XXXVIII.

SOME ELEMENTARY PROPERTIES OF CLOSED PLANE CURVES*.

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THE closed curves contemplated are supposed to have nothing higher than *double* points. By infinitesimal changes of position of the branches intersecting in it, a triple point is decomposable into 3 double points, a quadruple point into 6, and generally an x-ple point into $\frac{x(x-1)}{1 \cdot 2}$ double points.

I. A closed curve cuts any infinite unknotted line in an even number of points. [Infinite here implies merely that both ends are outside the closed curve.]

For, if it be broken anywhere, as at A (fig. 1), both free ends are on the same side of the infinite line.

II. The same is true if the infinite line be knotted.

For, as there is nothing higher than a double point, the knotted line may be opened up into an unknotted one (as in fig. 2) without changing the circumstances.

It is an interesting problem to find the number of such modes of opening a given knot. An extension of this problem leads to the question of the number of essentially distinct ways in which a closed curve may be broken up into separate closed curves, knotted or unknotted (fig. 3).

III. If any two closed curves cut one another, there is an even number of points of intersection.

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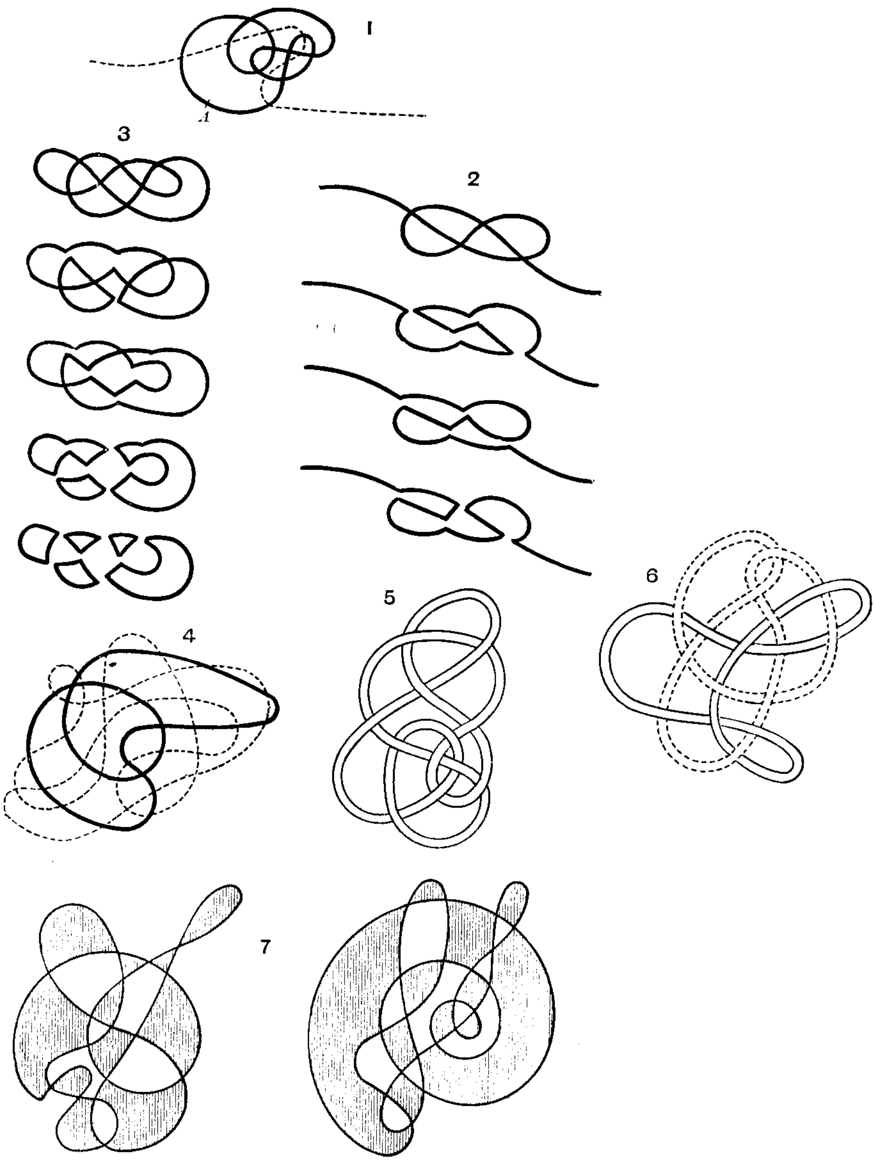
1
A
3
2
4
5
6
7

For there must be points of one of them at least in the outer boundary of the complex figure. Open it at such a point, and the line becomes infinite in the sense of I. above (fig. 4).

IV. In going continuously along a closed curve from a point of intersection to the same point again an even number of intersections is passed.

For (*a*) If the partial path cross itself, it must pass twice through each such intersection.

(*b*) As regards the rest, the two parts may be considered to be separate closed curves as in III.

V. Hence, in going round such a closed curve we may go alternately above and below the branches as we meet them (fig. 5). Strictly speaking, we have only now arrived at *Knots*; and, in what precedes, we ought to read 'autotomic' for 'knotted.'

VI. By III. the same proposition is true of a complex arrangement of any number of separate closed curves superposed in any manner (fig. 6).

VII. In passing from the interior of any one cell to that of any other—in any system of superposed closed curves—the number of crossings is always even or always odd, whatever path be taken.

For any path from the exterior, through each of these cells to the exterior again, has an even number of crossings. Varying only the part of this path between the two cells, it must have always an even or an odd number of crossings.

VIII. Hence, the cells may be coloured black and white in such a way that from white to white there is always an even number of crossings, and from white to black an odd number. Such closed curves therefore divide the plane as nodal lines do a vibrating plate (fig. 7).

The development of this subject promises absolutely endless work—but work of a very interesting and useful kind—because it is intimately connected with the theory of knots, which (especially as applied in Sir W. Thomson's Theory of *Vortex Atoms*) is likely soon to become an important branch of mathematics.