any closed loop f, and any representative g of $[A_i]$, we will show that the $(n + 1)$-link (L, fg) is homotopic to (L, f). It will follow by induction that $(L, fg_1 \cdots g_n)$ is homotopic to (L, f), where $g_1 \cdots g_n$ represents any element of $E(L)$. Thus loops which represent the same element of $\mathcal{G}$ give rise to homotopic $(n + 1)$-links. This will complete the proof.

Approximate the maps $l_1, \cdots, l_n$ by polygonal maps $l'_1, \cdots, l'_n$. This can be done so that the images $l'_1(C), \cdots, l'_n(C)$ and $f(C) \cup g(C)$ remain disjoint, and so that g represents an element of $[A_i(L')]$.

Let $a_i \in G(L')$ be a meridian to the i^{th} component of L'. By Lemma 4 every element of $A_i(L')$ can be written as a product of conjugates of a_i and a_i^{-1}. In other words every element of $A_i(L')$ can be written as a product of meridians and their inverses. Therefore the representative g of $[A_i(L')]$ is homotopic to a loop $(h_1 h_2 h_1^{-1} h_2^{-1}) \cdots (h_{r-1} h_r h_{r-1}^{-1} h_r^{-1})$, where the loops $h_1, \cdots, h_r$ represent meridians to the i^{th} component of L', corresponding to paths $p_1, \cdots, p_r$, or the inverses of such meridians. By Lemma 1 it may be assumed that the paths p_j are polygonal, and that the images $p_j([0, 1])$ and $p_k([0, 1])$, $j \neq k$, have only the base point in common.

Choose a 3-cell neighborhood of the set $p_1([0, 1]) \cup p_2([0, 1])$. In this neighborhood the configuration shown in Figure 1 will occur. (The heavy lines represent portions of $l'_i(C)$). Deforming $h_1 h_2 h_1^{-1} h_2^{-1}$ and $l'_i(C)$ as indicated in Figure 2, the loop $h_1 h_2 h_1^{-1} h_2^{-1}$ can be reduced to a point. It follows by induction that $(L', f h_1 h_2 h_1^{-1} h_2^{-1} \cdots h_{r-1} h_r h_{r-1}^{-1} h_r^{-1})$ is homotopic to (L', f). The proof is now completed by combining the homotopies

$$(L, fg) \sim (L', fg) \sim (L', fh_1 h_2 \cdots h_r^{-1}) \sim (L', f) \sim (L, f).$$

A link will be called *i-trivial* if, for some homotopic link $(l'_i, \cdots, l'_n)$, the set $l'_i(C)$ consists of a single point. Combining Theorems 1, 2 and 3, the following result is obtained.

COROLLARY 1. *Let* $L = (l_1, \cdots, l_n)$ *be a proper link. A closed loop* f *in* $M - |L|$ *represents the identity element of* $\mathcal{G}(L)$ *if and only if the link* $(l_1, \cdots, l_n, f)$ *is* $(n + 1)$-*trivial.*

This can also be stated as follows.

COROLLARY 2. *A link is i-trivial if and only if its* i^{th} *parallel* $\beta_i \mathfrak{A}_i$ *is equal to* $\mathfrak{A}_i$.

The proofs are evident.

3. An example

In order to give a concrete geometrical illustration of the preceding theory, the following theorem will be proved. Results in this section will not be used in the following sections.

A link is *trivial* if it is homotopic to some $(l'_1, \cdots, l'_n)$ where the components $l'_1(C), \cdots, l'_n(C)$ consist of single points. By a *knot* is meant a proper 1-*link*.

THEOREM 4. *If a polygonal knot in euclidean space is replaced by a collection of parallel knots lying within a tubular neighborhood of the original, and having linking numbers zero, then the resulting link is trivial.*

It is first necessary to make some remarks about the lower central series of a group. For any group G, the first lower central subgroup G_1 is the group itself. The n^{th} lower central subgroup G_n is generated by all commutators $aba^{-1}b^{-1}$ with $a \in G$ and $b \in G_{n-1}$.

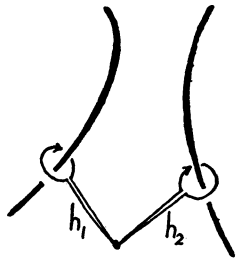

Fig 1

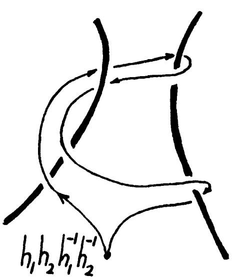

Fig 2a.

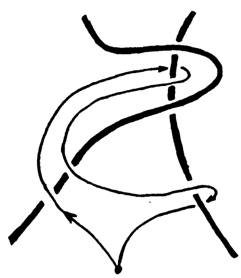

Fig 2b.

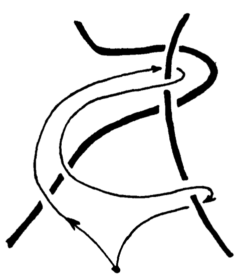

Fig 2c.

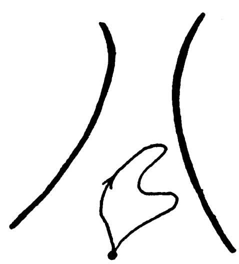

Fig 2d

LEMMA 5. *If L is a proper n-link in euclidean space, then the $(n + 1)$-st lower central subgroup $\mathcal{G}_{n+1}$ of $\mathcal{G}(L)$ contains only the identity element.*[2]

This is clear for the case $n = 0$. (The fundamental group of euclidean space contains only the identity element.) If it has been proved for the case $n - 1$,

[2] Hence $\mathcal{G}(L)$ can be considered as a factor group of $G(L)/G_{n+1}(L)$. This suggests some connection with the work of K. Chen [1] who has shown that, for a polygonal link L, the factor groups $G(L)/G_r(L)$ are invariants of the *isotopy* class of L for arbitrary values of the integer r.

then it follows that the n^{th} lower central subgroup of $\mathcal{G}(L^i) \approx \mathcal{G}(L)/\mathcal{A}_i(L)$ contains only the identity element, for each i. In other words the subgroup $\mathcal{G}_n$ of $\mathcal{G}(L)$ is contained in each $\mathcal{A}_i$. By the definition of the link group, the subgroups $\mathcal{A}_i$ are commutative. Since the groups $\mathcal{A}_1, \cdots, \mathcal{A}_n$ generate $\mathcal{G}$, it follows that elements of $\mathcal{G}_n$ commute with all elements of $\mathcal{G}$, which proves that $\mathcal{G}_{n+1} = 1$.

Let T denote the fundamental group of the complement $M - N$ of the tubular neighborhood. By the Alexander duality theorem, the abelianized group $T/T_2 \approx H_1(M - N)$ is infinite cyclic. It follows that $T_r = T_2$ for $r \geqq 2$. For if $a_0 \in T$ generates T/T_2, then every element of T can be written in the form $a_0^j\, b$ with $b \in T_2$. Modulo the subgroup T_3, all elements of T commute with elements of T_2. Since powers of a_0 commute with each other, it follows that T/T_3 is commutative. But this implies that $T_3 \supset T_2$, hence $T_2 = T_3 = T_4 = \cdots$.

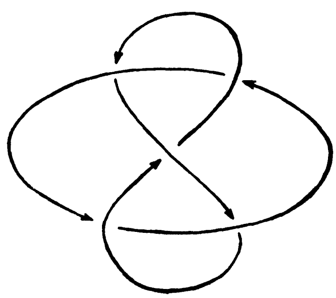

Fig 3

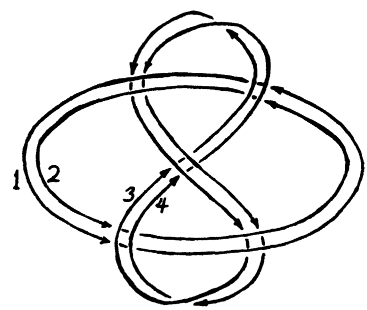

Fig 4

Consider the natural homomorphisms $T \to G(L) \to \mathcal{G}(L)$. The subgroup $T_2 = T_{n+1}$ of T is mapped into the subgroup $\mathcal{G}_{n+1} = 1$ of $\mathcal{G}$. This means that a closed loop in $M - N$ represents the identity element of $\mathcal{G}(L)$ whenever it represents the identity element of $T/T_2 \approx H_1(M - N)$: that is whenever it has linking number zero with N.

The parallel $\beta_i\mathcal{A}_i$ to any component of L can clearly be represented by a loop in $M - N$ having linking number zero with N. This means that $\beta_i\mathcal{A}_i$ equals $\mathcal{A}_i$, and hence that the link L is i-trivial. Since this is true for all values of i, it follows easily that L is trivial.

To conclude this section, a counter-example to a possible generalization of Theorem 4 will be given. The 2-link illustrated in Figure 3 is clearly trivial. If each component of this link is replaced by two parallel components, having linking numbers zero, then the resulting 4-link, illustrated in Figure 4, is not trivial. This can be proved by the techniques of Section 4. (Compare also Section 5.)

4. Trivial links

Consider the following two problems.

(I) To give a general procedure for deciding whether any given link is trivial.

(II) To solve the word problem for the link groups of all trivial links.

It follows from Corollary 1 that these problems are equivalent: Suppose that problem (I) has been solved; and Let L be any trivial proper link. Then a loop f in $M - |L|$ represents the identity element of $\mathcal{G}(L)$ if and only if the link (L, f) is trivial.

Now suppose that (II) has been solved, and suppose by induction that (I) has been solved for links with $n - 1$ components. Then a proper link $(l_1, \cdots, l_n)$ is trivial if and only if the sublink $(l_1, \cdots, l_{n-1})$ is trivial, and the loop l_n represents the identity element of $\mathcal{G}(l_1, \cdots, l_{n-1})$.

It will be shown in this section that problem (II) can be solved whenever the word problem for the fundamental group F of M can be solved.

For any proper link L let $JG(L)$ denote the integral group ring of $G(L)$. Let $K_i(L)$ denote the kernel of the natural homomorphism $JG(L) \to JG(L^i)$. Similarly let $\mathcal{K}_i(L)$ denote the kernel of the homomorphism $J\mathcal{G}(L) \to J\mathcal{G}(L^i)$. The two-sided ideals K_i^2 and $\mathcal{K}_i^2$ will be of particular interest.

LEMMA 6. *The natural homomorphism* $JG/K_1^2 + \cdots + K_n^2 \to J\mathcal{G}/\mathcal{K}_1^2 + \cdots + \mathcal{K}_n^2$ *is an isomorphism.*

The rings $JG/K_1^2 + \cdots + K_n^2 \approx J\mathcal{G}/\mathcal{K}_1^2 + \cdots + \mathcal{K}_n^2$ will be denoted by $\mathcal{R}(L)$.

It is evidently sufficient to prove that the subset E of JG is congruent to the identity modulo $K_1^2 + \cdots + K_n^2$. If a and a' are elements of A_i, then $aa' - a - a' + 1 = (a - 1)(a' - 1)$ is an element of K_i^2. Therefore $aa' \equiv a + a' - 1 \equiv a'a \pmod{K_i^2}$. This means that $[A_i] \equiv 1 \pmod{K_i^2}$; hence

$$E = [A_1] \cdots [A_n] \equiv 1 \qquad (\text{mod } K_1^2 + \cdots + K_n^2),$$

which completes the proof.

Let α_i be a meridian to the i^{th} component of L. An element s of $J\mathcal{G}$ is an expression of the form $\sum e_j \gamma_j$, where the e_j are integers and the γ_j are elements of $\mathcal{G}$. For each such s let α_i^s denote the product

$$\prod_j \gamma_j \alpha_i^{e_j} \gamma_j^{-1}.$$

Since all conjugates of α_i commute, this product is well defined.

LEMMA 7. *Every element of $\mathcal{A}_i(L)$ can be written in the form α_i^s with $s \in J\mathcal{G}(L)$. Two such elements are equal whenever their exponents are congruent modulo the ideal $\mathcal{K}_i + (\mathcal{K}_1^2 + \cdots + \mathcal{K}_n^2)$.*

By Lemma 4, every element of $\mathcal{A}_i(L)$ can be written as a product of conjugates of α_i and α_i^{-1}. Hence every element of $\mathcal{A}_i$ can be written in the form α_i^s with $s \in J\mathcal{G}$.

Every element of the ideal $\mathcal{K}_j$ can be written as a sum of expressions $\beta - \gamma$ with $\beta \equiv \gamma \pmod{\mathcal{A}_j}$. For such an exponent we have

$$\alpha_i^{\beta-\gamma} = \beta\alpha_i\beta^{-1}\gamma\alpha_i^{-1}\gamma^{-1} \equiv \beta\alpha_i\beta^{-1}\beta\alpha_i^{-1}\beta^{-1} = 1 \qquad (\text{mod } \mathcal{A}_j).$$

Hence $\alpha_i^s \in \mathcal{A}_j$ for all $s \in \mathcal{K}_j$.

For the special case $j = i$ we have

$$\alpha_i^{\beta-\gamma} = \beta\alpha_i(\beta^{-1}\gamma)\alpha_i^{-1}\gamma^{-1} = \beta(\beta^{-1}\gamma)\alpha_i\alpha_i^{-1}\gamma^{-1} = 1,$$

since both $\beta^{-1}\gamma$ and α_i belong to the commutative group $\mathcal{A}_i$. Therefore $\alpha_i^s = 1$ for all $s \in \mathcal{K}_i$.

Every element of $\mathcal{K}_j^2$ can be written as a sum of expressions $(\beta - \gamma)s$ with $\beta \equiv \gamma \pmod{\mathcal{A}_j}$ and $s \in \mathcal{K}_j$. For such an exponent we have

$$\alpha_i^{(\beta-\gamma)s} = \beta\alpha_i^s(\beta^{-1}\gamma)\alpha_i^{-s}\gamma^{-1} = \beta(\beta^{-1}\gamma)\alpha_i^s\alpha_i^{-s}\gamma^{-1} = 1,$$

since both α_i^s and $\beta^{-1}\gamma$ belong to $\mathcal{A}_j$. Therefore $\alpha_i^s = 1$ for all $s \in \mathcal{K}_j^2$. It follows that $\alpha_i^s = \alpha_i^{s'}$ whenever

$$s \equiv s' \pmod{\mathcal{K}_i + (\mathcal{K}_1^2 + \cdots + \mathcal{K}_n^2)}.$$

Note that the factor ring $J\mathcal{G}/\mathcal{K}_i + (\mathcal{K}_1^2 + \cdots + \mathcal{K}_n^2)$ is naturally isomorphic to $\mathcal{R}(L^i)$. Hence the expression α_i^σ is defined for elements σ of $\mathcal{R}(L^i)$.

THEOREM 5. *For an i-trivial link L, every element of* $\mathcal{A}_i(L)$ *can be expressed in one and only one way as* α_i^σ *with* $\sigma \in \mathcal{R}(L^i)$.

The group $\mathcal{G}(L)$ and the ring $\mathcal{R}(L^i)$ will not be changed if L is replaced by some homotopic link. Therefore, without loss of generality, we may assume that L is a polygonal link and that its i^{th} component is a small unknotted circle. Let N be a small euclidean neighborhood of this circle, and let G' be the subgroup of $G(L)$ generated by all closed loops in $M - |L| - N$. Then $G(L)$ is the free product of G' with the infinite cyclic group generated by a_i. (The group G' is naturally isomorphic to $G(L^i)$).

Following R. H. Fox [3] we introduce the concept of a derivation in the ring $JG(L)$. Define the homomorphism η: $JG \to J$ by

$$\eta(\textstyle\sum e_j g_j) = \sum e_j .$$

A derivation in JG is a map D: $JG \to JG$ such that

$$D(a + b) = D(a) + D(b) \tag{1}$$

$$D(ab) = D(a)\eta(b) + aD(b). \tag{2}$$

In particular the derivation D_i is defined by the further conditions

$$D_i(g) = 0 \qquad \text{for } g \in G' \tag{3}$$

$$D_i(a_i) = 1. \tag{4}$$

Since $G(L)$ is a free product, it is easily shown that there is a unique function D_i which satisfies these conditions.

It will next be proved that $D_i(K_j^2) \subset K_j^2$ for $j \neq i$, and that $D_i(K_i^2) \subset K_i$. For each $j \neq i$, a j^{th} meridian a_j to L can be chosen in the subgroup G' of G. It follows from Lemma 4 that every element of K_j is a sum of terms bk_jc, where $k_j = a_j - 1$. For such a term we have

$$D_i(bk_jc) = D_i(b)\eta(k_jc) + bD_i(k_j)\eta(c) + bk_j\,D_i(c) = bk_j\,D_i(c) \in K_j ,$$

since $\eta(k_j) = 0$ and $D_i(k_j) = 0$. Therefore $D_i(K_j) \subset K_j$ for $j \neq i$.

For any b, $c \in K_j$ we have $\eta(c) = 0$, hence

$$D_i(bc) = D_i(b)\eta(c) + bD_i(c) = bD_i(c).$$

For $j \neq i$, since both b and $D_i(c)$ are elements of K_j, this proves that $D_i(K_j^2) \subset K_j^2$. For $j = i$ we still have $b \in K_i$, which proves that $D_i(K_i^2) \subset K_i$. Since D_i is an additive homomorphism, it follows that D_i induces a homomorphism of $\mathcal{R}(L) = JG/K_1^2 + \cdots + K_n^2$ into $\mathcal{R}(L^i) \approx JG/K_i + (K_1^2 + \cdots + K_n^2)$.

Define the exponential homomorphism ε: $\mathcal{R}(L^i) \to \mathcal{A}_i(L)$ by $\varepsilon(\sigma) = \alpha_i^\sigma$. We must prove that the function ε is one-one. Let λ: $\mathcal{A}_i(L) \to \mathcal{R}(L)$ be the function which sends each $\gamma \in \mathcal{A}_i(L)$ into $\gamma - 1$ reduced modulo $\mathcal{K}_1^2 + \cdots + \mathcal{K}_n^2$. From the fact that $(\alpha_i^\sigma - 1)\ (\alpha_i^{\sigma'} - 1) \in \mathcal{K}_i^2$, it follows that $\alpha_i^{\sigma+\sigma'} - 1 \equiv \alpha_i^\sigma - 1 + \alpha_i^{\sigma'} - 1 \pmod{\mathcal{K}_1^2 + \cdots + \mathcal{K}_n^2}$, hence $\lambda\, \varepsilon(\sigma + \sigma') = \lambda\, \varepsilon(\sigma) + \lambda\, \varepsilon(\sigma')$.

It will now be proved that the composition $\delta\, \lambda\, \varepsilon$ of the maps

$$\mathcal{R}(L^i) \xrightarrow{\varepsilon} \mathcal{A}_i(L) \xrightarrow{\lambda} \mathcal{R}(L) \xrightarrow{\delta} \mathcal{R}(L^i)$$

is the identity map of $\mathcal{R}(L^i)$. Since $\delta\, \lambda\, \varepsilon$ is an additive homomorphism, it is sufficient to verify that $\delta\, \lambda\, \varepsilon(\gamma) = \gamma$ for an element γ of $\mathcal{R}(L^i)$ which come from $\mathcal{G}(L^i)$. Such an element γ is represented by an element g of G'. The element $\lambda\, \varepsilon(\gamma) = \gamma\alpha_i\gamma^{-1} - 1$ of $\mathcal{R}(L)$ is then represented by $ga_ig^{-1} - 1$ in $JG(L)$, and the element $\delta\, \lambda\, \varepsilon(\gamma)$ of $\mathcal{R}(L^i)$ by $D_i(ga_ig^{-1} - 1)$ in $JG(L)$. But

$$D_i(ga_ig^{-1} - 1) = D_i(g)\eta(a_ig^{-1}) + gD_i(a_i)\eta(g^{-1}) + ga_iD_i(g^{-1}) = g,$$

since $D_i(g) = D_i(g^{-1}) = 0$ and $D_i(a_i) = \eta(g^{-1}) = 1$. This implies that $\delta\, \lambda\, \varepsilon(\gamma) = \gamma$, hence that $\delta\, \lambda\, \varepsilon$ is the identity map of $\mathcal{R}(L^i)$. It follows that the function ε is one–one. Since ε maps $\mathcal{R}(L^i)$ onto $\mathcal{A}_i(L)$ by Lemma 7, this completes the proof of Theorem 5.

Since $\delta\, \lambda\, \varepsilon$ is an isomorphism and since ε is onto, it follows also that λ is one–one. This fact will be used to prove the following theorem.

THEOREM 6. *If the link L is trivial, then the composition of the natural homomorphisms $\mathcal{G}(L) \to J\mathcal{G}(L) \to \mathcal{R}(L)$ is an isomorphism of $\mathcal{G}(L)$ into $\mathcal{R}(L)$.*

For the case $n = 0$ this asserts that the natural map $F \to JF$ is an isomorphism which is clear. Suppose that the case $n - 1$ has been proved. Consider the multiplicative homomorphisms

$$\begin{array}{ccc} \mathcal{G}(L) & \xrightarrow{\rho} & \mathcal{R}(L) \\ \tau \big\downarrow & & \big\downarrow \tau' \\ \mathcal{G}(L^i) & \xrightarrow{\rho'} & \mathcal{R}(L^i); \end{array}$$

and let γ be any element of $\mathcal{G}(L)$ with $\rho(\gamma) = 1$. By the induction hypothesis, ρ' is an isomorphism. Since $\rho'\tau(\gamma) = \tau'\rho(\gamma) = 1$, this implies that $\tau(\gamma) = 1$, hence that $\gamma \in \mathcal{A}_i(L)$. Therefore the function $\lambda(\gamma) = \rho(\gamma) - 1$ is defined. Since λ is one-one and since $\rho(\gamma) - 1 = 0$, this implies that $\gamma = 1$, which completes the proof.

Theorems 5 and 6 suggest the importance of obtaining further information about the ring $\mathcal{R}(L)$. We will next study this ring for the special case of a trivial link. It is evidently sufficient to consider a link L for which each component is a small unknotted circle, so that $G(L)$ can be presented as a free product.

Suppose that $G(L)$ is the free product of F with the infinite cyclic groups generated by $a_1, \cdots, a_n$. Let k_i be the element $a_i - 1$ of JG. By a *canonical word* of JG will be meant a product of the form $\varphi_0 k_{j_1}\varphi_1 k_{j_2}\varphi_2 \cdots k_{j_p}\varphi_p$, $p \geqq 0$, where the φ_i are arbitrary elements of F and the j_i are distinct integers between 1 and n. By a *canonical sentence* of JG will be meant a sum or difference of any number of canonical words.

THEOREM 7. *Under these conditions, each element of* $\mathfrak{R} = JG/K_1^2 + \cdots + K_n^2$ *is represented by a unique canonical sentence in* JG.

Let $\mathfrak{S}$ be the ring whose elements are canonical sentences of JG, with addition defined in the ordinary way, but with multiplication defined by the rule

$$(\varphi_0 k_{h_1}\varphi_1 \cdots k_{h_p}\varphi_p)(\psi_0 k_{j_1} \cdots k_{j_q}\psi_q) = \begin{cases} 0 & \text{if } h_i = j_{i'} \text{ for some } i, i' \\ \varphi_0 k_{h_1}\varphi_1 \cdots k_{h_p}(\varphi_p\psi_0)k_{j_1} \cdots k_{j_q}\psi_q & \text{otherwise.} \end{cases}$$

Using the distributive laws, this multiplication for words extends to a unique multiplication for arbitrary elements of $\mathfrak{S}$. The associative law for multiplication is clear.

A homomorphism $\zeta: \mathfrak{S} \to \mathfrak{R}$ is obtained by mapping each canonical word $\varphi_0 k_{j_1} \cdots \varphi_p$ into the actual product $\varphi_0 k_{j_1} \cdots \varphi_p$ reduced modulo $K_1^2 + \cdots + K_n^2$. In order to veryfy that ζ is a homomorphism, it is only necessary to note that the identity

$$(\varphi_0 k_{h_1} \cdots \varphi_p)(\psi_0 k_{j_1} \cdots \psi_q) \equiv 0 \pmod{K_1^2 + \cdots + K_n^2}$$

holds in JG whenever $h_i = j_{i'}$ for some i, i'. Theorem 7 is clearly equivalent to the proposition that ζ is an isomorphism of $\mathfrak{S}$ onto $\mathfrak{R}$.

A homomorphism $\theta: JG \to \mathfrak{S}$ is defined by mapping each element φ of F into the canonical word φ, and mapping each meridian a_i into the canonical sentence $1 + 1k_i1$. Since G is a free product, and since the element $1 + 1k_i1$ of $\mathfrak{S}$ has an inverse $1 - 1k_i1$, this homomorphism is well defined. It clearly maps JG onto $\mathfrak{S}$. The composition $\zeta\theta$ of these two maps is the natural map $JG \to \mathfrak{R}$. (It is sufficient to verify this for elements of F and for the meridians a_i). Therefore ζ maps $\mathfrak{S}$ onto $\mathfrak{R}$; and the kernel of ζ is the image $\theta(K_1^2 + \cdots + K_n^2)$ of the kernel of $\zeta\theta$.

Every element of the ideal K_i is a sum of elements $g(a_i - 1)g'$. It follows by direct computation that every element of $\theta(K_i)$ is a sum of canonical words which contain k_i. Since the product of two such words is zero by the definition of multiplication in $\mathfrak{S}$, this means that $\theta(K_i^2) = 0$. It follows that the kernel $\theta(K_1^2 + \cdots + K_n^2)$ of ζ equals zero, which completes the proof of Theorem 7.

The preceding theorems give two separate solutions to the word problem for the link group of a trivial link, and therefore to the triviality problem for links. Using Theorems 6 and 7, each element of $\mathfrak{G}(L)$ corresponds to a unique element of $\mathfrak{R}(L)$ and therefore to a unique canonical sentence in $JG(L)$. Two elements in $\mathfrak{G}$ are equal if and only if their canonical sentences are equal.

A more efficient solution in practice is the following. Let L_n be a trivial n-link, and let L_r be the sublink formed by the first r components of L. Then $\mathcal{G}(L_{r-1})$ can be considered as a subgroup of $\mathcal{G}(L_r)$. Furthermore each element of $\mathcal{G}(L_r)$ can be expressed uniquely as the product of an element of $\mathcal{G}(L_{r-1})$ with an element of $\mathcal{a}_r(L_r)$. By Theorem 5, each element of $\mathcal{a}_r(L_r)$ can be expressed uniquely in the form $\alpha_r^{\sigma_{r-1}}$ with $\sigma_{r-1} \in \mathcal{R}(L_{r-1})$. It follows by induction that each element of $\mathcal{G}(L_n)$ can be expressed uniquely in the form

$$\varphi \alpha_1^{\sigma_0} \alpha_2^{\sigma_1} \cdots \alpha_n^{\sigma_{n-1}}$$

with $\varphi \in F$, $\sigma_0 \in \mathcal{R}(L_0) = JF$, $\sigma_1 \in \mathcal{R}(L_1)$, $\cdots$, $\sigma_{n-1} \in \mathcal{R}(L_{n-1})$. Since the word problems for these rings are solved by Theorem 7, this gives a second solution to the word problem for $G(L_n)$.

To conclude this section, the following theorem will be proved.

THEOREM 8. *If $\mathcal{G}(L)$ is isomorphic to the group $\mathcal{G}(\bar{L})$ of a trivial link, in an isomorphism which preserves the conjugate classes corresponding to meridians, then L is trivial.*

Let $(\alpha_i, \beta_i \mathcal{a}_i)$ be a meridian and parallel pair in $\mathcal{G}(L)$, and let $(\bar{\alpha}_i, \bar{\beta}_i \bar{\mathcal{a}}_i)$ be the pair in $\mathcal{G}(\bar{L})$ which correspond under the isomorphism. Then $\bar{\alpha}_i$ is a meridian in $\mathcal{G}(\bar{L})$ (but $\bar{\beta}_i \bar{\mathcal{a}}_i$ is not necessarily a parallel). We may assume that the link L is polygonal so that $l_i(C)$ has a smooth tubular neighborhood. If this neighborhood is orientable, then the identity $\alpha_i^{\beta_i - 1} = \beta_i \alpha_i \beta_i^{-1} \alpha_i^{-1} = 1$ holds. This implies that $\bar{\alpha}_i^{\bar{\beta}_i - 1} = 1$, hence by Theorem 5, that $\bar{\beta}_i$ represents the identity element of $\mathcal{R}(\bar{L}^i)$. It follows by Theorem 6 that $\bar{\beta}_i$ represents the identity element of $\mathcal{G}(\bar{L}^i)$, hence that $\bar{\beta}_i \bar{\mathcal{a}}_i = \bar{\mathcal{a}}_i$. This implies that $\beta_i \mathcal{a}_i = \mathcal{a}_i$, hence that L is i-trivial. Since this is true for all values of i, it follows that L is trivial.

If the neighborhood were non-orientable, then the identity $\alpha_i^{\beta_i + 1} = 1$ would hold. This would imply that $\bar{\alpha}_i^{\bar{\beta}_i + 1} = 1$, hence that $-\bar{\beta}_i$ represented the identity element of $\mathcal{R}(\bar{L}^i)$. Since this is impossible, the proof is complete.

5. Almost trivial links

Let L be an n-link in euclidean space such that every proper sublink is trivial. Such links will be called *almost trivial*. The n^{th} parallel $\beta_n \mathcal{a}_n$ in $\mathcal{G}(L)$ corresponds to an element β_n' of $\mathcal{G}(L^n)$. Since the link L^{n-1} is trivial, it follows that $\beta_n' \in \mathcal{a}_{n-1}(L^n)$; hence β_n' can be written in the form α_{n-1}^{σ} with $\sigma \in \mathcal{R}(L^{n-1,n})$. Since L^i is trivial for $i < n - 1$, it follows that every word of the canonical sentence corresponding to σ contains the factor k_i. Therefore σ can be written uniquely in the form

$$\sigma = \sum \mu(i_1 \cdots i_{n-2}, n-1\ n) k_{i_1} \cdots k_{i_{n-2}}$$

where the summation extends over all permutations $i_1 \cdots i_{n-2}$ of the integers $1, 2, \cdots, n - 2$.

The $(n - 2)!$ integers $\mu(i_1 \cdots i_{n-2}, n-1\ n)$ are homotopy invariants of L: By Theorem 2 it is sufficient to prove that they are not altered if each α_i and each $\beta_i \mathcal{a}_i$ is replaced by a conjugate. For $i = n$ this is true since β_n' is an element of the center of the group $\mathcal{G}(L^n)$. For $i < n$ it can be verified by a simple computation. On the other hand the homotopy class of L is completely specified by these integers μ. A homotopic link can be constructed from any trivial $(n-1)$-link by adjoining the loop α_{n-1}^{σ}, $\sigma = \sum \mu k_{i_1} \cdots k_{i_{n-2}}$. Thus we have obtained a complete set of homotopy invariants for almost trivial links in euclidean space.

Every link with two components is almost trivial. The single invariant $\mu(12)$ is clearly the linking number: hence the linking number is a complete homotopy invariant for 2-links. A link with three components is almost trivial if each pair of components has linking number zero. For such links we again obtain a single invariant $\mu(1, 23)$. The case $\mu = 1$ turns out to be the familiar link illustrated in Figure 5. Other values of μ may be obtained by traversing one component of this link μ times. The case $\mu = 3$ is illustrated in Figure 6. For links with four components we obtain two invariants. For example the link of Figure 4 has in-

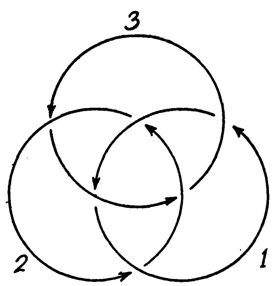

Fig 5

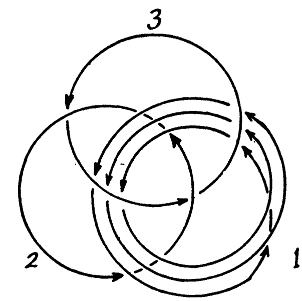

Fig 6

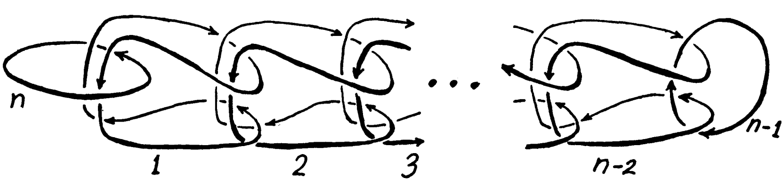

Fig 7

variants $\mu(12, 34) = \mu(21, 34) = 1$. As a final example, Figure 7 illustrates an n-link with invariants

$\mu(12 \cdots n-2, n-1\; n) = 1$,

$\mu(i_1 \cdots i_{n-2}, n-1\; n) = 0$ for all other permutations of $1, \cdots, n-2$.

The behavior of the invariants μ under simple transformations of the link will now be discussed. If the orientation of one component of L is reversed, then each invariant is multiplied by (-1). (Hence if the orientations of two components are reversed, then the resulting link is homotopic to the original). If the orientation of euclidean space is reversed, then each invariant is multiplied by $(-1)^{n-1}$.

In order to study the behavior of the invariants under permutation of the components of L, it is convenient to put the preceding discussion in the following more symmetrical form. For any two integers $r \neq s$ the parallel $\beta_s' \in \mathcal{G}(L^s)$ can be expressed in the form $\beta_s' = \alpha_r^{\sigma_{r,s}}$, where the element

$$\sigma_{r,s} = \sum \mu(i_1 \cdots i_{n-2}, r\, s) k_{i_1} \cdots k_{i_{n-2}}$$

of $\mathcal{R}(L^{r,s})$ is a complete invariant for L. We must now find the relations between the invariants $\sigma_{r,s}$ for different values of r and s.

If $\beta_s' = \prod(\gamma_j \alpha_r^{e_i} \gamma_j^{-1})$ then it can be shown by a simple geometric argument that $\beta_r' = \prod(\gamma_j^{-1} \alpha_s^{e_i} \gamma_j)$. The additive homomorphism $\sum e_j \gamma_j \rightarrow \sum e_j \gamma_j^{-1}$ of $J\mathcal{G}$ on itself induces an additive homomorphism ω: $\mathcal{R} \rightarrow \mathcal{R}$. It follows that

(1) $$\sigma_{s,r} = \omega(\sigma_{r,s}).$$

To find the relationship between $\sigma_{r,s}$ and $\sigma_{t,s}$ it is necessary to solve the equation $\beta_s' = \alpha_r^{\sigma_{r,s}} = \alpha_t^{\sigma_{t,s}}$. Consider the homomorphisms

$$\mathcal{R}(L^{t,s}) \xrightarrow{\varepsilon_t} \mathcal{A}_t(L^s) \xrightarrow{\lambda} \mathcal{R}(L^s) \xrightarrow{\delta_t} \mathcal{R}(L^{t,s})$$

of Theorem 5. From the identities $\delta_t \lambda \varepsilon_t(\sigma_{t,s}) = \sigma_{t,s}$ and $\varepsilon_t(\sigma_{t,s}) = \varepsilon_r(\sigma_{r,s})$ we obtain

(2) $$\sigma_{t,s} = \delta_t \lambda \varepsilon_r(\sigma_{r,s}).$$

In order to make specific computations, it is necessary to know the effects of the additive homomorphisms ω, δ_r, and $\lambda\varepsilon_r$ on canonical words of $\mathcal{R}$. These are given as follows

(3) $$\omega(k_{i_1} \cdots k_{i_m}) = (-1)^m k_{i_m} \cdots k_{i_1},$$

(4) $$\delta_r(k_{i_1} \cdots k_{i_m}) = \begin{cases} k_{i_1} \cdots k_{i_{m-1}} & \text{if} \quad i_m = r \\ 0 & \text{otherwise.} \end{cases}$$

The function $\lambda\varepsilon_r$ can be defined inductively by the rules

$$\lambda\varepsilon_r(1) = k_r$$

and

(5) $$\lambda\varepsilon_r(k_{i_1} k_{i_2} \cdots k_{i_m}) = k_{i_1}\tau - \tau k_{i_1}$$

where

$$\tau = \lambda\varepsilon_r(k_{i_2} \cdots k_{i_m}).$$

The composition $\delta_t \lambda \varepsilon_r$ can now be described by the rule

(6) $$\delta_t \lambda \varepsilon_r(k_{i_1} \cdots k_{i_m}) = -k_{i_1} \cdots k_{i_{p-1}} \lambda \varepsilon_r(k_{i_{p+1}} \cdots k_{i_m})$$

where

$$i_p = t.$$

Some simple examples will illustrate these formulas. For the case $n = 2$ the invariant $\sigma_{12} = \mu(12)$ is an integer. Therefore $\mu(21) = \omega(\mu(12)) = \mu(12)$. Thus the linking number is not changed if the components are interchanged.

For $n = 3$ the invariant σ_{23} is of the form μk_1. Therefore $\sigma_{32} = \omega(\mu k_1) = -\mu k_1$ and

$$\sigma_{13} = \delta_1 \lambda \varepsilon_2(\mu k_1) = \delta_1(\mu k_1 k_2 - \mu k_2 k_1) = -\mu k_2 .$$

Thus the invariant $\mu = \mu(1, 23)$ is skew-symmetric under permutations of the components.

For $n = 4$ the invariant σ_{34} has the form $\mu k_1 k_2 + \mu' k_2 k_1$, hence

$$\sigma_{43} = \omega(\sigma_{34}) = \mu' k_1 k_2 + \mu k_2 k_1$$

and

$$\sigma_{24} = \delta_2 \lambda \varepsilon_3(\sigma_{34}) = -\mu k_1 k_3 - \mu' k_1 k_3 + \mu' k_3 k_1 .$$

Thus the permutations (34) and (12) replace the invariants (μ, μ') by (μ', μ); while the permutation (23) replaces (μ, μ') by $(-\mu - \mu', \mu')$. This behavior can also be described by the following symmetry relations:

$$\mu(i_4 i_3 , i_2 i_1) = \mu(i_1 i_2 , i_3 i_4) = \mu(i_4 i_1 , i_2 i_3)$$

$$\mu(i_1 i_2 , i_3 i_4) + \mu(i_3 i_1 , i_2 i_4) + \mu(i_2 i_3 , i_1 i_4) = 0.$$

The complete set of symmetry relations for arbitrary values of n is given by the rules

$$\mu(i_1 i_2 \cdots i_{n-2} , i_{n-1} i_n) = \mu(i_n i_1 i_2 \cdots i_{n-3} , i_{n-2} i_{n-1}) \tag{7}$$

$$\mu(i_1 \cdots i_\nu r\, j_1 \cdots j_{n-\nu-2}\, s) = (-1)^{n-\nu} \sum \mu(h_1 \cdots h_{n-2}\, r\, s), \tag{8}$$

where the summation is extended over all sets $h_1 \cdots h_{n-2}$ of integers formed by intermeshing $i_1 \cdots i_\nu$ in that order with $j_{n-\nu-2} \cdots j_2 j_1$ in that order. (For example

$$\mu(1r\, 23s) = \mu(132rs) + \mu(312rs) + \mu(321rs)).$$

These relations are obtained by manipulation of the formulas (1) through (6). However the details are too involved to give here.

Remark. The preceding methods can also be used to define certain "self linking numbers", which are not invariant under homotopy. Let L be a trivial polygonal link, and let L' be the link obtained by replacing each component of L by a collection of parallel components having linking numbers zero (compare Section 3). Suppose that L' is almost trivial. Then the invariants μ of L' may be considered as describing the self linking of L. For example the link of Figure 3 has the self linking invariant $\mu(11, 22) = +1$.

6. Arbitrary links in euclidean space

The link group of an n-link in euclidean space has the presentation

$$(\alpha_1 , \cdots , \alpha_n / \alpha_1 w_1 \alpha_1^{-1} w_1^{-1} = 1, \cdots , \alpha_n w_n \alpha_n^{-1} w_n^{-1} = 1, E = 1),$$

where α_i is a meridian to the i^{th} component, and where w_i is a word in $\alpha_1 , \cdots , \alpha_n$ which represents the corresponding i^{th} parallel $\beta_i \mathfrak{A}_i$. The symbol "$E = 1$" denotes the set of relations which specify that conjugates of each α_i commute.

It is evidently sufficient to prove this for the special case of a polygonal link.

We will start with the Wirtinger presentation[3] of the fundamental group $G(L)$. This presentation is in terms of generators a_i^j; $i = 1, \cdots, n$; $j = 1, \cdots, r_i$, corresponding to the components of the projection of L on a plane, and relations

$$a_i^{j+1} = w_{i,j}^{-1} a_i^j w_{i,j} \tag{1}$$

$$a_i^1 = w_{i,r_i}^{-1} a_i^{r_i} w_{i,r_i}, \tag{2}$$

corresponding to the crossings of the projection. The following set of relations is clearly equivalent:

$$a_i^{j+1} = w_{i,j}^{-1} w_{i,j-1}^{-1} \cdots w_{i,1}^{-1} a_i^1 w_{i,1} \cdots w_{i,j} \tag{1'}$$

$$a_i^1 = w_{i,r_i}^{-1} w_{i,r_i-1}^{-1} \cdots w_{i,1}^{-1} a_i^1 w_{i,1} \cdots w_{i,r_i}. \tag{2'}$$

It is natural to try taking the relations (1′) as definitions of the a_i^{j+1}, $j \geqq 1$. The only difficulty which can occur is that the word $w_{i,1} \cdots w_{i,j}$ may contain a_i^{j+1} as a factor. But in the link group $\mathcal{G}(L)$, such factors may be cancelled without altering the relation (1′). It follows by an obvious double induction that all of the a_i^j can be defined in terms of the a_i^1. Furthermore the relations of type (1′) are completely exhausted during this process. The relations (2′) can be put in the form

$$\alpha_i w_i \alpha_i^{-1} w_i^{-1} = 1,$$

where $\alpha_i = a_i^1$ and $w_i = w_{i,1} \cdots w_{i,r_i}$. Since the word $w_i = w_{i,1} \cdots w_{i,r_i}$ clearly represents the i^{th} parallel $\beta_i \mathfrak{A}_i$, this completes the proof.

For example if L is a 2-link with linking number μ, then the parallels are represented by $\beta_1 = \alpha_2^\mu$ and $\beta_2 = \alpha_1^\mu$; hence $\mathcal{G}(L)$ has the presentation

$$(\alpha_1, \alpha_2 / \alpha_1 \alpha_2^\mu \alpha_1^{-1} \alpha_2^{-\mu} = \alpha_2 \alpha_1^\mu \alpha_2^{-1} \alpha_1^{-\mu} = E = 1).$$

For a trivial 2-link L′, every element of $G(L')$ can be put in the canonical form $\alpha_1^h \alpha_2^{i+jk_1}$, where h, i, j are arbitrary integers. For the link L, the additional relations $\alpha_1\alpha_2^\mu\alpha_1^{-1}\alpha_2^{-\mu} = \alpha_2\alpha_1^\mu\alpha_2^{-1}\alpha_1^{-\mu} = 1$ introduces the single relation $\mu k_1 = 0$ into this canonical form. (This means that the commutator subgroup of $\mathcal{G}(L)$ is a cyclic group of order $|\mu|$.)

A 3-link may be specified by choosing a conjugate class of elements in the link group of an arbitrary 2-link. An element β_3' of this conjugate class has the canonical form $\alpha_1^h\alpha_2^{i+jk_1}$, where j is only defined modulo the linking number $\mu(12)$. The integers h and i are clearly equal to $\mu(13)$ and $\mu(23)$. The effect of conjugating β_3' by α_1 or α_2 is to replace the integer j by $j + i$ or $j - h$. Hence an arbitrary 3-link is specified by giving the three linking numbers together with the number $j = \mu(123)$ which need only be defined modulo the greatest common divisor $\Delta = (h, i, \mu(12)) = (\mu(13), \mu(23), \mu(12))$. On the other hand the residue class of $\mu(123)$ modulo Δ is not changed if each meridian and parallel is replaced by a conjugate. Therefore the three integers $\mu(12)$, $\mu(13)$, $\mu(23)$ together with the residue class of $\mu(123)$ mod Δ give a complete homotopy invariant for 3-links.

[3] See for example [4] page 44.

For a given set of three linking numbers, the number of distinct links which occur is equal to Δ, unless $\Delta = 0$ in which case an infinite number of distinct links occur. For example for the linking numbers 0, 2, 4 there are two possible links. The case $\mu(123) \equiv 0 \pmod 2$ is illustrated in Figure 8; the case $\mu(123) \equiv 1 \pmod 2$ in Figure 9. The three linking numbers form a complete invariant by themselves only if they are relatively prime.

The procedures which have been used above to classify special types of links can be generalized to give a very rough description for arbitrary links in euclidean space. The general n-link can be built up as follows. Start with any proper loop l_1, and adjoin a loop $l_2 = \alpha_1^{\mu(12)}$. Then adjoin $l_3 = \alpha_1^{\mu(13)}\alpha_2^{\mu(23)+\mu(1,23)k_1}$. Continuing by induction, for each $i \leqq n$ it is necessary to adjoin a loop $l_i = \alpha_1^{\tau_1} \cdots \alpha_{i-1}^{\tau_{i-1}}$, where $\tau_j = \sum \mu(h_1 \cdots h_r, ji)k_{h_1} \cdots k_{h_r}$; the summation being extended over all ordered collections $h_1 \cdots h_r$ of integers between 1 and $j - 1$. (To make this construction precise it would be necessary to adopt some convention as to how the

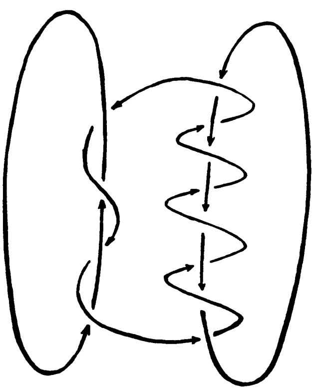

Fig 8

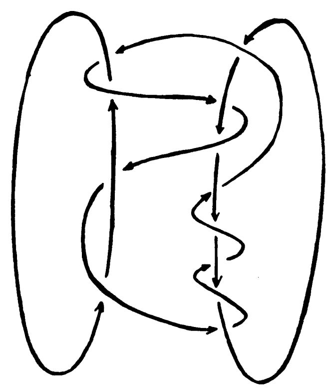

Fig 9

meridians are selected at each stage). Thus in order to specify an n-link it is sufficient to specify an integer $\mu(ji)$ for each 2-sublink; an integer $\mu(h, ji)$ for each 3-sublink; and in general $(m - 2)!$ integers $\mu(h_1 \cdots h_{m-2}, ji)$ for each of the $\binom{n}{m}$ sublinks with m components.

These integers μ are not invariants of the link. However let Δ denote the greatest common divisor of all integers $\mu(h_1 \cdots h_{m-2}, ji)$ with $m < n$. Then it can be proved that the residue classes $\mu(h_1 \cdots h_{n-2}, n-1\, n)$ modulo Δ are actually homotopy invariants. Thus we have constructed $(n - 2)!$ invariants for L, as well as similar invariants for all sublinks of L. Unfortunately these invariants are not strong enough to specify the homotopy class of L. It is to be hoped that some refinement of the above procedures will yield a complete homotopy classification of links in euclidean space. But since the present results are not too conclusive, and since the proofs are somewhat involved, further details will not be given.

7. Arbitrary manifolds

One rather interesting question has not been touched on so far. To each open 3-manifold there has been associated the following structure: To each conjugate class of elements in the fundamental group F there corresponds the link group $\mathcal{G}(l_1)$ together with the preferred class of elements $(\alpha_1, \beta_1\mathfrak{A}_1)$ and the natural homomorphism $\mathcal{G} \rightarrow F$. To each conjugate class of elements in each group $\mathcal{G}(l_1)$ there corresponds a group $\mathcal{G}(l_1, l_2)$; and so on. The huge collection of groups, homomorphisms, and preferred elements which results is a topological invariant of M. To what extent is this structure determined by the usual topological invariants, such as the fundamental group and linking invariants?

An example will illustrate this problem. Let M_1 be the product of a 2-sphere with a circle, and let M_2 be the product of an open 2-cell with a circle. Then M_1 and M_2 both have infinite cyclic fundamental groups. Let l_1 be a closed loop which generates the fundamental group. For the manifold M_1 the group $\mathcal{G}(l_1)$ is infinite cyclic, while for M_2 it is free abelian on two generators. This behavior can occur even for two manifolds having the same homotopy type. In fact let M'_i, $i = 1, 2$, be the manifold obtained by removing a single point from M_i. Then the link structure of M'_i is identical with that of M_i. But the manifolds M'_1 and M'_2 have the same homotopy type. (Both manifolds contain, as deformation retract, the union of a circle and a 2-sphere intersecting in a single point.)

The preceding example suggests the following question. Is the link structure of a *closed* 3-manifold completely determined by its homotopy type? (Or more generally let M be a compact manifold with boundaries B. Is the link structure of $M - B$ completely determined by the homotopy type of the pair (M, B)?) The answer to this question would be interesting whether positive or negative. In one case it would give a topological procedure for distinguishing between certain closed manifolds of the same homotopy type. In the other case it would give a non-abelian generalization of the duality theorems for homology groups.

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REFERENCES

1. CHEN, KUO-TSAI, *Isotopy invariants of links*, Ann. of Math. 56 (1952) pp. 343–353.
2. FOX, R. H., Bull. Amer. Math. Soc. 51 (1945), Abstract 140, pg. 526.
3. FOX, R. H., *Free differential calculus*, Ann. of Math. 57 (1953) pp. 547–560.
4. REIDEMEISTER, K., Knotentheorie, Julius Springer, Berlin, 1932.