ON THE 3-DIMENSIONAL BRIESKORN MANIFOLDS $M(p,q,r)$

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§1. Introduction

Let $M = M(p,q,r)$ be the smooth, compact 3-manifold obtained by intersecting the complex algebraic surface

$$z_1{}^p + z_2{}^q + z_3{}^r = 0$$

of Pham and Brieskorn with the unit sphere $|z_1|^2 + |z_2|^2 + |z_3|^2 = 1$. Here p, q, r should be integers ≥ 2. In strictly topological terms, M can be described as the r-fold cyclic branched covering of the 3-sphere, branched along a torus knot or link of type (p,q). See 1.1 below.

The main result of this paper is that M is diffeomorphic to a coset space of the form $\Pi \backslash G$ where G is a simply-connected 3-dimensional Lie group and Π is a discrete subgroup. In particular the fundamental group $\pi_1(M)$ is isomorphic to this discrete subgroup $\Pi \subset G$. There are three possibilities for G, according as the rational number $p^{-1} + q^{-1} + r^{-1} - 1$ is positive, negative, or zero. In the positive case discussed in Section 4, G is the unit 3-sphere group $SU(2)$, and Π is a finite subgroup of order $4(pqr)^{-1}(p^{-1}+q^{-1}+r^{-1}-1)^{-2}$. (See Section 3.2.) In the negative case discussed in Section 6, G is the universal covering group of $SL(2,R)$. The proof in this case is based on a study of automorphic forms of fractional degree. In both of these cases the discrete subgroup $\Pi \cong \pi_1(M)$ can be characterized as the commutator subgroup $[\Gamma,\Gamma]$ of a certain "centrally extended triangle group" $\Gamma \subset G$. [See Section 3. This result has also been obtained by C. Giffen (unpublished).] The centrally extended triangle group Γ has a presentation with generators $\gamma_1, \gamma_2, \gamma_3$ and relations

$$\gamma_1{}^p = \gamma_2{}^q = \gamma_3{}^r = \gamma_1\gamma_2\gamma_3 .$$

(Compare [Coxeter].) It follows that M is diffeomorphic to the maximal abelian covering space of the 3-manifold $\Gamma\backslash G$.

These statements break down when $p^{-1} + q^{-1} + r^{-1} = 1$. However, it is shown in Section 8 that M can still be described as a coset space $\Pi\backslash G$ where G is now a nilpotent Lie group, and Π is a (necessarily nilpotent) discrete subgroup. The proof is based on a more general fibration criterion. (Section 7.)

The author is indebted to conversations with J. -P. Serre, F. Raymond, and J. Joel.

HISTORICAL REMARKS. The triangle groups were introduced by H. A. Schwarz in the last century. [Three-dimensional analogues have recently been constructed by W. Thurston (unpublished).] The study in Section 5 of automorphic forms clearly is based on the work of Klein, Fricke, Poincaré and others. The manifolds $M = M(p, q, r)$ and their $(2n-1)$-dimensional analogues were introduced by [Brieskorn, 1966]. He computed the order of the homology group $H_1(M; Z)$, showing that M has the homology of a 3-sphere if and only if the numbers p, q, r are pairwise relatively prime. From the point of view of branched covering manifolds, this same result had been obtained much earlier by [Seifert, p. 222]. Those Brieskorn manifolds with $p^{-1} + q^{-1} + r^{-1} > 1$ have long been studied by algebraic geometers: Compare the discussion in [Milnor, 1968, §9.8] as well as [Milnor, 1974]. Those singular points of algebraic surfaces with finite local fundamental group have been elegantly characterized by [Prill] and [Brieskorn, 1967/68]. Those with infinite nilpotent local fundamental group have been elegantly classified by [Wagreich]. For other recent work on such singularities see [Arnol'd], [Conner and Raymond], [Orlik], [Saito], and [Siersma]. The work of [Dolgačev] and [Raymond and Vasquez] is particularly close to the present manuscript.

To conclude this introduction, here is an alternative description of $M(p, q, r)$. Recall that the *torus link* $L(p, q)$ of type (p, q) can be defined as the set of points (z_1, z_2) on the unit 3-sphere which satisfy the equation

$$z_1{}^p + z_2{}^q = 0 .$$

This link has d components, where d is the greatest common divisor of p and q. The n-th component, $1 \leq n \leq d$, can be parametrized by setting

$$z_1 = e(t/p), \quad z_2 = e((t+n+\tfrac{1}{2})/q)$$

for $0 \leq t \leq pq/d$, where $e(\alpha)$ stands for the exponential function $e^{2\pi i\alpha}$. Note that this link $L(p,q)$ has a canonical orientation.

LEMMA 1.1. *The Brieskorn manifold* $M(p,q,r)$ *is homeomorphic to the* r*-fold cyclic branched covering of* S^3, *branched along a torus link of type* (p,q).

Proof. Let $V \subset \mathbf{C}^3$ be the Pham-Brieskorn variety $z_1{}^p + z_2{}^q + z_3{}^r = 0$, non-singular except at the origin. Consider the projection map

$$(z_1, z_2, z_3) \mapsto (z_1, z_2)$$

from $V-0$ to $\mathbf{C}^2-0$. If we stay away from the branch locus $z_1{}^p + z_2{}^q = 0$, then clearly each point of $\mathbf{C}^2-0$ has just r pre-images in V. In fact these r pre-images are permuted cyclically by the group Ω of r-th roots of unity, acting on $V-0$ by the rule

$$\omega : (z_1, z_2, z_3) \mapsto (z_1, z_2, \omega z_3)$$

for $\omega^r = 1$. Thus the quotient space $\Omega \backslash (V-0)$ maps homeomorphically onto $\mathbf{C}^2-0$. It follows easily that $V-0$ is an r-fold branched cyclic covering of $\mathbf{C}^2-0$, branched along the algebraic curve $z_1{}^p + z_2{}^q = 0$.

Now let the group $\mathbf{R}^+$ of positive real numbers operate freely on $V-0$ by the rule

$$t : (z_1, z_2, z_3) \mapsto (t^{1/p}z_1, t^{1/q}z_2, t^{1/r}z_3)$$

for $t > 0$. Since every R^+-orbit intersects the unit sphere transversally and precisely once, it follows that $V - 0$ is canonically diffeomorphic to $R^+ \times M(p,q,r)$. Note that this action of R^+ on $V - 0$ commutes with the action of Ω.

Similarly, letting R^+ act freely on $C^2 - 0$ by the rule $t:(z_1, z_2) \mapsto (t^{1/p}z_1, t^{1/q}z_2)$, it follows that $C^2 - 0$ is canonically diffeomorphic to $R^+ \times S^3$. The projection map $V - 0 \to C^2 - 0$ is R^+-equivariant. Therefore, forming quotient spaces under the action of R^+, it follows easily that $M(p,q,r)$ is an r-fold cyclic branched covering of S^3 with branch locus $L(p,q)$. ∎ (Compare [Durfee and Kauffman], [Neumann].)

§2. The Schwarz triangle groups $\Sigma^* \supset \Sigma$

This section will be an exposition of classical material due to H. A. Schwarz and W. Dyck. (For other presentations see [Caratheodory], [Siegel], [Magnus].) We will work with any one of the three classical simply connected 2-dimensional geometries. Thus by the "plane" P we will mean either the surface of a unit 2-sphere, or the Lobachevsky plane [e.g., the upper half-plane $y > 0$ with the Poincaré metric $(dx^2 + dy^2)/y^2$], or the Euclidean plane. In different language, P is to be a complete, simply-connected, 2-dimensional Riemannian manifold of constant curvature $+1$, -1, or 0.

We recall some familiar facts. Given angles α, β, γ with $0<\alpha,\beta,\gamma<\pi$, there always exists a triangle T bounded by geodesics, in a suitably chosen plane P, with interior angles α, β, and γ. In fact P must be either spherical, hyperbolic, or Euclidean according as the difference $\alpha + \beta + \gamma - \pi$ is positive, negative, or zero. In the first two cases the area of the triangle T is precisely $|\alpha + \beta + \gamma - \pi|$, but in the Euclidean case the area of T can be arbitrary.

We are interested in a triangle with interior angles π/p, π/q, and π/r respectively, where $p, q, r \geq 2$ are fixed integers. Thus this triangle $T = T(p,q,r)$ lies either in the spherical, hyperbolic, or Euclidean plane according as the rational number $p^{-1} + q^{-1} + r^{-1} - 1$ is positive, negative, or zero.

DEFINITION. By the *full Schwarz triangle group* $\Sigma^* = \Sigma^*(p,q,r)$ we will mean the group of isometries of P which is generated by reflections $\sigma_1, \sigma_2, \sigma_3$ in the three edges of $T(p,q,r)$. We will also be interested in the subgroup $\Sigma \subset \Sigma^*$ of index 2, consisting of all orientation preserving elements of Σ^*.

REMARK 2.1. Before studying these groups further, it may be helpful to briefly list the possibilities. Let us assume for convenience that $p \le q \le r$ In the *spherical case* $p^{-1} + q^{-1} + r^{-1} > 1$, it is easily seen that (p,q,r) must be one of the triples

$$(2,3,3),\ (2,3,4),\ (2,3,5);\ \text{or}\ (2,2,r)$$

for some $r \ge 2$. The corresponding group $\Sigma(p,q,r)$ of rotations of the sphere is respectively either the tetrahedral, octahedral, or icosahedral group; or a dihedral group of order $2r$. The area of the associated triangle T can be any number of the form π/n with $n \ge 2$. In the *Euclidean case* $p^{-1} + q^{-1} + r^{-1} = 1$, the triple (p,q,r) must be either

$$(2,3,6),\ (2,4,4),\ \text{or}\ (3,3,3)\ .$$

For all of the infinitely many remaining triples, we are in the *hyperbolic case* $p^{-1} + q^{-1} + r^{-1} < 1$. The area of the hyperbolic triangle T can range from the minimum value of $(1 - 2^{-1} - 3^{-1} - 7^{-1})\pi = \pi/42$ to values arbitrarily close to π.

The structure of the full triangle group $\Sigma^* = \Sigma^*(p,q,r)$ is described in the following basic assertion. Recall that Σ^* is generated by reflections $\sigma_1, \sigma_2, \sigma_3$ in the three edges of a triangle $T \subset P$ whose interior angles are π/p, π/q, and π/r.

THEOREM 2.2 (Poincaré). *The triangle* T *itself serves as fundamental domain for the action of the group* Σ^* *on the "plane"* P. *In other words the various images* $\sigma(T)$ *with* $\sigma \in \Sigma^*$ *are mutually disjoint except for boundary points, and* *cover all of* P. *This group* Σ^* *has a presentation with generators* $\sigma_1, \sigma_2, \sigma_3$ *and relations*

$$\sigma_1^{\,2} = \sigma_2^{\,2} = \sigma_3^{\,2} = 1$$

and

$$(\sigma_1\sigma_2)^p = (\sigma_2\sigma_3)^q = (\sigma_3\sigma_1)^r = 1 \ .$$

Here it is to be understood that the edges are numbered so that the first two edges e_1 and e_2 enclose the angle of π/p, while e_2 and e_3 enclose the angle of π/q, and e_3, e_1 enclose π/r.

Proof of 2.2. Inspection shows that the composition $\sigma_1\sigma_2$ is a rotation through the angle $2\pi/p$ about the first vertex of the triangle T, so the relation $(\sigma_1\sigma_2)^p = 1$ is certainly satisfied in the group Σ^*. The other five relations can be verified similarly.

Let $\hat{\Sigma}$ denote the abstract group which is defined by a presentation with generators $\hat{\sigma}_1, \hat{\sigma}_2, \hat{\sigma}_3$ and with relations $\hat{\sigma}_i^{\,2} = 1$ and $(\hat{\sigma}_1\hat{\sigma}_2)^p = (\hat{\sigma}_2\hat{\sigma}_3)^q = (\hat{\sigma}_3\hat{\sigma}_1)^r = 1$. Thus there is a canonical homomorphism $\hat{\sigma} \mapsto \sigma$ from $\hat{\Sigma}$ onto Σ^*, and we must prove that this canonical homomorphism is actually an isomorphism.

Form a simplicial complex K as follows. Start with the product $\hat{\Sigma} \times T$, consisting of a union of disjoint triangles $\hat{\sigma} \times T$, one such triangle for each group element. Now for each $\hat{\sigma}$ and each $i = 1, 2, 3$ paste the i-th edge of $\hat{\sigma} \times T$ onto the i-th edge of $\hat{\sigma}\hat{\sigma}_i \times T$. More precisely, let K be the identification space of $\hat{\Sigma} \times T$ in which $(\hat{\sigma}, x)$ is identified with $(\hat{\sigma}\hat{\sigma}_i, x)$ for each $\hat{\sigma} \in \hat{\Sigma}$, for each $i = 1, 2, 3$, and for each $x \in e_i \subset T$. Using the relation $\hat{\sigma}_i^{\,2} = 1$, we see that precisely two triangles are pasted together along each edge of K.

Consider the canonical mapping $\hat{\Sigma} \times T \to P$ which sends each pair $(\hat{\sigma}, x)$ to the image $\sigma(x)$ (using the homomorphism $\hat{\sigma} \mapsto \sigma$ from $\hat{\Sigma}$ to the group Σ^* of isometries of P). This mapping is compatible with the identification $(\hat{\sigma}, x) \equiv (\hat{\sigma}\hat{\sigma}_i, x)$ for $x \in e_i$ since the reflection σ_i fixes e_i. Hence there is an induced map

$$f : K \to P \ .$$

We must prove that f is actually a homeomorphism.

First consider the situation around a vertex $(\hat{\sigma}, v)$ of K. To fix our ideas, suppose that v is the vertex $e_1 \cap e_2$ of T. Using the identifications

$$(\hat{\sigma}, v) \equiv (\hat{\sigma}\hat{\sigma}_1, v) = (\hat{\sigma}\hat{\sigma}_1\hat{\sigma}_2, v) = (\hat{\sigma}\hat{\sigma}_1\hat{\sigma}_2\hat{\sigma}_1, v) \cdots ,$$

together with the relation $(\hat{\sigma}_1\hat{\sigma}_2)^p = 1$, we see that precisely $2p$ triangles of K fit cyclically around the vertex $(\hat{\sigma}, v)$. (These $2p$ triangles are distinct since the $2p$ elements $\hat{\sigma}_1, \hat{\sigma}_1\hat{\sigma}_2, \hat{\sigma}_1\hat{\sigma}_2\hat{\sigma}_1, \cdots, (\hat{\sigma}_1\hat{\sigma}_2)^p$ of $\hat{\Sigma}$ map to distinct elements of Σ^*.) *Now inspection shows that the star neighborhood, consisting of* $2p$ *triangles fitting around a vertex of* K *maps homeomorphically onto a neighborhood of the image point* $\sigma(v)$ *in* P. The image neighborhood is the union of $2p$ triangles in P, each with interior angle π/p at the common vertex $\sigma(v)$.

Thus the canonical map $f : K \to P$ is locally a homeomorphism. But it is not difficult to show that every path in P can be lifted to a path in K. Therefore f is a covering map. *Since* P *is simply connected, this implies that* f *is actually a homeomorphism.* The conclusions that $\hat{\Sigma}$ maps isomorphically to the group Σ^*, and that the various images $\sigma(T)$ cover P with only boundary points in common, now follow immediately. ■

REMARK 2.3. More generally, following Dyck, one can consider a convex n-sided polygon A with interior angles $\pi/p_1, \cdots, \pi/p_n$. Again A is the fundamental domain for a group $\Sigma^* = \Sigma^*(A)$ of isometries which is generated by the reflections $\sigma_1, \cdots, \sigma_n$ in the edges of A with relations

$$\sigma_i^{\,2} = (\sigma_i \sigma_{i+1})^{p_i} = 1$$

for all i modulo n. In fact the above proof extends to this more general case without any essential change.

COROLLARY 2.4. *In the spherical case* $p^{-1} + q^{-1} + r^{-1} > 1$, *the full triangle group* $\Sigma^*(p,q,r)$ *is finite of order* $4/(p^{-1} + q^{-1} + r^{-1} - 1)$. *In the remaining cases* $p^{-1} + q^{-1} + r^{-1} \leq 1$, *the group* $\Sigma^*(p,q,r)$ *is infinite.*

Proof. Since the various images $\sigma(T)$ form a non-overlapping covering of P, the order of Σ^* can be computed as the area of P divided by the area of T. ∎

Recall that Σ denotes the subgroup of index 2 consisting of all orientation preserving isometries in the full triangle group Σ^*. Setting

$$\tau_1 = \sigma_1\sigma_2, \quad \tau_2 = \sigma_2\sigma_3, \quad \tau_3 = \sigma_3\sigma_1 ,$$

note that the product

$$\tau_1\tau_2\tau_3 = \sigma_1\sigma_2\sigma_2\sigma_3\sigma_3\sigma_1$$

is equal to 1.

COROLLARY 2.5. *The subgroup* $\Sigma(p,q,r)$ *has a presentation with generators* τ_1, τ_2, τ_3 *and relations* $\tau_1^p = \tau_2^q = \tau_3^r = \tau_1\tau_2\tau_3 = 1$.

Proof. This corollary can be derived, for example, by applying the Reidemeister-Schreier theorem.* [More generally, for the Dyck group described in 2.3 we obtain a presentation with generators $\tau_1, \cdots, \tau_n$ and relations

$$\tau_1^{p_1} = \cdots = \tau_n^{p_n} = \tau_1\tau_2\cdots\tau_n = 1.]$$

Details will be left to the reader. ∎

We conclude with three remarks which further describe these groups Σ

* See for example [Weir].

REMARK 2.6. Using 2.2, it is easy to show that an element of the group Σ has a fixed point in P if and only if it is conjugate to a power of τ_1, τ_2, or τ_3. *Hence every element of finite order in* Σ *is conjugate to a power of* τ_1, τ_2 *or* τ_3. Therefore the three integers p, q, r can be characterized as the orders of the three conjugate classes of maximal finite cyclic subgroups of Σ. (Caution: In the spherical case these three conjugate classes may not be distinct. In fact in the spherical case, since each vertex of our canonical triangulation of P is antipodal to some other vertex, it follows that each τ_i is conjugate to some τ_j^{-1} where j may be different from i.)

Here we have used the easily verified fact that every orientation preserving isometry of P of finite order has a fixed point.

THEOREM 2.7 (R. H. Fox). *The triangle group* $\Sigma(p, q, r)$ *contains a normal subgroup* N *of finite index which has no elements of finite order.*

[Fox] constructs two finite permutations of orders p and q so that the product permutation has order r. The subgroup N is then defined as the kernel of the evident homomorphism from Σ to the finite group generated by these two permutations. Using 2.6 we see that N has no elements of finite order. ■

Note also that N operates freely on P; that is, no non-trivial group element has a fixed point in P. Hence the quotient space $N\backslash P$ is a smooth compact Riemann surface which admits the finite group Σ/N as a group of conformal automorphisms. To compute the Euler characteristic $\chi(N\backslash P)$ of this Riemann surface, we count vertices, edges, and faces of the canonical triangulation of $N\backslash P$, induced from the triangulation of 2.2. This yields the formula

$$\chi(N\backslash P) \quad (p^{-1}+q^{-1}+r^{-1}-1)\ \text{order}\ (\Sigma/N)\ .$$

In the hyperbolic case $p^{-1} + q^{-1} + r^{-1} < 1$, it follows that the triangle group $\Sigma \supset N$ contains free non-abelian subgroups. For N is the fundamental group of a surface of genus $g \geq 2$, hence any subgroup of infinite index in N is the fundamental group of a non-compact surface and therefore is free.

Note that a given finite group Φ can occur as such a quotient Σ/N if and only if Φ is generated by two elements, and has order at least 3. For if Φ is generated by elements of order p and q, and if the product of these two generators has order r, then $\Sigma(p, q, r)$ maps onto Φ, and it follows from 2.6 that the kernel has no element of finite order. As an example, the triangle group $\Sigma(2, 3, 7)$ maps onto the simple group of order 168. (Compare [Klein and Fricke, pp. 109, 737] as well as [Klein, Entwicklung ⋯, p. 369].) Hence this simple group operates conformally on a Riemann surface $N \backslash P$ whose genus $g = 3$ can be computed from the equation $2 - 2g = 168(1 - 2^{-1} - 3^{-1} - 7^{-1})$.

More generally let Λ be any discrete group of isometries of P with compact fundamental domain. (That is, assume that there exists a compact set $K \subset P$ with non-vacuous interior so that the various translates of K by elements of Λ cover P, and have only boundary points in common.) *Then* Λ *also contains a normal subgroup* N *of finite index which operates freely on* P. (See [Fox] and [Bungaard, Nielsen]. A much more general theorem of this nature has been proved by [Selberg, Lemma 8].) Again the Euler characteristic $\chi(N \backslash P)$ of the smooth compact quotient surface is directly proportional to the index of N in Λ. In fact, the ratio $\chi(N \backslash P)/\text{order}(\Lambda/N)$ can be computed as a product $\chi(B_\Lambda)\chi(P)$ where the rational number $\chi(B_\Lambda)$ is the Euler characteristic of Λ in the sense of [Wall], and where $\chi(P) = \sum (-1)^n \text{ rank } H_n(P)$ is the usual Euler characteristic, equal to 1 or 2. Now assume that Λ preserves orientation.

The quotient $S = \Lambda \backslash P$ can itself be given the structure of a compact Riemann surface, even if Λ has elements of finite order. (Compare 6.3.) In general there will be finitely many ramification points, say $x_1, \cdots, x_k \in S$. Let $r_1, \cdots, r_k \geq 2$ be the corresponding ramification indices. Then classically the data $(S; x_1, \cdots, x_k; r_1, \cdots, r_k)$ provides a complete invariant for the group Λ. That is: a second such group Λ' is conjugate to Λ within the group of orientation preserving isometries of P if and only if the Riemann surface $S' = \Lambda' \backslash P$ is isomorphic to S under an isomorphism which preserves ramification points and ramification indices. The triangle group $\Sigma(p, q, r)$ corresponds to the special case where S has genus zero with three ramification points having ramification indices p, q, r.

REMARK 2.8. It is sometimes possible to deduce inclusion relations between the various groups $\Sigma(p, q, r)$ by noting that a triangle $T(p, q, r)$ can be decomposed into smaller triangles of the form $T(p', q', r')$. For example if $p = q$ one sees in this way that

$$\Sigma(p, p, r) \subset \Sigma(2, p, 2r)$$

as a necessarily normal subgroup of index 2. Similarly, taking $p = r$ one sees that

$$\Sigma(2, p, 2p) \subset \Sigma(2, 3, 2p)$$

as an abnormal subgroup of index 3. However, not all inclusions can be derived in this manner. A counterexample is provided by the inclusion $\Sigma(2, 3, 3) \subset \Sigma(2, 3, 5)$ of the alternating group on four letters into the alternating group on five letters.

§3. The centrally extended triangle group $\Gamma(p,q,r)$

As in the last section, let P denote either the Euclidean plane or the plane of spherical or hyperbolic geometry. Let $\overline{G}$ denote the connected Lie group consisting of all orientation preserving isometries of P. Then we can form the coset space $\overline{G}/\Sigma$ where

$$\Sigma = \Sigma(p, q, r) \subset \overline{G}$$

is the triangle group of Section 2. Clearly $\overline{G}/\Sigma$ is a compact 3-dimensional manifold. To compute the fundamental group $\pi_1(\overline{G}/\Sigma)$ it is convenient to pass to the universal covering group G of $\overline{G}$.

DEFINITION. The full inverse image in G of the subgroup $\Sigma \subset \overline{G}$ will be called the *centrally extended triangle group* $\Gamma = \Gamma(p, q, r)$.

Evidently the quotient manifold $\overline{G}/\Sigma$ can be identified with G/Γ, and hence has fundamental group $\pi_1(\overline{G}/\Sigma) \cong \Gamma$.

To describe the structure of Γ, let us start with the isomorphism $G/C \cong \overline{G}$ of Lie groups, where the discrete subgroup $C \cong \pi_1(\overline{G})$ is the center of G. In the spherical case, where $\overline{G}$ is the rotation group $SO(3)$, it is well known that this fundamental group C is cyclic of order 2. In the Euclidean and hyperbolic cases we will see that C is free cyclic.

Evidently Γ, defined as the inverse image of Σ under the surjection $G \to \overline{G}$, contains C as a central subgroup with $\Gamma/C \cong \Sigma$. [In fact one can verify that C is precisely the center of Γ.] The main object of this section is to prove the following.

LEMMA 3.1. *The centrally extended triangle group* $\Gamma = \Gamma(p,q,r)$ *has a presentation with generators* $\gamma_1, \gamma_2, \gamma_3$ *and relations* $\gamma_1{}^p = \gamma_2{}^q = \gamma_3{}^r = \gamma_1\gamma_2\gamma_3$.

Proof. We will make use of the following construction. Choose some fixed orientation for the "plane" P. Given a basepoint x and a real number θ, let

$$\overline{r}_x(\theta) \in \overline{G}$$

denote the rotation through angle θ about the point x. Thus we obtain a homomorphism $\overline{r}_x : R \to \overline{G}$ which clearly lifts to a unique homomorphism

$$r_x : R \to G$$

into the universal covering group. Since $\overline{r}_x(2\pi)$ is the identity element of $\overline{G}$, it follows that the lifted element

$$r_x(2\pi) \in G$$

belongs to the central subgroup C. We will use the notation $c = r_x(2\pi) \in C$. *In fact* C *is a cyclic group generated by* c, as one easily verifies by studying the fibration

$$\overline{G} \to P \cong \overline{G}/S^1$$

defined by the formula $\overline{g} \mapsto \overline{g}(x)$. Here S^1 denotes the group $\overline{r}_x(R) \subset \overline{G}$ consisting of all rotations about x. In the Euclidean and hyperbolic cases, since P is contractible, it follows that the fundamental group $\pi_1(S^1) \cong Z$ maps isomorphically onto $\pi_1(\overline{G}) \cong C$.

Note that this element $r_x(2\pi) \in C$ depends continuously on x, and therefore is independent of the choice of x.

Now recall that the subgroup $\Sigma \subset \overline{G}$ is generated by the three rotation:

$$\tau_1 = \overline{r}_{v_1}(2\pi/p),\ \ \tau_2 = \overline{r}_{v_2}(2\pi/q),\ \ \tau_3 = \overline{r}_{v_3}(2\pi/r)\ ,$$

where v_1, v_2, v_3 are the three vertices of T. It follows that the inverse image $\Gamma \subset G$ is generated by the three lifted rotations

$$\gamma_1 = r_{v_1}(2\pi/p),\ \ \gamma_2 = r_{v_2}(2\pi/q),\ \ \gamma_3 = r_{v_3}(2\pi/r)\ ,$$

together with the central element c. Clearly

$$\gamma_1{}^p = \gamma_2{}^q = \gamma_3{}^r = c\ .$$

Next consider the product $\gamma_1\gamma_2\gamma_3$. Since $\tau_1\tau_2\tau_3 = 1$, it is clear that $\gamma_1\gamma_2\gamma_3$ belongs to C, and hence is equal to c^k for some integer k. We must compute this unknown integer k.

It will be convenient to work with a more general triangle, with arbitrary angles. In fact, without complicating the argument, we can just as well consider an n-sided convex polygon $A \subset P$ with interior angles $\alpha_1, \cdots, \alpha_n$. Here we assume that $0 < \alpha_i < \pi$. If σ_i denotes the reflection in the i-th edge (suitably numbered), then $\sigma_i{}^2 = 1$, and therefore

$$(\sigma_1\sigma_2)(\sigma_2\sigma_3) \cdots (\sigma_{n-1}\sigma_n)(\sigma_n\sigma_1) = 1\ .$$

Lifting each rotation

$$\sigma_i\sigma_{i+1} = \bar{r}_{v_i}(2\alpha_i) \in \bar{G}$$

to the element

$$\gamma_i = r_{v_i}(2\alpha_i) \in G ,$$

it follows that the product $\gamma_1\gamma_2 \cdots \gamma_n$ belongs to the central subgroup C. Now as we vary the polygon A continuously, this central element $\gamma_1 \cdots \gamma_n$ must also vary continuously. But C is a discrete group, so $\gamma_1 \cdots \gamma_n$ must remain constant.

In particular we can shrink the polygon A down towards a point x, in such manner that the angles $\alpha_1, \cdots, \alpha_n$ tend towards the angles $\beta_1, \cdots, \beta_n$ of some Euclidean n-sided polygon. Thus the element $\gamma_i = r_{v_i}(2\alpha_i) \in G$ tends towards the limit $r_x(2\beta_i)$, while the product $\gamma_1 \cdots \gamma_n$ tends towards the product $r_x(2\beta_1 + \cdots + 2\beta_n)$. Therefore, using the formula

$$\beta_1 + \cdots + \beta_n = (n-2)\pi$$

for the sum of the angles of a Euclidean polygon, we see that the constant product $\gamma_1 \cdots \gamma_n$ must be equal to

$$r_x((n-2)2\pi) = c^{n-2} .$$

Finally, specializing to the case $n = 3$, we obtain the required identity $\gamma_1\gamma_2\gamma_3 = c$.

Thus we have proved that Γ is generated by elements $\gamma_1, \gamma_2, \gamma_3$, and c which satisfy the relations

$$\gamma_1{}^p = \gamma_2{}^q = \gamma_3{}^r = \gamma_1\gamma_2\gamma_3 = c .$$

Conversely, if $\hat{\Gamma}$ denotes the group which is defined abstractly by generators $\hat{\gamma}_1, \hat{\gamma}_2, \hat{\gamma}_3, \hat{c}$ and corresponding relations, then certainly the element $\hat{c} \in \hat{\Gamma}$ generates a central subgroup $\hat{C}$, with quotient $\hat{\Gamma}/\hat{C}$ isomorphic to Σ by Section 2.5. Thus we obtain the commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \hat{C} & \longrightarrow & \hat{\Gamma} & \longrightarrow & \Sigma & \longrightarrow & 1 \\ & & \downarrow\text{onto} & & \downarrow\text{onto} & & \downarrow\cong & & \\ 1 & \longrightarrow & C & \longrightarrow & \Gamma & \longrightarrow & \Sigma & \longrightarrow & 1\ . \end{array}$$

In the Euclidean and hyperbolic cases, C is free cyclic, hence $\hat{C}$ maps isomorphically to C, and it follows that $\hat{\Gamma}$ maps isomorphically to Γ.

In the spherical case, since C is cyclic of order 2, we must prove that $\hat{c}^2 = 1$ in order to complete the proof. This relation can be verified by a case by case computation. (Compare [Coxeter].) There is an alterna tive argument which can be sketched as follows.

To prove that $\hat{c}^2 = 1$, it suffices to show that $\hat{c}^2$ maps to 1 in the abelianized group $\hat{\Gamma}/[\hat{\Gamma},\hat{\Gamma}]$. For clearly $\hat{\Gamma}$ is a central extension of the form

$$1 \to \hat{C}^2 \to \hat{\Gamma} \to \Gamma \to 1\ .$$

Such a central extension is determined by a characteristic cohomology class in $H^2(\Gamma;\ \hat{C}^2)$. Consider the universal coefficient theorem

$$0 \to \operatorname{Ext}(H_1\Gamma, \hat{C}^2) \to H^2(\Gamma; \hat{C}^2) \to \operatorname{Hom}(H_2\Gamma, \hat{C}^2) \to 0$$

[Spanier, p. 243]. The group $H_2\Gamma$ is zero by Poincaré duality, since the finite group Γ is fundamental group of a closed 3-manifold. Therefore our extension is induced from an element of $\operatorname{Ext}(H_1\Gamma, \hat{C}^2)$, or in other words from an abelian group extension of the form

$$0 \to \hat{C}^2 \to A \to H_1\Gamma \to 0\ .$$

Thus we obtain a commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \hat{C}^2 & \longrightarrow & \hat{\Gamma} & \longrightarrow & \Gamma & \longrightarrow & 1 \\ & & \downarrow\cong & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \hat{C}^2 & \longrightarrow & A & \longrightarrow & H_1\Gamma & \longrightarrow & 0 \end{array}$$

with A abelian. *Therefore, in the spherical case, the group $\hat{C}^2$ generated by $\hat{c}^2$ maps injectively into the abelianized group $\hat{\Gamma}/[\hat{\Gamma},\hat{\Gamma}]$.*

But a straightforward matrix computation shows that $\hat{c}$ maps to an element of order $m(p^{-1}+q^{-1}+r^{-1}-1)$ in this abelianized group, where m is the least common multiple of p, q, r. In all of the spherical cases this product is 1 or 2, so $\hat{c}^2 = 1$. ∎

REMARK. Similarly for the Dyck group of Section 2.3 one obtains a central extension with generators $\gamma_1, \cdots, \gamma_n$ and with relations

$$\gamma_1^{p_1} = \cdots = \gamma_n^{p_n} = c \quad \text{and} \quad \gamma_1 \cdots \gamma_n = c^{n-2} .$$

COROLLARY 3.2. *The abelianized group* $\Gamma/[\Gamma,\Gamma]$ *has order* $|qr+pr+pq-pqr| = pqr\,|p^{-1}+q^{-1}+r^{-1}-1|$.

Here we adopt the usual convention that an infinite group has "order" zero. Thus the commutator subgroup has finite index in Γ if and only if $p^{-1}+q^{-1}+r^{-1} \neq 1$. To prove this corollary, we apply the usual theorem that the order of an abelianized group with n generators and n relations is equal to the absolute value of the determinant of the $n \times n$ relation matrix. Taking the three relations to be $\gamma_1\gamma_2\gamma_3\gamma_1{}^{-p} = 1$, $\gamma_1\gamma_2\gamma_3\gamma_2{}^{-q} = 1$, $\gamma_1\gamma_2\gamma_3\gamma_3{}^{-r} = 1$, the relation matrix becomes

$$\begin{bmatrix} 1-p & 1 & 1 \\ 1 & 1-q & 1 \\ 1 & 1 & 1-r \end{bmatrix}$$

with determinant $qr+pr+pq-pqr$, as required. ∎

In the spherical case $p^{-1}+q^{-1}+r^{-1} > 1$, since Γ has order $4/(p^{-1}+q^{-1}+r^{-1}-1)$ as a consequence of 2.4, it follows that the commutator subgroup $[\Gamma,\Gamma]$ has order $4/(pqr(p^{-1}+q^{-1}+r^{-1}-1)^2)$.

One case of particular interest occurs when p, q, r are pairwise relatively prime. In this case the index $i = |qr+pr+pq-pqr|$ of $[\Gamma,\Gamma]$ in Γ is relatively prime to pqr. Therefore, using 2.6, it follows that for any element γ of Γ which has finite order modulo the center C there exists an element γ^i of $[\Gamma,\Gamma]$ having the same finite order modulo $[\Gamma,\Gamma] \cap C$. It then follows that the three integers p, q, r are invariants of the group $[\Gamma,\Gamma]$. Namely, they can be characterized as the orders of the maximal finite cyclic subgroups of $[\Gamma,\Gamma]$ modulo its center $[\Gamma,\Gamma] \cap C$.

§4. The spherical case $p^{-1} + q^{-1} + r^{-1} > 1$

This section gives a concrete description of the Brieskorn manifolds M(p, q, r) in the spherical case. Since the conclusions are well known, the presentation is mainly intended as motivation for the analogous arguments in Section 6.

Let Γ be any finite subgroup of the group SU(2) of unimodular 2×2 unitary matrices, acting by matrix multiplication on the complex coordinate space $\mathbf{C}^2$. Note that SU(2) acts simply transitively on each sphere centered at the origin.

DEFINITION. A complex polynomial $f(z) = f(z_1, z_2)$ is Γ-invariant if

$$f(\gamma(z)) = f(z)$$

for all $\gamma \in \Gamma$ and all $z \in \mathbf{C}^2$. Let $H_\Gamma^{n,1}$ denote the finite dimensional vector space consisting of all homogeneous polynomials of degree n which are Γ-invariant. More generally, given any character of Γ, that is any homomorphism

$$\chi : \Gamma \to U(1) \subset \mathbf{C}^{\cdot} = \mathbf{C} - 0$$

from Γ to the unit circle, let $H_\Gamma^{n,\chi}$ denote the space of all homogeneous polynomials f of degree n which transform according to the rule

$$f(\gamma(z)) = \chi(\gamma) f(z) \ .$$

Note that the product of a polynomial in $H_\Gamma^{n,\chi}$ and a polynomial in $H_\Gamma^{m,\rho}$ belongs to the space $H_\Gamma^{n+m,\chi\rho}$. Thus the set of $H_\Gamma^{n,\chi}$ for all n and χ forms a bigraded algebra, which we denote briefly by the symbol $H_\Gamma^{*,*}$. This bigraded algebra possesses an identity element $1 \in H_\Gamma^{0,1}$.

LEMMA 4.1. *Let* $\Pi = [\Gamma, \Gamma]$ *be the commutator subgroup of* Γ. *Then the space* $H_\Pi^{n,1}$ *of* Π*-invariant homogeneous polynomials of degree* n *is equal to the direct sum of its subspaces* $H_\Gamma^{n,\chi}$ *as* χ *varies over all characters of* Γ.

Proof. Since every character of Γ annihilates Π, it follows that $H_\Gamma^{n,\chi} \subset H_\Pi^{n,1}$. On the other hand, since Π is normal in Γ, it follows that the quotient group Γ/Π operates linearly on $H_\Pi^{n,1}$. In fact, for each Π-invariant homogeneous polynomial f and each $\gamma \in \Gamma$ let $f\gamma$ denote the polynomial

$$z \mapsto f(\gamma(z)) .$$

(Thus Γ acts on the right.) This new polynomial is also Π-invariant since

$$(f\gamma)\pi = (f(\gamma\pi\gamma^{-1}))\gamma = f\gamma$$

for $\pi \in \Pi$. Clearly $f\gamma = f\gamma'$ whenever $\gamma \equiv \gamma' \bmod \Pi$. Since Γ/Π is finite and abelian, it follows that $H_\Pi^{n,1}$ splits as a direct sum of eigenspaces corresponding to the various characters of Γ/Π. ∎

Now consider a homogeneous polynomial $f \in H_\Gamma^{n,\chi}$ for some n and χ According to the fundamental theorem of algebra, f must vanish along n (not necessarily distinct) lines $L_1, \cdots, L_n$ through the origin in $\mathbb{C}^2$. Given these lines, the polynomial f is uniquely determined up to a multiplicative constant. Evidently each element of the group Γ must permute these n lines. Conversely, given n lines through the origin which are permuted by Γ, the corresponding homogeneous polynomial $f(z)$ of degree n clearly has the property that the rotated polynomial $f(\gamma(z))$ is a scalar multiple of $f(z)$ for each group element γ. Setting

$$f(\gamma(z))/f(z) = \chi(\gamma)$$

we obtain a character χ of Γ so that $f \in H_\Gamma^{n,\chi}$.

Let us apply these constructions to the centrally extended triangle group $\Gamma = \Gamma(p, q, r)$ of Section 3; where $p^{-1} + q^{-1} + r^{-1} > 1$. To do this we must identify SU(2) with the universal covering group G of Section 3. In fact, SU(2) operates naturally on the projective space $P = P^1(C)$ of lines through the origin in C^2. Or rather, since the central element $-I$ carries each line to itself, the quotient group $\overline{G} = SU(2)/\{\pm I\}$ operates on P, which is topologically a 2-dimensional sphere. Choosing a $\overline{G}$-invariant metric, we see easily that P will serve as model for 2-dimensional spherical geometry, with $\overline{G}$ as group of orientation preserving isometries and $G = SU(2)$ as universal covering group.

Let $k = 2/(p^{-1} + q^{-1} + r^{-1} - 1)$ denote the order of the quotient group $\Sigma = \Gamma/\{\pm I\}$. Then, by 2.6, nearly every orbit for the action of Σ on P contains k distinct points. The only exceptions are the three orbits containing the three vertices of the triangle T. These three exceptional orbits contain k/p, k/q, and k/r points respectively.

Let $f_1 \in H_\Gamma^{k/p,\chi_1}$, for appropriately chosen χ_1, be the polynomial which vanishes on the k/p lines through the origin corresponding to the orbit of the first vertex of T. Similarly construct the polynomials $f_2 \in H_\Gamma^{k/q,\chi_2}$ and $f_3 \in H_\Gamma^{k/r,\chi_3}$, each well defined up to a multiplicative constant. We will need some partial information about these three characters χ_1, χ_2, and χ_3.

LEMMA 4.2. *The three homomorphisms* $\chi_i : \Gamma \to U(1)$ *constructed in this way satisfy the relation* $\chi_1{}^p = \chi_2{}^q = \chi_3{}^r$.

Proof. Let $\gamma'_1, \cdots, \gamma'_k \in \Gamma$ be a set of representatives for the cosets of the subgroup $\{\pm I\} \subset \Gamma$. Then to each linear form $\ell(z) = a_1 z_1 + a_2 z_2$ we can associate the homogeneous polynomial

$$f(z) = \ell(\gamma'_1(z)) \cdots \ell(\gamma'_k(z))$$

of degree k. The argument above shows that $f \in H_\Gamma^{k,\chi_0}$ for some χ_0. Evidently this character χ_0 depends continuously on the linear form ℓ, and hence is independent of ℓ. Now specializing to the case where $\ell(z)$

vanishes at the line corresponding to one vertex of the triangle T, we see easily that $\chi_1{}^p = \chi_2{}^q = \chi_3{}^r = \chi_0$. ∎

REMARK. The characters χ_i themselves can be computed by the methods of Section 6.1. In fact, writing p_1, p_2, p_3 in place of p, q, r, the character $\chi_i(\gamma_j)$ is equal to $e(-k/2p_ip_j)$ for $i \neq j$ and to $e(1/p_j)e(-k/2p_jp_j)$ for $i = j$.

We are now ready to prove the following basic result.

LEMMA 4.3. *These three polynomials* f_1, f_2, f_3 *generate the bigraded algebra* $H_\Gamma^{*,*}$. *They satisfy a polynomial relation which, after multiplying each* f_i *by a suitable constant if necessary, takes the form* $f_1{}^p + f_2{}^q + f_3{}^r = 0$.

Proof. Let $f \in H_\Gamma^{n,\chi}$ be an arbitrary non-zero element of the bigraded algebra. Then f must have n zeros in $P = P^1(C)$. If one of these zeros lies at the i-th vertex of the triangle T, then clearly f is divisible by f_i. If f does not vanish at any vertex of T, then it must vanish at some point $x \in P$ which lies in an orbit with k distinct elements. Choose $\lambda \neq 0$ so that the linear combination $f_1{}^p + \lambda f_2{}^q \in H_\Gamma^{k,\chi_0}$ also vanishes at x, and hence vanishes precisely at the points of the orbit containing x. Then f is divisible by $f_1{}^p + \lambda f_2{}^q$. Now it follows easily by induction on the degree n that f can be expressed as a polynomial in the f_i.

A similar argument shows that the polynomial $f_3{}^r$ is divisible by $f_1{}^p + \lambda f_2{}^q$ for suitably chosen $\lambda \neq 0$, say

$$f_3{}^r = \lambda'(f_1{}^p + \lambda f_2{}^q) .$$

Multiplying each f_i by a suitable constant, we can put this relation in the required form $f_1{}^p + f_2{}^q + f_3{}^r = 0$. ∎

REMARK. More precisely, one can show that the ideal consisting of all polynomial relations between the f_i is actually generated by $f_1{}^p + f_2{}^q + f_3{}^r$ Compare 4.4 below.

Now let V denote the *Pham-Brieskorn variety* consisting of all triples $(v_1, v_2, v_3) \in \mathbf{C}^3$ with $v_1{}^p + v_2{}^q + v_3{}^r = 0$. Evidently the correspondence

$$z \mapsto (f_1(z), f_2(z), f_3(z))$$

maps $\mathbf{C}^2$ into V.

Let $\Pi = [\Gamma, \Gamma]$ denote the commutator subgroup of Γ. Since every character of Γ annihilates Π, we have $f_i(\pi(z)) = f_i(z)$ for $\pi \in \Pi$. Therefore (f_1, f_2, f_3) maps the orbit space $\Pi \backslash \mathbf{C}^2$ into V.

LEMMA 4.4. *In fact, this correspondence* $\Pi z \mapsto (f_1(z), f_2(z), f_3(z))$ *maps the orbit space* $\Pi \backslash \mathbf{C}^2$ *homeomorphically onto the Pham-Brieskorn variety* V.

Restricting to the unit sphere in $\mathbf{C}^2$, we will prove the following statement at the same time.

THEOREM 4.5. *The quotient manifold* $\Pi \backslash S^3$ *or* $\Pi \backslash SU(2)$ *is diffeomorphic to the Brieskorn manifold* $M(p, q, r)$.

The orbit space $\Pi \backslash S^3$ can be identified with the coset space $\Pi \backslash SU(2)$ since $SU(2)$ operates simply transitively on S^3.

Proof. First consider two points z' and z'' which do not belong to the same Π-orbit. Choose a (not necessarily homogeneous) polynomial $g(z)$ which vanishes at z'', but does not vanish at any of the images $\pi(z')$. Setting

$$h(z) = g(\pi_1(z)) g(\pi_2(z)) \cdots g(\pi_m(z))$$

where $\Pi = \{\pi_1, \cdots, \pi_m\}$, it follows that h is Π-invariant and $h(z') \neq h(z'')$. Expressing h as a sum of homogeneous polynomials and applying 4.1, we obtain a polynomial $f \in H_\Gamma^{n,\chi}$ for some n and χ satisfying the same condition $f(z') \neq f(z'')$. Finally, applying 4.3, we see that one of the f_i must satisfy $f_i(z') \neq f_i(z'')$. *Thus the mapping* (f_1, f_2, f_3) *embeds* $\Pi\backslash C^2$ *injectively into* V.

Note that each real half-line from the origin in C^2 maps to a curve

$$t \mapsto (t^{k/p}f_1(z), t^{k/q}f_2(z), t^{k/r}f_3(z))$$

in V which intersects the unit sphere of C^3 transversally and precisely once. Therefore we can map the unit sphere of C^2 into $M = M(p, q, r)$ by following each such image curve until it hits the unit sphere, and hence hits M. Thus we obtain a smooth one-to-one map from the quotient $\Pi\backslash S^3$ into M.

But a one-to-one map from a compact 3-manifold into a connected 3-manifold must necessarily be a homeomorphism. Therefore $\Pi\backslash S^3$ maps homeomorphically onto M. It follows easily that $\Pi\backslash C^2$ maps homeomorphically onto V, thus proving 4.4.

Now let us apply the theorem that a one-to-one holomorphic mapping between complex manifolds of the same dimension is necessarily a diffeomorphism. (See [Bochner and Martin, p. 179].) Since the complex manifold $\Pi\backslash C^2 - 0$ maps holomorphically onto $V - 0$, this mapping must have non-singular Jacobian everywhere. It then follows easily that the mapping $\Pi\backslash S^3 \to M$ is also a diffeomorphism. ■

§5. Automorphic differential forms of fractional degree

This section will develop some technical tools concerning functions of one complex variable which will be needed in the next section. Some of the concepts (e.g., "labeled" biholomorphic mappings) are non-standard.

It is common in the study of Riemann surfaces to consider abelian differentials (that is, expressions of the form $f(z)dz$) as well as quadratic differentials (expressions of the form $f(z)dz^2$). More generally, for any integer $k \geq 0$, a *differential* (= differential form) *of degree* k on an open set U of complex numbers can be defined as a complex valued function of two variables of the form

$$\phi(z,dz) = f(z)dz^k ,$$

where z varies over U and dz varies over $\mathbf{C}$.

To further explain this concept, one must specify how such a differential transforms under a change of coordinates. In fact, if $g : U \to U_1$ is a holomorphic map, and if $\phi_1(z_1, dz_1) = f_1(z_1)dz_1{}^k$ is a differential on U_1, then the *pull-back* $\phi = g^*(\phi_1)$ is defined to be the differential

$$\phi(z,dz) = \phi_1(g(z), dg(z)) = f_1(g(z))\dot{g}(z)^k dz^k$$

on U. Here $\dot{g}(z)$ denotes the derivative $dg(z)/dz$. This pull-back operation carries sums into sums and products into products.

We will need to generalize these constructions, replacing the integer k by an arbitrary rational number α. There are two closely related difficulties: If α is not an integer, then the fractional power dz^α is not uniquely defined, and similarly the fractional power $\dot{g}(z)^\alpha$ is not uniquely defined.

To get around the first difficulty we agree that the symbol dz is to vary, not over the complex numbers, but rather over the universal covering group $\widetilde{\mathbf{C}}^{\cdot}$ of the multiplicative group $\mathbf{C}^{\cdot}$ of non-zero complex numbers. Since every element of $\widetilde{\mathbf{C}}^{\cdot}$ has a unique n-th root for all n, it follows that the fractional power dz^α is always well defined in $\widetilde{\mathbf{C}}^{\cdot}$.

REMARK. This universal covering group $\widetilde{\mathbf{C}}^{\cdot}$ is of course canonically isomorphic to the additive group of complex numbers. In fact, the exponential homomorphism $e(z) = \exp(2\pi i z)$ from $\mathbf{C}$ to $\mathbf{C}^{\cdot}$ lifts uniquely to an isomorphism

$$\widetilde{e} : \mathbf{C} \to \widetilde{\mathbf{C}}^{\cdot}$$

of complex Lie groups. The kernel of the projection homomorphism $\widetilde{\mathbf{C}}^{\cdot} \to \mathbf{C}^{\cdot}$ is evidently generated by the image $\widetilde{e}(1)$.

We are now ready to describe our basic objects.

DEFINITION. A *differential* (= differential form) *of degree* α on an open set $U \subset \mathbf{C}$ is a complex valued function of the form

$$\phi(z, dz) = f(z)dz^{\alpha}$$

where z varies over U and dz varies over $\tilde{\mathbf{C}}^{\cdot}$. Here it is understood that the fractional power dz^{α} is to be evaluated in $\tilde{\mathbf{C}}^{\cdot}$ and then projected into $\mathbf{C}^{\cdot}$ to be multiplied by $f(z)$. In practice we will always assume that f is holomorphic, so that ϕ is holomorphic as a function of two variables. Note that the product of two holomorphic differentials of degrees α and β is a holomorphic differential of degree $\alpha + \beta$.

In order to define the pull-back $g^{*}(\phi)$ of a differential of fractional degree, we must impose some additional structure on the map g.

DEFINITION. By a *labeled holomorphic map* g from U to U_1 will be meant a holomorphic map $z \mapsto g(z)$ with nowhere vanishing derivative, together with a continuous lifting $\dot{g}$ of the derivative from $\mathbf{C}^{\cdot}$ to $\tilde{\mathbf{C}}^{\cdot}$. More precisely,

$$\dot{g} : U \to \tilde{\mathbf{C}}^{\cdot}$$

must be a holomorphic function whose projection into $\mathbf{C}^{\cdot}$ is precisely the derivative $dg(z)/dz$. (Alternatively, a labeling could be defined as a choice of one single valued branch of the many valued function $\log dg(z)/dz$ on U.) Given two labeled holomorphic maps

$$g : U \to U_1 \quad \text{and} \quad g_1 : U_1 \to U_2 ,$$

the *composition* $g_1 g : U \to U_2$ has a unique labeling which is determined by the requirement that the chain law identity

$$(g_1 g)^{\cdot}(z) = \dot{g}(z)\dot{g}_1(g(z))$$

should be valid in $\tilde{\mathbf{C}}^{\cdot}$.

Now consider a labeled holomorphic map $g : U \to U_1$ together with a differential

$$\phi_1(z_1, dz_1) = f_1(z_1)dz_1{}^{\alpha}$$

on U_1. The *pull-back* $g^*(\phi_1)$ is defined to be the differential

$$\phi(z, dz) = \phi_1(g(z), \dot{g}(z)dz)$$

on U. Note that this pull-back operation carries sums into sums and products into products. Furthermore, given any composition

$$U \xrightarrow{g} U_1 \xrightarrow{g_1} U_2$$

of labeled holomorphic maps, the pull-back $(g_1 g)^*(\phi_2)$ of a differential on U_2 is clearly equal to the iterated pull-back $g^*(g_1{}^*(\phi_2))$.

Let Γ be a discrete group of labeled biholomorphic maps of U onto itself.

DEFINITION. A holomorphic differential form $\phi(z, dz) = f(z)dz^a$ on U is *Γ-automorphic* if it satisfies

$$\gamma^*(\phi) = \phi$$

for every $\gamma \in \Gamma$. More generally, given any character $\chi : \Gamma \to U(1) \subset C^{\cdot}$, the form ϕ is called *χ-automorphic* if

$$\gamma^*(\phi) = \chi(\gamma)\phi$$

for every γ. (Thus the Γ-automorphic forms correspond to the special case $\chi = 1$.) Note that a form $\phi(z, dz) = f(z)dz^a$ is χ-automorphic if and only if f satisfies the identity

$$f(\gamma(z))\dot{\gamma}(z)^a = \chi(\gamma)f(z)$$

for all $\gamma \in \Gamma$ and all $z \in U$.

Evidently the χ-automorphic forms of degree a on U form a complex vector space which we denote by the symbol $A_\Gamma^{a,\chi}$. In this way we obtain a bigraded algebra $A_\Gamma^{*,*}$, where the first index a ranges over the additive group of rational numbers and the second index χ ranges over the multiplicative group $\mathrm{Hom}(\Gamma, U(1))$ of characters. This algebra possesses an identity element $1 \in A_\Gamma^{0,1}$. It is associative, commutative, and has no zero-divisors so long as the open set U is connected.

REMARK. The classical theory of automorphic forms of non-integer degree is due to [Petersson]. (Compare [Gunning], [Lehner].) It is based on definitions which superficially look rather different.

Suppose that we are given a normal subgroup of Γ.

LEMMA 5.1. *If* $N \subset \Gamma$ *is a normal subgroup, then the quotient* Γ/N *operates as a group of automorphisms of the algebra* $A_N^{*,1}$ *with fixed point set* $A_\Gamma^{*,1}$. *If the quotient group* Γ/N *is finite abelian of order* m, *then each* $A_N^{\alpha,1}$ *splits as the direct sum of its subspaces* $A_\Gamma^{\alpha,\chi}$ *as* χ *varies over the* m *characters of* Γ *which annihilate* N.

The proof is easily supplied. (Compare 4.1.) ■

COROLLARY 5.2. *If* $N \subset \Gamma$ *is a normal subgroup of finite index* m, *then every* $\phi \in A_N^{\alpha,1}$ *has a well defined "norm"* $(\gamma_1^*\phi)\cdots(\gamma_m^*\phi) \in A_\Gamma^{m\alpha,1}$. *Here* $\gamma_1,\cdots,\gamma_m$ *are to be representatives for the cosets of* N *in* Γ.

Again the proof is easily supplied. ■

It will be important in Section 6 to be able to extract n-th roots of automorphic forms.

LEMMA 5.3. *Let* $\phi(z,dz) = f(z)dz^\alpha$ *be a* χ*-automorphic form. If* f *possesses an* n-th *root*, $f(z) = f_1(z)^n$ *where* f_1 *is holomorphic, then the form* $\phi_1(z,dz) = f_1(z)dz^{\alpha/n}$ *is itself* χ_1*-automorphic for some character* χ_1 *of* Γ *satisfying* $\chi_1{}^n = \chi$.

Proof. For any group element γ, since the holomorphic forms ϕ_1 and $\gamma^*(\phi_1)$ both have degree α/n, the quotient $\gamma^*(\phi_1)/\phi_1$ is a well defined meromorphic function on U. Raising this function to the n-th power we obtain the constant function $\gamma^*(\phi)/\phi = \chi(\gamma)$. Therefore $\gamma^*(\phi_1)/\phi_1$ must itself be a constant function. Setting its value equal to $\chi_1(\gamma)$, it is easy to check that χ_1 is a character of Γ with $\chi_1{}^n = \chi$. ■

As open set U, let us take the upper half-plane P consisting of all $z = x + iy$ with $y > 0$. Then every biholomorphic map from U to itself has the form

$$z \mapsto z' = (g_{11}z + g_{12})/(g_{21}z + g_{22})$$

where

$$\begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix}$$

is an element, well defined up to sign, of the group $SL(2, \mathbb{R})$ of 2×2 real unimodular matrices. The derivative dz'/dz is equal to $(g_{21}z + g_{22})^{-}$

It follows easily that the group G consisting of all labeled biholomorphic maps from P to itself can be identified with the universal covering group of $SL(2, \mathbb{R})$. This group G contains an infinite cyclic central subgroup C consisting of group elements which act trivially on P. The generator c of C is characterized by the formulas

$$c(z) = z, \quad \dot{c}(z) = \widetilde{e}(1), \quad \dot{c}(z)^{\alpha} = \widetilde{e}(\alpha) \mapsto e^{2\pi i \alpha} \quad \text{in } \mathbb{C} .$$

A group $\Sigma \subset \overline{G}$ of conformal automorphisms of P is said to have *compact fundamental domain* if there exists a compact subset $K \subset P$ with non-vacuous interior so that the various images $\sigma(K)$ cover P, and are mutually disjoint except for boundary points. We will be interested in subgroups of G whose images in $\overline{G} = G/C$ satisfy this hypothesis.

LEMMA 5.4. *Let* $\Gamma \subset G$ *be such that the image* $\overline{\Gamma} = \Gamma/(\Gamma \cap C)$ *in* $\overline{G}$ *operates on the upper half-plane* P *with compact fundamental domain. Then* Γ *is discrete as a subgroup of the Lie group* G, *and the coset space* $\Gamma\backslash G$ *is compact. This group* Γ *necessarily intersects the center* C *non-trivially.*

Proof. As noted in Section 2.7 there exists a normal subgroup $N \subset \Gamma$ of finite index so that $\bar{N} = N/N \cap C$ operates freely on P. The orbit space under this action, denoted briefly by the symbol $\bar{N} \backslash P$, is then a smooth compact surface S of genus $g \geq 2$ with fundamental group $\pi_1(S) \cong \bar{N} \cong NC/C$. Here NC denotes the subgroup of G generated by N and C.

Since the group G/C operates simply transitively on the unit tangent bundle $T_1(P)$ of P, it follows easily that the coset space $(NC) \backslash G$ can be identified with the unit tangent bundle $T_1(S)$ of the quotient surface $\bar{N} \backslash P$. In particular this coset space is compact, with fundamental group

$$NC \cong \pi_1(T_1(S)) .$$

Hence the abelianized group $NC/[NC, NC] = NC/[N, N]$ can be identified with the homology group $H_1(T_1(S))$.

It follows that N must intersect C non-trivially. For otherwise NC would split as a cartesian product $N \times C$ with $N = N/N \cap C \cong \pi_1(S)$. Hence $T_1(S)$ would have first Betti number $2g + 1$, rather than its actual value of $2g$.

(Carrying out this argument in more detail and using the Gysin sequence of the tangent circle bundle (see [Spanier, p. 260] as well as [Milnor and Stasheff, pp. 143, 130]), one finds that the kernel of the natural homomorphism from $H_1(T_1(S))$ onto $H_1(S)$ is cyclic, with order equal to the absolute value of the Euler characteristic $\chi(S) = 2-2g$. Identifying these two groups with $NC/[N, N]$ and $\bar{N}/[\bar{N}, \bar{N}] \cong NC/[N, N]C$ respectively, we see that this kernel can be identified with $C/[N, N] \cap C$. *Therefore the element* c^{2-2g} *of* C *necessarily belongs to the commutator subgroup* $[N, N] \subset N$.)

Thus N has finite index in NC, so $N \backslash G$ is also compact, and it follows that $\Gamma \backslash G$ is compact. ∎

REMARK. Conversely, if $\Gamma \subset G$ is any discrete subgroup with compact quotient, then one can show that the hypothesis of 5.4 is necessarily satisfied. Such subgroups Γ can be partially classified as follows. Recall from Section 2.7 that the image $\overline{\Gamma} = \Gamma C/C$ is completely classified by the quotient Riemann surface $\overline{\Gamma}\backslash P$ together with a specification of ramification points and ramification indices. But Γ has index at most $2g-2$ in the full inverse image ΓC of $\overline{\Gamma}$. Therefore, for each fixed $\overline{\Gamma}$ there are only finitely many possible choices for Γ.

To show that automorphic forms really exist, we can proceed as follows Again let Γ satisfy the hypothesis of 5.4 and let $N \subset \Gamma$ be normal of finite index m, with $N/N \cap C$ operating freely on P.

LEMMA 5.5. *If a is a multiple of m, then the space $A_\Gamma^{a,1}$ is non-zero. In fact, this space contains a form ϕ which does not vanish throughout any prescribed finite (or even countable) subset of P.*

Proof. Recall that $A_N^{1,1}$ can be identified with the space of holomorphic abelian differentials $f(z)dz$ on the quotient surface $S = \overline{N}\backslash P$ of genus $g \geq 2$. By a classical theorem, this space has dimension g. Furthermore, using the Riemann-Roch theorem, the space of abelian differentials vanishing at some specified point of S has dimension $g-1$. (Compare [Springer, pp. 252, 270].) Clearly we can choose an element ψ of this g-dimensional vector space so as to avoid any countable collection of hyperplanes. Now the norm $\phi = \gamma_1{}^*(\psi)\cdots\gamma_m{}^*(\psi) \in A_\Gamma^{m,1}$ of Section 5.2 will be non-zero at any specified countable collection of points. Setting $a = km$, it follows that $\phi^k \in A_\Gamma^{a,1}$ has the same properties. ■

The density of zeros of an automorphic form can be computed as follows. We will think of the upper half-plane P as a model for the Lobachevsky plane, using the Poincaré metric $(dx^2+dy^2)/y^2$, and its associated area element $dxdy/y^2$.

Again let $\Gamma/\Gamma \cap C$ operate on P with compact fundamental domain. Let $\chi : \Gamma \to U(1)$ be a character of finite order. (The hypothesis that χ has finite order is not essential. It is made only to simplify the proof.)

LEMMA 5.6. *If* $\phi \in A_\Gamma^{\alpha,\chi}$ *is a non-zero automorphic form, then the density of zeros of* ϕ *is* $\alpha/2\pi$. *More explicitly: the number of zeros of* ϕ *in a large disk of Lobachevsky area* a, *each zero being counted with its appropriate multiplicity, tends asymptotically to* $a\alpha/2\pi$ *as* $a \to \infty$.

In particular it follows that $\alpha \geq 0$.

Proof. Again we may choose a normal subgroup N of finite index so that $\overline{N} = N/N \cap C$ operates freely on P. Furthermore, after raising ϕ to some power if necessary, we may assume that the character χ is trivial and that the degree $\alpha = k$ is an integer. By a classical theorem, the number of zeros of a k-th degree differential in a compact Riemann surface $\overline{N}\backslash P$ of genus $g \geq 2$ is equal to $(2g-2)k$, where k is necessarily non-negative. (For the case of an abelian differential $f(z)dz$, see for example [Springer, pp. 252, 267]. Given such a fixed abelian differential, any k-th degree differential on $\overline{N}\backslash P$ can be written uniquely as $h(z)f(z)^k dz^k$ where h is meromorphic on $\overline{N}\backslash P$, and hence has just as many zeros as poles.)

Since the quotient $\overline{N}\backslash P$ has area $(2g-2)2\pi$ by the Gauss-Bonnet theorem, it follows that the ratio of number of zeros to area is $k/2\pi$, as asserted. ■

REMARK. If $\phi \neq 0$ is a form in $A_\Gamma^{\alpha,\chi} \subset A_N^{\alpha,\chi|N}$, then it follows that the number of zeros of ϕ in $\overline{N}\backslash P$ is equal to $(2g-2)\alpha$. In particular, $(2g-2)\alpha$ is an integer. Thus we obtain a uniform common denominator for the rational numbers α which actually occur as degrees.

The algebra of N-automorphic forms can be described rather explicitly as follows. Let k be the order of the finite cyclic group $C/N \cap C$.

LEMMA 5.7. *If the rational number* α *is a multiple of* $1/k$, *then,*

$$\dim A_N^{\alpha,1} \geq (2g-2)\alpha + 1 - g ,$$

with equality whenever $a > 1$. *In particular, this vector space is non-zero whenever* $a > \frac{1}{2}$. *On the other hand, if* a *is not a multiple of* $1/k$, *then* $A_N^{a,1} = 0$.

It follows incidentally that $(2g-2)/k$ is necessarily an integer. The following will be proved at the same time.

LEMMA 5.8. *If* a *is a multiple of* $1/k$ *and* $a > g/(g-1)$, *then given two distinct points of* $\bar{N}\backslash P$ *there exists a form in* $A_N^{a,1}$ *which vanishes at the first point but not at the second.*

Proof. For any form ϕ of degree a the identity

$$c^*(\phi) = e(a)\phi$$

is easily verified. Thus if ϕ is N-automorphic and non-zero, with $c^k \in N$, then it follows that $e(ka) = 1$. Hence a must be a multiple of $1/k$.

Conversely, if a is a multiple of $1/k$, then it is not difficult to construct a complex analytic line bundle ξ^a over the surface $S = \bar{N}\backslash P$ so that the holomorphic sections of ξ^a can be identified with the elements of $A_N^{a,1}$. For example, the total space of ξ^a can be obtained as the quotient of $P \times \mathbb{C}$ under the group $N/N \cap C$ which operates freely by the rule $\nu : (z,w) \mapsto (\nu(z), \dot{\nu}(z)^{-a}w)$. Every holomorphic section $z \mapsto f(z)$ of the resulting bundle must satisfy the identity $f(\nu(z)) = \dot{\nu}(z)^{-a}f(z)$ appropriate to N-automorphic forms of degree a. Note that the tensor product $\xi^a \otimes \xi^\beta$ can be identified with $\xi^{a+\beta}$.

To compute the Chern class $c_1(\xi^a)$ we raise to the k-th tensor power so that holomorphic cross-sections exist as in 5.5, and then count the number of zeros of a holomorphic section as in 5.6. In this way we obtain the formula

$$c_1(\xi^a)[S] = (2g-2)a \ .$$

Now let us apply the Riemann-Roch theorem as stated in [Hirzebruch, p. 144]: For any analytic line bundle ξ over S,

$$\dim(\text{space of holomorphic sections}) \geq c_1(\xi)[S] + 1 - g \ .$$

Taking $\xi = \xi^\alpha$ this yields

$$\dim A_N^{\alpha,1} \geq (2g-2)\alpha + 1 - g$$

as asserted.

To decide when equality holds, and to prove 5.8, it is perhaps easier to use the older form of the Riemann-Roch theorem, as described in [Springer] or [Hirzebruch, p. 4]. Choosing some fixed $\phi \neq 0$ in $A_N^{\alpha,1}$, any element of $A_N^{\alpha,1}$ can be obtained by multiplying ϕ by a meromorphic function h on $\bar{N} \setminus P$ which has poles at most on the $(2g-2)\alpha$ zeros of ϕ. More precisely the divisors (h) and (ϕ) of h and ϕ must satisfy $(h) \geq (\phi)^{-1}$. According to Riemann-Roch, the number of linearly independent h satisfying this condition is $> \deg(\phi) + 1 - g$, with equality whenever the degree $(2g-2)\alpha$ of (ϕ) is greater than the degree $2g-2$ of the divisor of an abelian differential. This proves 5.7.

If we want this form $h\phi$ to vanish at z' [or at both z' and z''], then we must use the divisor $(\phi)^{-1}z'$ [respectively $(\phi)^{-1}z'z''$] in place of $(\phi)^{-1}$. A brief computation then shows the following. If the degree $(2g-2)\alpha - 2$ of the divisor $(\phi)z'^{-1}z''^{-1}$ satisfies

$$(2g-2)\alpha - 2 > 2g - 2 \ ,$$

or in other words if $\alpha > g/(g-1)$, then the space of forms in $A_N^{\alpha,1}$ which vanish at z' [respectively at z' and z''] is equal to $(2g-2)\alpha - g$ [respectively $(2g-2)\alpha - 1 - g$]. Since these two dimensions are different, there is a form which vanishes at z' but not z''. ∎

REMARK. More generally consider the vector space $A_N^{\alpha,\rho}$ where ρ is an arbitrary character of N. Suppose that $\gamma = c^j$ is an element of the intersection $N \cap C$. Then the appropriate equation

$$f(\gamma(z))\,\dot{\gamma}(z)^\alpha = f(z)\rho(\gamma)$$

takes the form $f(z)e(j\alpha) = f(z)\rho(c^j)$. Evidently there can be a solution $f(z) \neq 0$ only if the rational number α and the character ρ satisfy the relation

$$e(j\alpha) = \rho(c^j)$$

for every c^j in $N \cap C$. Conversely, if this condition is satisfied, then the argument above can easily be modified so as to show that

$$\dim A_N^{\alpha,\rho} \geq (2g-2)\alpha + 1 - g \ ,$$

with equality whenever $\alpha > 1$.

In the next section we will need a sharp estimate which says that "enough" automorphic forms exist. To state it we must think of an automorphic form ϕ explicitly as a function

$$\phi(z,w) = f(z)w^{\alpha}$$

of two variables, where $z \in P$ and $w \in \widetilde{C}^{\cdot}$. Let the groups $\Gamma \subset G$ operate freely on $P \times \widetilde{C}^{\cdot}$ by the rule

$$g(z,w) = (g(z), \dot{g}(z)w) \ .$$

With this notation, the statement that ϕ is Γ-automorphic can be expressed by the equation

$$\phi(\gamma(z,w)) = \phi(z,w)$$

for all $\gamma \in \Gamma$, $z \in P$, and $w \in \widetilde{C}^{\cdot}$.

THEOREM 5.9. *With Γ as in 5.4, two points (z', w') and (z'', w'') of $P \times \widetilde{C}^{\cdot}$ belong to the same Γ-orbit if and only if $\phi(z', w') = \phi(z'', w'')$ for every Γ-automorphic form ϕ.*

Proof. First consider the corresponding statement for the normal subgroup $N \subset \Gamma$ of Section 2.7. If $\phi(z', w') = \phi(z'', w'')$ for every $\phi \in A_N^{*,1}$ note that z' and z'' belong to the same N-orbit. For otherwise by 5.8 there would exist a form $\phi \in A_N^{3,1}$ which vanishes at z' but not z''.

Thus there exists $\nu \in N$ with $\nu(z'') = z'$. Note that

$$\phi(z', w') = \phi(z'', w'') = \phi(\nu(z'', w''))$$

for every N-automorphic form $\phi \in A_N^{\alpha,1}$. Defining the element $\tilde{e}(u) \in \tilde{C}^{\cdot}$ by the equation $\nu(z'', w'') = (z', w'\tilde{e}(u))$, note that

$$\phi(z', w'\tilde{e}(u)) = \phi(z', w')e(\alpha u) .$$

Setting this equal to $\phi(z', w')$, we see that $e(\alpha u) = 1$ whenever ϕ is non-zero at z'. By 5.8, α can be any sufficiently large multiple of $1/k$. Therefore u must be a multiple of k, say $u = nk$. Hence the corresponding power c^u is in N; completing the proof that (z', w') and (z'', w'') belong to the same N-orbit. In fact $c^{-u}\nu(z'', w'') = (z', w')$.

To prove the corresponding assertion for Γ we will make temporary use of inhomogeneous automorphic forms, that is, elements of the direct sum $\bigoplus A_N^{\alpha,1}$, to be summed over α. Given points (z', w') and (z'', w'') not in the same Γ-orbit, consider the m images $\gamma_j(z', z'')$ where $\gamma_1, \cdots, \gamma_m$ represent the cosets of N in Γ. The above argument constructs forms $\phi_j \in A_N^{*,1}$ with

$$\phi_j(z'', w'') \neq \phi_j(\gamma_j(z', w')) .$$

Subtracting the constant $\phi_j(z'', w'') \in A_N^{0,1} \cong C$ from each ϕ_j, we obtain an *inhomogeneous* form which vanishes at (z'', w') but not at $\gamma_j(z', w')$. Now almost any linear combination ϕ of $\phi_1, \cdots, \phi_m$ will vanish at (z'', w'') but not anywhere in the Γ-orbit of (z', w'). Hence the norm

$$\psi = \gamma_1{}^*(\phi) \cdots \gamma_m{}^*(\phi) \in \bigoplus A_\Gamma^{\alpha,1}$$

of Section 5.2 will vanish at (z'', w'') but not at (z', w'). Expressing ψ as the sum of its homogeneous constituents, clearly at least one must take distinct values at (z'', w'') and (z', w'). ∎

§6. The hyperbolic case $p^{-1}+q^{-1}+r^{-1}<1$

The computations in this section will be formally very similar to those of Section 4. However, automorphic forms will be used in place of homogeneous polynomials.

Let Γ be the extended triangle group $\Gamma(p,q,r)$ of Section 3, with $p^{-1}+q^{-1}+r^{-1}<1$ so that Γ can be considered as a group of labeled biholomorphic maps of the upper half-plane P. Recall that Γ has generators $\gamma_1,\gamma_2,\gamma_3$ which represent rotations about the three vertices of the triangle $T\subset P$. With this choice of Γ, the characters χ which actually occur for non-zero χ-automorphic forms can be described as follows. We continue to use the abbreviation $e(\alpha)=e^{2\pi i\alpha}$.

LEMMA 6.1. *Let χ be a character of the extended triangle group Γ. If $\phi\neq 0$ is a χ-automorphic form of degree α, then*

$$\chi(\gamma_1) = e((k+\alpha)/p)$$

where k is the order of the zero of ϕ at the first vertex of the triangle T. The values $\chi(\gamma_2)$ and $\chi(\gamma_3)$ can be computed similarly.

In particular, if ϕ does not vanish at the first vertex of T, then $\chi(\gamma_1)=e(\alpha/p)$.

Proof. Since $\gamma_1=r_{v_1}(2\pi/p)$ is a lifted rotation through the angle $2\pi/p$, the derivative $\dot\gamma_1(v_1)$ equals $\tilde e(1/p)$, hence the fractional power $\dot\gamma_1(v_1)^\alpha$ in $\tilde{C}^{\cdot}$ projects to the complex number $e^{2\pi i\alpha/p}=e(\alpha/p)$. Setting $\phi(z,dz)=f(z)\,dz^\alpha$, and substituting the Taylor expansion

$$f(z) = a(z-v_1)^k + b(z-v_1)^{k+1} + \cdots$$

in the identity

$$f(\gamma_1(z))\dot\gamma_1(z)^\alpha = \chi(\gamma_1)f(z)$$

we obtain

$$a(e(1/p)(z-v_1))^k e(\alpha/p) + \cdots = \chi(\gamma_1)a(z-v_1)^k + \cdots \ .$$

Hence $e(k/p)e(\alpha/p) = \chi(\gamma_1)$ as asserted. ■

Define a rational number s by the formula $s^{-1} = 1-p^{-1}-q^{-1}-r^{-1}$. Thus π/s is the Lobachevsky area of the base triangle T. Define a character χ_0 of Γ by the formulas

$$\chi_0(\gamma_1) = e(s/p), \ \chi_0(\gamma_2) = e(s/q), \ \chi_0(\gamma_3) = e(s/r) \ .$$

The necessary identities

$$\chi_0(\gamma_1{}^p) = \chi_0(\gamma_2{}^q) = \chi_0(\gamma_3{}^r) = \chi_0(\gamma_1\gamma_2\gamma_3)$$

are easily verified.

COROLLARY 6.2. *If the automorphic form $\phi \in A_\Gamma^{\alpha,\chi}$ does not vanish at any vertex of the triangle T, then the degree α must be a multiple of s, and the character χ must be equal to $\chi_0{}^{\alpha/s}$.*

Proof. By 6.1 we have $\chi(\gamma_1) = e(\alpha/p)$, $\chi(\gamma_2) = e(\alpha/q)$, $\chi(\gamma_3) = e(\alpha/r)$. Hence the relations

$$\gamma_1{}^p = \gamma_2{}^q = \gamma_3{}^r = \gamma_1\gamma_2\gamma_3$$

of Section 3.1 imply that $\chi(\gamma_1)^p = \chi(\gamma_2)^q = \chi(\gamma_3)^r = e(\alpha)$ must be equal to

$$\chi(\gamma_1)\chi(\gamma_2)\chi(\gamma_3) = e((p^{-1}+q^{-1}+r^{-1})\alpha) = e((1-s^{-1})\alpha) \ .$$

Therefore $e(\alpha/s) = 1$, or in other words α must be a multiple of s. The equation $\chi = \chi_0{}^{\alpha/s}$ clearly follows. ■

Now we can begin to describe the algebra $A_\Gamma^{*,*}$ more explicitly.

LEMMA 6.3. *With* Γ, s, *and* χ_0 *as above, the complex vector space* A_Γ^{s,χ_0} *has dimension* 2. *This space contains one and (up to a constant multiple) only one automorphic form which vanishes at any given point of* P.

Proof. We begin with the basic existence Lemma 5.5. For some α there exists a form $\phi \in A_\Gamma^{\alpha,1}$ which is non-zero throughout any specified finite subset of P. In particular we can choose ϕ to be non-zero on the three vertices of T. By 6.2, the degree α of this ϕ must be a multiple of s, say $\alpha = ks$.

Let us count the number of zeros of ϕ. Since the triangle T has Lobachevsky area $(1-p^{-1}-q^{-1}-r^{-1})\pi = \pi/s$, it follows that a fundamental domain $T \cup \sigma(T)$ for the action of $\Gamma/\Gamma \cap C$ on P has Lobachevsky area $2\pi/s$. But the number of zeros of ϕ per unit area is $ks/2\pi$ by 5.6. Therefore the number of zeros of ϕ in the fundamental domain $T \cup \sigma(T)$ is precisely equal to k. [Here each pair of zeros z and $\gamma(z)$ on the boundary of the fundamental domain must of course be counted as a single zero. Note that ϕ does not vanish at the corners of the fundamental domain.] In other words there are precisely k (not necessarily distinct) zeros of ϕ in the quotient space $\bar{\Gamma}\backslash P$.

Next note that this quotient space $\bar{\Gamma}\backslash P$ can be given the structure of a smooth Riemann surface. If we stay away from the three exceptional orbits, this is of course clear. To describe the situation near the vertex v_1 it is convenient to choose a biholomorphic map h from P onto the unit disk satisfying $h(v_1) = 0$. Then the coordinate $w = h(z)$ can be used as a local uniformizing parameter near v_1. Since the rotation γ_1 of P about v_1 corresponds to the rotation

$$h\gamma_1 h^{-1}(w) = e(1/p)w$$

of the unit disk about the origin, it follows that a locally defined holomorphic function of w is invariant under this rotation if and only if it is actually a holomorphic function of w^p. Hence w^p can be used as local uniformizing parameter for the quotient surface $\bar{\Gamma}\backslash P$ about the image of v_1. The other two vertices are handled similarly. Note that a meromorphic function on $\bar{\Gamma}\backslash P$ having a simple zero at the image of v_1 corresponds to a Γ-invariant meromorphic function on P having a p-fold zero at each point of the exceptional orbit Γv_1.

Topologically, this quotient $\bar{\Gamma}\backslash P$ can be identified with the "double" of the triangle T. Hence it is a surface of genus zero. More explicitly, following Schwarz, $\bar{\Gamma}\backslash P$ can be identified biholomorphically with the unit 2-sphere by using the Riemann mapping theorem to map T onto a hemisphere and then applying the reflection principle.

Since $\bar{\Gamma}\backslash P$ is a compact Riemann surface of genus zero, it possesses a meromorphic function with k arbitrarily placed zeros and k arbitrarily placed poles. Starting with the non-zero form $\phi \in A_\Gamma^{ks,1}$ constructed above, we can multiply by a Γ-invariant meromorphic function h which has poles precisely at the k zeros of ϕ, and thus obtain a new form $\psi = h\phi \in A_\Gamma^{ks,1}$ whose k zeros can be prescribed arbitrarily in $\bar{\Gamma}\backslash P$. In particular we can choose ψ so as to have a k-fold zero at one point of $\bar{\Gamma}\backslash P$, and no other zeros. (To avoid confusion, let us choose this point to be distinct from the three ramification points.) Then by 5.3 this form has a k-th root $\psi_1 \in A_\Gamma^{s,\chi}$ for some character χ, and by 6.2 the character χ must be precisely χ_0. Evidently the form ψ_1 has a simple zero at just one point of $\bar{\Gamma}\backslash P$.

Similarly we can choose $\psi_2 \in A_\Gamma^{s,\chi_0}$ which vanishes at a different point of $\bar{\Gamma}\backslash P$. Then ψ_1 and ψ_2 are linearly independent. A completely arbitrary element $\psi \neq 0$ of A_Γ^{s,χ_0} must have precisely one simple zero in $\bar{\Gamma}\backslash P$, using 5.6. Choosing a linear combination $\lambda_1\psi_1 + \lambda_2\psi_2$ which vanishes at this zero, we see that the ratio $\psi/(\lambda_1\psi_1 + \lambda_2\psi_2) \in A_\Gamma^{0,1}$ represents a holomorphic function defined throughout $\bar{\Gamma}\backslash P$, hence a constant. Thus ψ_1 and ψ_2 form a basis for A_Γ^{s,χ_0}, and this space contains precisely one 1-dimensional subspace consisting of forms which vanish at any prescribed point of $\bar{\Gamma}\backslash P$. ∎

The structure of $A_\Gamma^{*,*}$ can now be described as follows

LEMMA 6.4. *With $\Gamma = \Gamma(p,q,r)$ as above, the bigraded algebra $A_\Gamma^{*,*}$ is generated by three forms*

$$\phi_1 \in A_\Gamma^{s/p,\chi_1}, \quad \phi_2 \in A_\Gamma^{s/q,\chi_2}, \quad \phi_3 \in A_\Gamma^{s/r,\chi_3}$$

where χ_1, χ_2, χ_3 are characters satisfying

$$\chi_1{}^p = \chi_2{}^q = \chi_3{}^r = \chi_0 \ .$$

The automorphic form ϕ_i has a simple zero at each point of the orbit Γv_i, and no other zeros. These three forms satisfy a polynomial relation $\phi_1{}^p + \phi_2{}^q + \phi_3{}^r = 0$.

REMARK. The meromorphic function $-\phi_1{}^p/\phi_3{}^r$ is the Schwarz triangle function, which maps the quotient Riemann surface $\overline{\Gamma}\backslash P$ biholomorphically onto the extended complex plane, sending the three vertices of T to 0, 1 and ∞ respectively.

Proof of 6.4. To construct ϕ_1 we use 6.3 to construct a form ϕ in A_Γ^{s,χ_0} which vanishes only along the orbit of v_1. This form must have a p-fold zero at v_1 by 5.6 or by the proof of 6.3. Since P is simply connected, it follows that ϕ possesses a holomorphic p-th root ϕ_1. Then ϕ_1 is itself an automorphic form by 5.3. The rest of the proof is completely analogous to the proof of 4.3. ■

Let Π denote the commutator subgroup of $\Gamma(p,q,r)$. Then by 3.2, 5.1 and 6.4 the graded algebra $A_\Pi^{*,1}$ is generated by the three forms ϕ_1, ϕ_2, ϕ_3.

COROLLARY 6.5. *The coset space $\Pi\backslash G$ is diffeomorphic to the Brieskorn manifold $M(p,q,r)$.*

Proof. Let $V \subset \mathbf{C}^3$ be the Pham-Brieskorn variety $z_1{}^p + z_2{}^q + z_3{}^r = 0$, singular only at the origin. Since the three functions ϕ_1, ϕ_2, ϕ_3 on $P \times \widetilde{\mathbf{C}}^{\cdot}$ satisfy the relation $\phi_1{}^p + \phi_2{}^q + \phi_3{}^r = 0$, and are never simultaneously zero, they together constitute a holomorphic mapping $(\phi_1, \phi_2, \phi_3) : P \times \widetilde{\mathbf{C}}^{\cdot} \to V - 0 \subset \mathbf{C}^3$ between complex 2-dimensional manifolds

Recall from Section 5.9 that the groups $\Pi \subset G$ operate freely on $P \times \widetilde{\mathbf{C}}$ by the rule $g : (z, w) \mapsto (g(z), \dot{g}(z) w)$. Setting $z = x + iy$ and identifying w with dz, this action preserves the Poincaré metric $|dz|^2/y^2 = |w|^2/y^2$. In fact, G operates simply transitively on each 3-dimensional manifold $|w|/y =$ constant. Since Π is a discrete subgroup of G, it follows that the quotient $\Pi \backslash (P \times \widetilde{\mathbf{C}}^{\cdot})$ is again a complex 2-dimensional manifold.

Since each ϕ_i is Π-automorphic, the triple ϕ_1, ϕ_2, ϕ_3 give rise to a holomorphic mapping

$$\Phi : \Pi \backslash (P \times \widetilde{\mathbf{C}}^{\cdot}) \to V - 0$$

on the quotient manifold. By 5.9, since the ϕ_i generate $A_{\Pi}^{*,1}$, this mapping Φ is one-to-one. Hence by [Bochner and Martin, p. 179], Φ *maps* $\Pi \backslash (P \times \mathbf{C}^{\cdot})$ *biholomorphically onto an open subset of* $V - 0$. (It will follow in a moment that the image of Φ is actually all of $V - 0$.)

Choosing a base point $(z_0, 1)$ in $P \times \widetilde{\mathbf{C}}^{\cdot}$, map the coset space $\Pi \backslash G$ into the Brieskorn manifold $M(p, q, r) = V \cap S^5$ as follows. For each coset Πg the image $\Phi(\Pi g(z_0, 1))$ is a well defined point (z_1, z_2, z_3) of $V - 0$. Consider the curve

$$t \mapsto (t^{1/p} z_1, t^{1/q} z_2, t^{1/r} z_3) = \Phi(\Pi g(z_0, t^{1/s}))$$

through this point in $V - 0$, where $t > 0$. Intersecting this curve with the unit sphere, we obtain the required point $\Psi(\Pi g)$ of $M(p, q, r)$. It is easily verified that Ψ is smooth, well defined, one-to-one, and that its derivative has maximal rank everywhere. Since $\Pi \backslash G$ is compact while $M(p, q, r)$ is connected, it follows that Ψ is a diffeomorphism. ■

COROLLARY 6.6. *The Brieskorn manifold* $M(p, q, r)$ *has a finite covering manifold diffeomorphic to a non-trivial circle bundle over a surface.*

Proof. Choosing $N \subset \Pi$ as in 2.7, it is easily verified that $N\backslash G$ fibers as a circle bundle over the surface $\tilde{N}\backslash P$. ∎

This corollary remains true in the spherical and nilmanifold cases.

CONCLUDING REMARKS. It is natural to ask whether there is a generalization of 6.5 in which the group Π is replaced by an arbitrary discrete subgroup of $G = \widetilde{SL}(2, R)$ with compact quotient. It seems likely that such a generalization exists:

CONJECTURE. For any discrete subgroup $\Gamma \subset G$ with compact quotient, the algebra $A_\Gamma^{*,1}$ of Γ-automorphic forms is finitely generated.*

Choosing generators $\phi_1, \cdots, \phi_k$ for this algebra, it would then follow from 5.9 that the k-tuple $(\phi_1, \cdots, \phi_k)$ embeds the complex 2-manifold $\Gamma\backslash(P \times \tilde{C}^{\cdot})$ into the complex coordinate space C^k. It is conjectured that the image in C^k is of the form $V - 0$ where $V = V_\Gamma$ is an irreducible algebraic surface, singular only at the origin. Intersecting this image V_Γ with a sphere centered at the origin, we then obtain a 3-manifold diffeomorphic to $\Gamma\backslash G$.

In general it is not claimed that V_Γ embeds as a hypersurface. Presumably V_Γ can be embedded in C^3 only if the algebra $A_\Gamma^{*,1}$ happens to be generated by three elements.

Note that this surface V_Γ is *weighted homogeneous*. That is, if each variable z_j is assigned a weight equal to the degree of ϕ_j, then V_Γ can be defined by polynomial equations $f(z_1, \cdots, z_k) = 0$ which are homogeneous in these weighted variables.

Not every weighted homogeneous algebraic surface can be obtained in this way. Here is an interesting class of examples. Start with an algebraic curve S of genus $g \geq 2$ together with a complex analytic line bundle ξ over S with Chern number $c_1 < 0$. Let $V(\xi)$ be obtained from the total space $E(\xi)$ by collapsing the zero-section to a point. Applying the Riemann-Roch theorem to negative tensor powers ξ^{-n} one can presumably construct enough holomorphic mappings $V(\xi) \to C$ to embed $V(\xi)$ as a weighted homogeneous algebraic surface in some C^k.

CONJECTURE. The algebraic surface $V(\xi)$ obtained in this way is isomorphic to V_Γ for some discrete $\Gamma \subset G$ if and only if some tensor power $\xi^k = \xi \otimes \cdots \otimes \xi$ is isomorphic to the tangent bundle $\tau(S)$.

For each fixed S there are uncountably many line bundles ξ with negative Chern number. Only finitely many of these (the precise number is k^{2g} for each k dividing $2g-2$) satisfy the condition that the k-th tensor power is isomorphic to $\tau(S)$.

§7. A fibration criterion

In this section p, q, r may be any integers ≥ 2.

LEMMA 7.1. *If the least common multiples of* (p, q) *of* (p, r) *and of* (q, r) *are all equal:*

$$\ell.c.m.(p, q) = \ell.c.m.(p, r) = \ell.c.m.(q, r) \ ,$$

then the Brieskorn manifold $M(p, q, r)$ *fibers smoothly as a principal circle bundle over an orientable surface.* ■

The precise surface B and the precise circle bundle will be determined below.

At the same time we show that the complement of the origin in the Pham-Brieskorn variety $z_1{}^p + z_2{}^q + z_3{}^r = 0$ fibers complex analytically as a principal $C^{\cdot}$-bundle over the Riemann surface B. In other words this variety V can be obtained from a complex analytic line bundle ξ over B by collapsing the zero cross-section to a point.

One special case is particularly transparent. If $p = q = r$, then the hypothesis of 7.1 is certainly satisfied. The equation $z_1{}^p + z_2{}^p + z_3{}^p = 0$ is then homogeneous, and hence defines an algebraic curve B in the complex projective plane $P^2(C)$. Clearly the mapping $(z_1, z_2, z_3) \mapsto (z_1 : z_2 : z_3)$ fibers $M(p, q, r)$ as a circle bundle over B.

Proof of 7.1. Starting with any values of p, q, r, let m denote the least common multiple of p, q, and r. Then the group $C^{\cdot}$ of non-zero complex numbers operates on the variety $z_1{}^p + z_2{}^q + z_3{}^r = 0$ by the correspondence

$$u : (z_1, z_2, z_3) \mapsto (u^{m/p} z_1, u^{m/q} z_2, u^{m/r} z_3)$$

for $u \neq 0$. Restricting to the unit circle $|u| = 1$ and the unit sphere $|z_1|^2 + |z_2|^2 + |z_3|^2 = 1$, we obtain a circle action on $M = M(p, q, r)$.

Let us determine whether any group elements have fixed points in M or in $V - 0$. If

$$(u^{m/p} z_1, u^{m/q} z_2, u^{m/r} z_3) = (z_1, z_2, z_3) \in V - 0 ,$$

then at least two of the complex numbers z_1, z_2, z_3 must be non-zero, hence at least two of the numbers $u^{m/p}, u^{m/q}, u^{m/r}$ must equal 1. If the three integers $m/p, m/q, m/r$ happen to be pairwise relatively prime, then it clearly follows that $u = 1$.

Thus, if $m/p, m/q, m/r$ are pairwise relatively prime, we obtain a smooth free $C^{\cdot}$ action on $V - 0$ restricting to a smooth free circle action on $M = M(p, q, r)$. Evidently M fibers as a smooth circle bundle over the quotient space $S^1 \backslash M = B$, which must be a compact, orientable, 2-dimensional manifold. In fact, using the alternative description

$$B = C^{\cdot} \backslash (V - 0)$$

we see that B has the structure of a complex analytic 1-manifold. (The two quotient spaces can be identified since every $C^{\cdot}$-orbit intersects the unit sphere precisely in a circle orbit.)

Since an elementary number theoretic argument shows that $m/p, m/q, m/r$ are pairwise relatively prime if and only if the hypothesis of 7.1 is satisfied, this completes the proof. ∎

To compute the genus of the surface $B = S^1 \backslash M = C^{\cdot} \backslash (V - 0)$, we describe it as a branched covering of the 2-sphere $P^1(C)$ by means of the holomorphic mapping

$$f : (u^{m/p} z_1, u^{m/q} z_2, u^{m/r} z_3) \mapsto (z_1{}^p : z_2{}^q) .$$

Clearly f is well defined. A counting argument, which will be left to the reader, shows that the pre-image of a general point of $P^1(C)$ consists of precisely pqr/m points of B. Thus f is a map of degree pqr/m from B to $P^1(C)$.

There are just three branch points in $P^1(C)$, corresponding to the possibilities $z_1 = 0$, $z_2 = 0$, and $z_3 = 0$ respectively. The preimage of a branch point contains qr/m, or pr/m, or pq/m points respectively. Again the count will be left to the reader.

Now choose a triangulation of $P^1(C)$ with the three branch points $(0:1)$, $(1:0)$, and $(-1:1)$ as vertices. Counting the numbers of vertices, edges, and faces in the induced triangulation of B, we easily obtain the following.

LEMMA 7.2. *Let* p, q, r *be as in 7.1, with least common multiple* m. *Then the surface* $B = S^1 \backslash M$ *has Euler characteristic* $\chi(B) = (qr + pr + pq - pqr)/m = pqr(p^{-1} + q^{-1} + r^{-1} - 1)/m$.

In particular the sign of $\chi(B)$ is equal to the sign of $p^{-1} + q^{-1} + r^{-1} -$ The genus g can now be recovered from the usual formula $\chi = 2 - 2g$. Note that the genus satisfies $g > 2$, except in the four special cases $(2,2,2)$, $(2,3,6)$, $(2,4,4)$, and $(3,3,3)$. (Compare Section 2.1.)

To determine the precise circle bundle in question, we must compute the Chern class

$$c_1 = c_1(\xi) \in H^2(B; Z)$$

or equivalently the Chern number $c_1(\xi)[B]$ of the associated complex line bundle ξ. (The Chern class c_1 can also be described as the Euler class of ξ. Using the Gysin sequence ([Spanier, pp. 260-261], [Milnor and Stasheff, p. 143]), one sees that $H_1(M; Z)$ is the direct sum of a free abelian group of rank $2g$ and a cyclic group of order $|c_1(\xi)[B]|$.)

To compute c_1 we consider the map

$$F : (z_1, z_2, z_3) \mapsto (z_1{}^p, z_2{}^q)$$

from $V - 0$ to $C^2 - 0$. Thus we obtain a commutative diagram

$$\begin{tikzcd} V - 0 \arrow[r, "F"] \arrow[d] & C^2 - 0 \arrow[d] \\ B \arrow[r, "f"] & P^1(C) \end{tikzcd}$$

where the right hand vertical arrow is the canonical fibration $(z_1, z_2) \mapsto (z_1 : z_2)$ with Chern number -1, associated with the Hopf fibration $S^3 \to S^2$.

This map F is not quite a bundle map, since inspection shows that each fiber of the left hand fibration covers the corresponding fiber on the right m times. To correct this situation we must factor $V - 0$ by the action of the subgroup

$$\Omega \subset C^{\cdot}$$

consisting of all m-th roots of unity. Thus we identify (z_1, z_2, z_3) with $\omega^{m/p} z_1, \omega^{m/q} z_2, \omega^{m/r} z_3$ for each $\omega^m = 1$, obtaining a new commutative diagram

$$\begin{tikzcd} \Omega \backslash (V - 0) \arrow[r, "\overline{F}"] \arrow[d] & C^2 - 0 \arrow[d] \\ B \arrow[r, "f"] & P^1(C) \end{tikzcd}$$

where $\overline{F}$ is now a bundle map. Since f has degree pqr/m, it follows that the new $C^{\cdot}$-bundle $\Omega \backslash (V - 0) \to B$ has Chern number $-pqr/m$. But this new bundle can be described as the $C^{\cdot}$-bundle associated with the m-fold tensor product $\xi \otimes \cdots \otimes \xi$ of the original complex line bundle ξ. Therefore ξ has Chern number $c_1(\xi)[B] = -pqr/m^2$.

Recapitulating, we have proved the following.

THEOREM 7.3. *If the hypothesis*

$$m = \ell.c.m.(p, q) = \ell.c.m.(p, r) = \ell.c.m.(q, r)$$

of 7.1 is satisfied, then the Brieskorn manifold $M(p,q,r)$ *fibers as a smooth circle bundle with Chern number* $-pqr/m^2$ *over a Riemann surface of Euler characteristic* $pqr(p^{-1}+q^{-1}+r^{-1}-1)/m$.

The number pqr/m^2 can be described more simply as the greatest common divisor of p, q, r.

The negative sign of the Chern number has no particular topological significance, but is meaningful in the complex analytic context, since ξ is a complex analytic line bundle with no non-zero holomorphic cross-sections.

Note that the Euler characteristic of B is always a multiple of the Chern number of ξ. In general it is a large multiple, for it is not difficult to show that the ratio satisfies

$$\chi(B)/c_1(\xi)[B] = m(1-p^{-1}-q^{-1}-r^{-1}) \geq m/6$$

in the hyperbolic case. Hence this ratio tends to infinity with m. Therefore the genus of B also tends to infinity with m.

Here are two examples to illustrate 7.3.

EXAMPLE 1. For any $g \geq 0$, the manifold $M(2, 2(g+1), 2(g+1))$ fibers as a circle bundle with Chern number -2 over a surface of genus g. Similarly, for any $g \geq 1$, the manifold $M(2, 2g+1, 2(2g+1))$ fibers as a circle bundle with Chern number -1 over a surface of genus g.

EXAMPLE 2. The Brieskorn manifolds $M(p,q,r)$ are not all distinct. For example, $M(2,9,18)$ and $M(3,5,15)$ are diffeomorphic, since each fibers as a circle bundle with Chern number -1 over a surface of genus 4.

CONCLUDING REMARK. If it is known that $M(p,q,r)$ fibers as a circle bundle over a surface, does it follow that the hypothesis of 7.1 must be satisfied? The lens spaces $M(2,2,r)$ with $r \geq 3$ provide counter-examples. These fiber as circle bundles with Chern number $\pm r$ over a surface of genus zero. (Presumably there is no associated analytic fibration of $V-0$?) However, these are the only counter-examples. In the cases $p^{-1}+q^{-1}+r^{-1}\geq 1$, this can be verified by inspection. Thus we need only prove the following.

LEMMA 7.4. *In the hyperbolic case* $p^{-1}+q^{-1}+r^{-1}<1$, *if* $M(p,q,r)$ *has the fundamental group of a principal circle bundle over an orientable surface, then the hypothesis of 7.1 must be satisfied.*

The proof can be sketched as follows. First note that the fundamental group of a principal circle bundle over an orientable surface, modulo its center, has no elements of finite order. Now consider the fundamental group $\Pi=\Pi(p,q,r)$ of Section 6. The center of Π is precisely equal to $\Pi\cap C$. As noted in 2.6, an element of $\Gamma/C\supset\Pi/\Pi\cap C$ has finite order if and only if it is conjugate to a power of γ_1,γ_2, or γ_3 modulo C. To decide which powers of say γ_1 belong to Π, we carry out a matrix computation in the abelianized group Γ/Π. (Compare Section 3.2.) Setting $\mu=\ell.\mathrm{c.m.}(q,r)$, it turns out that the order k of γ_1 modulo Π is given by

$$k = p\mu(1-p^{-1}-q^{-1}-r^{-1}) \equiv -\mu(\mathrm{mod}\ p) .$$

Evidently the element $\gamma_1{}^k$ of Π belongs to $\Pi\cap C$ if and only if k is a multiple of p, or in other words if and only if μ is a multiple of p. Thus $\Pi/\Pi\cap C$ has no elements of finite order if and only if

$$\ell.\mathrm{c.m.}(q,r)\equiv 0 \qquad (\mathrm{mod}\ p) ,$$

and similarly

$$\ell.\mathrm{c.m.}(p,r)\equiv 0 \qquad (\mathrm{mod}\ q) ,$$
$$\ell.\mathrm{c.m.}(p,q)\equiv 0 \qquad (\mathrm{mod}\ r) .$$

Clearly these conditions are equivalent to the hypothesis of 7.1. ■

§8. **The nil-manifold case** $p^{-1} + q^{-1} + r^{-1} = 1$

As noted in 2.1, we are concerned only with three particular cases. The triple (p,q,r), suitably ordered, must be either $(2,3,6)$ or $(2,4,4)$ or $(3,3,3)$. Clearly each of these triples satisfies the hypothesis of 7.1. Hence by 7.3 the corresponding manifold $M = M(p,q,r)$ is a circle bundle over a torus. The absolute value of the Chern number of this circle bundle is the greatest common divisor of p, q, r which is either 1, or 2, or 3 respectively.

But any non-trivial circle bundle over a torus can also be described as a quotient manifold N/N_k as follows. Let N be the nilpotent Lie group consisting of all real matrices of the form

$$A = \begin{bmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix},$$

and let N_k be the discrete subgroup consisting of all such matrices for which a, b, and c are integers divisible by k. (Here k should be a positive integer.) Then the correspondence

$$A \mapsto (a \bmod k,\ b \bmod k)$$

maps N/N_k to the torus with a circle as fiber. The first homology group

$$H_1(N/N_k;\, Z) \cong N_k/[N_k, N_k]$$

is isomorphic to $Z \oplus Z \oplus (Z/k)$, so the Chern number of this fibration must be equal to $\pm k$. Thus we obtain the following three diffeomorphisms

$$\begin{aligned} M(2,3,6) &\cong N/N_1 \\ M(2,4,4) &\cong N/N_2 \\ M(3,3,3) &\cong N/N_3 \ . \end{aligned}$$

It must be admitted that this proof is rather ad hoc. I do not know whether there exists a more natural construction of these diffeomorphisms.

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Added in Proof:

Durfee, A., and Kauffman, L., Periodicity of branched cyclic covers, to appear.

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Note: I am informed by V. Arnold that all of the essential results of this paper have been obtained independently by I. V. Dolgačev. Parts of Dolgačev's work are described in his paper cited above, and in his paper "Automorphic forms and quasihomogeneous singularities," Funct. anal. 9:2 (1975), 67-68.