HOMOTOPY TYPE COMPARISON OF A SPACE WITH COMPLEXES ASSOCIATED WITH ITS OPEN COVERS[1]

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1. **Introduction.** This paper deals with the homotopy type comparison of a space with certain complexes (such as the nerve) associated with suitably "well pieced together" open covers of the space by homotopically trivial sets. The main theorem (Theorem 1) implies a slightly different version (Theorem 2) of a theorem of A. Weil [7]. The method of proof is quite different from Weil's. A simple use is made of a theorem in [6] of Alexandroff's [1] "discrete spaces" (spaces in which the intersection of any collection of open sets is open). Also use is made of a modification (Theorem 3) of a theorem of A. Dold and R. Thom [3].

It is conceivable that Theorem 1 could contribute to a positive solution of the unsolved problem: *Does every compact topological manifold have the homotopy type of a finite complex*? See the discussion following the statement of the theorem.

Other results to which Theorem 1 is related were obtained by J. Leray [5] and by K. Borsuk [2].

2. **Notation and terminology.** Suppose $\mathcal{U}$ is an open cover of a space X (a collection of nonempty open subsets of X whose union is X). Let $N(\mathcal{U})$ denote the nerve of $\mathcal{U}$. More than with $N(\mathcal{U})$, we shall be concerned with the following subcomplex of $N(\mathcal{U})$. Let $K(\mathcal{U})$ be the complex whose vertices are the members of $\mathcal{U}$ and whose simplexes are the finite totally ordered subcollections of $\mathcal{U}$ (where $\mathcal{U}$ is partially ordered by inclusion). In general $K(\mathcal{U})$ does not have the same homotopy type as $N(\mathcal{U})$.

The open cover $\mathcal{U}$ will be called *basis-like* if the intersection of any two members of $\mathcal{U}$ is a union of members of $\mathcal{U}$. This is equivalent to saying that $\mathcal{U}$ is a basis for a topology on X smaller than the given one. An open cover $\mathcal{U}$ is *point-finite* if each point of X is contained in only finitely many members of $\mathcal{U}$.

A map $f\colon X\to Y$ is a *weak homotopy equivalence* if the induced maps $f_*\colon \pi_i(X, x)\to\pi_i(Y, fx)$ are isomorphisms for all $x\in X$ and all $i\geqq 0$. A space X is *homotopically trivial* if $\pi_i(X, x)=0$ for all $i\geqq 0$.

If K is an (abstract) simplicial complex, $|K|$ denotes the underlying polyhedron with the weak topology.

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3. Statement of results.

THEOREM 1. *Let X be a space and let $\mathfrak{U}$ be a point-finite, basis-like, open cover of X by homotopically trivial sets. Then there exists a weak homotopy equivalence f: $|K(\mathfrak{U})| \to X$.*

REMARK. Of course if in addition we know that X has the homotopy type of a CW-complex (for example when X is a topological manifold), then we may conclude from a theorem of J. H. C. Whitehead [**8**] that f is an actual homotopy equivalence.

Thus the problem mentioned in §1 has a positive solution if the following question has an affirmative answer: *Does every compact topological manifold possess a finite, basis-like, open cover by contractible sets?* (In this context "contractible" is equivalent to "homotopically trivial.") Any counterexample to this would be a counterexample to the triangulation problem, because every finite polyhedron has such an open cover: the open stars of its simplexes. However, perhaps this question could be answered affirmatively without solving the triangulation problem.

In §5, Theorem 1 will be used to derive the following variation on a theorem of Weil.

THEOREM 2 (CF. WEIL [**7**, p. 141]). *Let X be a space and let $\mathfrak{V}$ be a point-finite open cover of X such that the intersection of any (finite) subcollection of $\mathfrak{V}$ is homotopically trivial. Then there exists a weak homotopy equivalence $|N(\mathfrak{V})| \to X$.*

In Weil's version, it is assumed that $X \times X \times [0, 1]$ is normal, that $\mathfrak{V}$ is locally finite, and that the intersections of subcollections of $\mathfrak{V}$ are solid; and it is concluded that there is an actual homotopy equivalence $|N(\mathfrak{V})| \to X$. If we replace the first of these assumptions by the assumption that X is separable metric, then we can derive Weil's conclusion, because then by a theorem of O. Hanner [**4**, p. 392], X is an ANR, so that Whitehead's theorem applies.

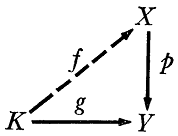
There are simple examples to which Theorem 1 applies, but where $|N(\mathfrak{U})|$ does not have the (weak) homotopy type of X.

The main tool in proving Theorem 1 is the following theorem used in [**6**]. This theorem follows from a modification of the proof of Satz 2.2 of Dold and Thom [**3**]. For details on the necessary modification, see [**6**].

THEOREM 3. *Suppose p is a map of a space X into a space Y and suppose there exists a basis-like open cover $\mathfrak{W}$ of Y satisfying the following condition: For each $W \in \mathfrak{W}$, the restriction $p|p^{-1}(W)$: $p^{-1}(W) \to W$ is a weak homotopy equivalence. Then p itself is a weak homotopy equivalence.*

4. **Proof of Theorem 1.** First let us state the following

LEMMA 1. *If in the diagram*



K is a CW-*complex and p is a weak homotopy equivalence, then f can be found making the diagram homotopy commutative. Hence if g is also a weak homotopy equivalence, then so is f.*

The proof, which is simple and involves the mapping cylinder of p, appears as a part of an argument on page 244 of [3].

Now suppose X and $\mathfrak{U}$ are as in the statement of Theorem 1. Using the fact that $\mathfrak{U}$ is partially ordered by inclusion and using the modification in [6] of Alexandroff's [1] procedure, we make $\mathfrak{U}$ into a topological space as follows. For each $U\in\mathfrak{U}$, let $[U]=\{V\in\mathfrak{U}: V\subset U\}$. Then as a basis for the required topology we take the collection $\{[U]: U\in\mathfrak{U}\}$.

As in [6], we define a map $g: |K(\mathfrak{U})|\to\mathfrak{U}$ as follows: If $x\in|K(\mathfrak{U})|$, then let $(U_0, \cdots, U_n)$ be the unique open simplex of $|K(\mathfrak{U})|$ to which x belongs, with $U_0\subset \cdots \subset U_n$. Then $g(x)=U_0$. In [6] it is shown that g is continuous and, by use of Theorem 3, that g is a weak homotopy equivalence.

Next we define a map $p: X\to\mathfrak{U}$. For each $x\in X$ let $p(x)$ be the smallest member of $\mathfrak{U}$ containing x. This exists since $\mathfrak{U}$ is point-finite and basis-like. The proof of the following lemma is straightforward.

LEMMA 2. *For each $U\in\mathfrak{U}$, $p^{-1}([U])=U$.*

Since the sets $[U]$ form (by definition) a basis for the space $\mathfrak{U}$, this implies that p is continuous.

Now we wish to apply Theorem 3 to show that p is a weak homotopy equivalence. For the basis-like open cover of $\mathfrak{U}$ we take simply the basis $\mathfrak{W}=\{[U]: U\in\mathfrak{U}\}$. It is shown in [6] that each $[U]$ is a *contractible* subset of $\mathfrak{U}$. By Lemma 2, $p^{-1}([U])=U$; and $p|U: U\to[U]$ is trivially a weak homotopy equivalence, since U is assumed to be homotopically trivial. Hence Theorem 3 implies that p is a weak homotopy equivalence. Now we apply Lemma 1 to obtain a weak homotopy equivalence $f: |K(\mathfrak{U})|\to X$.

5. **Proof of Theorem 2.** Let $\mathfrak{U}$ be the collection of all nonempty intersections of (finite) subcollections of $\mathfrak{V}$. Clearly $\mathfrak{U}$ satisfies the conditions of Theorem 1, so that we get a weak homotopy equiva-

lence $|K(\mathcal{U})| \to X$. From the facts that $\mathcal{V} \subset \mathcal{U}$ and that $\mathcal{U}$ is a refinement of $\mathcal{V}$, it is easy to see that $|N(\mathcal{V})|$ is a deformation retract of $|N(\mathcal{U})|$. (More generally, if two covers of a space refine each other, then their nerves have the same homotopy type.) Hence the proof of the theorem is completed by the following

LEMMA 3. *Let $\mathcal{U}$ be a cover of a space with the property that the intersection of any finite subcollection of $\mathcal{U}$ is either empty or a member of $\mathcal{U}$. Then $|K(\mathcal{U})|$ is a deformation retract of $|N(\mathcal{U})|$.*

PROOF. Abbreviate $K = K(\mathcal{U})$, $N = N(\mathcal{U})$, and let N' be the first barycentric subdivision of N. Let us define a simplicial map $\phi: N' \to K$ as follows. The vertices of N' are the barycenters $b(\sigma)$ of the simplexes σ of N. Define ϕ on these vertices by the equation

$$\phi(b(\sigma)) = \text{Carrier } \sigma$$

(By assumption Carrier σ, the intersection of the vertices of σ, being nonempty, is a member of $\mathcal{U}$, that is, a vertex of K.) Any simplex of N' is spanned by vertices $b(\sigma_0), \cdots, b(\sigma_n)$, where $\sigma_0 \subset \cdots \subset \sigma_n$. Then Carrier $\sigma_0 \supset \cdots \supset$ Carrier σ_n, so that these vertices span a simplex of K. Thus ϕ is simplicial.

Let us show that ϕ, as a map of $|N| = |N'|$ into $|K|$, is a deformation retraction. Clearly it is a retraction. It suffices then to show that for each simplex τ of N', both τ and $\phi(\tau)$ are subsets of some simplex σ of N. Let the vertices of τ be $b(\sigma_0), \cdots, b(\sigma_n)$, where $\sigma_0 \subset \cdots \subset \sigma_n$, and let the vertices of σ_n be $U_0, \cdots, U_r$. We can take σ to be the simplex of N spanned by the vertices

$$\{U_0, \cdots, U_r, \text{Carrier } \sigma_0, \cdots, \text{Carrier } \sigma_n\}.$$

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