ON SIMPLY CONNECTED 4-MANIFOLDS

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This paper will apply various known results to the problem of classifying simply connected 4-manifolds as to homotopy type. A complete classification is given for those manifolds with second Betti number $\leqq 7$, and a fairly good picture is obtained for the general case. But the problem is by no means solved.

The first section states results from the theory of quadratic forms. The author is indebted to O. T. O'Meara for assistance in the preparation of this section.

In §2 the quadratic form of a $4k$-manifold is defined, and elementary properties are verified.

In §3 it is shown (as a corollary to a theorem of J. H. C. Whitehead) that the homotopy type of a simply connected 4-manifold is determined by its quadratic form. The main problem is thus to decide whether or not a given quadratic form actually corresponds to some simply connected 4-manifold.

A subsequent paper will study the more general problem of classifying $2n$-manifolds which are $(n-1)$-connected as to homotopy type.

§1. Quadratic forms

Theorems 1 and 2 of this section will summarize known results concerning the classification of quadratic forms with determinant ± 1 over the ring of integers.

It will be convenient to define[2] a *quadratic form of rank r* over an integral domain D as a pair (A, ϕ) consisting of a free D-module A of rank r, and a non-singular, symmetric, bilinear pairing $\phi : A \times A \to D$. Two forms (A, ϕ) and (A', ϕ') are *equivalent* if there is an isomorphism of A onto A' which carries ϕ' onto ϕ. If D is contained in a larger integral domain D', note that every quadratic form over D gives rise to a quadratic form over D'.

Given a basis $(a_1, \cdots, a_r)$ for A, the form is completely described by the symmetric matrix $\|\phi(a_i, a_j)\|$. This section will only be concerned with quadratic forms over the integers such that this matrix has determinant ± 1.

We will say that a form is of *type* I (properly primitive) if some diagonal entry of its matrix is odd. If every diagonal entry is even, then the form is of *type* II (improperly primitive). Thus (A, ϕ) has type I if and only if $\phi(a, a)$ takes on odd values.

The index τ of a form is defined[3] as the number of positive diagonal entries minus the number of negative ones, after the matrix has been diagonalized over the real numbers.

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[2] This is compatible with the usual definition providing that $2 \neq 0$ in D.

[3] Topologists have called τ the "index," although "signature" is the classical term.

As examples, consider the quadratic forms corresponding to the following three matrices

$$T = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad V = \begin{pmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 \end{pmatrix}.$$

The first has type I and index 0; the second has type II and index 0; while the third has type II and index 8.

Two forms have the same *genus* if they are equivalent over the p-adic integers for every prime p, and if they are equivalent over the real numbers. (Compare [8] p. 106–107.)

THEOREM 1. *The rank r, the index τ, and the type* (I *or* II) *form a complete system of invariants for the genus of a quadratic form over the integers with determinant* ± 1. *A given rank, index and type actually occur if and only if the following three conditions are satisfied:*

(1) *r and τ are integers with*

$$-r \leqq \tau \leqq r, \qquad \tau \equiv r \pmod 2;$$

(2) *for forms of type* II, $\tau \equiv 0 \pmod 8$;

(3) *for forms of type* I, $r > 0$.

The following is an immediate consequence.

COROLLARY. *Every such form has the same genus as a form with matrix*

$$\operatorname{diag}(1, \cdots, 1, -1, \cdots, -1) \quad or \quad \pm \operatorname{diag}(U, \cdots, U, V, \cdots, V).$$

A form is called *definite* if $\tau = \pm r$. Otherwise it is *indefinite*.

THEOREM 2. *Two indefinite forms with determinant ± 1 are equivalent if and only if they have the same genus. This is also true for definite forms, providing the rank is* $\leqq 8$.

(This theorem would definitely be false for definite forms of rank $\geqq 9$. For example the two positive definite 9×9 matrices diag $(1, \cdots, 1)$ and diag $(V, 1)$ represent forms (A, ϕ) and (A', ϕ') of the same genus. These forms are not equivalent since the equation $\phi(a, a) = 1$ has eighteen solutions, while $\phi'(a', a') = 1$ has only two solutions.)

PROOF OF THEOREM 1. Let f_1, f_2 be two forms with determinant ± 1 having the same rank, index and type. Then f_1 is equivalent to f_2 over the p-adic integers, p odd, by Corollary 36b of [8]. They are equivalent over the real numbers since they have the same index. Therefore (see [8] p. 39) we have $c_\infty(f_1) = c_\infty(f_2)$, which implies that $c_2(f_1) = c_2(f_2)$. Now by Theorems 15 and 36 of [8], f_1 is equivalent to f_2 over the 2-adic integers. Therefore f_1 and f_2 have the same genus.

NECESSITY OF CONDITIONS (1), (2), (3). Conditions (1) and (3) are trivial. If f_1 has type II then Theorem 33a of [8] implies that f_1 is equivalent to diag $(U, \cdots, U)$ over the 2-adic integers (making use of the fact that ± 3 are not 2-adic squares).

Therefore the Gauss sum (see [2]) of f_1 modulo 8 is positive. Now the criterion (ε) of [2] implies that $\tau \equiv 0 \pmod 8$; which proves condition (2).

The sufficiency of conditions (1), (2), (3), is an easy exercise, using the forms mentioned in the Corollary. This completes the proof of Theorem 1.

PROOF OF THEOREM 2. For definite forms of rank $\leqq 8$, this result is due to Hermite [6] and Mordell [12]. (For further discussion see O'Connor and Pall [14].) For indefinite forms of type I, the result is due to Meyer [10].

For indefinite forms of type II the proof will be based on a theorem of Eichler ([4], [5]). We may assume that $r \geqq 4$, since the cases $r = 0, 2$ are easily taken care of.

Eichler considers a fixed quadratic form (V, ψ) over the rational numbers Q. A *lattice* L in V is a finitely generated subgroup of maximal rank. The *norm* $n(L)$ is the fractional ideal generated by all $\frac{1}{2}\psi(x, x)$ with $x \in L$. A lattice is *maximal* if no properly containing lattice has the same norm. Two lattices are *similar* if one is carried onto the other by a similarity transformation of V. An appropriate concept of *genus* is defined.

Eichler's theorem ([4] Satz 3) can be stated as follows. *Two maximal lattices having the same genus, in an indefinite vector space (V, ψ) of rank $\geqq 4$, are similar.*

To apply this theorem note that any free abelian group A can be considered as a lattice in the vector space $V = A \otimes Q$. A quadratic form (A, ϕ) gives rise to a form (V, ψ). If (A, ϕ) is of type II then it can be verified that the lattice A is always maximal.

This implies that if (A, ϕ) and (A', ϕ') are indefinite quadratic forms of type II and rank $\geqq 4$ which have the same genus, then they are "similar," in the sense that (A, ϕ) is equivalent to $(A', c\phi')$ for some constant c. Clearly c must be ± 1. If $\tau \neq 0$ then c must be $+1$, which completes the proof in this case.

For $\tau = 0$ the above argument shows that (A, ϕ) is equivalent to the form with matrix $\pm$ diag $(U, \cdots, U)$. Since U is equivalent to $-U$, this completes the proof.

§2. The quadratic form of a $4k$-manifold

All manifolds considered are to be closed and connected. A manifold M^n is *oriented* if it is orientable, and if one generator $\nu \in H_n(M^n)$ is distinguished; integer coefficients being understood. We will say that oriented manifolds M_1^n, M_2^n have the same *oriented homotopy type* if there is a homotopy equivalence $f : M_1^n \to M_1^n$ with $f_*(\nu_1) = \nu_2$.

To every oriented $4k$-manifold is associated its quadratic form $(B^{2k}(M^{4k}), \phi)$; where $B^j(X)$ denotes the "co Betti group" $H^j(X)$/(torsion subgroup); and where[4]

$$\phi(x, y) = \langle x \cup y, \nu \rangle.$$

Clearly manifolds with the same oriented homotopy type have equivalent quadratic forms.

LEMMA 1. *The quadratic form of an oriented $4k$-manifold has determinant* ± 1.

PROOF. (Compare Seifert and Threlfall [17] p. 252.) Consider the co Betti groups B^h, B^{n-h} of an oriented manifold M^n. The bilinear pairing $\phi : B^h \times B^{n-h} \to Z$

[4] Here $\langle \alpha, \beta \rangle$ denotes the Kronecker index of the cohomology class α and the homology class β.

defined by $\phi(x, y) = \langle x \cup y, \nu\rangle$ has a determinant which is well defined up to sign. Choose a basis $y_1, \cdots, y_r$ for B^{n-h}. By the Poincaré duality theorem (as stated in [20] p. 119–120) the cap product with ν defines an isomorphism of $H^{n-h}(M^n)$ onto $H_h(M^n)$. Therefore the elements $y_i \cap \nu (i = 1, \cdots, r)$ form a basis for the Betti group B_h. Choose a dual basis $\{x_i\}$ for B^h. Then the identity $(x_i \cup y_j) \cap \nu = x_i \cap (y_j \cap \nu)$ implies that $\phi(x_i, y_j) = \langle x_i, y_j \cap \nu\rangle = \delta_{ij}$. Therefore the determinant equals ± 1.

By a sum of two oriented n-manifolds M_1^n, M_2^n will be meant an oriented n-manifold $M_1^n + M_2^n$ obtained as follows. (Compare Seifert [16].) Choose smooth[5] closed n-cells $\bar{e}_i^n \subset M_i^n$, $i = 1, 2$, with boundaries $\dot{e}_i^n$ and interiors e_i^n. Choose a homeomorphism $f_2\ \dot{e}_1^n \to \dot{e}_2^n$ of degree -1. Now let $M_1 + M_2$ be the manifold obtained from $M_1^n - e_1^n$ and $M_2^n - e_2^n$ by matching the boundaries $\dot{e}_1^n$ and $\dot{e}_2^n$ under the homeomorphism f. This manifold has an orientation compatible with that of M_1^n and M_2^n.

The *sum* of two quadratic forms (A, ϕ) and (A', ϕ') will mean the form $(A \oplus A', \psi)$ where $\psi((x, x'), (y, y')) = \phi(x, y) + \phi'(x', y')$.

LEMMA 2. *The quadratic form of a sum of two oriented manifolds is naturally isomorphic to the sum of their quadratic forms.*

The proof is not difficult. (Compare [11] p. 400.)

The *index* or *type* of a manifold M^{4k} will mean the index or type of its quadratic form.

LEMMA 3. *A differentiable manifold M^{4k} which is $(2k - 1)$-connected has type* II *if and only if its Stiefel-Whitney class W_{2k} is zero.*

PROOF. Clearly M^{4k} has type II if and only if the homomorphism

$$\mathrm{Sq}^{2k} : H^{2k}(M^{4k}, Z_2) \to H^{4k}(M^{4k}, Z_2)$$

is zero. The conclusion now follows from Wu's formulas for the Stiefel-Whitney classes. (See [22].)

REMARK. In dimension 4 the following alternative interpretation holds. *A differentiable, simply connected manifold M^4 has type* II *if and only if, for every point p, the open manifold $M^4 - p$ is parallelizable.* The proof is not difficult.

§3. Simply connected 4-manifolds

In order to simplify the proofs in this section we assume that all manifolds considered are triangulable.

THEOREM 3. *Two oriented, simply connected 4-manifolds have the same oriented homotopy type if and only if their quadratic forms are equivalent.*

PROOF. Whitehead has shown ([21] Theorem 2) that the homotopy type of a finite, simply connected 4-dimensional polyhedron X is determined by the cohomology rings $H^*(X, Z_k)$, $k = 0, 1, 2, \cdots$; together with certain coefficient homomorphisms, Bochstein homomorphisms, and Pontrjagin squares. If $H^*(X,Z)$ has no torsion, then all of this structure is clearly determined by $H^*(X, Z)$.

For simply connected 4-manifold, the Poincaré duality theorem implies that

[5] By "smooth" we mean that for some neighborhood U of $\bar{e}_i$ the pair $(U, \bar{e}_i)$ is homeomorphic to the pair consisting of euclidean n-space R^n and the unit ball in R^n.

there is no torsion. Furthermore the integral cohomology ring is completely described by the quadratic form. Therefore the quadratic form determines the homotopy type. Using Theorem 3 of [21] we see that the quadratic form actually determines the oriented homotopy type.

In view of Theorems 1, 2 and Lemma 1 this implies:

COROLLARY 1. *The oriented homotopy type of a simply connected 4-manifold is determined by its second Betti number r, its index τ and its type,* I *or* II; *except possibly in the case of a manifold with definite quadratic form of rank $r \geqq 9$.*

(Note that the Betti number, index, and type are subject to the restrictions given by conditions (1), (2), (3) of Theorem 1.)

REMARK. The well-known classification of 2-manifolds is somewhat analogous to the description given by Corollary 1. Define the *type* of a surface to be II or I according as it is orientable or not; and let r denote the 1-dimensional mod 2 Betti number. (That is r is the dimension of $H_1(M^2, Z_2)$ over Z_2.) These two invariants characterize the surface, and are subject to the following relations: (1) r is a non-negative integer; (2) for surfaces of type II, $r \equiv 0 \pmod 2$; and (3) for surfaces of type I, $r > 0$.

The most familiar examples of simply connected 4-manifolds are the product $S^2 \times S^2$ and the complex projective plane $P_2(C)$. Let $\bar{P}_2(C)$ denote the complex projective plane with reversed orientation. The matrices of the corresponding quadratic forms are U, (1), and (—1) respectively. (Here U and V will denote the same matrices as in §1.) The following is a consequence of Corollary 1 and Lemma 2.

COROLLARY 2. *A simply connected 4-manifold of type* I *has the homotopy type of a sum of copies of $P_2(C)$ and $\bar{P}_2(C)$; except possibly when its quadratic form is definite of rank $\geqq 9$.*

(For example, the sum $P_2(C) + (S^2 \times S^2)$ must have the same homotopy type as $P_2(C) + P_2(C) + \bar{P}_2(C)$. The author does not know whether these two manifolds are homeomorphic. However, it is interesting to recall that the surface $P_2(R) + (S^1 \times S^1)$ is homeomorphic to $P_2(R) + P_2(R) + P_2(R)$.)

COROLLARY 3. *A simply connected 4-manifold of type* II *with index zero has the homotopy type of a sum of copies of $S^2 \times S^2$.*

Corollaries 2 and 3 take care of all possible homotopy types with Betti number $r \leqq 7$. The discussion could be completed very neatly if we could give an example of a simply connected 4-manifold with quadratic form V. However the following theorem asserts that such a manifold would be rather pathological.

THEOREM OF ROHLIN [15]. *If a differentiable simply connected 4-manifold has type* II, *then its index must satisfy*

$$\tau \equiv 0 \pmod{16}.$$

(Instead of $\tau \equiv 0 \pmod 8$ as in Theorem 1.) This restriction of Rohlin applies also to combinatorial[6] manifolds, since Cairns has proved ([3]) that every combinatorial 4-manifold possesses a differentiable structure.

[6] By a *combinatorial manifold* we mean a manifold in the sense of Newman [13] and Alexander [1].

The following example, suggested to the author by Hirzebruch, shows that the case $\tau = 16$ can actually occur. Let M^4 be any nonsingular algebraic surface of degree 4 in $P_3(C)$. *Then M^4 is simply connected, has second Betti number 22, index —16, and has type* II.

(Exactly the same description would hold for the Kummer surfaces, studied by Spanier in [18]. The relationship between these two examples might be interesting to study.)

PROOF. According to a remark of Lefschetz ([9] p. 57), M^4 is simply connected. Let α denote the canonical generator of $H^2(P_3(C))$, and let $i : M^4 \to P_3(C)$ denote the inclusion map. Since M^4 has degree 4 we have

$$\langle i^*(\alpha^2), \nu\rangle = 4.$$

Furthermore[7] the Chern class of the normal bundle of M^4 is $1 + 4i^*(\alpha)$. Recall that the Chern class of $P_3(C)$ is $(1 + \alpha)^4$. The Chern class $1 + c_1 + c_2$ of M_4 can now be computed by the product theorem. Solving the equation

$$(1 + c_1 + c_2)(1 + 4i^*(\alpha)) = i^*(1 + \alpha)^4,$$

we obtain

$$c_1 = 0, \qquad c_2 = 6i^*(\alpha^2).$$

Since the Stiefel-Whitney class W_2 of M^4 is equal to c_1 reduced modulo 2, Lemma 3 implies that M^4 has type II. The Euler characteristic of M^4 is equal to $\langle c_2, \nu\rangle = 24$, which implies that the middle Betti number r equals 22. Finally the formulas

$$\tau = \tfrac{1}{3}\langle p_1, \nu\rangle, \qquad p_1 = c_1^2 - 2c_2$$

imply that $\tau = -16$; which completes the proof.

We conclude by asking several questions. The basic question of which quadratic forms are actually represented by simply connected 4-manifolds appears very difficult. However the following easier version would still be interesting. *Which genera of quadratic forms are represented by differentiable, simply connected 4-manifolds?* The first unanswered case of this is the following. *Does there exist a differentiable, simply connected 4-manifold of type* II *with index* 16 *and Betti number* 16? A positive answer to this question would imply that every genus compatible with the Rohlin theorem actually occurs.

Another interesting problem would be the following. *Which genera are represented by simply connected non-singular algebraic surfaces?*

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[7] The basic reference for the following discussion is Hirzebruch [7] pp. 66–73, 85. See also Steenrod [19] p. 212.

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