

# $Spin^c$ -MANIFOLDS

BLAKE MELLOR

## 1. INTRODUCTION

$Spin^c$ -structures on manifolds are a complex analogue to the more common notion of spin structures on manifolds. They have been known since the 1960's (see [A-B-S]), but they had no real importance (as far as I can tell), until the recent announcement of the Seiberg-Witten equations for 4-manifolds in [W]. These equations promise to vastly simplify the study of smooth 4-manifolds, and their definition requires the presence of a  $spin^c$ -structure. In this paper I will review the definition of  $spin^c$ -structures on manifolds from both a geometric and algebraic point of view, and prove their existence in some important cases. I will conclude by looking at how they appear in the formulation of the Seiberg-Witten equations.

## 2. GEOMETRIC FORMULATION OF $Spin_n^c$

In one sense,  $spin$  and  $spin^c$  structures are just generalizations of orientations. Consider a smooth manifold  $M^n$  with tangent bundle  $TM$ . This vector space bundle gives rise to a principal  $O(n)$ -bundle of frames, which we denote  $P_O(TM)$ . Recall that the manifold is said to be *orientable* if this bundle can be reduced to an  $SO(n)$ -bundle  $P_{SO}(TM)$ , making the fibers connected. This means that any trivialization of the bundle over the (disconnected) 0-skeleton of  $M$  can be extended to a trivialization over the (connected) 1-skeleton. The next step is to make the fiber simply connected (where possible). This will mean that a trivialization over the 1-skeleton of  $M$  can be extended over the 2-skeleton. Recalling that, for  $n \geq 3$ ,  $\pi_1(SO(n)) = \mathbb{Z}_2$ , we define  $Spin_n$  to be the double cover of  $SO(n)$ . For  $n \geq 3$ , this is the universal (i.e. simply-connected) cover; in the exceptional cases we have  $Spin_2 = S^1$  and  $Spin_1 = S^0$ . We then say that the manifold is *spin* if the bundle  $P_{SO}(TM)$  has a double cover by a principal  $Spin_n$ -bundle  $P_{Spin}(TM)$ .

To find the complex analogue, we replace  $SO(n)$  by the group  $SO(n) \times U(1)$ , and consider its double cover. With this in mind, we define:

$$Spin_n^c = (Spin_n \times U(1)) / \{\pm(1, 1)\} = Spin_n \times_{\mathbb{Z}_2} U(1)$$

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This is the desired double cover of  $SO(n) \times U(1)$  via the map  $[A, \lambda] \mapsto [p(A), \lambda^2]$ , where  $p$  is the double cover of  $SO(n)$  by  $Spin_n$ . Finally, we define  $M$  to be  $spin^c$  if given the bundle  $P_{SO}(TM)$ , there are principal bundles  $P_{U(1)}(TM)$  and  $P_{Spin^c}(TM)$  with a  $spin^c$ -equivariant bundle map:

$$\xi : P_{Spin^c}(TM) \longrightarrow P_{SO}(TM) \times P_{U(1)}(TM).$$

This definition of  $Spin_n^c$  leads to a very nice geometric criterion for the existence of a  $spin^c$ -structure ([K2]). Since  $U(1) = SO(2)$ , there is a natural map  $SO(n) \times U(1) \rightarrow SO(n+2)$  which extends (via Whitney sum) to a map of bundles. We can define  $Spin_n^c$  as the pullback by this map of the covering map  $Spin_{n+2} \rightarrow SO(n+2)$ :

$$\begin{array}{ccc} Spin_n^c & \longrightarrow & Spin_{n+2} \\ \downarrow & & \downarrow \\ SO(n) \times U(1) & \longrightarrow & SO(n+2) \end{array}$$

Therefore, a  $spin^c$ -structure on  $TM$  consists of a complex line bundle  $L$  and a  $spin$ -structure on  $TM \oplus L$ . We can restate this as:

**Theorem 1.** *A manifold  $M$  is  $spin^c$  (i.e.  $TM$  has a  $spin^c$ -structure)  $\Leftrightarrow$  there is a complex line bundle  $L$  over  $M$  such that  $TM \oplus L$  has a  $spin$ -structure.*

So  $M$  is  $spin^c$  if the obstruction to extending a trivialization of the tangent bundle over the 2-skeleton can be removed by adding a complex line bundle.

### 3. EXAMPLES OF $Spin^c$ -MANIFOLDS

We start with examples of manifolds which have canonical  $Spin^c$ -structures.

**Theorem 2.** *If  $M$  is a  $spin$  manifold, then  $M$  has a canonical  $spin^c$ -structure.*

PROOF: We simply extend the  $spin$  structure by taking the fiber product with the trivial  $U(1)$ -bundle  $U_1$ , letting

$$P_{Spin^c}(TM) = P_{Spin}(TM) \times_{M, \mathbb{Z}_2} U_1. \square$$

**Theorem 3.** *If  $M$  has an almost complex structure, then  $M$  has a canonical  $spin^c$ -structure.*

PROOF: Let  $j : U(k) \rightarrow SO(2k)$  denote the natural homomorphism. Then we can define a homomorphism  $g : U(k) \rightarrow SO(2k) \times U(1)$  by  $g(A) = (j(A), \det(A))$ . Although  $j$  does not lift to  $Spin_{2k}$ ,  $g$  does lift to  $Spin_{2k}^c$ . Denote this lift  $\gamma$ . An almost complex structure on  $M$  means  $TM$  can be viewed as a complex vector bundle, and so  $M$  has

an unitary frame bundle  $P_{U(n)}(TM)$ . We now construct the desired  $Spin^c$  bundle as an associated bundle:

$$P_{Spin^c}(TM) = P_{U(n)}(TM) \times_{\gamma} Spin_{2k}^c. \quad \square$$

In fact, we can give another, more algebraic, general criterion for whether a manifold has a  $Spin^c$ -structure:

**Theorem 4.** *An orientable manifold  $M$  can be given a  $Spin^c$ -structure  $\Leftrightarrow$  the second Stiefel-Whitney class  $w_2(M)$  is the mod 2 reduction of an integral class.*

PROOF: Recall that a manifold  $M$  has a  $spin$ -structure  $\Leftrightarrow$  the second Stiefel-Whitney class  $w_2(M)$  is 0 (see [L-M] and [K2]). So we apply our geometric criterion from the last section, which says that  $M$  can be given a  $spin^c$ -structure  $\Leftrightarrow$  there is a complex line bundle  $L$  such that  $TM \oplus L$  is spin, which means  $w_2(TM \oplus L)$  is 0. But, since the Stiefel-Whitney classes are stable, we have:

$$w_2(TM \oplus L) = w_2(TM) + w_2(L) + w_1(TM)w_1(L) = 0$$

Both these bundles are orientable, so the first Stiefel-Whitney classes are both 0, which means  $w_2(TM) + w_2(L) = 0$ . Since these are mod 2 classes,  $w_2(TM) = w_2(L)$ .  $w_2(L)$  has an integral lift, the first Chern class of the line bundle, so  $w_2(TM) = w_2(M)$  also has an integral lift, which proves the theorem in one direction. To go the other way, we can follow the same argument backwards, since if  $w_2(TM)$  lifts to an integral class  $e$ , we can always find a complex line bundle with first Chern class  $e$ , which will be the line bundle we need for our  $spin^c$ -structure.  $\square$

In particular, by [M], this means that any orientable four manifold can be given a  $Spin^c$ -structure, which will be crucial to the formulation of the Seiberg-Witten equations.

#### 4. CLASSIFICATION OF $spin^c$ -STRUCTURES OF A MANIFOLD

We will classify  $spin^c$ -structures by using classifying spaces, an important tool from algebraic topology. Our discussion here follows [?]. We start with a basic definition:

DEFINITION: A *classifying space* for a group  $G$  is a CW-complex  $BG$  and principal  $G$ -bundle  $EG$  over  $BG$  such that given any space  $X$  and a principal  $G$ -bundle  $E$  over  $X$ , there is a map  $f : X \rightarrow BG$  such that  $E = f^*(EG)$ .

It is not hard to show that  $BG$  is unique up to homotopy equivalence. From our definition and discussion of  $Spin_n^c$  we have the following commutative diagram of groups, with rows and columns exact:

$$\begin{array}{ccccc} \mathbb{Z}_2 & \subset & U(1) & \rightarrow & U(1) \\ \parallel & & \downarrow & & \downarrow \\ \mathbb{Z}_2 & \subset & Spin_n^c & \rightarrow & SO(n) \times U(1) \\ & & \downarrow & & \downarrow \\ & & SO(n) & = & SO(n) \end{array}$$

This diagram induces a similar commutative diagram of classifying spaces (by, for example, Milgram's construction of the classifying space in **[P]**). Therefore, we can view  $BSpin_n^c$  as a bundle over  $BSO(n)$  with fiber  $BU(1)$ .

Now we view the tangent bundle of a manifold  $M$  as a map  $\eta : M \rightarrow BSO(n)$ . A  $spin^c$ -structure on the tangent bundle is then a lift of this map to  $BSpin_n^c$ , giving a commutative diagram:

$$\begin{array}{ccc} BU(1) & \rightarrow & BSpin_n^c \\ & \nearrow & \downarrow \\ M & \xrightarrow{\eta} & BSO(n) \end{array}$$

**Theorem 5.** *The set of lifts of  $\eta$  is in bijective correspondence with  $[M, BU(1)]$ .*

PROOF: Let  $h_p$  denote the homeomorphism from  $BU(1)$  to the fiber of  $BSpin_n^c$  over the point  $p \in BSO(n)$ . Given a map  $\lambda \in [M, BU(1)]$ , define the lift  $\eta_\lambda$  by  $\eta_\lambda(x) = h_{\eta(x)} \circ \lambda(x)$ . This is clearly an injective map from  $[M, BU(1)]$  into the set of lifts; it is also surjective, since two different lifts will have to disagree on at least one fiber.  $\square$

Since  $[M, BU(1)]$  is just the set of complex line bundles over  $M$ , which are classified by their first Chern class, the theorem implies that the set of lifts (and hence the  $spin^c$ -structures on  $M$ ) is in correspondence with the second cohomology group  $H^2(M; \mathbb{Z})$ . (Alternatively, we note from **[P]** that  $BU(1) = BS^1 = \mathbb{C}P^\infty$ . Since  $\mathbb{C}P^\infty = K(\mathbb{Z}, 2)$ , the Eilenberg-MacLane space, this means  $[M, BU(1)] = [M, K(\mathbb{Z}, 2)] = H^2(M; \mathbb{Z})$ , by **[K1]**.) We can combine this group structure with the correspondence to define a simply transitive group action of  $[M, BU(1)] = H^2(M; \mathbb{Z})$  on the set of lifts:

$$\begin{aligned} \gamma \cdot \eta_\lambda &= \eta_{\gamma \cdot \lambda} \\ \gamma, \lambda &\in H^2(M; \mathbb{Z}) \end{aligned}$$

We also want to consider our geometric criterion identifying a  $spin^c$ -structure on  $M$  with a complex line bundle  $L$  over  $M$  and a  $spin$ -structure on  $TM \oplus L$ . The first question is whether the  $spin^c$ -structure

determines the complex line bundle in this description. The answer is “Yes.” From the commutative diagram of groups drawn above, we can induce the following commutative diagram:

$$\begin{array}{ccccc}
 & & BSpin_n^c & & \\
 & \nearrow \mu & & \searrow pr & \\
 M & & \downarrow & & B(SO(n) \times U(1)) \\
 & \searrow \eta & & \swarrow & \\
 & & BSO(n) & & 
 \end{array}$$

where the map  $\mu : M \rightarrow BSpin_n^c$  is a lift of the map  $\eta : M \rightarrow BSO(n)$ , and the maps on the right-hand side of the diagram are projections induced from our commutative diagram of groups. So the lift  $\mu$  of  $\eta$  canonically gives us a lift  $pr \circ \mu : M \rightarrow B(SO(n) \times U(1))$ . This lift is the complex line bundle desired.

We can also ask the question in reverse: does the complex line bundle determine the  $spin^c$ -structure? Here, the answer is unsurprisingly “No.” Recall from the proof of Theorem 4 in Section 3 that we must have  $w_2(TM) = w_2(L) = c_1(L) \pmod 2$ . Hence there are strictly less than  $|H^2(M; \mathbb{Z})|$  possible line bundles, so these cannot determine the  $|H^2(M; \mathbb{Z})|$   $spin^c$ -structures in a one-to-one fashion. The question now becomes: given a complex line bundle, how many different  $spin^c$ -structures are associated with that bundle?

As a first approximation, we compute the number of  $spin$ -structures on  $TM \oplus L$ . As above, the  $spin$ -structures on  $TM \oplus L$  correspond to lifts of a map  $\eta : M \rightarrow BSO(n+2)$  to  $BSpin_{n+2}$ , so we have a diagram:

$$\begin{array}{ccc}
 B\mathbb{Z}_2 & \rightarrow & BSpin_{n+2} \\
 & \nearrow & \downarrow \\
 M & \xrightarrow{\eta} & BSO(n+2)
 \end{array}$$

Exactly as in the previous theorem, we find that the set of lifts is in bijective correspondence with  $[M, B\mathbb{Z}_2]$ . [P] proves that  $B\mathbb{Z}_2 = \mathbb{R}P^\infty$ . But  $\mathbb{R}P^\infty$  is just the Eilenberg-MacLane space  $K(\mathbb{Z}_2, 1)$ , so we have  $[M, B\mathbb{Z}_2] = [M, K(\mathbb{Z}_2, 1)] = H^1(M; \mathbb{Z}_2)$  (the last equality is proved in [K1]). Therefore, the set of  $spin$ -structures on  $TM \oplus L$  corresponds to  $H^1(M; \mathbb{Z}_2)$ .

While each of these  $spin$ -structures pulls back to a different lift from  $B(SO(n) \times U(1))$  to  $BSpin_n^c$ , they are not all different when considered as lifts from  $BSO(n)$  to  $BSpin_n^c$ . We will not completely answer the question of when they are or are not different, but we will show:

**Theorem 6.** *Two lifts which differ by the action of an element in  $H^1(M; \mathbb{Z}_2)$  which comes from  $H^1(M; \mathbb{Z})$  give the same  $spin^c$ -structure, assuming the complex line bundles are the same.*

PROOF: As above, we have that  $H^1(M; \mathbb{Z}) = [M, K(\mathbb{Z}, 1)] = [M, S^1]$ . It will clearly suffice to show that a lift corresponding to an element in  $H^1(M; \mathbb{Z}_2)$  which comes from  $H^1(M; \mathbb{Z})$  gives the same  $spin^c$ -structure as the lift corresponding to the 0 element. Such a lift would factor through  $S^1$  in each fiber; i.e. the image of the lift in each fiber  $B\mathbb{Z}_2 = \mathbb{R}\mathbb{P}^\infty$  lies in the canonical copy of  $S^1$  embedded in  $\mathbb{R}\mathbb{P}^\infty$  as  $\mathbb{R}\mathbb{P}^1$ . However, when we view  $BSpin_n^c$  as a bundle over  $BSO(n)$ , the fiber is  $BU(1) = \mathbb{C}\mathbb{P}^\infty$ , which is simply-connected. Therefore the copies of  $S^1$  can all be homotoped to a point in these fibers (simultaneously, since the homotopy is the same in each fiber), which means the lift is the same as the 0-lift.  $\square$

Hence, the number of  $spin^c$ -structures on  $M$  associated with each complex line bundle over  $M$  is at most

$$|H^1(M; \mathbb{Z}_2) \text{ mod those elements coming from } H^1(M; \mathbb{Z})|.$$

## 5. A DESCRIPTION OF $Spin_n^c$ VIA CLIFFORD MODULES

In this section I will give a much more algebraic formulation of the groups  $Spin_n$  and  $Spin_n^c$ . This formulation will give us information about the structure of these groups which is very useful in studying vector bundles. However, before diving into a sea of algebra, I will try to give some geometrical motivation, following [K2].

Recall that an element of the orthogonal group  $O(n)$  can always be written as a product of reflections  $\rho_i$  across hyperplanes through the origin. Each such reflection is determined by a unit normal  $v_i$  to the hyperplane; note that  $v_i$  and  $-v_i$  determine the same reflection. So we can write an element of  $O(n)$  as a “product”  $[v_1 \cdot v_2 \cdots v_k]$ , where each equivalence class contains a product and its negative, and  $0 \leq k \leq n$ . Then the double cover of  $O(n)$  is just the group of signed products, which is called  $Pin_n$  (a play on  $SO(n)$  and  $Spin_n$  which stuck). We will define the *Clifford algebra*  $Cl_n$  so that it contains  $Pin_n$  in a natural way.

DEFINITION: Given a real vector space  $V$  with an inner product  $Q$ , the *Clifford algebra*  $Cl(V, Q)$  is the quotient algebra  $\mathcal{T}(\mathcal{V})/\mathcal{I}(\mathcal{V})$ , where  $\mathcal{T}(\mathcal{V})$  is the tensor algebra  $\otimes V$ , and  $\mathcal{I}(\mathcal{V})$  is the ideal generated by elements of the form  $v \otimes v - Q(v, v)$ .

To increase the resemblance to our geometric motivation (and to make things easier to write) we will usually write products as  $vw$  rather than  $v \otimes w$ . The relation given in the definition can be rewritten as  $vw + wv = 2Q(v, w)$ . These relations have a particularly nice form when we consider an orthonormal basis  $\{e_1, \dots, e_n\}$  for  $V$ , and assume that  $Q$

is positive definite. Then we have that  $e_i e_j = -e_j e_i$  and  $e_i e_i = 1$ . From these, we can see that a basis for  $Cl(V, Q)$  is  $\{e_I = e_{i_1} \dots e_{i_k} \text{ where } i_1 < i_2 < \dots < i_k, \text{ and } 0 \leq k \leq n\}$  (when  $k = 0$  we get the identity  $1 = e_\emptyset$ ). Therefore, the dimension of  $Cl(V, Q)$  is  $2^n$ , where  $n$  is the dimension of  $V$ .

$Cl(V, Q)$  has a natural  $\mathbb{Z}_2$ -grading  $Cl(V, Q) = Cl^0(V, Q) \oplus Cl^1(V, Q)$  where the first term is generated by products of an even number of elements of  $V$ , and the second is generated by products of an odd number of elements of  $V$ . We consider the multiplicative group of units in the Clifford algebra, denoted  $Cl^\times(V, Q)$ . This group has a natural representation in the Clifford algebra, called the *adjoint* representation:

$$Ad : Cl^\times(V, Q) \longrightarrow Aut(Cl(V, Q))$$

$$Ad(\varphi)(x) = \varphi x \varphi^{-1}$$

If  $v \in V$  with  $Q(v, v) \neq 0$ , then  $v$  is a unit ( $v^{-1} = -v/Q(v, v)$ ), and  $Ad(v)$  preserves the inner product ( $Q(Ad(v)(w), Ad(v)(w)) = Q(w, w)$ ); so  $Ad$  restricts to a representation of  $P(V, Q) = \{v \in V \text{ s.t. } Q(v, v) \neq 0\}$  in  $O(V, Q) = \{\lambda \in GL(V) \text{ preserving } Q\}$ . Now we define:

$Pin(V, Q) \subset P(V, Q)$  is the subgroup generated by  $v \in V$  with  $Q(v, v) = \pm 1$

$$Spin(V, Q) = Pin(V, Q) \cap Cl^0(V, Q)$$

We can show that these groups (for a real vector space) are double covers of  $O(V, Q)$  and  $SO(V, Q)$  respectively, so this agrees with our geometric definition of the spin groups.

We are particularly interested in the case when  $V = \mathbb{R}^n$ , and  $Q$  is the usual positive definite inner product (dot product). Then we define  $Cl_n = Cl(V, Q)$ ,  $Spin_n = Spin(V, Q)$ , etc. We now define the groups  $Spin_n^c$  as before:

$$Spin_n^c = Spin_n \times_{\mathbb{Z}_2} U(1)$$

We associate with  $Cl_n$  a *volume element*  $\omega = e_1 e_2 \dots e_n$ , where  $\{e_1, \dots, e_n\}$  is an orthonormal basis for  $\mathbb{R}^n$  (with a given orientation).  $\omega$  is independent of the choice of this basis (in  $Cl_n$ ), and we have the relation:

$$\omega^2 = (-1)^{n(n+1)/2}$$

Similarly, we consider the case when  $V$  is a complex vector space and define  $\mathbb{C}l_n$  to be  $Cl_n \otimes \mathbb{C}$ . Notice that  $Spin_n^c \subset \mathbb{C}l_n$ . Again, we define a volume element  $\omega_{\mathbb{C}} = i^{[(n+1)/2]} \omega$ . In this case, we find the square of the volume element is always 1.

These volume elements give us useful decompositions of vector spaces which have  $Cl_n$ -representations.

**DEFINITION:** A  $Cl_n$ -module is a real vector space  $W$  together with a representation  $\rho : Cl_n \rightarrow Hom_{\mathbb{R}}(W, W)$ . We often denote  $\rho(\varphi)(w)$  by

$\varphi \cdot w$ , and call this operation *Clifford multiplication*. Similarly, in the complex case we define  $\mathbb{C}l_n$ -modules.

If  $W$  is a  $Cl_n$ -module, and  $\omega^2 = 1$ , then we get a decomposition  $W = W^+ \oplus W^-$  into the eigenspaces of  $\omega$ , so  $W^\pm = (1/2)(1 \pm \omega)W$ . In the complex case, the square of the volume element is always 1, so the decomposition always exists.

We say that the representation  $\rho$  is *reducible* if  $W$  can be written  $W_1 \oplus W_2$ , where  $\rho(\varphi)(W_i) \subseteq W_i$  for every  $\varphi \in Cl_n$ . Otherwise, we call the representation *irreducible*. We call two representations  $\rho_j : Cl_n \rightarrow Hom(W_j, W_j)$  *equivalent* if there is a linear isomorphism  $F : W_1 \rightarrow W_2$  such that  $F \circ \rho_1(\varphi) \circ F^{-1} = \rho_2(\varphi)$  for every  $\varphi \in Cl_n$ . There is a well-understood classification of Clifford algebras (see [L-M]) which gives us the following fact:

**Theorem 7.** *The number of inequivalent irreducible real representations of  $Cl_n$  is 2 if  $n+1 \equiv 0 \pmod{4}$ , and 1 otherwise. The number of inequivalent irreducible complex representations of  $\mathbb{C}l_n$  is 2 if  $n$  is odd and 1 if  $n$  is even.*

Finally, we will introduce one more type of bundle - the *spinor bundles* of a manifold:

**DEFINITION:** If the manifold  $M$  has a spin structure  $\xi : P_{Spin}(TM) \rightarrow P_{SO}(TM)$ , a *real spinor bundle* is an associated bundle  $S(M) = P_{Spin}(TM) \times_{\mu} W$ , where  $W$  is a left module for  $Cl_n$  and  $\mu : Spin_n \rightarrow SO(W)$  is the representation given by Clifford multiplication by elements of  $Spin_n \subset Cl_n^0$ . Similarly, we define a *complex spinor bundle*, with  $W$  a complex left module for  $\mathbb{C}l_n = Cl_n \otimes \mathbb{C}$ .

We easily generalize this definition to *spin<sub>c</sub>*-manifolds by defining the spinor bundle  $S(M) = P_{Spin^c}(TM) \times_{\Delta} V$ , where  $V$  is a complex  $Cl_n$ -module, and  $\Delta : Spin_n^c \rightarrow GL(V)$  is the restriction of the  $Cl_n$  representation to  $Spin_n^c \subset Cl_n \otimes \mathbb{C}$ . If this representation is irreducible, we say that the spinor bundle is *fundamental*. So by the theorem above, there is one fundamental spinor bundle if  $n$  is even, and two if  $n$  is odd. However, in the odd case the two bundles are equivalent when restricted to  $Spin_n^c$ , so in fact there is always a unique fundamental spinor bundle, which we denote  $S(M)$ . Since we are in the complex case, we can use the volume element  $\omega_{\mathbb{C}}$  to decompose  $S(M)$  into two bundles  $S^{\pm}(M) = (1/2)(1 \pm \omega_{\mathbb{C}})S(M)$ . We will use these bundles in the next section to define the Seiberg-Witten equations.



## 6. THE SEIBERG-WITTEN EQUATIONS

To define the Seiberg-Witten equations, we specialize to the case of orientable 4-manifolds, following [T] and [A]. We know, from section 3, that any orientable 4-manifold has a *spin*<sup>c</sup>-structure. We also know, from the classification of Clifford algebras in [L-M], that  $\mathcal{Cl}_4 = \mathbb{C}(4)$ , the algebra of  $4 \times 4$  complex matrices. The unique irreducible complex representation is the natural representation of this group on  $\mathbb{C}^4$ , so the fundamental spinor bundle  $S(M)$  is a  $\mathbb{C}^4$ -bundle, which splits (as described in section 4) into two  $\mathbb{C}^2$ -bundles  $S^\pm(M)$ . By restricting this representation to the natural copy of  $\mathbb{R}^4$  lying inside  $\mathcal{Cl}_4$ , Clifford multiplication gives us a map  $c$  from the cotangent bundle  $T^*(M)$  into the skew-adjoint endomorphisms of  $S(M) = S^+(M) \oplus S^-(M)$  (skew-adjoint because of the relation  $vv = -Q(v, v)$ ).  $c$  induces the following map by duality:

$$\begin{aligned}\sigma : S^+(M) \otimes T^*(M) &\rightarrow S^- \\ \sigma(s \otimes v) &= p_-(c(v)(s, 0))\end{aligned}$$

where  $p_-$  is the projection  $S(M) \rightarrow S^-(M)$ .

We will construct the fundamental spinor bundles  $S^\pm(M)$  explicitly as associated bundles to representations. First, we recall the following Lie group isomorphisms:

$$\begin{aligned}Spin_4 &= SU(2) \times SU(2) \\ SO(4) &= (SU(2) \times SU(2))/\{\pm 1\} \\ Spin_4^c &= (SU(2) \times SU(2) \times U(1))/\{\pm 1\}\end{aligned}$$

These give us two natural actions of  $SO(4)$  on  $\mathbb{R}^3$ :

$$\begin{aligned}\lambda_\pm : SO(4) \times \mathbb{R}^3 &\longrightarrow \mathbb{R}^3 \\ \lambda_+ : ([p, q], x) &\longmapsto Im(px) \\ \lambda_- : ([p, q], x) &\longmapsto Im(qx)\end{aligned}$$

where we are identifying  $SU(2) = S^3$  with the unit quaternions, and  $\mathbb{R}^3$  with the imaginary quaternions. The associated  $\mathbb{R}^3$ -bundles to these representations are isomorphic to  $\Lambda_+$  (the self-dual two-forms) and  $\Lambda_-$  (the anti-self-dual two-forms) respectively. We extend these actions to actions of  $Spin_4^c$  on the quaternions  $\mathbb{H}$ :

$$\begin{aligned}s_\pm : Spin_4^c \times \mathbb{H} &\longrightarrow \mathbb{H} \\ s_+ : ([p, q, \lambda], x) &\longmapsto px\lambda^{-1} \\ s_- : ([p, q, \lambda], x) &\longmapsto qx\lambda^{-1}\end{aligned}$$

We view the associated  $\mathbb{R}^4$ -bundles to these actions as  $\mathbb{C}^2$ -bundles, and by **[A]** these are the spinor bundles  $S^+(M)$  and  $S^-(M)$ , respectively. Then we have a pairing:

$$(\cdot, \cdot) : S^+(M) \otimes S^+(M)^* \longrightarrow \Lambda_+$$

which is the equivariant extension of the map on fibers given by:

$$(\cdot, \cdot) : x \otimes y \longmapsto \text{Im}(xiy)$$

where the bundle of imaginary quaternions is identified with  $\Lambda_+$  as before.

Our penultimate step is to introduce the complex line bundle  $L = \det(S^+(M))$ , together with a connection  $A$ . Together with the riemannian connection on  $T^*(M)$ ,  $A$  induces a covariant derivative  $\nabla_A$  on  $S^+(M)$  which maps sections of  $S^+(M)$  to sections of  $S^+(M) \otimes T^*(M)$ . We define the Dirac operator  $D_A$  as the composition of this map with  $\sigma$ :

$$\begin{aligned} D_A : \Gamma(S^+(M)) &\rightarrow \Gamma(S^-(M)) \\ D_A(s)(m) &= \sigma(\nabla_A(s)(m)) \end{aligned}$$

We are now ready to state the Seiberg-Witten equations. The data for these equations is a pair  $(A, \psi)$  where  $A$  is a connection on  $L$  and  $\psi$  is a section of  $S^+(M)$ , and we let  $F_A^+$  denote the self-dual part of the curvature of  $A$ :

$$\begin{aligned} D_A(\psi) &= 0 \\ F_A^+ &= (\psi, \psi^*) \end{aligned}$$

The Seiberg-Witten invariant is given by properly counting the solutions to these equations, as described in **[T]**. Taubes also states the fundamental theorem:

**Theorem 8.** *If  $M$  is a compact, oriented, connected 4-manifold with  $b_2^+ > 1$ , then the Seiberg-Witten invariant  $SW$  is a map from the space of  $\text{spin}^c$ -structures on  $M$  to the integers  $\mathbb{Z}$  which depends only on the underlying smooth structure of  $M$ .*

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