The Geometry and Physics of Knots

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When a mathematician addresses a general scientific audience, even as enlightened an audience as attends a Royal Institution Discourse, he faces a daunting task. Mathematics can be such a highly technical and abstract subject that communicating its latest developments to a lay public presents formidable difficulties. Bearing this in mind I selected this evening's topic according to the following criteria. It should have:

(1) a simple visual content;
(2) an interesting historical background;
(3) a relation to physics;
(4) a recent exciting story.

I picked on the subject of knots because it seemed to satisfy all these criteria, and on 11th July 1988 I sent my title off to the Royal Institution. Two weeks later, on 26th July, there was a spectacular further break-through which will be my main focus and makes the story even more exciting and topical than I had originally anticipated.

Knots, Links and Braids

Let me begin by introducing the dramatis personnae of the evening: *knots* and their close relatives, *links* and *braids*. For a mathematician a knot is a closed piece of string, with no loose ends, exemplified in Figure 1.

These pictures represent the piece of string laid flat on a table, so that at each crossing point one piece is 'under' and the other piece 'over' as indicated. Two knots are considered the same if we can manipulate one so that it looks like the other. The precise length (and thickness) of the string is irrelevant. Only its

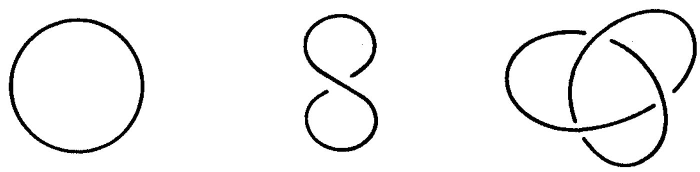


Figure 1

'knottedness' is important. For example, the first two pictures (the circle and the figure-eight) represent the same knot – a rather uninteresting one referred to (for obvious reasons) as the 'unknot'. On the other hand, the trefoil knot cannot be disentangled and is not the same as the unknot. Of course in manipulating knots we are not allowed to cut and rejoin the string.

In every day parlance, a knotted piece of string usually has two free ends and we can with skill, untie the knot by threading the free ends through the knot. However, for a long piece of string this is a lengthy process and we might prefer (or be forced by other factors) to keep the free ends fixed and work directly on disentangling the knot. By this stage we are essentially back to the problem of knots in closed strings, which is why mathematicians have focussed on this point of view.

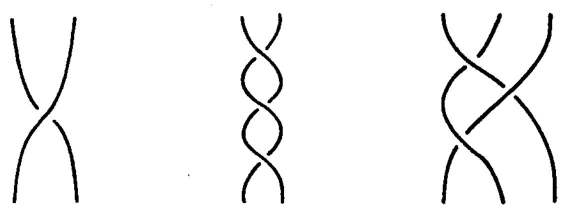
A link is like a knot except that it is made up of several closed pieces of string. Each piece may itself be knotted but, in addition, the different pieces may be 'linked' as illustrated by the following simple cases (Figure 2).

Figure 2

One reason why it is necessary to consider links as well as knots is that it is not easy, when presented with a diagrammatic picture (or by a real string tangle) to decide whether it consists of one or more pieces.

Finally a *braid* is a collection of strings, or 'strands', (with free ends) which may be entangled but which all 'move' in the same direction. Examples of braids with two or three strands are illustrated in Figure 3.

Figure 3



Note that each strand is always moving upwards. As with knots and links two braids are considered the same if we can manipulate one into the other while keeping the ends fixed and always maintaining the upwardness of each braid. The exact positions of the end points of the braid are also considered irrelevant.

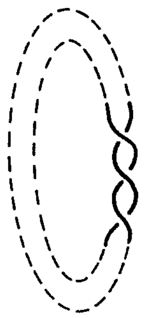


Figure 4

Any braid can be converted into a link in a standard manner by simply connecting up the initial and final points as indicated in Figure 4.

A little manipulation shows that this particular example represents a trefoil knot. More generally every knot or link arises this way from a suitable braid.

As these few examples illustrate, the problem of deciding whether two plane diagrams represent the same knot or link is a difficult one. It corresponds essentially to the practical difficulty in trying to disentangle a complicated piece of string. By disregarding questions concerning the length and thickness of the string the problem is not so much one of *geometry* as of *topology*. In fact the study of knots is the archetype of a topological problem.

History of Knot Theory

Knots have attracted attention since the earliest times, as in the classical story of Alexander the Great and the Gordian knot. It was not however until the nineteenth century that it began to be considered scientifically. The notion of linkage is of fundamental importance in connection with electromagnetic induction, a fact which was fully appreciated by Maxwell. It is appropriate to recall that, here at the Royal Institution, Faraday demonstrated that an electric current along a wire produces an external magnetic field whose lines of forces link around the wire (Figure 5).

Such ideas may have been, in part, behind the ambitious theory of *Vortex Atoms* put forward by Lord Kelvin around 1867 [2]. At this period the ultimate nature of matter was a great mystery (it still is!) and Kelvin had the magnificent idea that

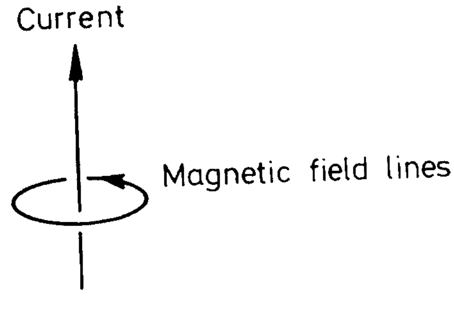


Figure 5

THE FIRST SEVEN ORDERS OF KNOTTINESS.

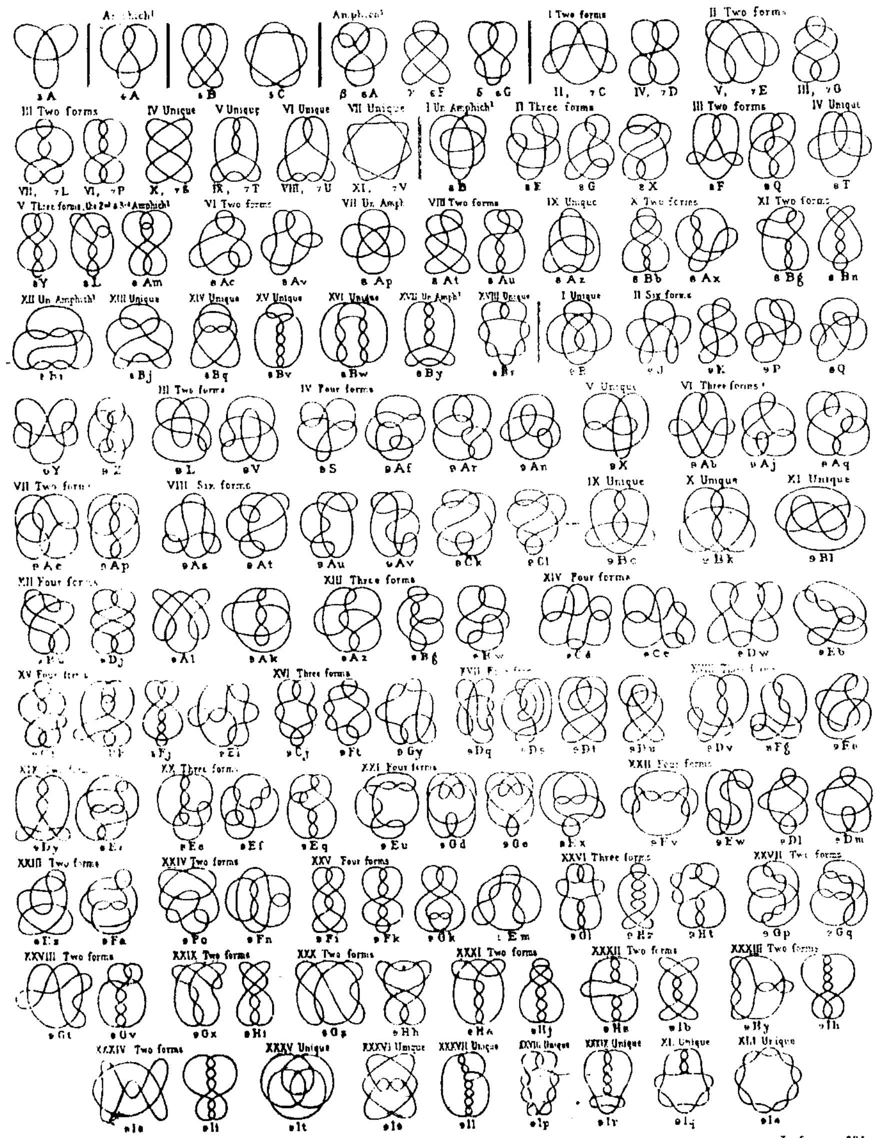


Figure 6

atoms might consist of knotted vortex tubes of the ether. His arguments in favour of this possibility can be summarised as follows:

(1) *Stability*. The stability of matter could be explained by the topological stability of knots, *i.e.* under continuous deformation knots remain essentially the same.

(2) *Variety*. The large number (actually infinite) of different knots could account for the different chemical elements.

(3) *Vibrations*. The vortex tubes could presumably vibrate and this might explain spectral lines.

As a twentieth century footnote we might add a further argument:

(4) *Transmutation*. At very high energies atoms can change into other atoms just as knots can, if we allow some cutting and recombination, change into other knots.

Kelvin's theory was, for a decade or so, taken very seriously and Maxwell's verdict was that "it satisfies more of the conditions than any atom hitherto imagined". If Kelvin's theory was on the right lines then a classification of knots was clearly going to be an essental ingredient and P.G. Tait, one of Kelvin's collaborators, spent more than 10 years studying and tabulating knots. He enumerated knots by the number of crossings of a plane diagram and produced tables of the distinct knots arising. This turned out to be a monumental task. Tait studied knots with up to 11 crossings (a sample page of his tables is copied in Figure 6). For 10 crossings there are 165 different knots while more recent computer tabulations for 13 crossings produce over 10,000 different knots. Perhaps it is fortunate for chemists that Kelvin's theory was eventually discarded!

In the course of his investigations Tait made a number of empirical discoveries which have subsequently been christened as Tait's conjectures. These conjectures appear highly plausible but resisted all attempts at proof by mathematicians for a whole century. Very recently, as a result of the exciting new ideas I am reporting on, many of Tait's conjectures have now been established. I will explain the simplest of Tait's conjectures. For this I need two notions. First an *alternating* knot diagram is one where, in following the path of knot, we meet crossings which are alternately 'over' and 'under'. A diagram which is not alternating tends to be one that can be simplified (see Figure 7), so that it might seem reasonable to concentrate on knots given by alternating diagrams (non-alternating knots, *i.e.* knots which cannot be represented by an alternating diagram, do exist but require many crossings and are hard to draw). Next, consider a schematic knot diagram as in Figure 8 in which a single crossing point separates the knot into two separate parts, schematically indicated by boxes. Clearly such a diagram can be simplified by a simple half-twist which removes the central crossing, to give Figure 9.

Figure 7