FINITE SPACES AND SIMPLICIAL COMPLEXES

NOTES FOR REU BY J.P. MAY

1. Statements of results

Finite simplicial complexes provide a general class of spaces that is sufficient for most purposes of basic algebraic topology. There are more general classes of spaces, in particular the finite CW complexes, that are more central to the modern development of the subject, but they give exactly the same collection of homotopy types. The relevant background on simplicial complexes will be recalled as we go along and can be found in most textbooks in algebraic topology (but not in my own book [6]). We write $|K|$ for the geometric realization of $K$.

We recall the definition of the homotopy groups $\pi_n(X, x)$ of a space $X$ at $x \in X$. When $n = 0$, this is just the set of path components of $X$, with the component of $x$ taken as a basepoint (and there is no group structure). When $n = 1$ it is the fundamental group of $X$ at the point $x$. For all $n \geq 0$, it can be described most simply by considering the standard sphere $S^n$ with a chosen basepoint $\ast$. One considers all maps $\alpha: S^n \to X$ such that $f(\ast) = x$. One says that two such maps $\alpha$ and $\beta$ are based homotopic if there is a based homotopy $h: \alpha \simeq \beta$. Here a homotopy $h$ is based if $h(\ast, t) = x$ for all $t \in I$. If $n = 1$, the map $\alpha$ is a loop at $x$, and we can compose loops to obtain a product which makes $\pi_1(X, x)$ a group. The homotopy class of the constant loop at $x$ gives the identity element, and the loop $\alpha^{-1}(t) = \alpha(1 - t)$ represents the inverse of the homotopy class of $\alpha$. There is a similar product on the higher homotopy groups, but, in contrast to the fundamental group, the higher homotopy groups are Abelian.

A map $f: X \to Y$ induces a function $f_*: \pi_n(X, x) \to \pi_n(Y, f(x))$. One just composes maps $\alpha$ and homotopies $h$ as above with the map $f$. If $n \geq 1$, $f_*$ is a homomorphism.

Definition 1.1. A map $f: X \to Y$ is a weak homotopy equivalence if

$$f_*: \pi_n(X, x) \to \pi_n(Y, f(x))$$

is an isomorphism for all $n \geq 0$. If $n = 0$, this means that components are mapped bijectively. Two spaces $X$ and $Y$ are weakly homotopy equivalent if there is a finite chain of weak homotopy equivalences $Z_i \to Z_{i+1}$ or $Z_{i+1} \to Z_i$ starting at $X = Z_1$ and ending at $Z_q = Y$.

The definition may seem strange at first sight, but it has gradually become apparent that the notion of a weak homotopy equivalence is even more important in algebraic topology than the notion of a homotopy equivalence. The notions are related. We state some theorems that the reader can take as reference points. Proofs can be found in [6].
Theorem 1.2. A homotopy equivalence is a weak homotopy equivalence. Conversely, a weak homotopy equivalence between CW complexes (for example, between simplicial complexes) is a homotopy equivalence.

Theorem 1.3. Spaces $X$ and $Y$ are weakly homotopy equivalent if and only if there is a space $Z$ (in fact a CW complex $Z$) and weak homotopy equivalences $Z \longrightarrow X$ and $Z \longrightarrow Y$.

That is, the chains that appear in the definition need only have length two. For those who know about homology and cohomology, we record the following result.

Theorem 1.4. A weak homotopy equivalence induces isomorphisms of all singular homology and cohomology groups.

Following McCord [7], we are going to relate finite spaces with finite simplicial complexes, explaining the following two theorems. Since any finite space is homotopy equivalent to a $T_0$-space, there is no loss of generality if we restrict attention to finite $T_0$-spaces. McCord actually deals more generally with $A$-spaces, but the arguments are no different.

Theorem 1.5. For a finite $T_0$-space $X$, there is a finite simplicial complex $\mathcal{K}(X)$ with vertex set $X$, and there is a weak homotopy equivalence $\psi = \psi_X: |\mathcal{K}(X)| \longrightarrow X$.

A map $f: X \longrightarrow Y$ of finite spaces defines a simplicial map $\mathcal{K}(f): \mathcal{K}(X) \longrightarrow \mathcal{K}(Y)$ such that $f \circ \psi_X = \psi_Y \circ |f|$.

The essential point in the proof, which we will take for granted, is that weak homotopy equivalence is a local notion in the sense of the following theorem. McCord [7] relies on point-by-point comparison with arguments in the early paper [1], which doesn’t prove the result but comes close. More modern references are [5, 11].

Theorem 1.6. Let $p: E \longrightarrow B$ be a continuous map. Suppose that $B$ has an open cover $\mathcal{O}$ with the following two properties.

(i) If $b$ is in the intersection of sets $U$ and $V$ in $\mathcal{O}$, then there is some $W \in \mathcal{O}$ with $x \in W \subset U \cap V$; that is, $\mathcal{O}$ is a basis for a possibly smaller topology than that originally given on $B$.

(ii) For each $U \in \mathcal{O}$, the restriction $p: p^{-1}U \longrightarrow U$ is a weak homotopy equivalence.

Then $p$ is a weak homotopy equivalence.

Theorem 1.5 is itself used to obtain the following complementary result.

Theorem 1.7. For a finite simplicial complex $K$, there is a finite $T_0$-space $\mathcal{X}(K)$ whose points are the barycenters of the simplices of $K$, and there is a weak homotopy equivalence $\phi = \phi_K: |K| \longrightarrow \mathcal{X}(K)$.

For a simplicial map $g: K \longrightarrow L$, there is a map $\mathcal{X}(g): \mathcal{X}(K) \longrightarrow \mathcal{X}(L)$ such that $\mathcal{X}(g) \circ \phi_K \simeq \phi_L \circ |g|$.
Remark 1.8. Writing $K'$ for the barycentric subdivision of $K$, so that $|K| = |K'|$, we will have $\mathcal{K}(K) = K'$. The map $\phi_K$ will be $\psi_{\mathcal{K}(K)}$, and Theorem 1.5 will apply to show that it is a weak homotopy equivalence.

As a warm-up exercise, we will consider suspensions of spaces and give a finite model for the $n$-sphere before turning to the proofs of these general results.

2. Problems

Also before turning to the proofs, we list a few problems that spring immediately to mind. To the best of my knowledge, none of them have been studied.

Problem 2.1. For small $n$, determine all homotopy types and weak homotopy types of spaces with at most $n$ elements.

Addendum 2.2. This has been done in class or by students in the cases $n \leq 6$. Nearly all finite spaces with so few points are disjoint unions of (weakly) contractible spaces.

Problem 2.3. Is there an effective algorithm for computing the homotopy groups of $X$ in low degrees in terms of the increasing chains in $X$?

Remark 2.4. The dimension of the simplicial complex $\mathcal{K}(X)$ is the maximal length of a sequence $x_0 < \cdots < x_n$ in $X$. A map $g: K \rightarrow L$ of simplicial complexes of dimension less than $n$ is a homotopy equivalence if and only if it induces an isomorphism of homotopy groups in dimension less than $n$ and an epimorphism of homotopy groups in dimension $n$.

Problem 2.5. Let $X$ be a minimal finite space. Give a descriptive interpretation of what this says about $|\mathcal{K}(X)|$.

Addendum 2.6. There is a nice paper of Osaki [9] that interprets Stong’s process of passing from a finite $T_0$-space $X$ to its core $Y$. He shows that $\mathcal{K}(X)$ is obtained from $\mathcal{K}(Y)$ by a sequence of elementary simplicial collapses, so that $|\mathcal{K}(X)|$ and $|\mathcal{K}(Y)|$ have the same simple homotopy type. It follows that if $X$ and $Y$ are homotopy equivalent finite $T_0$-spaces, then $\mathcal{K}(X)$ and $\mathcal{K}(Y)$ have the same simple homotopy type. If $K$ is not collapsible, then $\mathcal{K}(K)$ is a minimal finite space. He displays non-collapsible triangulations $K_1$ and $K_2$ of $S^1$ such that $\mathcal{K}(K_1)$ and $\mathcal{K}(K_2)$ are not homeomorphic and therefore, being minimal, not homotopy equivalent. This provides a simple example of weak homotopy equivalent finite spaces that are not homotopy equivalent.

Problem 2.7. Let $f: X \rightarrow Y$ be a weak homotopy equivalence between minimal finite spaces. Is $f$ necessarily a homotopy equivalence and hence a homeomorphism?

Problem 2.8. Suppose that two finite spaces $X$ and $Y$ are weakly homotopy equivalent. Are they weakly homotopy equivalent via a chain in which all of the $Z_i$ are again finite spaces?

Addendum 2.9. The answer is yes, as an application of the simplicial approximation theorem for finite spaces of Hardie and Vermeulen [2]. It is discussed below.

Problem 2.10. Are there computationally effective algorithms for enumerating the homotopy types and, presumably much harder, the weak homotopy types of finite spaces?
Addendum 2.11. Osaki [9] has given two theorems that describe when one can shrink a finite $T_0$-space, possibly minimal, to a smaller weakly homotopy equivalent space. He asks whether all weak homotopy equivalences are generated by the simple kinds that he describes.

**Problem 2.12.** Is there a combinatorial way of determining when a weak homotopy equivalence of finite spaces is a homotopy equivalence?

**Problem 2.13.** Rather than restricting to finite simplicial complexes, can we model the world of finite CW complexes in the world of finite spaces. The discussion of spheres and cones in the next section gives a starting point.

### 3. The non-Hausdorff suspension

The suspension is one of the most basic constructions in all of topology. Following McCord [7], we show that it comes in two weakly equivalent versions, the classical one and a non-Hausdorff analogue that preserves finite spaces. For the purposes of these notes, we shall use the following unbased variant of the classical suspension.

**Definition 3.1.** The suspension $SX$ of a space $X$ is the quotient space obtained from $X \times [-1, 1]$ by identifying $X \times \{-1\}$ to a single point $-$ and identifying $X \times \{1\}$ to a single point $+$. Thus $SX$ can be thought of as obtained by gluing together the bases of two cones on $X$. For a map $f: X \to Y$, define $Sf: SX \to SY$ by $(Sf)(x, t) = (f(x), t)$.

We defined the non-Hausdorff cone $CX$ by adjoining a new cone point $\ast$ and letting the proper open subsets of $CX$ be all of the open subsets of $X$, and we saw that $CX$ is contractible. We now change notation and call the added point $\ast$.

**Definition 3.2.** Define the non-Hausdorff suspension $S\ast X$ by adjoining two new points, denoted $\ast$ and $\ast$, and letting the proper open subsets be the open sets in $X$ and the sets $X \cup \ast$ and $X \cup \ast$. Again, $S\ast X$ can be thought of as obtained by gluing together two copies of $CX$. If $f: X \to Y$ is a map, define maps $Cf$ and $Sf$ by using $f$ on $X$ and sending $\ast$ to $\ast$ and $\ast$ to $\ast$.

Observe that if $X$ is a $T_0$-space, then so are $CX$ and $SX$.

**Definition 3.3.** Define a comparison map

$$\gamma = \gamma_X: SX \to S\ast X$$

by $\gamma(x, t) = x$ if $-1 < t < 1$, $\gamma(+) = +$ and $\gamma(-) = -$. Observe that, for a map $f: X \to Y$, $\gamma_Y \circ Sf = Sf \circ \gamma_X$. Inductively, define $S^nX = SSS^{n-1}X$ and $S^nX = SSS^{n-1}X$ and let $\gamma^n: S^nX \to S^nX$ be the common composite displayed in the commutative diagram

$$
\begin{array}{ccc}
S^nX & \xrightarrow{S\gamma^{n-1}} & SSS^{n-1}X \\
\gamma \downarrow & & \downarrow \gamma \\
SSS^{n-1}X & \xrightarrow{S\gamma^{n-1}} & S^nX
\end{array}
$$

**Theorem 3.4.** For any space $X$, the map $\gamma: SX \to S\ast X$ is a weak homotopy equivalence. For any weak homotopy equivalence $f: X \to Y$, the maps $Sf: SX \to SY$ and $Sf: S\ast X \to S\ast Y$ are weak homotopy equivalences. Therefore $\gamma^n: S^nX \to S^nX$ is a weak homotopy equivalence for any space $X$. 

Proof. This is an application, or rather several applications, of Theorem 1.6. Take the three subspaces $X$, $X \cup \{+\}$, and $X \cup \{-\}$ as our open cover of $\mathbb{S}X$ and observe that the latter two subspaces are copies of $C^X$ and are therefore contractible. The respective inverse images under $\gamma$ of these three subspaces are the images in $S\times \{-1,1\}$, $X \times \{-1,1\}$, and $X \times [-1,1]$. The restrictions of $\gamma$ on these subspaces are homotopy equivalences, hence weak homotopy equivalences. Similarly, taking the three subspaces $Y$, $Y \cup \{+\}$, and $Y \cup \{-\}$ as our open cover of $\mathbb{S}Y$, their inverse images under $\mathbb{S}f$ are $X$, $X \cup \{+\}$, and $X \cup \{-\}$, and the restrictions of $\mathbb{S}f$ on these three subspaces are weak homotopy equivalences. Finally, take the images in $\mathbb{S}Y$ of $Y \times (-1/2, 1/2)$, $Y \times [-1, 1/2]$, and $Y \times (-1/2, 1]$ as our open cover of $\mathbb{S}Y$. Their inverse images under $Sf$ are the corresponding subspaces of $\mathbb{S}X$, and the restrictions of $Sf$ to these subspaces are weak homotopy equivalences. \qed

Example 3.5. Let $X = S^0$, a two-point discrete space. Then $S^n \times X$ is homeomorphic to the $n$-sphere $S^n$, while $S^n \times X$ is a $T_0$-space with $2n + 2$ points. Thus we have a weak homotopy equivalence $\gamma^n$ from $S^n$ to a finite space with $2n + 2$ points.

Proposition 3.6. Each $S^n S^0$ is a minimal finite space.

Proof. Certainly $S^n S^0$ is $T_0$, and it has no upbeat or downbeat points since each point has incomparable points above or below it in the partial ordering. \qed

Problem 3.7. Is $S^n S^0$ the finite space with the smallest number of points that is weakly homotopy equivalent to an $n$-sphere? The answer is probably yes and probably known, but I don’t know how to prove it and can’t find it in the literature.

4. Recollections about simplicial complexes

Definition 4.1. An abstract simplicial complex $K$ is a set $V = V(K)$, whose elements are called vertices, together with a set $\mathcal{K}$ of (non-empty) finite subsets of $V$, whose elements are called simplices, such that every vertex is an element of some simplex and every subset of a simplex is a simplex; such a subset is called a face of the given simplex. We say that $K$ is finite if $V$ is a finite set. The dimension of a simplex is one less than the number of vertices in it. A map $g: K \rightarrow L$ of abstract simplicial complexes is a function $g: V(K) \rightarrow V(L)$ that takes simplices to simplices. We say that $K$ is a subcomplex of $L$ if the vertices and simplices of $K$ are some of the vertices and simplices of $L$. We say that $K$ is a full subcomplex of $L$ if, further, every simplex of $K$ whose vertices are in $K$ is a simplex of $K$.

Definition 4.2. A set $\{v_0, \ldots, v_n\}$ of points of $\mathbb{R}^n$ is geometrically independent if the equations $\sum t_i v_i = 0$ and $\sum t_i = 0$ for real numbers $t_i$ imply $t_1 = \cdots = t_n = 0$. It is equivalent that the vectors $v_i - v_0$, $1 \leq i \leq n$, are linearly independent. The $n$-simplex $\sigma$ spanned by $\{v_0, \ldots, v_n\}$ is then the set of all points $x = \sum t_i v_i$, where $\sum t_i = 1$. The $t_i$ are called the barycentric coordinates of the point $x$. When each $t_i = 1/n + 1$, the point $x$ is called the barycenter of $\sigma$. The points $v_i$ are the vertices of $\sigma$. A simplex spanned by a subset of the vertices is a face of $\sigma$; it is a proper face if the subset is proper. The standard $n$-simplex $\Delta_n$ is the $n$-simplex spanned by the standard basis of $\mathbb{R}^{n+1}$.

Definition 4.3. A simplicial complex, or geometric simplicial complex, $K$ is a set of simplices in some $\mathbb{R}^N$ such that every face of a simplex in $K$ is a simplex in $K$ and the intersection of two simplices in $K$ is a simplex in $K$. The set of vertices of...
$K$ is the union of the sets of vertices of its simplexes. The notions of subcomplex and full subcomplex are evident.

**Definition 4.4.** The geometric realization $|K|$ is the the union of the simplices of $K$, each regarded as a subspace of $\mathbb{R}^N$, with the topology whose closed sets are the sets that intersect each simplex in a closed subset. If $K$ is finite, but not in general otherwise, this is the same as the topology of $|K|$ as a subspace of $\mathbb{R}^N$. The open simplices of $|K|$ are the interiors of its simplices (where a vertex is an interior point of its 0-simplex), and every point of $|K|$ is an interior point of a unique simplex. The boundary $\partial \sigma$ of a simplex $\sigma$ is the subcomplex given by the union of its proper faces. The closure of a simplex is the union of its interior and its boundary.

**Definition 4.5.** A map $g: K \rightarrow L$ of simplicial complexes is a function from the vertex set $V(K)$ to the vertex set $V(L)$ such that, for each subset $S$ of $V(K)$ that spans a simplex, the set $g(S)$ is the set of vertices of a simplex of $L$. Then $g$ determines the continuous map $|g|: |K| \rightarrow |L|$ that sends $\sum t_i v_i$ to $\sum t_i g(v_i)$. Note that we do not require $g$ to be one–to–one on vertices, but $|g|$ is nevertheless well-defined and continuous. If $g$ is a bijection on vertices and simplices, we say that it is an isomorphism, and then $|g|$ is a homeomorphism. It is usual to abbreviate $|g|$ to $g$ and to refer to it as a simplicial map.

**Definition 4.6.** The abstract simplicial complex a$K$ determined by a geometric simplicial complex $K$ has vertex set the union of the vertex sets of the simplices of $K$. Its simplices are the subsets that span a simplex of $K$. An abstract finite simplicial complex $K$ determines a geometric finite simplicial complex $gK$ by choosing any bijection between the vertices of $K$ and a geometrically independent subset of some $\mathbb{R}^N$. For specificity, we can take the standard basis elements of $\mathbb{R}^N$ where $N$ is the number of points in the vertex set $V(K)$. The geometric simplices are spanned by the images under this bijection of the simplices of $K$. For an abstract simplicial complex $K$, a$gK$ is isomorphic to $K$, the isomorphism being given by the chosen bijection. Similarly, for a finite geometric simplicial complex $K$, ga$K$ is isomorphic to $K$.

We could remove the word finite from the previous definition by defining geometric simplicial complexes more generally, without reference to a finite dimensional ambient space $\mathbb{R}^N$, but there is no point in going into that here. We also note that we do not have to realize in such a high dimensional Euclidean space. The following result holds no matter how many vertices there are. It is rarely used, but is conceptually attractive. A proof can be found in [4, 1.9.6]

**Theorem 4.7.** Any simplicial complex $K$ of dimension $n$ can be geometrically realized in $\mathbb{R}^{2n+1}$.

In view of the discussion above, abstract and geometric finite simplicial complexes can be used interchangeably. In particular, the geometric realization of an abstract simplicial complex is $K$ is understood to mean the geometric realization of any $gK$.

We need a criterion for when the geometric realizations of two simplicial maps are homotopic.

**Proposition 4.8.** Let $f$ and $g$ be maps from a topological space $X$ to $|K| \subset \mathbb{R}^N$. Say that $f$ and $g$ are simplicially close if, for each $x \in X$, both $f(x)$ and $g(x)$ are
in the closure of some simplex $\sigma(x)$ of $L$. If $f$ and $g$ are simplicially close, then they are homotopic.

**Proof.** Define $h: X \times I \longrightarrow \mathbb{R}^N$ by

$$h(x, t) = (1 - t)f(x) + tg(x).$$

Since $h(x, t)$ is in the closure of $\sigma(x)$ and therefore in $|K|$, it specifies a homotopy as required. \hfill \square

5. **Cones and subdivisions of simplicial complexes**

Let $K$ be a finite geometric simplicial complex in $\mathbb{R}^N$.

**Definition 5.1.** Define the cone $CT$ of a topological space $T$ to be the quotient space $T \times I / T \times \{1\}$.

**Definition 5.2.** Let $x$ be a point of $\mathbb{R}^N - K$ such that each ray starting at $x$ intersects $|K|$ in a single point. Observe that the union of $\{x\}$ and the set of vertices of a simplex of $K$ is a geometrically independent set. Define the cone $K \ast x$ on $K$ with vertex $x$ to be the geometric simplicial complex whose simplices are all of the faces of the simplices spanned by such unions. Then $K$ is a subcomplex of $K \ast x$, $x$ is the only vertex not in $K$, and $|K \ast x|$ is homeomorphic to $C|K|$.

**Example 5.3.** A simplex is the cone of any one of its vertices with the subcomplex spanned by the remaining vertices (the opposite face).

**Definition 5.4.** A subdivision of $K$ is a simplicial complex $L$ such that each simplex of $L$ is contained in a simplex of $K$ and each simplex of $K$ is the union of finitely many simplices of $L$.

**Lemma 5.5.** If $L$ is a subdivision of $K$, then $|L| = |K|$ (as spaces).

The $n$-skeleton $K^n$ of $K$ is the union of the simplices of $K$ of dimension at most $n$. It is a subcomplex.

**Construction 5.6.** We construct the (first) barycentric subdivision $K'$ of $K$. We subdivide the skeletons of $K$ inductively. Let $L_0 = K^0$. Suppose that a subdivision $L_{n-1}$ of $K^{n-1}$ has been constructed. Let $b_\sigma$ be the barycenter of an $n$-simplex $\sigma$ of $K$. The space $|\partial\sigma|$ coincides with $|L_\sigma|$ for a subcomplex $L_\sigma$ of $L_{n-1}$, and we can define the cone $L_\sigma \ast b_\sigma$. Clearly $|L_\sigma \ast b_\sigma| = |\sigma|$ and $|L_\sigma \ast b_\sigma| \cap |L_{n-1}| = |L_\sigma| = |\partial\sigma|$. If $\tau$ is another $n$-simplex, then $|L_\sigma \ast b_\sigma| \cap |L_\tau \ast b_\tau| = |\sigma \cap \tau|$, which is the realization of a subcomplex of $L_{n-1}$ and therefore of both $L_\sigma$ and $L_\tau$. Define $L_n$ to be the union of $L_{n-1}$ and the complexes $L_\sigma \ast b_\sigma$, where $\sigma$ runs over all $n$-simplices of $K$.

Our observations about intersections show that $L_n$ is a simplicial complex which contains $L_{n-1}$ as a subcomplex. The union of the $L_n$ is denoted $K'$ and called the barycentric subdivision of $K$.

The second barycentric subdivision of $K$ is the barycentric subdivision of the first barycentric subdivision, and so on inductively. We can enumerate the simplices of $K'$ explicitly rather than inductively.

**Proposition 5.7.** Define $\sigma < \tau$ if $\sigma$ is a proper face of $\tau$. Then $K'$ is the simplicial complex whose simplices $\sigma'$ are the spans of the geometrically independent sets $\{b_{\sigma_1}, \ldots, b_{\sigma_n}\}$, where $\sigma_1 > \cdots > \sigma_n$. In particular, the barycenters $b_\sigma$ are the vertices of $K'$. The vertex $b_{\sigma_1}$ is called the leading vertex of the simplex $\sigma'$. 
Proposition 5.8. There is a simplicial map \( \sigma \) interior point of the simplex \( K \) of possibly lower dimension than \( g \) map each vertex \( b \sigma \) of the span a face of \( \sigma \) are barycenters of faces of \( \sigma \), hence are mapped under \( \xi \) to vertices of \( \sigma \). Therefore the images under \( \xi \) of the vertices of \( \sigma \) span a face of \( \sigma \), so that \( \xi \) is a simplicial map. If \( x \in |K| \) is an interior point of the simplex \( \sigma \), then it is mapped under \( \xi \) to a point of \( \sigma \), and we let \( \sigma(x) = \sigma_1 \). Since \( \xi \) maps every vertex of \( \sigma \) to a vertex of \( \sigma_1 \), \( x \) and \( \xi(x) \) are both in the closure of \( \sigma_1 \).

Remark 5.9. In the cases of interest to us, there is a partial order on the simplices of \( K \) that restricts to a total order on each simplex of \( K \). In that case, we have the standard simplicial map \( \xi: K' \to K \) specified by letting \( \xi(b_\sigma) \) be the maximal vertex \( x_\sigma \) of the simplex \( \sigma = \{x_0, \ldots, x_n\} \).

Proposition 5.10. A simplicial map \( g: K \to L \) induces a subdivided simplicial map \( \begin{array}{c} g': K' \to L' \end{array} \) whose realization is simplicially close to \( |g| \) and hence homotopic to \( |g| \).

Proof. The images under \( g \) of the vertices of a simplex \( \sigma \) of \( K \) span a simplex \( g(\sigma) \), of possibly lower dimension than \( \sigma \), and we define \( g'(b_\sigma) = b_{g(\sigma)} \) on vertices. If \( b_\sigma \) is the leading vertex of a simplex \( \sigma' \) of \( K' \), then all other vertices of \( \sigma' \) are barcenters of faces of \( \sigma_1 \). Their images under \( g' \) are barcenters of vertices of \( g(\sigma_1) \). If \( x \) is an interior point of \( \sigma' \), then both \( g(x) \) and \( g'(x) \) are in the closure of \( g(\sigma_1) \).

6. The Definition and Properties of \( \mathcal{X}(X) \) and \( \mathcal{X}(K) \)

Let \( X \) be a finite \( T_0 \)-space.

Definition 6.1. Define \( \mathcal{X}(X) \) to be the abstract simplicial complex whose vertex set is \( X \) and whose simplices are the finite totally ordered subsets of the poset \( X \). Since a map \( f: X \to Y \) is an order-preserving function, it may be regarded as a simplicial map \( \mathcal{X}(f): \mathcal{X}(X) \to \mathcal{X}(Y) \).

Lemma 6.2. If \( V \) is a subspace of \( X \), then \( \mathcal{X}(V) \) is a full subcomplex of \( \mathcal{X}(X) \).

Proof. The ordering \( \leq \) on \( V \) is the restriction of the ordering \( \leq \) on \( X \). Every totally ordered subset of \( X \) whose points are in \( V \) is a totally ordered subset of \( V \).

Definition 6.3. Define \( \psi: |\mathcal{X}(X)| \to X \) as follows. Each point \( u \in |\mathcal{X}(X)| \) is an interior point of a simplex \( \sigma \) spanned by some strictly increasing sequence \( x_0 < x_1 < \cdots < x_n \) of points of \( X \). Define \( \psi(u) = x_0 \).

It is convenient to start with the proof of the last statement of Theorem 1.5.

Lemma 6.4. If \( f: X \to Y \) is a map of finite \( T_0 \)-spaces, then \( f \circ \psi_X = \psi_Y \circ |f| \).
Proposition 6.5. Let $\psi \in \mathcal{K}(X)$. Then
\[
\psi^{-1}(V) = \bigcup \{ \text{star}(v) | v \in V \},
\]
where $\text{star}(v)$ is the union of the open simplices of $|\mathcal{K}(X)|$ that have $v$ as a vertex. Therefore $\psi$ is continuous.

Proof. If $\psi(u) = v \in V$, then $u$ is an interior point of a simplex $\sigma$ of which $v$ is the initial vertex $x_0$. Thus $u \in \text{star}(v)$. Conversely, suppose that $u \in \text{star}(v)$ with $v \in V$. Then $u$ is an interior point of a simplex $\sigma$ determined by an increasing sequence $x_0 < x_1 < \cdots < x_n$ such that some $x_i = v \in V$. Since $x_0 \leq v$, $x_0 \in U_v$. Since $V$ is open, $U_v \subset V$. Thus $\psi(u) = x_0$ is in $V$. □

Corollary 6.6. $|\mathcal{K}(V)|$ is a deformation retract of $\psi^{-1}(V)$.

Proof. By Lemma 6.2, $\mathcal{K}(V)$ is a full subcomplex of $\mathcal{K}(X)$. It follows that $|\mathcal{K}(V)|$ is a deformation retract of its open star in $|\mathcal{K}(X)|$. This is a standard fact in the theory of simplicial complexes, and a more detailed proof is given in [8, 70.1 and p. 427]. Consider a simplex $\sigma$ that is in the open star of $V$ but is not contained in $V$. Then $\sigma$ has vertex set the disjoint union of a set of vertices in $V$ and a set of vertices in $X - V$. Each point $u$ of $\sigma$ that is neither in the span $s$ of the vertices in $V$ nor in the span $t$ of the vertices not in $V$ is on a unique line segment joining a point in $t$ to a point in $s$. Define the required retraction $r$ by sending $u$ to the end point in $s \subset V$ of this line segment, letting $r$ be the identity map on $V$ and thus on $s$. Deformation along such line segments gives the required homotopy showing that $i \circ r$ is the identity, where $i$ is the inclusion of $|\mathcal{K}(V)|$ in its open star. □

Recall that each open subset $U_x$ of $X$ is contractible.

Proposition 6.7. For $x \in X$, $\psi^{-1}(U_x)$ is contractible.

Proof. By the previous corollary, it suffices to show that $|\mathcal{K}(U_x)|$ is contractible. Let $V_x = U_x - x$. We claim that $|\mathcal{K}(U_x)|$ is isomorphic to the cone $\mathcal{K}(V_x) \ast x$. Indeed, a simplex of $\mathcal{K}(V_x)$ is given by an increasing sequence $x_0 < x_1 < \cdots < x_n$. The increasing sequence $x_0 < x_1 < \cdots < x_n < x$ gives a simplex of $\mathcal{K}(U_x)$, and every simplex of $\mathcal{K}(U_x)$ not in $\mathcal{K}(V_x)$ is of this form. □

The proof that $\psi$ is a weak homotopy equivalence. Theorem 1.6 applies to the minimal open cover of $X$. If $x$ is in $U_y \cap U_z$, then $x$ is in $U_y$ or $U_z$, so that $U_x$ is contained in $U_y$ or $U_z$. This verifies the first hypothesis of the cited theorem, and the second hypothesis holds by the previous result. □

Now let $K$ be a finite geometric simplicial complex with first barycentric subdivision $K'$.

Definition 6.8. Define a finite $T_0$-space $\mathcal{K}(K)$ as follows. The points of $\mathcal{K}(K)$ are the barycenters $b_\sigma$ of the simplices of $K$, that is, the points of $K'$. The required partial order $\leq$ is defined by $b_\sigma \leq b_\tau$ if $\sigma \subset \tau$. The open subspace $U_{b_\sigma}$ coincides with $\mathcal{K}(\sigma)$, where $\sigma$ (together with its faces) is regarded as a subcomplex of $K$. 


The proof of Theorem 1.7. Using the barycenters themselves to realize the vertices geometrically, we find that \( \mathcal{X}(K) = K' \), by Proposition 5.7. Define

\[
\phi_K = \psi_{\mathcal{X}(K)}: |K| = |K'| = |\mathcal{X}(K)| \to \mathcal{X}(K).
\]

Then \( \phi_K \) is a weak homotopy equivalence by Theorem 1.5. For a simplicial map \( g: K \to L \), define \( \mathcal{X}(g) = g' \) on barycenters and note that this function is order-preserving and therefore continuous. Clearly \( |\mathcal{X}(g)| = |g'| \) and therefore, by Theorem 1.5 and Proposition 5.10, \( \mathcal{X}(g) \circ \phi_K = \phi_L \circ |g'| \simeq \phi_L \circ |g| \).

\[ \square \]

7. Mapping spaces

For completeness, we record results of Stong [10] that were obtained about the same time as the results of McCord recorded above and that give a quite different approach to the relationship between finite simplicial complexes and finite spaces. Since the proofs are fairly long and combinatorial in flavor, and since the statements do not have quite the same immediate impact as those in McCord’s work, we shall not work through the details here.

Rather than constructing finite models for finite simplicial complexes, Stong studies all maps from simplicial complexes \( K \) into finite spaces \( X \) by studying the properties of the function space \( X^K \). More generally, he fixes a subcomplex \( L \) of \( K \) and a basepoint \( * \in X \) and studies the subspace \( (X, *)^{(K, L)} \) of maps \( f: K \to L \) such that \( f(L) = * \). Homotopies relative to \( L \) between such maps are homotopies \( h \) such that \( h(k, t) = * \) for all \( k \in L \).

**Theorem 7.1.** Let \( L \) be a subcomplex of a finite simplicial complex \( K \), let \( X \) be a finite space with basepoint \( * \), and let \( F = (X, *)^{(K, L)} \) denote the subspace of \( X^L \) consisting of those maps \( f: K \to L \) such that \( f(L) = * \).

(i) For any \( f \in F \), there is a map \( g \in F \) such that the set \( V = \{ h | h \leq g \} \subset F \) is a neighborhood of \( f \) in \( F \); that is, there is an open subset \( U \) such that \( f \in U \subset V \).

(ii) If \( f \simeq f' \) relative to \( L \), then there is a sequence of elements \( \{g_1, \ldots, g_s\} \) in \( F \) such that \( g_1 = f \), \( g_s = h \), and either \( g_i \leq g_{i+1} \) or \( g_{i+1} \leq g_i \) for \( 1 \leq i < s \).

The essential point of this analysis is the following consequence.

**Corollary 7.2.** The path components and components of \( F \) coincide. That is, the homotopy classes of maps \( f: (K, L) \to (X, *) \) are in bijective correspondence with the components of \( F \).

8. The simplicial approximation theorem

The classical point of barycentric subdivision is its use in the simplicial approximation theorem, which in its simplest form reads as follows. Starting with \( K^{(0)} = K \), let \( K^{(n)} = K'K^{(n-1)} \) be the \( n \)th barycentric subdivision of a simplicial complex \( K \). By iteration of \( \xi: K' \to K \), we obtain a simplicial map \( \xi^{(n)}: K^{(n)} \to K \) whose geometric realization is a homotopy equivalence.

**Theorem 8.1.** Let \( K \) be a finite simplicial complex and \( L \) be any simplicial complex. Let \( f: |K| \to |L| \) be any continuous map. Then, for some sufficiently large \( n \), there is a simplicial map \( g: K^{(n)} \to L \) such that \( f \) is homotopic to \( |g| \).
This means that, for the purposes of homotopy theory, general continuous maps may be replaced by simplicial maps. We shall explain the proof shortly.

There are two papers, [2, 3], that start with the simplicial approximation theorem and take up where McCord and Stong leave off. In view of the explicit constructions of $\mathcal{H}(X)$ and $\mathcal{D}(K)$, the following definition is reasonable.

**Definition 8.2.** Define the (first) barycentric subdivision of a finite $T_0$-space $X$ to be $X' = \mathcal{D} \mathcal{H}(X)$. For a map $f: X \to Y$, define $f': X' \to Y'$ to be $\mathcal{D} \mathcal{H}(f)$.

Iterating the construction, define $X^{(n)} = (X^{(n-1)})'$. where $X^{(0)} = X$. Observe inductively that $\mathcal{H}(X^{(n)}) = (sK(X))^{(n)}$ since $\mathcal{H}(K) = K'$.

**Proposition 8.3.** There is a map $\zeta = \zeta_X: X' \to X$ that makes the following diagram commute, and $\zeta$ is a weak homotopy equivalence.

\[
\begin{array}{ccc}
|\mathcal{H} \mathcal{D} \mathcal{H}(X)| & \xrightarrow{\psi_{X \mathcal{D} \mathcal{H}(X)}} & |\mathcal{H}(X)'| \xrightarrow{|\xi_{X(X)}|} |\mathcal{H}(X)| \\
\psi_{X \mathcal{D} \mathcal{H}(X)} & & \psi_X \\
X' = \mathcal{D} \mathcal{H}(X) & \xrightarrow{\zeta_X} & X.
\end{array}
\]

The simplicial map $\xi_{X(X)}$ coincides with $\mathcal{H}(\zeta_X): \mathcal{H}(X') \to \mathcal{H}(X)$. For any map $f: X \to Y$, $\xi_Y \circ f' = f \circ \xi_X$.

**Proof.** The vertices of $\mathcal{D} \mathcal{H}(X)$ are the barycenters of the simplices of $\mathcal{H}(X)$. These simplices $\sigma$ are spanned by increasing sequences $x_0 < \cdots < x_n$ of elements of $X$. Let $\zeta(b_\sigma) = x_n$. Since $b_\sigma \leq b_\tau$ implies $\sigma \subset \tau$ and thus $\zeta(b_\sigma) \leq \zeta(b_\tau)$, $\zeta$ is continuous. Inspection of definitions shows that $\xi_{X(X)} = \mathcal{H}(\zeta_X)$, and the commutativity of the diagram follows from the “naturality” of $\psi$ with respect to the map $\zeta_X$. That $\zeta_X$ is a weak homotopy equivalence now follows from the diagram, since all other maps in it are weak homotopy equivalences. The last statement is clear by inspection of definitions. \qed

**Theorem 8.4.** Let $X$ and $Y$ be finite $T_0$-spaces and let $f: |\mathcal{H}(X)| \to |\mathcal{H}(Y)|$ be any continuous map. Then, for some sufficiently large $n$ there is a continuous map $g: X^{(n)} \to Y$ such that $f$ is homotopic to $|\mathcal{H}(g)|$.

**Proof.** By the classical simplicial approximation theorem above, for a sufficiently large $n$ there is a simplicial approximation

\[ j: \mathcal{H}(X^{(n-1)}) = (\mathcal{H}(X))^{(n-1)} \to \mathcal{H}(Y) \]

to $f$. Let $g = \zeta_Y \circ j$. Since $\mathcal{H}(g) = \mathcal{H}(\zeta_Y) \circ \mathcal{H}(j) = \mathcal{H}(\zeta_Y) \circ j'$ and since $|j'| \simeq |j| \simeq f$ and $|\mathcal{H}(\zeta_Y)| = |\xi_{\mathcal{H}(Y)}| \simeq \mathrm{id}$, we have $|\mathcal{H}(g)| \simeq f$, as required. \qed

The point is that finite models for spaces have far too few maps between them. For example, $\pi_n(S^n, \ast) = \mathbb{Z}$, but there are only finitely many distinct maps from any finite model for $S^n$ to itself. The theorem says that, after subdividing the domain sufficiently, we can realize any of these homotopy classes in terms of maps between (different) finite models for $S^n$.

**Sketch proof of the simplicial approximation theorem.** We are given $f: |K| \to |L|$. Give $|K|$ the open cover by the sets $f^{-1}(\text{star}(w))$, where $v$ runs over the vertices of $L$. Since $|K|$ is a compact subspace of a metric space, the “Lebesgue lemma” ensures that there is a number $\delta$ such that any subset of $|K|$ of diameter less than
δ is contained in one of the open sets \(\text{star}(w)\). The diameter of a (closed) simplex is easily seen to be the maximal length of a one-dimensional face. Each barycentric subdivision therefore has the effect of cutting the maximal diameter of a simplex in half, so that there is an \(n\) such that every simplex of \(K^{(n)}\) has diameter less than \(\delta/2\). Then each \(\text{star}(v)\) for a vertex \(v\) of \(K\) has diameter less than \(\delta\), and we conclude that \(f(\text{star}(v)) \subset \text{star}(w)\) for some vertex \(w\) of \(L\). Define \(g: V(K) \to V(L)\) by letting \(g(v) = w\) for some \(w\) such that \(f(\text{star}(v)) \subset \text{star}(g(v))\). One checks that \(g\) maps simplices to simplices and so specifies a map of simplicial complexes. If \(u\) is an interior point of a simplex \(\sigma\) of \(K\), then \(f(x)\) is an interior point of some simplex \(\tau\) of \(L\). One can check that \(g\) maps each vertex of \(\sigma\) to a vertex of \(\tau\). This implies that \(|g|\) is simplicially close to \(f\) and therefore homotopic to \(f\). □

9. Really finite \(H\)-spaces

The circle is a topological group. If we regard it as a the subspace of the complex plane consisting of points of norm one, then complex multiplication gives the product \(S^1 \times S^1 \to S^1\). How can we model such a basic structure in terms of a map of finite spaces?

Stong proved a rather amazing negative result about this problem. We will not go into the combinatorial details of his proof, contenting ourselves with the statement.

**Definition 9.1.** Let \((X, e)\) be a finite space with a basepoint \(e\). Suppose given a map \(\phi: X \times X \to X\). Say that \(X\) is an \(H\)-space of type I if multiplication by \(e\) on either the right or the left is homotopic to the identity. That is, the maps \(x \to \phi(e, x)\) and \(x \to \phi(x, e)\) are each homotopic to the identity. Say that \(X\) is an \(H\)-space of type II if the em shearing maps \(X \times X \to X \times X\) defined by sending \((x, y)\) to either \((x, \phi(x, y))\) or \((y, \phi(x, y))\) are homotopy equivalences.

A topological group is an \(H\)-space of both types, but it is much less restrictive for a space to be an \(H\)-space than for a space to be a topological group. By definition, the notion of \(H\)-space is homotopy invariant in the sense that if one defines an \(H\)-space structure on \((X, e)\) to be a homotopy class of products \(\phi\), then one has the following result.

**Proposition 9.2.** If \((X, e)\) and \((Y, f)\) are homotopy equivalent, then \(H\)-space structures on \((X, e)\) correspond bijectively to \(H\)-space structures on \((Y, f)\).

This motivates Stong to study \(H\)-space structures on minimal finite spaces. Here it is easy to see the following result.

**Proposition 9.3.** Let \((X, e)\) be a finite \(H\)-space of either type. Then the maps \(X \to X\) that send \(x\) to either \(\phi(x, e)\) or \(\phi(x, e)\) are homeomorphisms.

Examining the combinatorial relationship of general points of \(X\) to the point \(e\), Stong then arrives at the following striking conclusion.

**Proposition 9.4.** If \((X, e)\) is an \(H\)-space of either type, then the point \(e\) is both maximal and minimal under \(\leq\).

This means that \(e\) is a component of \(X\). Stong shows that this implies the following conclusions for general finite \(H\)-spaces.

**Theorem 9.5.** Let \(X\) be a finite space and let \(e \in X\). Then there is a product \(\phi\) making \((X, e)\) an \(H\)-space of type I if and only if \(e\) is a deformation retract of its
component in $X$. Therefore $X$ is an $H$-space for some basepoint $e$ if and only if some component of $X$ is contractible.

**Theorem 9.6.** Let $X$ be a finite space. Then there is a product $\phi$ making $X$ an $H$-space of type II if and only if every component of $X$ is contractible.

**Corollary 9.7.** A connected finite space $X$ is an $H$-space of either type if and only if $X$ is contractible.

So there is no way that we can model the product on $S^1$ by means of an $H$-space structure on some finite space $X$. Our standard model $\mathbb{T} = \mathbb{S}S^0$ of $S^1$ can be embedded in $\mathbb{C}$ as the four point subgroup $\{\pm 1, \pm i\}$, but then the complex multiplication is not continuous. However, the multiplication can be realized as a map $\mathbb{T} \times \mathbb{T}^{(n)} \to \mathbb{T}$ for some finite $n$, by the simplicial approximation theorem for finite spaces. It is natural to expect that some small $n$ works here. The following result is proven in [3].

**Theorem 9.8.** Choosing minimal points $e$ in $\mathbb{T}$ and $f \in \mathbb{T}'$ as basepoints, there is a map

$$\phi : \mathbb{T}' \times \mathbb{T}' \to \mathbb{T}$$

such that $\phi(f, f) = e$ and the maps $x \mapsto \phi(x, f)$ and $x \mapsto \phi(f, x)$ from $\mathbb{T}'$ to $\mathbb{T}$ are weak homotopy equivalences.

That is, we can realize a kind of $H$-space structure after barycentric subdivision. The proof is horribly unilluminating. The space $\mathbb{T}'$ has eight elements, the space $\mathbb{T}$ has four elements. One writes down an $8 \times 8$ matrix with values in $\mathbb{T}$, choosing it most carefully so that when the 8 point and 4 point spaces are given the appropriate partial order, and the 64 point product space the product order, the function represented by the matrix is order preserving. Then one checks the row and column corresponding to multiplication by the basepoint.

Several other interesting spaces and maps are modelled similarly in the cited paper, for example $\mathbb{R}P^2$ and $\mathbb{C}P^2$.

**References**