The method of infinite repetition in pure topology:
II. Stable applications

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1. Introduction

We shall apply the theory developed in Part I of this paper to obtain rather strong theorems regarding the stable nature of topological manifolds. In particular, (§ 5), the stable homeomorphism theorem is proven (originally reported for differentiable manifolds in [SE1]): If \( f: M_1^s \to M_2^s \) is a tangential homotopy equivalence, then there is an integer \( k \geq 0 \) and a topological isomorphism \( F \) such that

\[
\begin{array}{ccc}
M_1^s \times R^k & \xrightarrow{F} & M_2^s \times R^k \\
\downarrow \pi & & \downarrow \pi \\
M_1^s & \xrightarrow{f} & M_2^s
\end{array}
\]

is homotopy-commutative. Here "tangential" is a concept which makes sense in terms of the tangent microbundles of \( M_j^s, j = 1, 2 \). This result works for differentiable, combinatorial, and topological manifolds, yielding the appropriate sort of isomorphism in each case. This theorem is proved for separable metric manifolds without boundary.

In § 6 we reduce "all" stable topological problems to problems in the theory of stable topological microbundles, by means of the stable homeomorphism theorem.

Along the way, we are led to prove a stable isotopy theorem for purely topological manifolds (§ 3).

2. Foundational facts and definitions

We shall rely heavily on Milnor's theory of microbundles [MB 2–6]. The reader is also referred to [PLMB] for the piecewise linear theory.

**Lemma 1.** Let \( X \) be a (separable metric) topological manifold. Then \( X \) is of the homotopy type of a countable locally finite simplicial complex.

**Proof.** See [M]1.

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1 We may sharpen Lemma 1 to say that there is an open subset \( E \) of \( R^M \) (\( M \) large) of the homotopy type of \( X \). One may give a direct argument for this using the existence of stable normal microbundles, or more quickly, one may use [K]. Namely, take \( E \) to be a euclidean bundle over \( X \) contained in a normal microbundle of some imbedding \( X \subset R^M \). This sharpened fact we use in Propositions 2, 3 of §6.
Let $f: M \to M$ be a continuous map of topological, combinatorial, or differentiable manifolds.

**DEFINITION.** The map $f$ is said to be *(differentiably, combinatorially, or topologically) tangential* if

$$f^*\tau_k = \tau_1,$$

where $\tau_k$ is the (differentiable, combinatorial, or topological) stable tangent microbundle class of $M_k$, $k = 1, 2$.

If $M$ is a topological manifold with boundary, $\partial M$, then by Theorem 2 of [Br]:

**PROPOSITION 1.** The boundary $\partial M$ is collared in $M$. *(That is, there is a neighborhood $\partial M \times [0, 1) \subset M$ about $\partial M = \partial M \times \{0\} \subseteq M$.)

From this follows

**COROLLARY 2.** If $i: \partial M \to M$ is the natural injection, then

$$\tau_{\partial M} = i^*\tau_M.$$

**COROLLARY 3.** Let $f: (M_1, \partial M_1) \to (M_2, \partial M_2)$ be a map such that $f: M_1 \to M_2$ is a tangential map of topological manifolds. Then the restriction

$$\partial f: \partial M_1 \to \partial M_2$$

is also tangential.

**DEFINITION.** A *topological (combinatorial, or differentiable) $\pi$-manifold* $M$ is a topological (combinatorial, or differentiable) manifold whose stable topological (combinatorial, or differentiable) tangent microbundle class, $\tau_M$, is zero.

**EXAMPLE 1.** Any $n$-dimensional submanifold of $R^n$ (or of any other $n$-dimensional $\pi$-manifold) is a $\pi$-manifold.

**EXAMPLE 2.** Let $V^{n-1}$ be a compact manifold without boundary which admits a locally flat imbedding in (is a topological submanifold of) $R^n$. Then $V^{n-1}$ is a $\pi$-manifold.

Example 2 is non-trivial, and makes use of Proposition 1.

**LEMMA 2.** If $M_1, M_2$ are $\pi$-manifolds, any continuous map $f: M_1 \to M_2$ is tangential.

**PROOF.** Since the tangent microbundle classes $\tau_1 = \tau_2 = 0$, one clearly has $f^*\tau_2 = \tau_1$.

3. **Stable isotopy theorem (topological manifolds)**

Using the theory of topological microbundles, we will prove a foundational theorem about topological isotopies. An isotopy, here, is a level-preserving imbedding $\Phi: X \times I \to Y \times I$.

**THEOREM (Stable isotopy for topological manifolds).** Let $f_0, f_1: M \to W$
be open imbeddings of the (unbounded) manifold \( M \) in \( W \), which are homotopic. Then there is a \( q \geq 0 \) and a topological isotopy \( \Phi_t: M \times R^t \to W \times R^t \), \( 0 \leq t \leq 1 \) such that \( \Phi_0 = f_0 \times \{1\} \), and \( \Phi_1 = f_1 \times \{0\} \) on \( M \times \{0\} \).

REMARK. The number \( q \) may have to be taken quite a bit larger than the customary \( 2 \dim M + 1 \).

The maps \( f \) need not really be open imbeddings. However, it is this case that will be useful to us.

**Lemma 4.** There is a continuous map \( p: M \times I \to R^{k+1} \) such that \( p(M \times \{0\}) = (0) \in R^{k+1} \), and \( p \) is a closed imbedding of \( M \times (0, 1] \) in \( R^{k+1} - \{0\} \).

**Proof.** Represent \( R^{k+1} = R^k \times R \), and let \( p': M \to R^k \) be a closed imbedding \( (k > 2 \dim M + 1) \). This exists by [H-W]. Take \( p(m, t) = (t \cdot p'(m), t) \in R^k \times R \). Let

\[
f: M \times I \longrightarrow W \times I
\]

be a continuous homotopy between \( f_0, f_1 \).

**Lemma 5.** The normal microbundle of the imbedding

\[
f \times p: M \times I \to W \times I \times R^{k+1}
\]

is trivial.

**Proof.** Consider the diagram

\[
\begin{array}{ccc}
W \times \{0\} & \longrightarrow & W \times I \times R^{k+1} \\
\uparrow f_0 & & \downarrow f \times p \\
M \times \{0\} & \longrightarrow & M \times I.
\end{array}
\]

Since \( f_0 \) is an open imbedding, its normal microbundle is trivial. Therefore all imbeddings of the above diagram have trivial normal microbundle with the possible exception of \( f \times p \). Consequently \( f \times p \), also, has trivial normal microbundle.

**Proof of the Theorem.** Since \( f \times p \) has trivial normal microbundle, there is a diagram:

\[
\begin{array}{ccc}
W \times I \times R^{k+1} \times \{0\} & \longrightarrow & W \times I \times R^{k+1} \times R^{k'} \\
\uparrow f \times p & & \uparrow \subseteq \text{(open)} \\
M \times I \times \{0\} & \longrightarrow & M \times I \times R^{k+k'+1}
\end{array}
\]

for some \( k' \geq 0 \).

Now consider the natural imbeddings
Now, let $s \in R^{k+k'+1}$, and define

$$\Phi_t(m, r, s) = (m, r + t, s)$$

for

$$(m, r, s) \in M \times R \times R^{k+k'+1} = M \times R^{k+k'+2} \subseteq W \times R^{k+k'+2}.$$

It is immediate that $\Phi_t: M \times R^q \to M \times R^q (q = k + k' + 2)$ satisfies the requirements of the theorem.

4. Isotopy and infinite repetition spaces

In spirit, this section belongs to Part I of this paper. As in Part I, if $a: E \to E$ is an open imbedding, we let $E(a)$ denote the injective limit of the sequence

$$E \xrightarrow{a} E \xrightarrow{a} \cdots$$

(and we call it the infinite repetition space formed via $a$).

We shall study the relations between infinite repetition spaces formed via isotopic maps.

**Proposition 2.** Let $F$ be a separable metric manifold without boundary. Let $a_i: F \to F$ be an isotopy of open imbeddings. Then there is a topological isomorphism

$$\varphi: F(a_0) \times R \xrightarrow{\simeq} F(a_1) \times R.$$

We shall prepare to prove Proposition 2. First let us remark that the isotopy $a_i$ gives rise to an open imbedding,

$$F \times R \xrightarrow{a} F \times R$$

(One extends $a$ from $F \times I$ to $F \times R$ by taking $a_t = a_0$ if $t \leq 0$, and $a_t = a_1$ if $t \geq 1$.) By a shrinking map

$$s: F \times R \to F \times R$$

we shall mean one of the form: $s = 1 \times \sigma$ where $\sigma: R \subseteq R$ is an orientation-
preserving homeomorphism of $R$ onto some bounded open interval.

**Proposition 3.** Let

$$a: F \times R \to F \times R$$

be as above. Let

$$s: F \times R \to F \times R$$

be a shrinking map. Then there is a topological isomorphism

$$\varphi: (F \times R)(as) \approx (F \times R)(a) .$$

Let us take Proposition 3 for granted, for a moment, and we shall prove Proposition 2.

Let $\sigma_0, \sigma_1: R \to R$ be shrinking maps whose images lie to the left of 0 and to the right of 1, respectively. Let $s_0, s_1: F' \times R \to F' \times R$ be the corresponding imbeddings. Notice

$$as_0 = (a_0 \times 1) \cdot s_0$$
$$as_1 = (a_1 \times 1) \cdot s_1 .$$

We may now form an isomorphism

$$\varphi: F(a_0) \times R \approx F(a_1) \times R$$

as a composite,

$$F(a_0) \times R = (F \times R)(a_0 \times 1) \leftarrow\approx (F \times R)((a_0 \times 1) \cdot s_0) = (F \times R)(as_0)$$

$$\varphi \approx$$

$$\Downarrow \varphi_1$$

$$= (F \times R)(a)$$

$$\Downarrow \varphi_2$$

$$F(a_1) \times R = (F \times R)(a_1 \times 1) \leftarrow\approx (F \times R)((a_1 \times 1) \cdot s_1) = (F \times R)(as_1) .$$

**Proof of Proposition 3.** Since $a$ is level-preserving, $a$ commutes with all shrinking maps $s = 1 \times \sigma$.

Suppose we find a sequence of imbeddings $\{\beta_j\}$ such that
\[ R \xrightarrow{a} R \xrightarrow{a} R \xrightarrow{a} \cdots \]

\[ R \xrightarrow{\beta_0} R \xrightarrow{\beta_1} R \xrightarrow{\beta_2} \cdots \]

is commutative, and the \(\beta_j\) induced an isomorphism
\[ \beta: R(\sigma) \approx R(1). \]

Then set \(b_j = 1 \times \beta_j\), and
\[ F \times R \xrightarrow{as} F \times R \xrightarrow{as} F \times R \xrightarrow{as} \cdots \]

\[ F \times R \xrightarrow{a} F \times R \xrightarrow{a} F \times R \xrightarrow{a} \cdots \]

is commutative and induces an isomorphism,
\[ b: F \times R(as) \approx F \times R(a). \]

It remains to define the \(\beta_j\). Each \(\beta_j\) will be a shrinking homeomorphism with image \(B_j\), such that \(\bigcup B_j = R\). These will be defined inductively by taking \(\beta_0 = \sigma\), say, and using the following lemma.

**Lemma 3.** If \(\sigma_1, \sigma_2: R \to R\) are two shrinking maps, and \(B \subset R\) is any compact set, there is a shrinking map
\[ \sigma_3: R \to R \]

such that
(a) \(B \subset \sigma R\)
(b) \(\sigma_3\sigma_1 = \sigma_2\).

**Proof.** If \(B_1, B_2 \subset R\) are the closures of \(\sigma_1 R, \sigma_2 R\) respectively, define \(\sigma_3: B_1 \to B_2\) by \(\sigma_3 = \sigma_2\sigma_1^{-1}\) on \(\sigma_1 R\), and extended in the unique way on the endpoints of \(\sigma, R\) to get a homeomorphism from \(B_1\) to \(B_2\). Now extend \(\sigma_3\) to a homeomorphism on all of \(R\) with bounded image containing \(B\).

This lemma may be used as follows. Suppose \(\beta_{j-1}\) constructed; let \(\sigma_1 = \sigma, \sigma_2 = \beta_{j-1}, B = [-j, +j]\). Define \(\beta_j\) to be \(\sigma_3\) as provided by the lemma.

5. The stable homeomorphism theorems

Assume as in §4, that all manifolds are separable metric with boundary.

**Theorem I.** Let
\[ M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_1 \]

be open imbeddings of topological (or combinatorial, or differentiable) mani-
folds which are homotopy-inverses of one another. Then there is a $q > 0$ and homeomorphisms $F, G$ such that
\[ M_1 \times \mathbb{R}^q \xrightarrow{F} M_2 \times \mathbb{R}^q \cong M_1 \times \mathbb{R}^q \]
\[ M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_1 \]
is homotopy-commutative.

REMARKS. 1. In the differentiable and combinatorial cases, $F, G$ may be chosen to be differentiable or combinatorial isomorphisms.

2. There is the following extension to manifolds with boundary:

(Relative version) let
\[ (M_1, \partial M_1) \xrightarrow{f_1} (M_2, \partial M_2) \xrightarrow{f_2} (M_1, \partial M_1) \]
be open imbeddings which are homotopy-inverses of one another (in the sense of pairs).

Then there is a $q > 0$, and homeomorphisms $F, G$ such that
\[ (M_1, \partial M_1) \times \mathbb{R}^q \xrightarrow{F} (M_2, \partial M_2) \times \mathbb{R}^q \cong (M_1, \partial M_1) \times \mathbb{R}^q \]
\[ (M_1, \partial M_1) \xrightarrow{\pi} (M_2, \partial M_2) \xrightarrow{\pi} (M_1, \partial M_1) \]
is homotopy commutative.

Some indications of how to generalize all theorems of this theory to pairs (and to general categories of topological diagrams) have been given in [I, § 9].

3. The $q > 0$ may have to be chosen much larger than the customary $2 \operatorname{dim} M_j, j = 1, 2$.

THEOREM II.* Let
\[ f; M_1^n \longrightarrow M_2^n \]
be a tangential homotopy equivalence between the (differential, combinatorial, or topological) manifolds $M_j^n$. Then there is a $q > 0$, and an isomorphism $F$ such that
\[ M_1 \times \mathbb{R}^q \xrightarrow{F} M_2 \times \mathbb{R}^q \]
\[ M_1 \xrightarrow{f} M_2 \]

PROOF (of Theorem II, granted Theorem I). Let $g$ be a homotopy-inverse to $f$. Let $k > 0$ be large enough so that there is an imbedding with normal microbundle,

* M. Hirsch has also obtained a version of this theorem [H].
Since $f$ is tangential, so is $f_1$, and an elementary calculation shows that the normal microbundle of $f_1$, $\nu(f_1)$, is stably trivial. (Namely: The standard fact about normal microbundles,

$$\tau_1 + \nu(f_1) = f_1^*\tau_2,$$

coupled with tangentiality of $f_1$,

$$\tau_1 = f_1^*\tau_3,$$

gives $\nu(f_1) = 0$.)

Since $\nu(f_1)$ is stably trivial, there is a $k' > 0$ and a commutative diagram [MB, Lem. 5.1]:

$$
\begin{array}{c}
M_2 \times \mathbb{R}^k \times \{0\} \\
\downarrow f_i \quad \downarrow f' \subseteq \text{open} \\
M_1 \times \mathbb{R}^k \times \mathbb{R}^k' \\
\end{array}
$$

Setting

$$M_1' = M_1 \times \mathbb{R}^{k+k'},$$

$$M_2' = M_2 \times \mathbb{R}^k \times \mathbb{R}^{k'},$$

we have

$$
\begin{array}{c}
M_1' \quad \xrightarrow{f'} \quad M_2' \\
\quad \quad \quad \quad \equiv \quad \equiv \\
M_1 \times \mathbb{R}^{k+k'} \quad M_2 \times \mathbb{R}^k \times \mathbb{R}^{k'} \\
\downarrow \pi \quad \downarrow \pi \\
M_1 \quad \xrightarrow{f} \quad M_2
\end{array}
$$

is homotopy-commutative.

A similar construction for $g$ allows us to replace the map $f, g$ with open imbeddings $f', g'$:

$$
\begin{array}{c}
M_1 \times \mathbb{R}^j \quad \xrightarrow{f'} \quad M_2 \times \mathbb{R}^j \quad \xrightarrow{g'} \quad M_1 \times \mathbb{R}^j \\
\downarrow \text{open} \quad \downarrow \text{open} \quad \downarrow \text{open} \\
M_1 \quad \xrightarrow{f} \quad M_2 \quad \xrightarrow{g} \quad M_1
\end{array}
$$
Thus, applying Theorem I to the pair $f', g'$ yields Theorem II.

We now prepare for the proof of Theorem I.

**Lemma 5.** There is a $k > 0$, such that

$$M_1 \times R^k \xrightarrow{f \times 1} M_2 \times R^k \xrightarrow{g \times 1} M_1 \times R^k$$

has the following property:

There are imbeddings

$$u_j: M_j \times R^k \longrightarrow M_j \times R^k \quad (j = 1, 2)$$

such that

(i) $(gf) \times 1 \approx u_1; \ (fg) \times 1 \approx u_2$

(ii) $u_j = 1$ on $(M_j \times \{0\}), \ j = 1, 2.$

(iii) $u_j(M_j \times D_k) \subset M_j \times R^k$ is closed.

**Proof.** Concentrate on the construction of $u_1$, the problem being identical to that of the construction of $u_2$.

Since $gf: M_1 \rightarrow M_1$ is homotopic to 1, let $\Phi'_1$ be the isotopy between $gf \times 1$, and $1 \times \{0\}: M_1 \times R^k \rightarrow M_1 \times R^k$ as guaranteed by the stable isotopy theorem (§ 3). Take $u'_1 = \Phi'_1 \cdot (gf \times 1)$.

Then (i), (ii) follows immediately. Assertion (iii) is not yet true, but $u'_1$ is easily modified by an isotopy (which keeps $M_1 \times \{0\}$ fixed) to an imbedding $u_1$ which does satisfy (iii) as guaranteed by the next lemma. Thus $u_1$ satisfies the requirements of Lemma 5, the proof of which will be complete (after Lemma 6, below). Denote by $\Phi$, the isotopy between $u_1$ and $gf \times 1$. Thus $u_1 = \Phi_1 \cdot (gf \times 1)$.

If $M$ is a separable metric manifold, let $\varepsilon(M)$ be a function such that

$$\varepsilon(M) > 0.$$ 

Let $\gamma = \gamma(\varepsilon): M \times R^k \rightarrow M \times R^k$ be given by $\gamma(m, r) = (m, \varepsilon(m) \cdot r)$.

**Lemma 6.** Let $W$ be a metric space, and $f: M \times R^k \rightarrow W$ an imbedding such that $f(M \times \{0\})$ is closed. Then there is a function $\varepsilon$, giving rise to a $\gamma = \gamma(\varepsilon)$ such that the imbedding $g = f \gamma: M \times R^k \rightarrow W$ has the following property:

$$(*) \ g(M \times D^k_r) \subset W \text{ is closed for all } 0 \leq r \leq 1.$$ 

Such an imbedding $g$ we call admissible. $f$ is isotopic to $g$ by an $f_i$ which is stationary on $M \times \{0\}$.

**Proof.** Let $\varepsilon > 0$ be a positive function on $M$, and set

$$\delta_i(m) = \text{diam } f(\{m\} \times D^k_{\varepsilon(m)}$$

as computed in the metric of $W$. 
Clearly, given any continuous positive function \( \tilde{\delta}(m) \), there is a positive function \( \epsilon(m) \) such that \( \tilde{\delta}_\epsilon(m) \leq \tilde{\delta}(m) \) for all \( m \in M \). Let \( \tilde{\delta} \) possess the following property:


(\text{**}) The sets \( M_\epsilon = \{ m \in M \mid \tilde{\delta}(m) \geq \epsilon \} \) are compact for all \( \epsilon > 0 \).

(Since \( M \) is separable metric, such a function \( \tilde{\delta} \) exists). Then there is an \( \epsilon > 0 \) such that \( \tilde{\delta}_\epsilon \leq \tilde{\delta} \), and consequently \( \tilde{\delta}_\epsilon \) also satisfies (\text{**}). Set \( \gamma = \gamma(\epsilon) \), and let \( g: M \times R^k \to W \) be the imbedding \( g = f \cdot \gamma \). If \( \tilde{\delta}' \) is the function,

\[
\tilde{\delta}'(m) = \text{diam } g(\{m\} \times D^k),
\]

then clearly \( \tilde{\delta}' = \tilde{\delta}_\epsilon \). Thus \( \tilde{\delta}' \) satisfies (\text{**}), whence (\text{*)} easily follows. It is also easily seen that \( \gamma \) is isotopic to 1 by an isotopy leaving \( M \times \{0\} \) fixed. Hence \( f \approx g \), where the isotopy may be chosen, again to leave \( M \times \{0\} \) fixed. This proves Lemma 6. To conclude Lemma 5, let \( f = u_1 \), and take \( u_\alpha \) to be the imbedding \( g \) given by Lemma 6.

Consider the dilation spaces \( \tilde{\mathcal{S}}_j = (\tilde{E}_j, X_j, \tilde{\iota}_j) \) \( j = 1, 2 \) where \( \tilde{E}_j = M_j \times R^k; X_j = M_j \times \{0\} \), and \( \tilde{\iota}_j: M_j \times R^k \to M_j \times R^k \) is the radial dilation given by \( 1 \times \rho \).

Set \( v_j = u_j \cdot (1 \times \rho) = u_j \tilde{\iota}_j \).

**Lemma 7.** The \( v_j \) are extendably bounded maps with respect to the dilation spaces \( \tilde{\mathcal{S}}_j \) (\( j = 1, 2 \), respectively).

**Proof.**

\[
\begin{array}{ccc}
E_j & \xrightarrow{v_j} & E_j \\
\downarrow & \equiv & \downarrow \\
X_j
\end{array}
\]

is commutative, by \( \text{ii) above} \). The \( v_j \) are open imbeddings, and since \( u_j \) is admissible on \( M_j \times D^k \), \( v_j = u_j \tilde{\iota}_j \) is extendably bounded. Notice that

\[
v_1 = u_1 \cdot (1 \times \rho) = \Phi_1(gf \times \rho) = \Phi_1(g \times 1)(f \times \rho) \\
v_2 = u_2 \cdot (1 \times \rho) = \Phi_2(fg \times \rho) = \Phi_2(f \times \rho)(g \times 1).
\]

Let \( \alpha = f \times \rho; \beta = g \times 1 \). Set \( E_{1,2} \) to be the double injective limit space

\[
\begin{array}{ccc}
E_1 & \xrightarrow{\alpha} & E_2 \\
\Psi_1 \times 1 \approx & \equiv & \Psi_2 \times 1 \approx \\
E_1(v_1) \times R & \xrightarrow{\approx} & E_1(\beta \alpha) \times R \\
\varphi_1 \approx & \equiv & \varphi_2 \approx \\
E_2(v_2) \times R.
\end{array}
\]

Then we have the sequence of isomorphisms

\[
E_1(i_1) \times R \xrightarrow{\sigma_1 \times 1 \approx} E_1 \times R \xrightarrow{F \approx} E_2 \times R \xrightarrow{\sigma_2 \times 1 \approx} E_2(v_2) \times R
\]

In the above, the \( \Psi_j \) comes from \([1, 6, \text{Th. 1}]\), since \( v_j \) is extendably bounded.
The $\varphi_i$ comes from Proposition 2 of § 4. The isomorphism
\[
F: M_1 \times R^k \to M_2 \times R^k
\]
\[
\text{homotopy-commutative properties,}
\]
could hardly avoid having the requisite homotopy-commutativity properties, and $G: M_2 \times R^k \to M_1 \times R^k$ may be taken to be $F^{-1}$, proving Theorem I.

6. Stable topological questions

A “stable problem” in topology is one which is *insensitive* to cartesian product by euclidean space. Given any problem in topology, it is usually quite easy to state a “stable analogue”. For example, we shall state stable analogues of four well-known topological problems.

(1) *The stable homeomorphism problem.* Given a continuous map
\[
f: M_1^n \to M_2^n
\]
when is there an integer $k \geq 0$, and an isomorphism $F$, such that
\[
M_1^n \times R^k \xrightarrow{F} M_2^n \times R^k
\]
is homotopy-commutative? If such an $F$ exists, then $M_1, M_2$ will be called *stably isomorphic*.

(2) *The stable triangulation problem.* Given a topological manifold $M$, is there an integer $k \geq 0$ such that $M \times R^k$ can be given the structure of combinatorial manifold?

(This is a more optimistic version of the following: Find necessary and sufficient *conditions* on $M$ so that an integer $k \geq 0$ with the above property may be found.)

(3) *The stable Hauptvermutung.* Let $f: M_1 \to M_2$ be a topological isomorphism between the combinatorial manifolds $M_1, M_2$. Is there an integer $k \geq 0$ and a combinatorial isomorphism $F$ such that
\[
M_1 \times R^k \xrightarrow{F} M_2 \times R^k
\]
is homotopy-commutative? (That is, are the manifolds $M_1, M_2$ stably *combinatorially* isomorphic?)

Problem (1) is answered by the “stable homeomorphism theorem” of § 5.
It should be noticed that necessary and sufficient conditions are that \( f \) be a tangential homotopy equivalence. The notion of “homotopy-equivalence” is, of course, well-known.

To determine whether a given map \( f \) is “tangential” in the differentiable case reduces to questions about \( BO \), which is a space quite well understood. (Its homotopy is periodic of order 8 \( \cdots \) see [B]).

In the combinatorial case, “tangentially” reduces to questions about \( BPL \), about which much is known. (One has an exact sequence \( 0 \to \pi_n(BO) \to \pi_n(BPL) \to \Gamma_{n+1} \to 0 \).

After the work of Kervaire-Milnor, Smale, and the recent work of J. Cerf, the groups \( \Gamma_n \) may be regarded as well-known. Consequently, much information may be deduced about \( \pi_n(BPL) \). For example \( \pi_n(BPL) \) is free of rank 1 or 0 depending whether or not \( n \equiv 0 \pmod{4} \).

In the topological case, however, “tangentiality” may be regarded (at present) as an uncomputable notion. Thus in the topological case, the stable homeomorphism theorem does not actually solve problem (1), but rather, it reduces problem (1) to a question about topological microbundles (the question of “tangentiality” of \( f \)).

In this section our attempt will be to show that these stable topological problems (2), (3), (4) may be reduced to problems in the theory of topological microbundles as well.

Consider the natural map

\[
    k_{PL}(X) \xrightarrow{\gamma} k_{TOP}(X)
\]

where \( X \) ranges over all finite dimensional, locally finite, countable, simplicial complexes.

**PROPOSITION 2.** These are equivalent:

(i) The stable triangulation conjecture is true.

(ii) The mapping \( \gamma \) is surjective (for all such \( X \)).

**PROOF.** (ii) \( \Rightarrow \) (i). Let \( M \) be a topological manifold, and \( \varepsilon: K \to M \) a homotopy equivalence between \( M \), and an open subset of euclidean space \( K \) (see § 2, footnote to Lemma 1) we may assume \( K \) to be a combinatorial manifold without boundary. Consider \( \varepsilon^*(\tau_{\text{ad}}) \in k_{TOP}(K) \). Since \( \gamma \) is surjective, there is a PL-microbundle, \( \tau \in k_{PL}(K) \) such that \( \gamma(\tau) = \varepsilon^*(\tau_{\text{ad}}) \). Let \( \tau \) be represented by \( N \xrightarrow{\varepsilon} K \). We may do this by the open Whitehead regular neighborhood \( W \) of \( K \subseteq N \). Then

\[
    W \xleftarrow{\pi} K
\]
is a homotopy equivalence, and the combinatorial tangent microbundle of $W$ is given by
\[ \tau_w = \pi^*\tau_k + \pi^*\tau. \]
This shows that
\[ W \times R^{\dim M} \xrightarrow{p_1} W \xrightarrow{i_3} M \xrightarrow{\pi} M \times R^{\dim W}, \]
is a tangential homotopy equivalence between the $n$-dimensional topological manifolds $W \times R^{\dim M}$ and $M \times R^{\dim W}$ ($n = \dim W + \dim M$).

Consequently, by the stable homeomorphism theorem, there is a topological isomorphism
\[ F: W \times R^{\dim M + k} \xrightarrow{\approx} M \times R^{\dim W + k}. \]
Since $W$ is a combinatorial manifold, $F$ transports a combinatorial structure to $M \times R^q$, for $q = \dim W + k$.

(i) $\implies$ (ii). Given $X$, a simplicial complex (again countable, locally finite, finite dimensional), and $\tau \in k_{\text{top}}(X)$, assuming the stable triangulation problem we must show
\[ \tau = \gamma(\tau') \quad \text{for } \tau' \in k_{\text{pl}}(X). \]
Replacing $X$ by its open Whitehead regular neighborhood in some large dimensional euclidean space, we may assume $X$ to be a combinatorial $\pi$-manifold. Let $W \xrightarrow{\pi} X$ be a euclidean bundle representation of the microbundle class $\tau$ (see [K]). Then $W$ is a topological manifold whose tangent microbundle class is given by
\[ \tau_w = \pi^*\tau \in k_{\text{top}}(W). \]
(Since $\tau_x = 0$). But, assuming (i), $W \times R^q$ may be given a combinatorial structure. Consequently $\tau_w = \gamma(\tau'_w)$ where $\tau'_w \in k_{\text{pl}}(W)$ is the image of
\[ \tau_{(w \times R^q)} \in k_{\text{pl}}(W \times R^q). \]
Take $\tau' = i^*(\tau'_w) \in k_{\text{pl}}(X)$. It follows that $\tau = \gamma(\tau')$.

**Proposition 3.** These are equivalent statements:

(i) The stable Hauptvermutung is true.

(ii) The mapping $\gamma$ is injective. (Again for all countable, locally finite, finite dimensional $X$.)

**Proof.** The proof of the above is similar to that of Proposition 2.

(i) $\implies$ (ii). Assume the contrary of (ii). Thus we have a (countable, locally finite, finite dimensional) simplicial complex $X$ such that $\gamma: k_{\text{pl}}(X) \rightarrow k_{\text{top}}(X)$ is not injective. We may again replace $X$ by a combinatorial manifold (which is
an open set in $R^n$). Consequently take $\xi_1, \xi_2$ to be PL-microbundles, inequivalent in $k_{pl}(X)$ yielding the same topological microbundle $\xi_j: M_j \subset X (j = 1, 2)$. We may replace $M_j$ by the (open) Whitehead regular neighborhood of $X \subset M_j$. Thus, the $M_j$ are combinatorial manifolds and

$$\tau_{pl}(M_j) = \tau(M_j) = \xi_j + \tau(X).$$

Since $X$ is open in $R^n$, we have

$$(\ast) \quad \tau(M_j) = \xi_j \quad \quad (j = 1, 2).$$

Now regard the $M_j$ as topological manifolds. Consider the map

$$\begin{array}{ccc}
M_1 & \overset{f}{\longrightarrow} & M_2 \\
\pi_1 & \downarrow & \pi_2 \\
X & \longleftarrow &
\end{array}$$

which is clearly a homotopy-equivalence by properties of the Whitehead regular neighborhood. Since

$$\eta_1(\xi_1) = \eta_1(\xi_2),$$

we have

$$f^*\tau_{top}(M_2) = \tau_{top}(M_1),$$

by $(\ast)$ above. Thus $f$ is topologically tangential, and hence, by the stable homeomorphism theorem, there is a topological isomorphism

$$\begin{array}{ccc}
M_1 \times R^q & \overset{F}{\longrightarrow} & M_2 \times R^q \\
\pi & \downarrow & \pi \\
M_1 & \overset{f}{\longrightarrow} & M_2 \\
\end{array}$$

making the above commutative.

Consequently if

$$W_j = M_j \times R^q \quad \quad j = 1, 2$$

are the combinatorial manifolds, one has that $W_1$ is topologically isomorphic to $W_2$ but

$$\tau_{pl}(W_1) "= " \tau_{pl}(M_1) \neq \tau_{pl}(M_2) "= " \tau_{pl}(W_2),$$

and so $W_1$ is definitely not combinatorially isomorphic with $W_2$.

(ii) $\Rightarrow$ (i). Assume the contrary of (i). Thus we have two combinatorial manifolds $M_1, M_2$ which are topologically isomorphic, and yet $M_1 \times R^q$ is combinatorially distinct from $M_2 \times R^q$ (in the particular sense of Problem 3) for all $q$. Let $f: M_1 \rightarrow M_2$ be the topological isomorphism. Then $f$ is clearly a homotopy equivalence. Is $f$ combinatorially tangential? No, because then the
combinatorial version of the stable homeomorphism theorem would apply, denying our assumption that there is no combinatorial isomorphism making

\[ \begin{align*}
M_1 \times \mathbb{R}^q & \xrightarrow{F} M_2 \times \mathbb{R}^q \\
\downarrow & \quad \downarrow \\
M_1 & \xrightarrow{f} M_2
\end{align*} \]

homotopy commutative (for any \( q \geq 0 \)). Thus we may conclude

\[ f^* \tau_{PL}(M_2) \neq \tau_{PL}(M_1). \]

Let \( \hat{z}_1 = \tau_{PL}(M_1); \hat{z}_2 = f^* \tau_{PL}(M_2). \)

Clearly \( f \) is topologically tangential, since \( f \) is a topological isomorphism. Consequently

\[ f^* \tau_{TOP}(M_2) = \tau_{TOP}(M_1) \]

or

\[ f^* \gamma \tau_{PL}(M_2) = \gamma \tau_{PL}(M_1) \]

or

\[ \gamma(\hat{z}_1) = \gamma(\hat{z}_2). \]

Thus \( \gamma \) is not injective.

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References


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