PROPER SURGERY GROUPS FOR NON-COMPACT
MANIFOLDS OF FINITE DIMENSION

by Serge Maumary*

Introduction

This work first appeared in preprint form in 1972, with the goal of "computing" the formal open surgery obstruction groups (cf. Taylor [10]) in terms of the projective Wall groups introduced by Novikov [5]. The theory turned out to be quite complicated, both algebraically and geometrically. Despite its complexity the theory plays a role in at least two beautiful classical processes:

i) The transfer process, going from a surgery problem on a manifold $M$ to one on a covering $\tilde{M}$ of $M$. A typical case arises in the study of the $L$-groups of infinite groups. For a normal map $(f,b):M \rightarrow X$ from a compact $n$-manifold $M$ to a finite $n$-dimensional Poincaré complex $X$ with $\pi_1(X) = \pi \times \mathbb{Z}$ the transfer map $t : L^n_h(\pi \times \mathbb{Z}) \rightarrow L^n_{n-1}(\pi)$ sends the finite surgery obstruction $\sigma^h_x(f,b) \in L_h^n(\pi \times \mathbb{Z})$ in the finite Wall group of Shaneson [13] to the proper surgery obstruction $\sigma^p_x(\tilde{f},\tilde{b}) \in L^p_{n-1}(\pi)$ of the covering map $(\tilde{f},\tilde{b}):\tilde{M} \rightarrow \tilde{X}$, with $\tilde{X}$ the infinite cyclic covering of $X$ such that $\pi_1(\tilde{X}) = \pi$. Note that $\tilde{M}$ is not compact and $\tilde{X}$ is not finite, and that there is a dimension shift in the proper surgery obstruction.

ii) The deleting (or removing) process, going from a problem on a compact pair $(M,K)$ to one on $M-K$ with "conditions at $\infty$" or "boundary conditions". Typical cases arise in the study of knots and singularities, especially in dimension 4 (cf. the work of Cappell-Shaneson, Casson, Freedman etc.).

*Supported by the Fonds National Suisse, credit SG58.
These notes may serve as a general framework for particular cases.

On the algebraic side, the projective L-groups \( L^P_\pi(\pi) \) appear in the analogue of the splitting theorem of Shaneson [13]

\[
L^S_n(\pi \times \mathbb{Z}) = L^S_n(\pi) \oplus L^h_{n-1}(\pi)
\]

obtained by Novikov [5] and Ranicki [14]

\[
L^h_n(\pi \times \mathbb{Z}) = L^h_n(\pi) \oplus L^P_{n-1}(\pi)
\].

Work of Bak, Carlsson, Hambleton, Kolster, Milgram and Pardon (in various combinations) has shown that the computation of the projective L-groups \( L^P_\pi(\pi) \) for finite groups \( \pi \) is easier than the computation of the finite L-groups \( L^h_\pi(\pi) \) and of the original simple L-groups \( L^S_\pi(\pi) \) of Wall [11], reducing to class group theory.

Pedersen and Ranicki [15] give a different geometric interpretation of the projective L-groups \( L^P_\pi(\pi) \), in terms of normal maps \((f,b): M \longrightarrow X\) from compact \( n \)-dimensional manifolds \( M \) to finitely dominated \( n \)-dimensional Poincaré complexes \( X \) with \( \pi_1(X) = \pi \).

A brief account of the main results of this paper may be found in Maumary [12].
Summary.

We consider non-compact connected manifolds $M$ of finite dimension, which are countable union of compact subsets, and proper maps $f$ of such manifolds ($f^{-1}$ (compact) $=$ compact). Given a proper normal map of open manifolds $f: M \rightarrow X$, we look for the obstruction to having a proper normal cobordism from $f$ to some proper homotopy equivalence at $\infty$: $f': M' \rightarrow X$ (see [9] for definition). We shall need extensively mapping cylinder constructions, which change $X$ into a properly homotopy equivalent CW-complex. So we have to study the proper homotopy invariant properties of the classical Poincaré duality in a non-compact manifold: this is taken care in Chapter I. Then we make $f$ as connected as possible at $\infty$, by doing a sequence of ordinary surgeries $\rightarrow \infty$ and carving out a sequence of properly embedded q-spheres piped to $\infty$ as in Chapter II. Then, when $m = 2q+1$, we show (Th. III, 9) that for some sequence of cocompact submanifolds $M_n \rightarrow \infty$ the intersection pairing on the boundary, induces a non-singular quadratic form $\sigma_n \in L_{2q}(\pi_1 X_n)$ on a projective quotient of a submodule of $K_q(\partial M_r)^\#$ (coefficients $\pi_1 X_n, r > n$), and that the extension $\sigma_n^\#$ of $\sigma_n$ to $\pi_1 X_{n-1}$ is canonically equivalent to $\sigma_{n-1}$. This is obtained by finding adequate cocompact subcomplexes $X_n \rightarrow \infty$ in $X$ (up to mapping cylinder constructions) and an extensive use of Poincaré duality. The case $m = 2q+2$ can be divided in two cobordisms with common boundary $U^{2q+1}$, such that for some sequence of cocompact submanifolds $U_n \rightarrow \infty$ in $U$, the intersection form on $K_q(\partial U_r)^\#$ is canonically free hyperbolic and contains a distinguished projective Lagrangian
plane $l_n \in L_{2q+1}(\pi_1 X)$ (see notations and Th. IV. 4). Moreover, there is an essentially canonical equivalence between $l_n$ and $l_{n-1}$. More precisely, we get in this way an exact sequence

$$\lim_{\rightarrow} L_m(\pi_1 X) \oplus L_{m-1}(\pi_1 X)$$

where $L_m(\pi_1 X)$ is the proper surgery obstruction group at $\infty$ and $\lim_{\rightarrow}$ is as usual the cokernel of the map $1-s: \prod_{n \geq 1} L_m(\pi_1 X_n)$ given by

$$(1-s)(a_1, a_2, a_3, \ldots) = (a_1 - a_2^#, a_2 - a_3^#, \ldots).$$

This can be globalized to the whole proper surgery group $L_m(X)$ (see e.g. [10]) as an exact sequence

$$1-s \to \prod_{m} \to L_m(\pi_1 X) \oplus \prod_{m} \to L_m(X) \to \prod_{m-1} \to L_m(\pi_1 X) \oplus \prod_{m}$$

where $\prod_{m} = \prod_{n \geq 1} L_m(\pi_1 X_n)$ and $(1-s)(a_1, a_2, a_3, \ldots) = (-a_1^#, a_1 - a_2^#, a_2 - a_3^#, \ldots)$. Observe that although the map $1-s$ is in terms of $\pi_1 X_n$ for all $n$, nevertheless, Ker($1-s$) and Coker($1-s$) only depend on the equivalence class of the system $\pi_1 X_1 + \pi_1 X_2 + \pi_1 X_3 + \ldots$. This exact sequence is the hermitian analog of a 5-terms exact sequence for $K$-theory obtained in [2] and [9]. Our method is geometric and uses a minimum of algebra (concentrated in Chapter V).

Let me thank W. Browder who encouraged me when I started this work at the I.A.S. (1969-71), Princeton. Let me thank also J. Wagoner for his helpful suggestions when I achieved this paper at U.C., Berkeley (1972). I also owe to R. Lee some useful conversations.

Berkeley
March, 1972.
Notations and conventions.

1) For connected CW-complexes, all chain and cochain complexes, homology and cohomology modules are with universal coefficients. For non-connected CW-complexes, they are direct sum over the components. # means with some understood extended coefficients.

2) Our main geometrical situation will be the mapping cylinder of a map \( f: M \to X \), with some understood subcomplexes \( X_n \) and \( M_n \subseteq X_n \). If \( M_n \equiv X_n \cap M \), \( M_n \equiv X_n \cap M \), we write \( K_k(M_n) \) for \( H_k(X_n, M_n) \), \( K_k(M_n, M_n) \) for \( H_k(X_n, X_n \cup M_n) \), \( K_k(M_n) \) for \( H_k(X_n, M_n) \) and similarly for cohomology:

![Diagram](image)

\[ / / / \text{area} \quad \text{mod} \quad \times \times \times \times \text{area} \]

One should always remember what the \( X_n \) are, as we shall have various \( X_n \) intersecting \( M \) along the same \( M_n \).

3) For cocompact subcomplex (with relatively compact complement) a square ■ will mean a compact subcomplex containing
4) All L-groups are Wall-Novikov's groups (see [5]). Namely, given a group $G$, $L_{2q}(G)$ denotes the group of equivalence classes of quadratic finitely generated projective $ZG$-modules (with the properties of the intersection pairing in a closed 2q-manifold). The null element is represented by a quadratic module $\langle P \oplus X \rangle$ such that $\langle P, P \rangle = 0$, $\langle X, X \rangle = 0$ the induced composite isomorphism $P \cong X^* \cong P^{**}$, $X \cong P^* \cong X^{**}$ being $(-1)^q$ the evaluation map. Note that the dual is taken w.r.t. the involution $g \mapsto \omega(g)g^{-1}$ of $ZG$ for some homomorphism $\omega: G \to \pm 1$. This is also called a projective $(-1)^q$-hyperbolic module, and if $P$ is free, a free $(-1)$-hyperbolic module. The opposite of a quadratic module $\langle Q \rangle$ is represented by $Q$ with the opposite form $\langle x, y \rangle' = -\langle x, y \rangle$. Now, $L_{2q+1}(G)$ denotes the group of equivalence classes of projective Lagrangian planes $\ell$ in the standard free $(-1)^q$-hyperbolic module $\langle P \oplus X \rangle$ ($\ell$ is defined as a maximal direct summand of $P \oplus X$ such that $\langle \ell, \ell \rangle = 0$). The null element is represented by a Lagrangian plane $\ell$ which takes the trivial form $\ell_p \oplus \ell_x$ ($\ell_p, \ell_x$ = direct summand of $P, X$ respectively) after some Lagrangian transformation of $\langle P \oplus X \rangle$. 

\[ X_n = / / / / \text{ area} \]
\[ \cdot \]
\[ X_n = \backslash \backslash \backslash \backslash \text{ area} \]
The latter is defined as follows: let \( \langle t \Theta H \rangle \) be a hyperbolic module \((H \cong t^*, t \cong H^*)\), where \( t \) is projective, and \( X \subseteq H, t \Phi H \) be linear maps, such that via \( H \cong t^* \), \( \phi \) becomes a \((-1)^{q+1}\) symmetric bilinear form on \( t \) (similar to the intersection pairing on a \( 2q+2 \)-manifold with boundary). A Lagrangian transformation of \( \langle P \Theta X \rangle \) is the quadratic automorphism of 
\( \langle P \Theta X \rangle + \langle t \Theta H \rangle \) defined by \((p, x, t, h) \rightarrow (pt^* t, x, t, h - x - \phi t)\)
where \( t^* P \) is the dual of \( \gamma \), and \( p \in P, x \in X, t \in t, h \in H \). Note that the Lagrangian plane \( \ell_0 \equiv P \Theta H = \{(p,0,0,h)\} \)
is left fixed, while the image of the Lagrangian plane 
\( X \Theta t = \{(0,x,t,0)\} \) is \( \ell_1 \equiv \{(pt^* t, x, t, -x - \phi t)\} \). These 
planes \( \ell_0, \ell_1 \) are considered as new "trivial" Lagrangian planes 
in \( \langle P \Theta X \rangle \Theta \langle t \Theta H \rangle \) and the Lagrangian plane \( \ell \) in \( \langle P \Theta X \rangle \)
represents \( 0 \) is \( \ell \Theta t \) takes a trivial form w.r.t. \( \ell_0 \) and 
\( \ell_1 \). The opposite of a Lagrangian plane \( \ell \) in \( \langle P \Theta X \rangle \) is 
represented by \( \ell^* \) in \( \langle P \Theta X \rangle^* \), where \( \langle \ell \Theta l^* \rangle = \langle P \Theta X \rangle \)
and \( \langle p, x \rangle^* = -\langle p, x \rangle \). When \( \ell \) is free, then \( \ell \) and \( \ell^* \) in 
\( \langle P \Theta X \rangle \) are equivalent, hence in this case (Wall groups) the 
inverse of \( \ell \) in \( \langle P \Theta X \rangle \) is also represented by \( \ell \) in \( \langle P \Theta X \rangle^* \).

5) We shall often agree to reorder a sequence of integers 
\( r_n \to \infty \) by replacing \( r_n \) by \( n \).
TABLE OF CONTENTS

Chapter I: Poincaré duality at $\infty$
Chapter II: Proper surgery at $\infty$
Chapter III: The open odd dimensional case
Chapter IV: The open even dimensional case
Chapter V: The algebra of inverse and direct systems

References
CHAPTER I. POINCARE' DUALITY AT $\infty$

1. We work with the category of connected CW-complex $X$ of finite dimension admitting a countable sequence $X_1 \supset X_2 \supset \cdots$ of subcomplexes, which is a fundamental system of ngbd of $\bigcap_{n=1}^{\infty} X_n$ is compact and $\bigcap_{n=1}^{\infty} X_n = \emptyset$. By choosing a base point in each connected component of $X_n$, we let $\tilde{X}_n$ be the union of the universal covering of each pointed component. Then the $\pi_1 X_n$-chain complex $C(X_n)$ of cellular chains of $\tilde{X}_n$ is determined. Note that $X_n$ has finitely many components.

Let $X_n$ denote any finite subcomplex of $X_n$ containing the frontier of $X_n$ in $X$. We have a relative chain complex $C(X_n, X_n)$ by taking $\tilde{X}_n$ mod the induced covering of $X_n$. Similarly we have relative chain complexes $C(X_n, X_n \cup X_r)$ for $r \geq n$, and
we define \( C^*_c(X_n, X_n) \) by \( \lim_{r \to \infty} C^*(X_n, X_n \cup X_r) \), where the dual is taken w.r.t. the anti-automorphism \( g \mapsto \omega(g)g^{-1} \) of \( Z \pi_1 X_n \), for some fixed homomorphism \( \pi_1 X \to \pm 1 \). By joining the base points in \( X_{n+1} \) to the base points in \( X_n \) (forming a tree growing in each non-compact component of \( X_1 \)), we get by excision canonical inverse systems of chain complexes \( \{C(X_n)\} \) and \( \{C^*_c(X_n, X_n)\} \) well-defined up to an obvious notion of conjugate equivalence (see Chapter V). Given an element \([X] \in \lim_{+} H_m(X, X_n; \mathbb{Z})\) (coefficients extended by \( \pi_1 X \not\to \mathbb{Z} \)), we find by excision \([X_{n,r}] \in H_m(X_n, \mathbb{Z} \cup X_r; \mathbb{Z})\). The cap products by these latter homology classes induce a morphism of inverse systems \( \{H^K_c(X_n, X_n)\} \to \{H_{m-k}(X_n)\} \) (see Chap. V and [1]). We shall say that \([X] \) is a m-fundamental class for \( X \) at infinity if \( \cap [X] \) is an equivalence of inverse systems (see Chapter V). Observe that by taking a subsequence of \( (X_n) \) one can assume to have commutative diagrams

\[
\begin{array}{ccc}
H^K_c(X_n, X_n) & \overset{\cap}{\longrightarrow} & H(X_n) \\
\uparrow \psi_{X} & & \uparrow \psi_{X} \\
H^K_c(X_{n+1}, X_{n+1}) & \overset{\cap}{\longrightarrow} & H(X_{n+1})
\end{array}
\]

2. **Lemma.** Let \( f: X \to X' \) be a proper homotopy equivalence. If \([X] \) is a m-fundamental class at \( \infty \), then so is \([X'] = f_*[X] \).

For instance, if \( X \) has the proper homotopy type of a m-manifold, then \( X \) has a m-fundamental class at \( \infty \). The proof of the lemma is clear.

3. If \( M, X \) are provided with m-fundamental classes at \( \infty \) \([M], [X] \), then we say that a proper map \( f: M \to X \) is of degree 1 if \( f_*[M] = [X] \). As \( f \) is proper we can find convergent sequences
of ngbd of $\omega$, $M_n$, $X_n$, such that $f(M_n) \subset X_n$ and choose $M_n$, $X_n$ such that $f(M_n) \subset X_n$. Then we have the modules $K_n(M_n)$ and $K_{c}(M_n, M_n) \equiv H^{k+1}_{c}(X_n, X_n \cup M_n)$ (see notations), which also form inverse systems, well-defined up to conjugate equivalence. When $M$ is a manifold, we can choose the $M_n$ to be cocompact submanifolds with boundary $\partial M_n$ = closed bicollared submanifold. By enlarging $X_n$, we can assume that $f(\partial M_n) \subset X_n$. Now we identify $X$ with the mapping cylinder of $M \times X$, so $X_n \cap M = M_n$ and $X_n \cap M = \partial M_n$.

4. Lemma. Let $M$ be an open manifold and $f: M \times X$ be a proper map of degree 1. Assume that $X_n \cap M = \partial M_n$. Then the composition $K_{m-k}(M_n) \otimes H^{m-k}(M_n) \equiv H^{k}_{c}(M_n, \partial M_n) \otimes K_{c}(M_n, M_n)$ is a canonical equivalence of inverse systems, say $\psi: \{K_{m-k}(M_n)\} \rightarrow \{K^{a}_{c}(M_n, \partial M_n)\}$.

Proof. Choose an equivalence $\psi_x: H_{m-k}(X_{n+1}) \rightarrow H^{k}_{c}(X_{n}, X_{n})$ inverse to $\cap[X]$, and let $\alpha_n$ be the composition of morphisms

$$H^{k}_{c}(M_{n+1}, M_{n+1}) \otimes[H] \stackrel{\cap[M]}{\rightarrow} H^{k}_{a}(M_{n+1}) \stackrel{f_{*}}{\rightarrow} H^{k}_{a}(X_{n+1}) \stackrel{\psi_{x}}{\rightarrow} H^{k}_{c}(X_{n}, X_{n}).$$

Then the square

$$
\begin{array}{ccc}
H^{k}_{c}(M_{n+1}, \partial M_{n+1}) & \stackrel{\alpha_{n}}{\rightarrow} & H^{k}_{c}(X_{n}, X_{n}) \\
\approx \downarrow \cap[M] & & \downarrow \cap[X] \\
H^{k}_{a}(M_{n+1}) & \stackrel{f_{*}, n}{\rightarrow} & H^{k}_{a}(X_{n+1}) \\
H^{k}_{a}(M_{n}) & \downarrow f_{*}, n & \\
\end{array}
$$

is commutative, hence provides an equivalence $\text{Ker} \alpha_{n} \rightarrow \text{Ker} f_{*}, n$. Moreover, the composition $\alpha_{n} \circ f_{*}$ is just the canonical map $1$. 
Hence the map \( \beta = 1 - f^* \circ \alpha^* : H_c^*(M_{n+2}, \mathbb{A}^M_{n+2}) \to H_c^*(M_{n+1}, \mathbb{A}^M_{n+1}) \)
induces a morphism \( \text{Coker } f_{n+2}^* \circ \text{Ker } \alpha_n^* \) which turns out to be
inverse to the morphism \( \text{Ker } \alpha_n^* \to H_c^*(M_{n+1}, \mathbb{A}^M_{n+1}) \to \text{Coker } f_{n+1}^* \),
hence is an equivalence. The composition \( \beta \psi_x \) (we skip some
obvious map) reduces to \( 1_0 \psi_x \), because \( f^* \circ \alpha^* \circ \gamma_{ol} = f^* \psi_x \circ \theta_n^* \).

Hence \( \beta \circ \beta \circ \psi : K^*_n(M_n) \to \text{Ker } f^* \) is the canonical map. But
the latter turns out to be an equivalence, by introducing the
composition of morphisms

\[
\begin{array}{cccc}
\alpha^*_{n} : H^*_c(X_{n+1}) \xrightarrow{\psi_x} H^*_c(X_n, X_n) \xrightarrow{f_n^*} H^*_c(M_n, M_n) \xrightarrow{\cap [M]} H^*_c(M_n)
\end{array}
\]

which satisfies \( f_n^* \circ \alpha^*_{n} = \text{canonical map } \) (use Chapter V). Similarly
\( \text{Coker } f_n^* \to K^*_c(M_n, \mathbb{A}_n^M) \) is an equivalence.

**Addendum:** \( \psi \) has an inverse equivalence \( K^*_c(M_n, \mathbb{A}_n^M) \to K^*_{-k}(M_n) \).

**Proof.** By using \( \alpha^* \) and \( \alpha^*_n \), check that the maps in the
kernel systems of \( K^*_n(M_n) \to \text{Ker } f^*_n \) and the cokernel system
of \( \text{Coker } f_n^* \to K^*_c(M_n, \mathbb{A}_n^M) \) vanish.

5. The above Poincaré duality has its dual counterpart.
Namely, for a proper map \( f:M \to X \) of degree 1, we have also
the module \( K^k_c(M_n) \equiv H_c^{k+1}(X_n, M_n) \) (see notations). If now \# means
with coefficients \( \pi \chi_n \), \( n \) fixed, then for \( r > n \) \( \{K^k_c(M_r)^\#\}_n \)
and \( \{K^k(M_r, M_r)^\#\}_n \) are canonical direct systems (the latter by
excision). Then the following holds.

6. **Lemma.** (Dual to lemma 4) With the above setting,
if \( M_n \equiv X_n \cap M = \mathbb{A}^M_n \), then the composition
\[ K_{m-k}(M_r, \partial M_r)^\# \to H_{m-k}(M_r, \partial M_r)^\# \cong H^k_c(M_r)^\# \to k^k_c(M_r)^\# \] is a canonical equivalence of direct systems, say \( \psi: \{ K_{m-k}(M_r, \partial M_r)^\# \}_n \to \{ k^k_c(M_r)^\# \}_n \).

**Proof.** First we show that \([X]\) induces by cap products an equivalence of direct systems \( \{ K^k_c(X_r)^\# \}_n \to \{ K_{m-k}(X_r, X_r)^\# \}_n \). The dual of the cochain complex \( C^*(X_r, X_r)^\# \) is the chain complex \( \overline{C}(X_r, X_r) \equiv \lim_{s \to} C(X_r, X_r \cup X_s)^\# \) of locally finite \( \pi_1 X_r \)-cellular chains.

Now \([X]\) comes from \( \overline{c}(X, \mathbb{Z}) \), because so does \([M]\). Then we get two morphisms \( \{ C^*(X_r)^\# \}_n \to \{ \overline{C}(X_r, X_r)^\# \}_n \), either by taking induced chain cap products \( \overline{e} \), or by dualizing the induced former chain cap products \( \{ C^*(X_r, X_r)^\# \}_n \to \{ C(X_r)^\# \}_n \). On homology level, they are the same up to sign, hence \( \{ H^*(X_r)^\# \}_n \to \{ H^*(X_r, X_r)^\# \}_n \) is an equivalence of direct systems (See V, 12). In particular, \( \lim_{r \to} H^*(X_r)^\# \cong \lim_{r \to} H^*_c(X_r, X_r)^\# \). The first member is the end cohomology \( \overline{H}^*_c(X_n) \), determined by the chain complex \( \lim_{r \to} C^*(X_r)^\# \), and the second member is say the end homology \( \overline{H}^*_c(X_n) \), determined by the chain complex \( C^c(X_n) \equiv \lim_{r \to} \overline{C}(X_r, X_r)^\# \) which is nothing but the quotient

\[ \overline{C}(X_n, X_n) \to C(X_n, X_n) \) (take \( \lim_{r \to} \lim_{s \to} \lim_{t \to} C(X_n, X_n) \to C(X_n, X_n) \to C(X_n, X_n) \to 0 \) where \( X_n-X_r \) is the subcomplex \( (X_n-X_r) \cup X_r \). Then we have an exact commutative ladder (see [1])
We have seen that the middle rung is an equivalence of direct systems, hence so is the left rung by V. 8. Now, we can dualize the proof of lemma 4 to get the assertion.

7. By taking a subsequence of \((X_n)\), we can assume to have simultaneous equivalences \(\psi_x: H_c(X_{n+1}) \rightarrow H_c(X_n; \mathbb{F}_p)\) and \(\overline{\psi}_x: H_c(X_r, X_r) \rightarrow H_c(X_{r+1})\). Then \(\psi: K_c(M_n) \rightarrow K_c(M_n; \mathbb{F}_p)\) and \(\overline{\psi}: K_c(M_r, \mathbb{F}_p) \rightarrow K_c(M_{r+1}, \mathbb{F}_p)\) have inverses \(K_c(M_n, \mathbb{F}_p) \rightarrow K_c(M_{n-1}, \mathbb{F}_p)\) and \(K_c(M_r, \mathbb{F}_p) \rightarrow K_c(M_{r+1}, \mathbb{F}_p)\). Hence, by taking again a subsequence of \((X_n)\), we can assume that \(\psi\) and \(\overline{\psi}\) have inverses \(K_c(M_n, \mathbb{F}_p) \rightarrow K_c(M_{n-1}, \mathbb{F}_p)\) and \(K_c(M_r, \mathbb{F}_p) \rightarrow K_c(M_{r+1}, \mathbb{F}_p)\). Another important observation is that the square

\[
\begin{array}{c}
\psi \\
\overline{\psi}
\end{array}
\]

\[
\begin{array}{c}
K_c(M_r, \mathbb{F}_p) \\
K_c(M_{r+1}, \mathbb{F}_p)
\end{array}
\]

is commutative. Now, let \(\mathcal{C}(X_r, M_r)\) be the chain complex dual to \(C_c(X_r, M_r)\), and \(K_r(M_r)\) its k+1-homology. We have a canonical
map $\overline{K}_k(M_r)^\# + (K^c_k(M_r)^\#)^*$, hence, by composition with the dual of $\overline{\psi}$, a map $\overline{\psi}^*: \overline{K}_k(M_r)^\# \to K^{m-k}(M_r, \partial M_r)^\#$, which is a morphism of inverse systems. Similarly, the dual of $\psi$ provides a morphism of direct systems $\overline{\psi}^*: \overline{K}_k(M_r, \partial M_r)^\# \to K^{m-k}(M_r)^\#$. By taking the direct limit of the latter for $r \to \infty$, we get $K^e_k(M_r, \partial M_r)^\# \to K^{m-k}_e(M_r)$.

The exact ladders

\[
\begin{array}{cccccccc}
\longrightarrow & K^{m-k}(M_r)^\# & \longrightarrow & K^e_k(M_r)^\# & \longrightarrow & K^{m-k+1}_c(M_r)^\# & \longrightarrow \\
\uparrow \psi^* & \uparrow \lim \psi^* & \uparrow & \overline{\psi} & \\
\longrightarrow & \overline{K}_k(M_r, \partial M_r)^\# & \longrightarrow & K^e_k(M_r, \partial M_r)^\# & \longrightarrow & K^{m-k+1}_k(M_r, \partial M_r)^\# & \longrightarrow \\
\end{array}
\]

and

\[
\begin{array}{cccccccc}
\longrightarrow & K^{m-k}(M_r, \partial M_r)^\# & \longrightarrow & K^e_k(M_r, \partial M_r)^\# & \longrightarrow & K^{m-k+1}_c(M_r, \partial M_r)^\# & \longrightarrow \\
\uparrow \overline{\psi}^* & \uparrow \lim \psi^* & \uparrow & \psi & \\
\longrightarrow & \overline{K}_k(M_r)^\# & \longrightarrow & K^e_k(M_r)^\# & \longrightarrow & K^{m-k+1}_k(M_r)^\# & \longrightarrow \\
\end{array}
\]

where $K^e_k(M_r)^\# \equiv K^e_k(M_r, \partial M_r)^\#$ and $K^{m-k}_e(M_r)^\# \equiv K^{m-k}_e(M_r, \partial M_r)^\#$ by definition of $H^*_e$ and $H^*_k$, are $+$ commutative. In general one knows nothing about $\psi^*$ and $\overline{\psi}^*$. 
CHAPTER II. PROPER SURGERY AT $\infty$

The data is a proper normal map $f: M \to X$ of degree 1, where $M$ is a smooth open (oriented) m-manifold and $X$ a complex with fundamental class $[X] = f_*[M]$ at $\infty$. Of course, "normal" means as in [1] that for some stable vector bundle $\zeta$ over $X$, $f$ is covered by a map $\nu + \zeta$, where $\nu$ is the stable normal bundle of $M$ in euclidian space. A cobordism of such a data is the obvious thing (see III, 9 and IV, 3), and we look for the obstruction for $f$ to be cobordant to a proper map $f': M' \to X$ such that

i) $f'$ induces a bijection of ends spaces

ii) the morphism $f_*': \{\pi_1 M'_n\} \to \{\pi_1 X'_n\}$ of inverse systems of groups is an equivalence

iii) all inverse systems $\{K_k(M'_n)\}$ are equivalent to 0.

Geometrically, this means that $f'$ is a proper homotopy equivalence at $\infty$ (see [9]).

Recall first (see [11]) that, if $f: M \to X$ maps a bicollared closed submanifold $\bullet M_1$ of $M$ into a finite subcomplex $X_1$ of $X$, then the restriction $\bullet f: \bullet M_1 \to X_1$ is normal, and every surgery on it extends to a surgery of $f$:

\[
\begin{array}{c}
\bullet M_1 \times I \\
\downarrow \quad \downarrow \\
M \quad \partial M_1
\end{array}
\]
By doing this on a divergent sequence \( \mathbf{M}_n + \mathbf{X}_n \) we get obviously a cobordism of \( f: M \times X \). If \( \mathbf{M}_n + \mathbf{X}_n \) and \( \mathbf{M}_{n+1} + \mathbf{X}_{n+1} \) bound a restriction \( \mathbf{M}_n - \mathbf{M}_{n+1} + \mathbf{X}_n - \mathbf{X}_{n+1} \) then every surgery on the latter rel. \( \mathcal{A} \mathbf{M}_n \cup \mathcal{A} \mathbf{M}_{n+1} \) extends also to a surgery of \( f \).

\[
\begin{array}{cc}
\mathbf{M} & \\
\mathcal{A} \mathbf{M}_n & \mathcal{S}_n & \mathcal{A} \mathbf{M}_{n+1}
\end{array}
\]

By doing this for each \( n \), we get also a cobordism of \( f \).

We consider still another particular kind of surgery. Suppose we have a proper embedding \( \phi: \mathbb{R}^q \to M \) and a proper extension \( \psi: \mathbb{R}^{q+1}_+ \to X \) of \( f \circ \phi \), where \( \mathbb{R}^{q+1}_+ = \mathbb{R}^q \times [1, \infty) \). Then, if \( E \) is a tubular nbhd of \( \phi(\mathbb{R}^q) \) in \( M \), we have a trivialization \( E \cong \mathbb{R}^q \times D^{m-q} \) (by contracting \( \mathbb{R}^q \) into 0). Similarly, we have a trivialization of \( \phi^*v \) which extends to a trivialization of \( \psi^*v \). Hence we can make a cobordism on \( f \) by gluing \( \mathbb{R}^{q+1}_+ \times D^{m-q} \) to \( M \times I \) along \( E \):

\[
\begin{array}{c}
\mathbb{R}^{q+1}_+ \times D^{m-q} \\
\downarrow \\
E \\
\mathbb{R}^{q+1}_+ \times D^{m-q} \end{array}
\]

and mapping the resulting \( (m+1) \)-manifold \( W \) to \( X \times I \) by

\[
P = \begin{cases} 
  f \times \text{id.} \text{ on } M \times I \\
  \mathbb{R}^{q+1}_+ \times D^{m-q} \xrightarrow{\text{proj.}} \mathbb{R}^{q+1}_+ \xrightarrow{\psi} X \text{ on } \mathbb{R}^{q+1}_+ \times D^{m-q}
\end{cases}
\]
$W$ is a cobordism from $M$ to $M' \equiv M - \phi(\mathbb{R}^q)$, and both inclusions $M + W + M' \cup D^{m-q}$ are homotopy equivalences ($D^{m-q}$ is a fiber of $E$). Observe that $W$ is constructed from $M'$ by attaching first a $(m-q)$-handle along a framed sphere transverse to $\phi(\mathbb{R}^q)$ and then carving out $\phi(\mathbb{R}^{m-q})$ in the result, i.e., attaching $(\phi'(\mathbb{R}^{m-q}) \times D^q) \times \mathbb{R}_+$.

If $M_1 \supset M_2 \supset \ldots$ is a fundamental system of nbhd of $\infty$ in $M$, then $W_n \equiv (M_n \times I) \cup \left( \mathbb{R}_+^{q+1} - nD_+^{q+1} \times D^{m-q} \right)$ is such a system in $W$, where $nD_+^{q+1}$ is the half disc of radius $n$ in $\mathbb{R}_+^{q+1}$. If $E \subset M_k$ but $E \not\subset M_{k+1}$, it is more convenient to replace above $nD_+^{q+1}$ by $(n-k)D_+^{q+1}$. Then

$$M'_n \equiv W_n \cap M' = \begin{cases} M_n - \phi(\mathbb{R}^q) & \text{for } n \leq k \\ (M_n \cup q\text{-handle}) - \phi(\mathbb{R}^q) & \text{for } n > k. \end{cases}$$
This implies that for \( q \geq 2 \) and \( m - q \geq 2 \) the ends spaces of 
\( M \) and \( M' \) are the same. Moreover, for \( q \geq 3 \) and \( m - q \geq 3 \),
\[ \pi_1 M_n \cong \pi_1 M'_n. \]

We shall only use this kind of surgery in the case where
\( (\phi, \psi) \) comes from an embedding \( S^q \phi' \hookrightarrow M \) and an extension
\( D^{q+1} \psi' \times X \) of \( f \circ \phi' \), by piping \( \phi(S^q) \) to \( \infty \) along a proper
embedding \([0, \infty) \to M\):
In this case, $\pi_1 M_n = \pi_1 M'_n$ already for $q \geq 2, m-q \geq 3$:

$$W_n = [0,1] \times M_n \cup_{1 \times C_n^q \times D_n^{n-q}} (1,\infty) \times C_n^q \times D_n^{n-q} \cup_{(n,\infty) \times D_n^q \times D_n^{n-q}} \hat{W}_n = [0,1] \times \hat{M}_n \cup_{1 \times \hat{D}_n^q \times D_n^{n-q}} [1,\infty) \times \hat{D}_n^q \times D_n^{n-q} \cup_{n \times D_n^q \times D_n^{n-q}} n \times D_n^q \times D_n^{n-q}.$$
CHAPTER III. THE OPEN ODD DIMENSIONAL CASE

1. Let $M$ be an open manifold of dimension $2q+1 \geq 7$, and $f: M \to X$ be a proper normal map of degree 1. We assume that $X$ is connected, and choose a sequence of cocompact subcomplexes $X_n \to \infty$ in $X$, such that $X_n$ has only non-compact components. Moreover, we can choose finite subcomplexes $\bullet X_n$ of $X_n$ containing the frontier such that $\bullet X_n \cap \text{(component of } X_n\text{)}$ is connected. If $Y_n = (X-X_n) \cup X_n$, then by replacing $X$ by $(X_n \times I) \cup (Y_n \times I)$ as follows:

\[
\begin{array}{c}
\begin{array}{c}
\bullet X_n \times I \\
Y_n \times I
\end{array}
\end{array}
\xrightarrow{\begin{array}{c}
\begin{array}{c}
\bullet X_n \\
X_n \times I
\end{array}
\end{array}}
\begin{array}{c}
\begin{array}{c}
\bullet X_n \\
X_n \times I
\end{array}
\end{array}
\]

we can assume that each $\bullet X_n$ is bicollared in $X$. Putting $f$ transverse on each $\bullet X_n$, we get submanifolds $M_n = f^{-1}(X_n)$ with boundary $\partial M_n = f^{-1}(\bullet X_n)$, such that $M_n \to \infty$. After surgering each map $\partial M_n \cong X_n$ and $\overline{M_n-M_{n+1}} \to \overline{X_n-X_{n+1}}$, we can assume that they are $q$-connected. In particular, $f$ is bijective on ends spaces, by interpreting an end of $X$ as a function $\{X_n\} \not\in \{\pi_0 X_n\}$ such that $\varepsilon(X_{n+1}) \subseteq \varepsilon(X_n)$ and similarly for $M$. By van Kampen and Mayer-Vietoris, each map $M_n \to X_n$ is $q$-connected. Now, in the homotopy exact sequence

\[
\pi_{q+1}(\overline{X_{k'-k+1}}, \overline{M_{k'-k+1}}) \to \pi_{q+1}(\overline{X_{k'-k+1}}, \overline{X_{k'+k+1} \cup M_{k'-k+1}}) \to \pi_{q+1}(\overline{X_{k'+k+1}}, \overline{M_{k'+k+1}})
\]
the last term vanishes by Hurewicz isomorphism and excision. The middle term is finitely generated because by the Hurewicz isomorphism it is the lowest homology of a finite complex. Hence each generator is represented by a map \((D^{q+1}, S^q) \to (X_k - X_{k+1}, M_k - M_{k+1})\), and moreover, \(S^q\) can be embedded by general position. We can pipe the image of \(S^q\) to \(\infty\) by a proper embedded pipe line.

![Diagram](image)

to get a proper map \((\mathbb{R}^{q+1}, \mathbb{R}^q) \to (X_k, M_k)\) which is an embedding on \(\mathbb{R}^q\). Let us do surgery on this map, as in Chapter III. In the diagramm

\[
\begin{array}{c}
\mathcal{M}_n \\
\cap \\
\circ W_n \xrightarrow{F} X_n \\
\cup \\
\circ \mathcal{M}_n'
\end{array}
\]

\((W_n = \text{frontier})\)

we have

\[
\begin{cases}
W_n = \mathcal{M}_n \times I & \text{if } n < k \\
(\mathcal{M}_n \lor S^q) \sim W_n \sim (\mathcal{M}_n' \cup D^{q+1}) & \text{if } n \geq k
\end{cases}
\]
(we can assume that \( \phi(D^q) \) meets \( \partial M_n \) along the sphere of radius \( n-k \), when \( n > k \)). Hence the maps \( W_n \rightarrow X_n \) and \( \partial M_n \rightarrow X_n \) are also \( q \)-connected, for any \( n \). In the diagram

\[
\begin{array}{ccc}
M_n & \xrightarrow{f} & X_n \\
\downarrow \text{incl} & & \downarrow \\
W_n & \xrightarrow{F} & X_n \\
\downarrow \text{incl} & & \\
M_n & \xrightarrow{f'} & \\
\end{array}
\]

\( W_n \) has the homotopy type of \( M_n \) and \( M_n \cup D^{q+1} \) for \( n < k \), and of \( M_n \vee S^q \) and \( M_n \cup D^{q+1} \) for \( n \geq k \). Hence \( F_n \) and \( f'_n \) are also \( q \)-connected. Now, if we write \( K_q(M_k',M_{k+1}') = H_{q+1}(X_k',X_{k+1} \cup M_k') \) in the mapping cylinder of \( f' \), we have

\[
K_q(M_n',M_{n+1}') = \begin{cases}
0 & \text{for } n = k \\
K_q(M_n,M_{n+1}) & \text{for } n \neq k,
\end{cases}
\]

as easily verified. By induction on \( k \), we can assume that \( K_q(M_n,M_{n+1}) = 0 \) for each \( n \). An immediate consequence is that \( K^q_c(M_n) = 0 \), hence the direct system \( \{K_{q+1}(M_r,\partial M_r)\} \) is equivalent to \( \{0\} \), by duality. Another consequence is that \( \tilde{K}_q(M_n) \equiv \tilde{H}_{q+1}(X_n,M_n) \) vanishes: because \( K_k(M_n) = 0 \) for \( k < q \), each \( n \), one can eliminate by Whitehead's trick (see [6]) all cells of dimension \( \leq q \) in \( X_n - M_n \) and this by a proper (simple) homotopy equivalence of \( X \) rel. \( M \). Moreover, for each \((n,r)\), separately, one can also eliminate the \( q+1 \)-cells in \( X_n - (M_n \cup X_r) \), because \( K_q(M_n,M_r) = 0 \). Hence each chain complex
C(X_n, M_n U X_r) has the chain homotopy type of one chain complex C(n,r) which vanishes in dimensions ≤q+1. Moreover, we can get commutative squares

\[
\begin{array}{ccc}
C(X_n, M_n U X_r) & \xrightarrow{\text{h.e.}} & C(n,r) \\
\uparrow & & \uparrow \\
C(X_n, M_n U X_{r+1}) & \xrightarrow{\text{h.e.}} & C(n,r+1)
\end{array}
\]

as follows: having eliminated in X_n - (M_n U X_r), getting X'_n, we choose the elimination in X_n - (M_n U X_{r+1}) by first eliminating in X_r - (M_r U X_{r+1}) getting X''_r, and then extending this formal deformation to X'_n, getting X''_n. This provides the required commutative diagram

\[
\begin{array}{ccc}
C(X_n, M_n U X_r) & \xrightarrow{\text{can.}} & C(X'_n, M_n U X_r) \cong C(n,r) \\
\downarrow \cong \downarrow \text{can.} & & \downarrow \text{can.} \\
C(X_n, M_n U X_{r+1}) & \xrightarrow{\text{can.}} & C(X''_n, M_n U X_{r+1}) \cong C(n,r+1)
\end{array}
\]

Now, the chain mapping cone of each homotopy equivalence $C(X_n, M_n U X_r) \to C(n,r)$ is free acyclic, and for $n$ fixed, $r$ variable, they form an induced inverse system. Because each cone splits completely, their inverse limit is an acyclic chain complex, which is nothing but the chain mapping cone of $\lim_{r} C(X_n, M_n U X_r) \to \lim_{r} C(n,r)$, hence the latter map is a homology isomorphism. This proves that the $(q+1)$-dimensional homology
of \( \lim_{r} C(X_{n}, M_{n} \cup X_{r}) \) vanishes, i.e. \( K_{q}(M_{n}) = 0 \). By duality, this implies that the inverse system \( \{K_{q+1}^{+1}(M_{n}, \mathcal{M}_{n})\} \) is equivalent to \( \{0\} \).

2. **Proposition:** The inverse system \( \{K_{q}(M_{n})\} \) and the direct system \( \{K_{q}(M_{r}, \mathcal{M}_{r})^\#\}_{n} \) (in the latter \# means with \( \pi_{1}M_{n} \) coefficients, for \( n \) fixed) are equivalent to systems of projective countably generated modules.

**Proof.** By using the duality equivalence, one has to prove the same assertion for \( \{K_{q+1}^{+1}(M_{n}, \mathcal{M}_{n})\} \) and \( \{K_{q+1}(M_{r})^\#\} \). As above, we can assume that \( X_{n} \to M_{n} \) contains no cells of dimension \( \leq q \), for each \( n \). Moreover, for each \( (n,r) \) separately, one can eliminate the \((q+1)\)-cells in \( X_{n} - (X_{n} \cup M_{n} \cup X_{r}) \), because

\[
K_{q}(M_{n}, \mathcal{M}_{n} \cup M_{r}) \cong H_{q+1}(X_{n}, X_{n} \cup M_{n} \cup X_{r}) = 0
\]

in virtue of the homology exact sequence

\[
0 \to K_{q}(M_{n}, M_{r}) \to K_{q}(M_{n}, \mathcal{M}_{n} \cup M_{r}) \to K_{q-1}(\mathcal{M}_{n})^\#
\]

Hence each chain complex \( C(X_{n}, X_{n} \cup M_{n} \cup X_{r}) \) has the chain homotopy type of one chain complex \( C(n,r) \) say, which vanishes in dimension \( \leq q+1 \). Moreover, we can get commutative diagram

\[
\begin{array}{ccc}
C(X_{n}, X_{n} \cup M_{n} \cup X_{r}) & \xrightarrow{h.e.} & C(n,r) \\
\uparrow & & \uparrow \\
C(X_{n}, X_{n} \cup M_{n} \cup X_{r+1}) & \xrightarrow{h.e.} & C(n,r+1)
\end{array}
\]

as follows: having eliminated in \( X_{n} - (X_{n} \cup M_{n} \cup X_{r}) \) getting \( X_{n}' \), choose the elimination in \( X_{n} - (X_{n} \cup M_{n} \cup X_{r+1}) \) by first eliminating in \( X_{r} - (M_{r} \cup X_{r+1}) \) getting \( X_{r}' \), and then extending
this formal deformation to $X_n'$, getting $X_n''$. This provides the required commutative diagram

$$
\begin{array}{c}
C(X_n, X_n \cup M_n \cup X_r) \longrightarrow C(X_n', X_n \cup M_n \cup X_r) \cong C(n,r) \\
\downarrow \cong \\
C(X_n'', X_n \cup M_n \cup X_r') \longrightarrow C(X_n', X_n \cup M_n \cup X_r') \\
\uparrow \text{can.} \\
C(X_n, X_n \cup M_n \cup X_{r+1}) \longrightarrow C(X_n', X_n \cup M_n \cup X_{r+1}) \cong C(n,r+1)
\end{array}
$$

If $C^*(n) \cong \lim C^*(n,r)$, we then have a chain map $C_c^*(X_n, X_n \cup M_n) \rightarrow C^*(n)$ which is a homology isomorphism. But the above maps $C(n,r+1) \rightarrow C(n,r)$ are such that $C^*(n)$ is free of countable rank (up to isomorphism of $C(n,r)$, they are cellular embeddings), and $C_c^*(X_n, X_n \cup M_n)$ is also free. So actually the map $C_c^*(X_n, X_n \cup M_n) \rightarrow C^*(n)$ is a chain homotopy equivalence. Using homotopy inverse maps, we get an inverse system $C^*(n+1) \rightarrow C^*(n)$ whose associated homology systems are isomorphic to $\{K_c^*(M_n, \partial M_n)\}$ (although the diagram

$$
\begin{array}{c}
C_c^*(X_n, X_n \cup M_n) \longrightarrow C^*(n) \\
\downarrow \text{can.} \\
C_c^*(X_{n+1}, X_{n+1} \cup M_{n+1}) \longrightarrow C^*(n+1)
\end{array}
$$

is only chain homotopy commutative). Hence Prop. V, 9 applies to $\{C^*(n)\}$, proving the assertion for $\{K_q(M_n)\}$. For the other system, the proof is similar.
3. We are now at the point where we cannot do further surgeries, but we can still work on the subcomplexes \( (X_n, X'_n) \) to improve the canonical square

\[
\begin{array}{ccc}
K^{q+1}(M_r, \alpha M_r) & \xrightarrow{\psi} & K^{q+1}(M_r) \\
\uparrow & & \uparrow \\
K_q(M_r) & \rightarrow & K_q(M_r, \alpha M_r)
\end{array}
\]

that we have so far.

4. Lemma. Ker \( \psi \) and Ker \( \bar{\psi} \) are finitely generated.

Proof. In the proof of Proposition 2 above, we have shown that \( \{K^{q+1}_c(M_r, \alpha M_r)\}_n \) is the top homology system associated to some system of free chain complexes \( \{C(r)\} \) (this is not so for \( C^*_c(X_r, \alpha X_r \cup M_r) \) as \( C^{q+1}_c(X_r, \alpha X_r \cup M_r) \neq 0 \)). Then V. 10 applies to \( \{C(r)\} \), giving an equivalence \( \{K^{q+1}_c(M_r, \alpha M_r)\} \cong \{P_r\} \) which is injective for each \( r \), where each \( P_r \) is projective, as well as the image \( P'_r \) of \( P_{r+2} \). Hence the composition

\[
K_q(M_r) \xrightarrow{\psi} K^{q+1}_c(M_r, \alpha M_r) \xrightarrow{\psi} P_r
\]

has kernel equal to Ker \( \psi \). Moreover, its image is \( P'_r \), because \( \psi \) and the injection into \( P_r \) are both equivalences, hence we have the commutative diagram

\[
\begin{array}{ccc}
K^q(M_r) & \xrightarrow{\psi} & P_r \\
\downarrow \text{surj.} & & \downarrow \text{surj.} \\
K_q(M_{r+2}) & \xrightarrow{\psi} & P_{r+2}
\end{array}
\]
Then the exact sequence \( 0 \to \text{Ker } \psi_r \to K_q(M_r)^\# \to P'_r \to 0 \) splits. In particular, \( \text{ker } \psi_r \) is a retract of \( K_q(M_r)^\# \). But the commutative triangle

\[
\begin{array}{c}
K_q(M_r)^\# \\
\downarrow \psi \\
K_{q+1}(M_r,\partial M_r)^\# \\
\end{array} \quad \begin{array}{c}
\to \\
\uparrow \\
\to \\
\end{array} \quad \begin{array}{c}
K_q(M_{r-1})^\# \\
\end{array}
\]

shows that \( \text{ker } \psi_r \subseteq \text{ker } \iota = \partial K_{q+1}(M_{r-1},M_r)^\# \). As \( K_{q+1}(M_{r-1},M_r)^\# \) is a finitely generated module, so is \( \text{Ker } \iota \).

But \( \text{ker } \psi_r \) becomes a retract of \( \text{Ker } \iota \), hence is also finitely generated. The same argument applies to \( \text{ker } \overline{\psi} \).

\underline{Remarks 1}. If above we knew that \( \psi \) was already injective, then \( K_q(M_r)^\# \) is isomorphic to \( P'_r \), hence is projective. Moreover, by V. 10, one can assume that \( P'_r \) is a direct summand of \( P_r \), hence \( \psi \) splits. Similarly for \( \overline{\psi} \).

\underline{2}. We have shown that for each \( n \), there is some \( r_n > n \) such that \( K_q(M_{r_n})^\# \psi K_{q+1}(M_{r_n},\partial M_{r_n})^\# \) and

\[
K_q(M_{r_n},\partial M_{r_n})^\# \overline{\psi} K_{q+1}(M_{r_n})^\# (\pi L_n - \text{coefficients}) \text{ have finitely generated kernels. Up to taking a subsequence, one can assume that } r_n = n + 1.
\]
5. **Main Lemma.** By enlarging $X_{n+1}$ inside $X_n$, and $X_{n+1}$ inside $X_n - X_{n+2}$, one can get commutative squares

$$
\begin{array}{cccc}
\longrightarrow & K_{q+1}^c(M_n, M_n) & \longrightarrow & K_{q+1}^c(M_n) \\
\uparrow & \psi & \uparrow & \bar{\psi} \\
K_q(M_n) & \longrightarrow & K_q(M_n, M_n) \\
\end{array}
$$

where $K_q(M_n)$ and $K_q(M_n, M_n)$ are projective (countably generated) and $\psi$ is bijective.

**Proof.** Our starting situation is as in §3

$$
X \left\{ \begin{array}{ccc}
& X_n & \\
M & \downarrow & \\
& X_n & \\
\end{array} \right. \\
\begin{array}{ccc}
& X_n & \\
M_n & \downarrow & M_{n+1} \\
& X_n & \\
\end{array} \\
\begin{array}{ccc}
& X_n & \\
M_{n+1} & \downarrow & M_{n+2} \\
& X_n & \\
\end{array}
$$

$X_n \cap M = M_n$, $X_n \cap M = \emptyset M_n$, a square

$$
\begin{array}{cccc}
\longrightarrow & K_{q+1}^c(M_n, \emptyset M_n) & \longrightarrow & K_{q+1}^c(M_n) \\
\uparrow & \psi & \uparrow & \bar{\psi} \\
K_q(M_n) & \longrightarrow & K_q(M_n, \emptyset M_n) \\
\end{array}
$$
and inverses $k^{q+1}_c(M_n, M_n) \to k^q(M_{n-1})$, for $\psi$, and $k^{q+1}_c(M_n) \to k^q(M_{n+1}, M_{n+1})$ for $\overline{\psi}$. Choose new $X'_n, M'_n$ as follows

$$X'_n \equiv X_{n+1} \cup M_n \quad X'_n \equiv X_{n+1} \cup \overline{M_n}$$

Then we get a new square

$$
\begin{array}{c}
K^{q+1}_c(M_n, M_n) \quad \longrightarrow \quad K^{q+1}_c(M_n) \\
\uparrow \psi' \quad \quad \quad \quad \quad \uparrow \psi \\
\downarrow \quad \quad \quad \quad \quad \quad \downarrow \\
K_q(M_n) \quad \longrightarrow \quad K_q(M_n, M_n)
\end{array}
$$

by taking the old square for $n+1$, with $\pi_nX_n$-coefficients. By §4, ker $\psi'$ is finitely generated. Each generator can be represented by a map $(D^{q+1}, S^q) \to (X'_n, X'_n \cup M_n)$, by Hurewicz. But the inverse of $\overline{\psi}$ shows that $\alpha$ represents 0 in $K_q(M_{n+1}, M_{n+1})$, i.e. $\alpha$ can be deformed into $X_{n+1} \to X_{n+2} \cup M_n$.

By mapping cylinder constructions, one can assume that $\alpha$ is the characteristic map of a cell $e^{q+1}$ in $X_{n+1} \to X_{n+2}$, attached to $X'_n \cup \overline{M_n} \to X'_n$ for some $r$ large enough (good for a finite
set of generators of $\operatorname{Ker} \tilde{\psi}'$). Choose new $X''_n, X''_n$ as follows

$$X''_n \equiv X'_n \quad X''_n = X'_n \cup \overline{M_n - M_r} \cup e^{q+1}. $$

By passing to the quotient, $\tilde{\psi}'$ induces now injections

$\tilde{\psi}'' : \operatorname{K}_q(M_n, M_n)'' \rightarrow \operatorname{K}_c^{q+1}(M_n)''$. This is still an equivalence with

inverse $\operatorname{K}_c^{q+1}(M_n)'' \rightarrow \operatorname{K}_q(M_{n+1}, M_{n+1})''$. Consider the diagram of exact sequences

$$
\begin{array}{cccccc}
0 & \longrightarrow & \operatorname{K}_c^q(M_n, M_n)'' & \longrightarrow & \operatorname{K}_e^q(M_n, M_n)'' & \longrightarrow & \operatorname{K}_c^{q+1}(M_n, M_n)'' \\
& & \uparrow \tilde{\psi}'' \ast & & \uparrow \lim \psi \ast & & \\
& \longrightarrow & \operatorname{K}_q^{q+1}(M_n)'' & \longrightarrow & \operatorname{K}_q^e(M_n)'' & \longrightarrow & 0
\end{array}
$$

where $\operatorname{K}_e^q(M_n, M_n)'' \equiv \operatorname{K}_e^q(M_n)$, $\operatorname{K}_e^{q+1}(M_n)'' \equiv \operatorname{K}_e^{q+1}(M_n, M_n)$ by definition of $\operatorname{K}_e^\ast$ and $\operatorname{K}_e^\ast$ (see I.7). Claim: $\psi^\ast$ and $\tilde{\psi}''$ are equivalences.

In fact, by using the proof of V.6, the canonical map

$\overline{K}_q^{q+1}(M_n, \mathfrak{A}_M, \mathfrak{A}_M) \rightarrow (\operatorname{K}_c^{q+1}(M_n, \mathfrak{A}_M))^\ast$ is an equivalence, and by V. 9, the dual of $\psi$ is an equivalence hence so is the composed map $\psi^\ast$.

This implies that $\lim \psi^\ast$ is an isomorphism. Similarly, $\tilde{\psi}''^\ast$
is an equivalence. We get an induced equivalence

\[ \psi^\# : K_q(M_n) \to K^q(M_n) \] (apply V.5) and a commutative square

\[
\begin{array}{ccc}
K^q(M_n) & \to & K^q(M_n) \\
\psi^\# & \downarrow & \psi^\# \text{ (injective)} \\
K_q(M_n) & \to & K_q(M_n) \\
\end{array}
\]

Observe that \( K^q(M_n) \xrightarrow{\text{restr.}} K^q(M_n) \to K_q(M_n-1) = K_q(M_n) \) is an inverse for \( \psi^\# \), because the square

\[
\begin{array}{ccc}
K^q(M_n) & \to & K^q(M_n) \\
\psi^\# & \downarrow & \psi' \\
K_q(M_n) & \to & K_q(M_n) \\
\end{array}
\]

is commutative. Now, by §4, ker \( \psi^\# \) is finitely generated.

As \( \psi^\# \) is injective, it is certainly contained in the image of \( K_q(M_n) \# \), but also in the image of \( K^q(M_n-1) \) in virtue of the inverse \( K^q(M_n) + K_q(M_n-1) \) for \( \psi^\# \). By Hurewicz and mapping cylinder construction each generator of ker \( \psi^\# \) can be represented by a cell \( e^{q+1} \) in \( X_n \) attached to \( M \), which is the boundary mod \( M \) of a cell \( e^{q+2} \) in \( X_{n-1} \). Choose new \( X_n', X_n'' \) as follows
\[ X''_n = X'_{n+1} \cup \overline{M}_n \cup e^{q+2}, \quad X'' = X'_{n+1} \cup \overline{M}_n \cup e^{q+2}. \]

By passing to quotient, \( \psi'' \) induces an injective equivalence
\[ \psi'' : K_q(M_n)'' \to K_{q+1}^{q+1}(M_n, \overline{M}_n)''. \]

Then again the duality argument used in the previous step provides an equivalence
\[ \bar{\psi}'' : K_{q}(M_n, \overline{M}_n)'' \to K_{q+1}^{q+1}(M_n)'' . \]

Claim: \( K_q(M_n)'' \) and \( K_q(M_n, \overline{M}_n)'' \) are projective. In fact, as \( \psi'' \) and \( \bar{\psi}'' \) are injective,
\( K_q(M_n)'' \) and \( K_q(M_n, \overline{M}_n)'' \) are projective, by §4. But
\[ K_q(M_n, \overline{M}_n)'' \cong K_q(M_{n+1}, \overline{M}_{n+1})'' \] by excision. Then the exact sequence
\[ K_q(M_n)'' \to K_q (M_n)'' \to K_q (M_n, \overline{M}_n)'' \to 0 \]

implies that the image of the first map is projective, hence its kernel is a retract of \( K_q(M_n)'' \), in particular, finitely generated. By Hurewicz and mapping cylinder constructions, each generator of this kernel can be represented by a cell \( e^{q+1} \) in \( X''_n \) attached to \( M_n \), which is the boundary mod \( M \) of a cell \( e^{q+2} \) in \( X'' \) (meeting \( X'_{n+1} \) only along \( M \)). Choose new \( X''_n, X'_{n+1} \) as follows.
$\mathbf{X}_n \equiv \mathbf{X}_n''$, $\mathbf{X}_n' \equiv \mathbf{X}_n'' \cup e^{q+2}$. We get the same $K$-groups as before, except that we have an injective restriction

\[ K_c^{q+1}(M_n, M_n)'' \rightarrow K_c^{q+1}(M_n, M_n)''. \]

But \( \psi' \) factors through this injection, because so does

\[ K_c(M_s)'' \xrightarrow{\text{surj.}} K_c^{q+1}(M_s, M_s)''. \]

Restricting to $K_c^{q+1}(M_n, M_n)''. \]

for large $s$. So we get a final square

\[
\begin{array}{ccc}
K_c^{q+1}(M_n, M_n)'' & \rightarrow & K_c^{q+1}(M_n)'' \\
\uparrow \psi' & & \uparrow \overline{\psi}' \\
K_q(M_n)'' & \rightarrow & K_q(M_n, M_n)''
\end{array}
\]

where $\psi'$ is injective. **Claim:** $\psi'$ is also surjective. As $\psi$ is an equivalence, it suffices to show that the maps

\[ K_c^{q+1}(M_{n+1}, M_{n+1})'' \rightarrow K_c^{q+1}(M_n, M_n)'' \]

are surjective. In the exact sequence

\[
\begin{array}{ccc}
K_{q+1}(M_n)'' & \rightarrow & K_{q+1}(M_n, M_n)'' \\
\rightarrow & & \delta \\
K_q(M_n)'' & \rightarrow & K_q(M_n, M_n)''
\end{array}
\]

the first map vanishes, because the inverse system $\{K_{q+1}(M_n)''\}$ is equivalent to 0, and we have commutative squares
This implies $K_{q+1}(M, M) = 0$, hence also $K_{q+1}^{\ast}(M, M) = 0$

because $K_{q}(M, M)$ is projective. Then we have the exact sequence

$$0 \rightarrow K_{q}^{\ast}(M, M) \rightarrow K_{e}^{\ast}(M, M) \rightarrow K_{c}^{\ast}(M, M) \rightarrow 0.$$  

This implies that $K_{c}^{q+1}(M+1, M+1) \rightarrow K_{c}^{q+1}(M, M)$ is surjective, because $K_{e}^{q}(M+1, M+1) \rightarrow K_{e}^{q}(M, M)$ by definition of $K_{e}^{q}$.

6. **Lemma.** If in the squares

$$K_{c}^{q+1}(M, M) \rightarrow K_{c}^{q+1}(M, M)$$  

$$\Psi$$  

$$K_{q}(M, M) \rightarrow K_{q}(M, M)$$

\(\Psi\) is bijective, then \(\overline{\Psi}\) is injective.

**Proof.** By considering the diagram
it suffices to show that $\psi^*$ is a surjective equivalence, i.e. that the canonical map $K_{q+1}(M_n, \cdot) \to (K_{c}^{q+1}(M_n, \cdot))$ is onto. This is an equivalence by V.9 and V.12, and it is surjective, because (with notations as in V.7) $K_{c}^{q+1}(M_n, \cdot) \cong P_n'$ is a retract of $P_n$, which is a retract of $E(n)$, hence $K_{c}^{q+1}(M_n, \cdot)$ is a retract of $C(n) \subset E(n)$. In particular, all linear forms on $K_{c}^{q+1}(M_n, \cdot)$ extend. The middle map $\lim \psi^*$ is an isomorphism (because $\overline{\psi}^*$ is an equivalence of direct systems), hence $\overline{\psi}$ is injective.

7. Proposition. Let us come back to the initial situation of lemma 5, obtained after preliminary surgery: $\mathfrak{M}_n \cap M = \partial M_n$.

For each $n$, and sufficiently large $r > n$ there is a certain non-trivial submodule $A \subset K_{q}(\partial M_r)^\# (\Pi X_n$-coefficients) such that the restriction to $A$ of the intersection pairing $$\psi: K_{q}(\partial M_r)^\# \to H_{q}(\partial M_r)^\# \cong H_{q}(\partial M_r)^\# \to K^q(\partial M_r)^\#$$ induces a non-singular quadratic form on a projective finitely generated quotient of $A$.

Proof. By the two preceding lemma, we can assume that in the square
\[ \begin{array}{cc}
K^{q+1}(\mathbb{M}_n, \mathbb{M}_n) & \rightarrow K^{q+1}(\mathbb{M}_n) \\
\uparrow \psi & \uparrow \psi \\
K_q(\mathbb{M}_n) & \rightarrow K_q(\mathbb{M}_n, \mathbb{M}_n)
\end{array} \]

\( \psi \) is an isomorphism, \( \overline{\psi} \) injective, \( K_q(\mathbb{M}_n) \) and \( K_q(\mathbb{M}_n, \mathbb{M}_n) \) are projective, and \( K^{q+1}(\mathbb{M}_n, \mathbb{M}_n) = 0 \). The horizontal maps are part of the exact sequences of \( (\mathbb{M}_n, \mathbb{M}_n) \):

\[ 0 \rightarrow K^q(\mathbb{M}_n)^\# \rightarrow K^{q+1}(\mathbb{M}_n, \mathbb{M}_n) \rightarrow K^{q+1}(\mathbb{M}_n) \]

\[ 0 \rightarrow K_q(\mathbb{M}_n)^\# \rightarrow K_q(\mathbb{M}_n) \rightarrow K_q(\mathbb{M}_n, \mathbb{M}_n) \rightarrow 0 \]

hence we get an induced isomorphism \( \psi \) of \( K_q(\mathbb{M}_n) \) with its dual, i.e. a non-singular bilinear form on \( K_q(\mathbb{M}_n)^\# \). The lower exact sequence shows that \( K_q(\mathbb{M}_n)^\# \) is projective (finitely generated). One should remember that all the above K-groups refer to the last choice \( \mathbb{X}_n^{\text{iv}}, \mathbb{X}_n^{\text{w}} \) in the proof of 5. But we are interested in the initial choice \( \mathbb{X}_n, \mathbb{X}_n^{\text{iv}} \). Choose \( r \) so large that \( \mathbb{X}_r \) meet \( \mathbb{X}_n^{\text{iv}} \) only along \( \mathbb{M}_n \), or not at all. Then by excision we have a canonical map

\[ H^{q+1}(\mathbb{X}_n^{\text{iv}}, \mathbb{X}_n^{\text{w}} \cup \mathbb{M}_n) \rightarrow H^{q+1}(\mathbb{X}_r^{\text{iv}}, \mathbb{X}_r^{\text{w}} \cup \mathbb{M}_r)^\#. \]

In the exact sequence

\[ H^{q+1}(\mathbb{X}_r^{\text{iv}}, \mathbb{X}_r^{\text{w}} \cup \mathbb{M}_r)^\# \rightarrow H^{q+1}(\mathbb{X}_n^{\text{iv}}, \mathbb{X}_n^{\text{w}} \cup \mathbb{M}_n) \rightarrow H^{q+1}(\mathbb{X}_n^{\text{iv}}, \mathbb{X}_n^{\text{w}} \cup \mathbb{M}_n) \rightarrow 0 \]

\[ K_q(\mathbb{M}_r)^\# \quad \quad K_q(\mathbb{M}_n)^\# \]
the last term is $H_{q+1}(X_r, X_r \cup M_r)$ by Mayer-Vietoris, because $H_{q+1}(X_r, X_r \cup M_r) = 0$. Hence the second map factors through $K_q(M_n, M_n)$ hence $K_q(M_n)$ is contained in the image of $K_q(\partial M_n)$. Let $A$ be the preimage $i^{-1}(K_q(M_n))$ in $K_q(\partial M_n)$, and consider the diagram

![Diagram](image)

The fact that the two maps $A \to K_q(M_n)$ are equal is a result of diagram chasing $A \to K_q(M_n) \to K_q(\partial M_n)$. 
8. Trivial surgery. Let us come back again to the situation obtained after preliminary surgery. Choose a proper embedding \( \phi_k: \mathbb{R}^{q+1}_+ \to M_k \) and let us do surgery on \( (f \circ \phi_k, \phi_k \mid \mathbb{R}^q) \) as in Chapter II. From the picture

we see that

\[
K_q(M_n') = \begin{cases} 
K_q(M_n) \oplus [e] & \text{for } n \leq k \\
K_q(M_n) \oplus [e] \oplus [f] & \text{for } n > k 
\end{cases}
\]

where \([e],[f]\) denote free modules of rank 1 generated by \(e,f\).

Moreover, the map \(K_q(M_{n+1}') \to K_q(M_n')\) sends \(e\) to \(e\), for all \(n\),
and $f$ to $f$ for $n > k$, and $f$ to 0 for $n = k$. Similarly,

we see that $K_q(M'_n, M_n) = \begin{cases} K_q(M'_n, \mathcal{M}_n) \oplus [e] & \text{for } n \leq k \\ K_q(M'_n, \mathcal{M}_n) & \text{for } n > k \end{cases}$.

Hence, if we do this operation for $k \to \infty$, we get

$K_q(M'_n) = K_q(M_n) \oplus E \oplus F_n$, $K_q(M'_n, \mathcal{M}_n) = K_q(M_n, \mathcal{M}_n) \oplus E_n$, where

$E$ is a free module of countable rank, $E_n$ the free module generated by all but a finite number say $s_n$ of basis elements of $E$, $F_n$ a free module of finite rank $s_n$. The map $K_q(M_{n+1}) \to K_q(M_n)$ sends $E$ to $E$ identically $F_{n+1}$ onto $F_n$ with a basis element mapped to itself or to 0. The map $K_q(M'_n, \mathcal{M}_n) \to K_q(M'_{n+1}, \mathcal{M}_{n+1})$ is onto, a basis element being mapped to itself or to 0. Now, each $e \times (I, \partial I)$ introduces a new basis element in

$K_q^{q+1}(M'_n, \mathcal{M}_n)$ for $n > k$, and each $f \times (I, \partial I)$ also, for all $n$, hence $K_q^{q+1}(M'_n, \mathcal{M}_n) = K_q^{q+1}(M_n, \mathcal{M}_n) \oplus (E/E'_n)^* \oplus F_c^*$, where

$F_c^* = \lim_{s} F_s^*$ is a free module of countable rank. Similarly we have $K_q^{q+1}(M'_n) = K_q^{q+1}(M_n) \oplus (F_c^*/F_n^*)$. The canonical map

$\psi: K_q(M'_n) \to K_q^{q+1}(M'_n, \mathcal{M}_n)$ induces an isomorphism

$E \oplus F_n \to \frac{(E/E'_n)^*}{F_c^*}$. Hence, on the kernel of

$K_q(M'_n) \to K_q(M'_n, \mathcal{M}_n)$ we have added the free hyperbolic module $(E/E'_n) \oplus F_n$. The reciprocal trivial surgery consists in the following: do surgery on a trivial $(q-1)$-sphere in $\mathcal{M}_k$, getting $M'$ by extending it to $M$, then carve out the core $S^q$ (piped to $\infty$) of the $q$-handle in $M_k'$, getting $M''$: 
If $e$ is the transverse $q$-sphere to the core $S^q$ and $f$ the $q$-sphere parallel to this core; we have the same situation as above, with $e$ and $f$ exchanged (note that $e$ bounds a transverse $q+1$-disc in $M''_{k-1}$).

9. Cobordism invariance. Suppose we have a proper normal cobordism $F: W^{2q+2} \to Y$ between $f^+: M^+ \to X^+$ and $f^-: M^- \to X^-$ ($Y$ has a $2q+2$-fundamental class mod $X^+ \cup X^-$ at $\infty$ and the inclusion $X^+ \subset Y$ are simple homotopy equivalences). Choose $(X^+_n, X^-_n)$ arbitrarily in $X^\pm$. By using a collar along $X^\pm$, we can find nbhd of $\infty$ $Y_n$ in $Y$, and finite subcomplexes $Y_n$ containing the frontier, such that $Y_n \cap X^\pm = X^\pm_n$, $Y_n \cap X^\pm = X^\pm_n$. 

Now, by a standard construction (see beginning of §1) we can assume that \( X_n^+ \) is bicollared in \( X^+ \) and \( Y_n \) is bicollared in \( Y \). We can put then \( f^\pm \) and \( F \) transverse on these subcomplexes. Then \( F^{-1}(Y_n) \) is a submanifold \( W_n \) (ngbd of \( \infty \)) with boundary
\[
\partial W_n = M_n^+ \cup \hat{\partial} W_n \cup M_n^-.
\]

where the frontier \( \hat{\partial} W_n \) is a compact bicollared submanifold with boundary \( \partial W_n = \partial M_n^+ \cup \partial M_n^- \), and \( M_n^+ = f^{\pm-1}(X_n^+) \). The relativization of Chapter I is clear and we get canonical squares

\[
\begin{array}{ccc}
K^2q+2-k(W_n, \partial W_n) & \xrightarrow{\psi} & K^2q+2-k(W_n, M_n^+ \cup M_n^-) \\
\uparrow \psi & & \uparrow \psi \\
K_k(W_n) & \xrightarrow{} & K_k(W_n, \hat{\partial} W_n) \\
\end{array}
\quad
\begin{array}{ccc}
K^2q+2-k(W_n, \hat{\partial} W_n) & \xrightarrow{\psi_\infty} & K^2q+2-k(W_n, \partial W_n) \\
\uparrow \psi & & \uparrow \psi_\infty \\
K_k(W_n, \hat{\partial} W_n, M_n^+ \cup M_n^-) & \xrightarrow{} & K_k(W_n, \partial W_n) \\
\end{array}
\]

where \( \psi \), \( \psi_\infty \) are equivalences of inverse system and \( \psi_\| \), \( \psi_\infty \) equivalences of direct systems (all with inverse shifting the indice by \( \pm 1 \)).
We can assume that the preliminary surgery on \( f^+ : M^+ \rightarrow X^+ \) (see §1), are already done. Then by doing surgery on \( W_n \rightarrow Y_n \) \( W_n \rightarrow W_{n+1} + Y_n \rightarrow Y_{n+1} \) rel \( M^+ \cup M^- \), one can assume that \( W_n \rightarrow Y_n \) is q-connected and \( W_n \rightarrow W_{n+1} + Y_n \rightarrow Y_{n+1} \) \( q+1 \)-connected. Now, by handles subtraction in \( W_n \) (see [11]) extended to \( W \), and carving out construction (see §7) one divides the cobordism invariance problem in two cases:

1\textsuperscript{st} case: invariance by trivial surgery and \( X^+ \equiv X^- \)

2\textsuperscript{nd} case: invariance by cobordism satisfying the additional condition \( K_q (\partial W_n, \partial M_n^+ \cup \partial M_n^-) = 0 \). Schematically:
Let us concentrate on the 2\textsuperscript{nd} case. **Claim:** the construction in the proof of lemma 5 extend to $W \rightarrow Y$. We have to follow the whole proof of lemma 5, and we use the same notations, with an additional $\pm$. The first operation $X^\pm_n = X^\pm_{n+1} \cup M^\pm_n$, $X^\pm_{n+1} = X^\pm_n \cup M^\pm_n - M^\pm_{n+1}$ is induced by $Y^\prime_n = Y^\prime_{n+1} \cup W_n$, $Y^\prime_{n+1} = Y^\prime_n \cup W_n - W_{n+1}$

We get the $K^\prime$-squares by taking the above $K$-square for $n+1$ with $\mu_n^\prime$-coefficients.

\[
\begin{align*}
K^{q+1}(W_n, \delta W_n) &\rightarrow K^{q+1}(W_n, M^+_n \cup M^-_n) \\
K^{q+1}(W_n, \delta W_n) &\rightarrow K^{q+1}(W_n, \delta W_n) \\
K^{q+1}(W_n, \delta W_n) &\rightarrow K^{q+1}(W_n, \delta W_n) \\
K^{q+1}(W_n, \delta W_n) &\rightarrow K^{q+1}(W_n, \delta W_n)
\end{align*}
\]

\[
\begin{align*}
K^{q+1}(W_n, \delta W_n) &\rightarrow K^{q+1}(W_n, \delta W_n) \\
K^{q+1}(W_n, \delta W_n) &\rightarrow K^{q+1}(W_n, \delta W_n) \\
K^{q+1}(W_n, \delta W_n) &\rightarrow K^{q+1}(W_n, \delta W_n) \\
K^{q+1}(W_n, \delta W_n) &\rightarrow K^{q+1}(W_n, \delta W_n)
\end{align*}
\]

where $\delta W_n = (X^+_n \cup Y^-_n \cup X^-_n) \cap W = M^+_n \cup W_n - W_{n+1} \cup M^-_n$
The second operation is $X_n^{*'} = X_n^*$, $X_n^* = X_n^* \cup \overline{M_n^*} \cup e^{q+1}$, where $e^{q+1}$ describes generators of $\ker (K_q(M_n^*, M_n^*)' + K_c(M_n^-))$ contained in $X_n^{*'} \cup \overline{M_n^*}$. By connectivity of $f$, $e^{q+1}$ bounds a cell $e^{q+2}$ in $X_n^{*'} \cup \overline{M_n^*} \equiv Y_n$ (up to mapping cylinder constructions). Then take $Y_n^{*'} = Y_n$, $Y_n^{*'} = Y_n \cup \overline{M_n^*} \cup e^{q+2} \cup e^{q+1}$.

As $Y_n$ collapses onto $Y_n \cup \overline{M_n^*}$, we have $K_q(M_n^*, M_n^*)' \equiv K_q(M_n^*, M_n^*)'$.

So the map $\psi : K_q(M_n^*, M_n^*)' \to K^{q+1}(M_n^*, M_n^*$ becomes a map $\psi : K_q(M_n^*, M_n^*)' \to K^{q+1}(M_n^*, M_n^*)'$. We get $\psi$ by duality:

$$
\begin{array}{cccc}
0 & \to & K^{q+1}(M_n^*, M_n^*)' & \to & K^{q+1}(M_n^*, M_n^*)' & \to & K^{q+1}(M_n^*, M_n^*)' \\
\uparrow & & \uparrow & & \uparrow & & \lim \downarrow \\
\psi' & & \psi & & \psi & & \psi \\
K_q(M_n^*, M_n^*)' & \to & K_q(M_n^*, M_n^*)' & \to & K_q(M_n^*, M_n^*)' & \to & K_q(M_n^*, M_n^*)'.
\end{array}
$$
Then \( \psi_{\pm} \) induces a \( \psi'' \):

\[
0 \to K_{c}^{q+1}(W_n, \emptyset W_n)'' \to K_{c}^{q+1}(W_n, W_n)'' \to \oplus K_{c}^{q+1}(M_n, M_n)'' \to K_{q}^{+1}(W_n, W_n)'' \to K_{q}^{+1}(W_n, M_n^+ \cup M_n^-)'' \to \oplus K_{q}^{+1}(M_n)''
\]

and then we get \( \psi_{\pm}'' \) by duality again. This provides the convenient \( K'' \)-squares. Before extending the third operation, we need some preparation inside \( Y, \text{ rel } X^\pm \). Observe that the direct system \( \{K_{q+1}(W_s, \emptyset W_s)''\}_n \) is composed of surjections. Then we can apply the argument of §4 to see that \( \ker \psi'' \) is finitely generated. As in the operation " of lemma 5, we can add cells \( e^{q+2} \) to \( Y_n'' \) to get \( \psi'' \) (split) injective, without altering anything on \( X^+ \cup X^- \). When \( \psi'' \) is (split) injective, so is \( \psi_{\pm}'' \) in virtue of the diagram

\[
0 \to K_{c}^{q+1}(W_n, M_n^+ \cup M_n^-) \to K_{c}^{q+1}(W_n)'' \to \oplus K_{c}^{q+1}(M_n^+)'' \to K_{q+1}(W_n, \emptyset W_n)'' \to K_{q+1}(W_n, M_n^-)'' \to K_{q}^{+1}(M_n)''.
\]

Then the duality diagram
shows that $\psi$ is surjective. Now, we are ready to extend the third operation $X^+_n = X_n^{+'} \cup M_n^+ \cup e_+^{q+2}$, $X_{n+1}^+_n = X_n^{+'} \cup M_n^- \cup e_+^{q+2}$
where $e_+^{q+2}$ is a null homotopy of a generator $e_{q+1}$ of ker $\psi^+_c: K_q(M_{n+1}^+) \to K_q^c(M_{n+1}^+, M_{n+1}^-)$". Actually $e_{q+1}$ lies in $K_q(M_{n+1}^+)"$ and we look at its image $e_{q+1}$ in $K_q(W_{n+1})"$. The exact sequence $K_{q+1}(W_{n+1}, M_{n+1}^+) \to K_q(W_{n+1})" \to 0$, shows that $e_{q+1}$ comes from some $e_{q+2} \in K_{q+1}(W_{n+1}, W_{n+1})"$. Under the composition $K_{q+1}(W_{n+1}, W_{n+1})" \xrightarrow{\psi"} K_{q+1}(W_{n+1}, M_{n+1}^+ \cup M_{n+1}^-)" + K_{q}^c(W_{n+1}, M_{n+1}^+, M_{n+1}^-)" \equiv K_{q+1}(W_{n+1}, M_{n+1}^+, M_{n+1}^-)"$, $e_{q+2}$ is mapped to 0 (so does $e_{q+1}$), hence $\psi\vert_{(e_{q+2})}$ lies in $K_{q+1}(W_{n+1}, M_{n+1}^+)"$ (see top exact sequence below). But we have by the above preparation the diagram

\[
\begin{array}{ccccccc}
0 & \to & K_{q+1}^c(W_{n+1}, M_{n+1}^+, M_{n+1}^-) & \xrightarrow{\psi}" & K_{q+1}(W_{n+1}, M_{n+1}^+, M_{n+1}^-) & \xrightarrow{\psi}" & K_q(W_{n+1})" \\
& & & (surjective) & & & \\
0 & \to & K_{q+1}^c(W_{n+1}, W_{n+1}) & \xrightarrow{\psi}" & K_{q+1}(W_{n+1}, W_{n+1}) & \xrightarrow{\psi}" & K_q(W_{n+1})"
\end{array}
\]
from which one deduces that $e^{q+2}$ comes from $K_{q+1}(W_{n+1})^\ast$, i.e. $e^{q+1} = 0$. As a result, $e^{q+1} = e^{q+2}$ with $e^{q+2} \in K_{q+1}(W_{n+1}, M_{n+1}^+ \cup M_{n+1}^-)^\ast$, because of the exact sequence:

$$K_{q+1}(W_{n+1}, M_{n+1}^+ \cup M_{n+1}^-)^\ast \xrightarrow{\partial} K_q(M_{n+1}^\pm)^\ast \xrightarrow{\psi} K_q(W_{n+1})^\ast \rightarrow 0.$$ 

Now, $K_{q+1}(W_{n+1}, M_{n+1}^+ \cup M_{n+1}^-)^\ast \xrightarrow{\psi} K_q(W_{n+1})^\ast$ maps $e^{q+2}$ to 0 in virtue of the following diagram:

$$
\begin{array}{ccc}
K_q(W_{n+1})^\ast & \xrightarrow{\text{inj.}} & \oplus K_q(M_{n+1}^\pm)^\ast \\
\uparrow \psi & & \uparrow \oplus \\
K_{q+1}(W_{n+1}, M_{n+1}^+ \cup M_{n+1}^-)^\ast & \xrightarrow{\Theta} & \oplus K_q(M_{n+1})^\ast \\
\end{array}
$$

In particular, the image of $e^{q+2}$ in $K_{q+1}(W_{n+1}, M_{n+1}^+ \cup M_{n+1}^-)^\ast$ is mapped to 0 by $\psi$, and by using an inverse $K_{q+1}^c(W_{n+1}, W_{n+1})^\ast \rightarrow K_{q+1}(W_{n+1}, M_{n+1}^+ \cup M_{n+1}^-)^\ast$ we see that $e^{q+2}$ vanishes in $K_{q+1}(W_{n+1}, M_{n+1}^+ \cup M_{n+1}^-)^\ast$. This means that the cell $e^{q+2}$ in $Y_{n+1}$ can be deformed over a cell $e^{q+3}$ in $Y_n$ into a cell $e^{q+2}_\pm$ in $X_n^\pm$, that we can assume to coincide with the initial ones.
Take \( Y'''_n = Y''_n \cup W_n \cup e^{q+2}_\pm \cup e^{q+3}_\pm \), \( Y''_n = Y''_n \cup \overline{W_n - W} \cup e^{q+2}_\pm \cup e^{q+3}_\pm \).

As \( Y''_n \) collapses on \( Y''_{n+1} \cup W_n \), we have \( K_{q+1}(W_n)^{'''} = K_{q+1}(W_{n+1})^{''''} \) and by excision \( K_{q+1}(W_n, W_n)^{''''} = K_{q+1}(W_{n+1}, W_{n+1})^{''''} \).

\( K_{q+1}(W_n, \mathcal{W}_n)^{''''} = K_{q+1}(W_{n+1}, \mathcal{W}_{n+1})^{''''} \). Then \( \psi', \psi'' \) become \( \psi''', \psi''' \), and \( \psi'' \) passes to the quotient by \( e^{q+2} \), to give \( \psi''' \).

We get \( \psi''' \) by duality. To extend the last operation \( \psi ''' \), we just add the same cells \( e^{q+2} \) to \( Y''_n \) as to form \( X^{'''}_n \) as to form \( X^{'''}_n \) this doesn't change the \( K_\ast \) and \( K_\ast \)-modules of \( W_n \), \( (W_n, \mathcal{W}_n), (W_n, M_n^+ \cup M_n^-) \) and we get \( \psi ''' \) from the diagram.

\[
0 \to K^{q+1}_c(W_n, M_n^+ \cup M_n^-)^{'''} \to K^{q+1}_c(W_n)^{'''} \oplus K^{q+1}_c(M_n^+)^{'''} \to K^{q+1}_c(W_n, \mathcal{W}_n)^{'''} \oplus K^{q+1}_c(M_n^+, \mathcal{W}_n)^{'''}.
\]
Now that we have proved that the operations of lemma 5 extend, we can assume that the subcomplexes $Y_n, \bar{Y}_n$ of $Y$ intersect $X^\pm$ along $\bar{X}_n^\pm X_n^\pm$, which satisfy the conditions of lemma 5, and moreover, that we have the squares

\[
\begin{array}{ccc}
K_{q+1}(W_n, \bar{w}_n) & \longrightarrow & K_{q+1}(W_n, M_n^+ \cup M_n^-) \\
\uparrow \psi & & \uparrow \psi \\
K_{q+1}(\bar{W}_n) & \longrightarrow & K_{q+1}(\bar{W}_n, \bar{w}_n) \\
\end{array}
\]

\[
\begin{array}{ccc}
K_{q+1}(W_n, \bar{w}_n) & \longrightarrow & K_{q+1}(W_n) \\
\uparrow \psi & & \uparrow \psi \\
K_{q+1}(\bar{W}_n, \bar{w}_n) & \longrightarrow & K_{q+1}(\bar{W}_n, \bar{w}_n) \\
\end{array}
\]

with inverse for $\psi, \psi_1, \psi_\bar{w}, \psi$ shifting the indice by $\pm 1$.

Claim: By changing the $Y_n, \bar{Y}_n$ rel. $X^+ \cup X^-$ one can assume that $\psi, \psi_\bar{w}$ are isomorphisms and $\psi_1, \psi_\bar{w}$, injective. We proceed as in lemma 5, but skip quite a bit through it. The direct system 

$\{K_{q+1}(W_n, \bar{w}_n)\}_n$ is composed of surjections, hence by the argument of §4, ker $\psi_1$ is finitely generated and we can kill it by enlarging $Y_n$ (keeping the squares as above). Once $\psi_1$ is injective (split by §4), so is $\psi_\bar{w}$ by a previous argument, and moreover, by duality, $\psi$ and $\psi_\bar{w}$ are (split) surjective.

In particular, ker $\psi_\bar{w}$ is a retract. But it is contained in the image of $K_{q+1}(W_n, M_n^+ \cup M_n^-)^\#$, in virtue of the diagram

\[
\begin{array}{ccc}
K_{q+1}(W_n, \bar{w}_n) & \longrightarrow & K_{q+1}(W_n, \bar{w}_n) \\
\uparrow \psi & & \uparrow \psi \text{ (injective)} \\
K_{q+1}(W_n, M_n^+ \cup M_n^-)^\# & \longrightarrow & K_{q+1}(W_n, M_n^+ \cup M_n^-) \\
\end{array}
\]

$\longrightarrow K_{q+1}(W_n, \bar{w}_n) \rightarrow 0$
hence is finitely generated. Actually, ker $\psi_\sim$ comes from $K_{q+1}(w_n)^\#$ in virtue of the diagram

\[
\begin{array}{c}
K^q(w_n)^\# \xrightarrow{\oplus} K^q(M_n^+)^\# \\
\downarrow \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \downarrow \sim \\
K_{q+1}(w_n)^\# \xrightarrow{\oplus} K_{q+1}(w_n, M_n^+ \cup M_n^-)^\# \xrightarrow{\oplus} K_q(M_n^+)^\#
\end{array}
\]

Hence, as in the operation "" of lemma 5, we can add cells $e^{q+3}$ to both $Y_n$ and $\tilde{Y}_n$ to get $\psi_\sim$ bijective

The same argument also applies to ker $\psi$: it is a retract, and contained in the image of $\bigoplus K_{q+1}(M_n^\pm)$ in virtue of the diagram
\[ K_{q+1}^c(W_n, \mathfrak{M}_n) \rightarrow K_{q+1}^c(W_n, \mathfrak{M}_n) \]
\[ \oplus K_{q+1}(M_n^\pm) \rightarrow K_{q+1}(W_n) \rightarrow K_{q+1}(W_n, M_n^+ \cup M_n^-) \]

But, because \( K_{q+1}(M_n^+, M_n^-) = 0 \), \( K_{q+1}(M_n^+) \) is a quotient of \( K_{q+1}(M_n^+) \), and the latter is finitely generated because \( K_{q+1}(M_n^+) \) is projective (if in a finite chain complex the lowest homology \( H_k \) is projective then \( H_{k+1} \) is finitely generated because the \( k+1 \)-cycles are direct summand). Now that we have shown how to prove the second claim, we can assume to have the following diagram,

\[ 0 \rightarrow K_{q+1}^c(W_n, \mathfrak{M}_n^0) \rightarrow K_{q+1}^c(W_n, \mathfrak{M}_n^+ \cup \mathfrak{M}_n^-) \rightarrow K_{q+1}^c(W_n, \mathfrak{M}_n^+ \cup \mathfrak{M}_n^-) \]
\[ \oplus \quad \psi \quad \text{(isom.)} \quad \psi \quad \text{(inj.)} \]
\[ K_{q+1}(W_n) \rightarrow K_{q+1}(W_n) \rightarrow K_{q+1}(W_n, \mathfrak{M}_n^0) \rightarrow K_{q+1}(W_n, \mathfrak{M}_n^0) \rightarrow 0 \]

Observe that the exact sequence

\[ K_{q+1}(W_n) \rightarrow K_{q+1}(W_n, \mathfrak{M}_n^0) \rightarrow K_{q+1}(W_n, \mathfrak{M}_n^+ \cup \mathfrak{M}_n^-) \rightarrow K_{q+1}(W_n, \mathfrak{M}_n^+ \cup \mathfrak{M}_n^-) \rightarrow K_{q+1}(W_n, \mathfrak{M}_n^0) \rightarrow 0 \]

remains exact is one replaces \( K_{q+1}(W_n) \) by its image \( 0 \) in \( K_{q+1}(W_n) \), and \( K_{q+1}(W_n, \mathfrak{M}_n^+ \cup \mathfrak{M}_n^-) \) by its image \( E \) in
$K_{q+1}(\mathbb{W}_n, M^+_n \cup M^-_n)$, in virtue of the diagram

$$
\begin{array}{c}
K_{q+2}(\mathbb{W}_n, \mathbb{W}_n) \longrightarrow K_{q+2}(\mathbb{W}_n, \mathcal{W}_n) \longrightarrow \bigoplus_{\pm} \bigoplus_{q+1}(M^+_n, M^-_n) \\
\Downarrow \\
K_{q+1}(\mathbb{W}_n) \longrightarrow K_{q+1}(\mathbb{W}_n, M^+_n \cup M^-_n) \longrightarrow \\
\Downarrow \\
K_{q+1}(\mathbb{W}_n, M^+_n \cup M^-_n).
\end{array}
$$

But the former diagrams provides isomorphisms $E \cong K^q(\mathbb{W}_n)^\#$ and $K_q(\mathbb{W}_n) \cong E^\ast$. Hence, putting $K_q(\mathbb{W}_n)^\# \equiv F$, the above exact sequence reduces to $0 \to E \to \bigoplus_{\pm} K(M^\pm)_n \to F \to 0$ where the quadratic form on the middle module induces isomorphism $E \cong F^\ast$, $F \cong E^\ast$. **Claim:** this sequence splits. To construct a section $F \to \bigoplus_{\pm} K(M^\pm)_n$, consider the diagram of exact sequences

$$
\begin{array}{c}
K_{q+1}(\mathbb{W}_n)^\# \longrightarrow K_{q+1}(\mathbb{W}_n, M^+_n \cup M^-_n)^\# \longrightarrow \bigoplus_{\pm} K(M^\pm)_n \\
\Downarrow \\
K_{q+1}(\mathbb{W}_n) \longrightarrow K_{q+1}(\mathbb{W}_n, M^+_n \cup M^-_n) \longrightarrow \bigoplus_{\pm} K(M^\pm)_n \\
\Downarrow \\
0 \to K_{q+1}(\mathbb{W}_n, \mathcal{W}_n) \longrightarrow K_{q+1}(\mathbb{W}_n, \mathcal{W}_n) \longrightarrow \bigoplus_{\pm} K(M^\pm, M^\pm)_n \\
\Downarrow \\
K(\mathbb{W}_n)^\# \longrightarrow 0 \\
\Downarrow \\
0
\end{array}
$$
It contains a commutative triangle

\[ K_{q+1}(W_n) \quad \xrightarrow{\text{(injective)}} \quad K_{q+1}(W_n, \overline{M}_n^+ \cup M_n^-) \]

\[ \rightarrow K_{q+1}(W_n, \overline{\mathcal{W}}_n). \]

In particular, \( K_{q+1}(W_n) \) is a submodule of \( K_{q+1}(W_n, \overline{M}_n^+ \cup M_n^-) \) which meets the image \( E \) of \( K_{q+1}(W_n, \overline{M}_n^+ \cup M_n^-) \) only at 0. But, as \( K_{q+1}(W_n, \overline{\mathcal{W}}_n) \) and \( K_q(M_n^+ \cup M_n^-) \) are projective, \( E \) and \( K_q(W_n) \) are direct summands of \( K_{q+1}(W_n, \overline{M}_n^+ \cup M_n^-) \). Hence, the preimage of \( K_{q+1}(W_n) \) by the map

\[ K_{q+1}(W_n, \overline{M}_n^+ \cup M_n^-) \rightarrow K_{q+1}(W_n, \overline{\mathcal{W}}_n) \]

is \( E \oplus K_q(W_n) \). We construct a map \( K_q(W_n)^\# \oplus K_q(M_n^+ \cup M_n^-)^\# \) by representing \( x \in K_q(W_n)^\# \) into \( K_{q+1}(W_n, \overline{\mathcal{W}}_n) \), taking the image \( x' \) into \( K_{q+1}(W_n, \overline{\mathcal{W}}_n) \), then a section \( K_{q+1}(W_n, \overline{\mathcal{W}}_n) \rightarrow K_{q+1}(W_n, \overline{M}_n^+ \cup M_n^-) \), and taking the image of \( x'' = s(x') \) in \( E \oplus K_q(M_n^+ \cup M_n^-) \). This does not depend on the way of representing \( x \), because if \( x' \) comes from \( K_{q+1}(W_n) \), then on one side \( s(x') \in E \oplus K_q(W_n) \), but on the other side the \( E \)-component of \( s(x') \) is 0, hence \( s(x') \) comes from \( K_q(W_n) \), i.e., vanishes in \( \bigoplus K_q(M_n^+ \cup M_n^-) \). This achieves the proof that, in the second case of cobordism, the quadratic module \( \langle K_q(M_n^+) \oplus K_q(M_n^-) \rangle \) is isomorphic to a projective hyperbolic module \( \langle E \oplus F \rangle \), i.e., \( K_q(M_n^+ \cup M_n^-)^\# \) and \( K_q(M_n^-)^\# \) are equivalent. As for the first case of cobordism, i.e., a trivial surgery from \( f^+: M^+ \to X \) to \( f^-: M^- \to X \), observe that the same operations on \( X \) can be
used to satisfy lemma 5 for both $f^+$, and then one readily sees the equivalence.

10. **Theorem.** Let $M$ be an open manifold of dim $2q+1 \geq 7$ and $f: M \rightarrow X$ a proper normal map of degree 1. Then, to the cobordism class $[f]$ of $f$ are associated canonically a sequence $(\mathcal{Q}_n) \in \lim_+ L_{2q}(\pi_1 X_n)$ and, if all $\mathcal{Q}_n$ vanish, an element $(\ell_n) \in \lim_+ L_{2q+1}(\pi_1 X_n)$, such that $[f]$ contains a proper homotopy equivalence at $\infty$ iff all $\mathcal{Q}_n = 0$ and $(\ell_n) = 0.

By definition, if $\{A_n\}$ is an inverse system of abelian groups, $\lim_+ A_n$ is the cokernel of the map $\prod_{n \geq 1} A_n \xrightarrow{1-S} \prod_{n \geq 1} A_n$ sending $(a_1, a_2, a_3, \ldots)$ to $(a_1-a_2^\#, a_2-a_3^\#, a_3-a_4^\#, \ldots)$, where $a_n^\#$ is the image of $a_n$ in $A_{n-1}$. A subsequence gives the same result, e.g. $\lim_+ A_{2n+1} \cong \lim_+ A_n$ by sending $(a_1, a_2, a_3, \ldots)$ to $(a_1+a_2^\#, a_3+a_4^\#, \ldots)$ in the range product. Note that the choice of base points and paths has no influence on the inverse system $\{L_*(\pi_1 X_n)\}$ because an inner automorphism of a group $G$ induces $\pm$ identity on $L_*(G)$, according to whether $\omega: \pi_1 X_n \rightarrow \pm 1$ is trivial or not.

**Proof.** Define $\mathcal{Q}_n$ by the quadratic module $K_q(M_n)^\#$ obtained in Proposition 7. A canonical equivalence between $\mathcal{Q}_{n+1}$ and $\mathcal{Q}_n$ is given by the exact sequence

$$K_{q+1}(\overline{M_n-M_{n+1}}, \overline{M_n} \cup \overline{M_{n+1}})^\# \rightarrow K_q(M_n)^\# \otimes K_{q+1}(M_{n+1}) \rightarrow K_q(M_n-M_{n+1})^\# \rightarrow 0.$$  

Actually, $K_q(M_n-M_{n+1})^\#$ is projective in virtue of the exact sequence
\[ K_{q+1}(M_{n+1} \backslash M_n) \# \rightarrow K_q(M_{n-M_{n+1}}) \# \rightarrow K_q(M_n) \rightarrow K_q(M_{n+1} \backslash M_{n+1}) \# \rightarrow 0. \]

A reciprocal duality between \( F \equiv K_q(M_{n-M_{n+1}}) \# \) and the image \( E \) of \( K_{q+1}(M_{n-M_{n+1}}, M_n \cup M_{n+1}) \# \) comes from the diagram

\[
\begin{array}{cccccc}
0 & \rightarrow & K^q(M_{n-M_{n+1}}) \# & \rightarrow & K^q(M_n) \# & \oplus & K^q(M_{n+1}) \# & \rightarrow & K^{q+1}(M_{n-M_{n+1}}, M_n \cup M_{n+1}) \\
& & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
& K_{q+1}(M_{n-M_{n+1}}, M_n \cup M_{n+1}) \# & \rightarrow & K_q(M_n) \# & \oplus & K_q(M_{n+1}) \# & \rightarrow & K_q(M_{n-M_{n+1}}) \# & \rightarrow 0
\end{array}
\]

and its dual. This exhibits \( K_q(M_n) \# \oplus K_q(M_{n+1}) \# \) as the hyperbolic module \( E \oplus F \), i.e., \( \mathcal{C}_n \# \oplus \mathcal{C}_{n+1} \# = \mathcal{C}_n \). By §8, the element \( \mathcal{C}_n \) is independent of all choices and invariant by cobordism. If all \( \mathcal{C}_n = 0 \), then by choosing a trivialization \( E_n \oplus F_n \) for \( K_q(M_n) \# \), the plane \( K_q(M_{n-M_{n+1}}) \# \) becomes a Lagrangian plane \( L_n \) in the standard module \( (E_n \oplus E_{n+1}) \oplus (F_n \oplus F_{n+1}) \), i.e., \( L_n \in L_{2q+1}(\pi_1 X_n) \). Another choice of trivializations modify the sequence \( (L_n) \) by a sequence in the image of \( L-S \). The same is true if one alters \( f \) by a cobordism, and we sketch the proof as follows. Let \( f^\pm: M^\pm \rightarrow X^\pm \) be cobordant by \( F: W \rightarrow Y \), in the final setting of §8. Then we have quadratic modules \( K_q(M_n^+) \# \), trivial by assumption, and Lagrangian planes \( K_q(M_n^-) \# \) in \( \langle K_q(M_n^+) \# \rangle \oplus \langle K_q(M_n^-) \# \rangle \), and \( K_q(M_{n-M_{n+1}}^+) \# \) in \( \langle K_q(M_n^+) \# \rangle \oplus \langle K_q(M_{n+1}^-) \# \rangle \). By choosing a trivialization \( \langle E_n^+ \oplus F_n^+ \rangle \) of \( \langle K_q(M_n^+) \# \rangle \), the planes
$K_q(\tilde{M}_n)$ and $K_q(M^{\pm}_{n-M^{\pm}_{n+1}})$ become elements $\omega_n, l_n \in L_{2q+1}(\pi_1 X_n)$.

We have to show that $\ell_n^+ - \ell_n^- = \omega_{n+1} - \omega_n$, i.e., that the Lagrangian plane $\ell_n = K_q(\tilde{M}_n) + K_q(M^{\pm}_{n-M^{\pm}_{n+1}}) + K_q(M^{-}_{n-M^{-}_{n+1}}) + K_q(\tilde{W}_{n+1})$ in $\langle \mathcal{H} \rangle \oplus \langle \mathcal{I} \rangle$ is equivalent to 0 where

$$\langle \mathcal{H} \rangle \equiv \langle K_q(M^+) \rangle \oplus \langle K_q(M^-) \rangle \oplus \langle K_q(M^+_{n+1}) \rangle \oplus \langle K_q(M^-_{n+1}) \rangle$$

is equivalent to a trivial one. We consider this problem as the bounded case of Chapter 1. For this, we need to choose the very initial $X^+, X^+_{n, n}$, $X^+_n$, $Y, Y_n$ as follows. By infinite simple homotopy type theory, $X^\pm$ is simply homotopy equivalent to a CW-complex of the form $X^0 \pm \cup H^\pm$, where $H^\pm$ is a locally finite $2q+1$-handlebody of 0 and 1-handels, which is a thickened tree (see [10]). Moreover, $Y$ is simply homotopy equivalent rel $X^+\cup X^-$ to a CW-complex of the form $Y^0 \cup H$, where $H$ is a locally finite $2q+2$-handlebody of 0 and 1-handels, such that $H \cap X^+ = H^+, \quad \exists H = H^+ \cup H^- \cup \exists H$, $Y^0 \cap X^+ = X^0 \pm$. 

\[
\begin{array}{c}
Y \\
X^+
\end{array}
\]

\[
\begin{array}{c}
H_n
\end{array}
\]

\[
\begin{array}{c}
X^-
\end{array}
\]
As ngbd of $\infty$ in $H$, we take subhandlebodies $H_n$ with relative frontier a disjoint union of $D^2 \times I$ (see figure). Choose ngbd of $\infty$ $Y^0_n$ in $Y^0$, and finite subcomplexes $Y^0_n$ containing the frontier. The $X^{0\pm}_n \equiv Y^0_n \cap X^\pm$ are ngbd of $\infty$ in $X^{0\pm}$, and let $X^{0\pm}_n \equiv Y^0_n \cap X^\pm$. By the construction in §1, we can assume that $Y^0_n$ is bicollared in $Y^0$ (and $X^{0\pm}_n$ bicollared in $X^\pm$). Now, $X^{0\pm}_n \cup (H_n \cap X^\pm)$ is a ngbd of $\infty$ $X^\pm_n$ in $X^\pm$, and we choose $X^\pm_n \equiv X^{0\pm}_n$. Similarly, $Y_n \equiv Y^0_n \cup H_n$ is a ngbd of $\infty$ in $Y$ such that $Y_n \cap X^\pm = X^\pm_n$, and by using a collar along $\partial H$, we can assume that $Y_n \equiv Y^0_n \cup H_n$ is bicollared in $Y$. Then we do all the necessary preliminary surgery (as in §1) first on $M^+ \xrightarrow{f^\pm} X^\pm$, then on $W \xrightarrow{F} Y$ rel. $M^+ \cup M^-$. Then one meets the modules $K_q(M^+_{n+1}M^-_{n+1})$ and $K_q(\hat{W}_n)$. Represent each generator by an embedded $q$-sphere, and extend them into immersed $q+1$-discs in $\hat{W}_n - \hat{W}_{n+1}$ (see IV. 1). Then pipe the left discs and upper and lower discs to $\infty$ as in the figure.
We also connect up by path all the \( q \)-spheres so obtained in a connected component of \( \dot{W}_{n+1} \). Then take a regular ngbd \( V \) of this connected union of images of immersions. Let \( V_n = V \cap W_n \), 
\[
\begin{align*}
U_n^+ &= V_n \cap M_n^+, \\
U_n^- &= V_n \cap M_n^-, \\
\partial^+ V_n &= \partial V \cap W_n - U_n^+ \cup U_n^-,
\end{align*}
\]
\[
\partial^+ U_n^\pm = \partial U_n^\pm \cap M_n^\pm, \\
M_0^\pm = M_n^\pm - U_n^\pm, \\
W^0 = \overline{W-V}.
\]
Then, as in IV. 1, \( \overline{V_n - V_{n+1}} \) is a handlebody on \( U_{n+1} \uplus \overline{U_n^+ - U_n^-} \cup U_{n+1} \) composed of 1 and \( q+1 \)-handles. This allows (by standard geometrical arguments like in [7]) to arrange \( F \) and \( f^\pm \) so that they induce maps 
\[
\begin{align*}
M_0^{0\pm} + X_0^{0\pm}, \\
U^+ + H^+, \\
3U^+ + 3H^+, \\
W^0 + Y^0, \\
V + H, \\
\partial^+ V + \partial^+ H
\end{align*}
\]
(now, \( H \) may be smaller). Apply §5 to \( M_0^{0\pm} + X_0^{0\pm} \) and §9 to \( W^0 + Y^0 \) rel \( \partial W^0 \). Then, by IV. 2,
we get a projective Lagrangian plane $K_q(\tilde{w}_n^0)$ in $K_q(\mathfrak{g}U_n)$. By
the argument of [11, lemma 7.2], the Lagrangian plane
$K_q(\tilde{w}_n^0) \oplus K_q(\tilde{w}_{n+1}^0)^*$ in $\langle K_q(\mathfrak{g}U_n) \oplus K_q(\mathfrak{g}U_{n+1}) \rangle'$ is equivalent
to $L_n$ in $\langle H \oplus \langle H \rangle \rangle'$. But the former is equivalent to 0 by
IV. 4. Hence $\lambda_n \in \lim_{\leftarrow} L_{2q+1}(\pi_1 x_n)$ associated to $[f]$ in a
well-defined way. If all $\sigma_n = 0$ and $(\lambda_n) = 0$, then by
[11] (realizing Lagrangian transformation) one can arrange so
that actually the plane $K_q(\overline{M_n-M_{n+1}})$ is actually "trivial",
inj.
for all $n$. This means that the map: $K_q(\overline{M_n}) \oplus K_q(\overline{M_{n+1}}) \to K_q(\overline{M_n-M_{n+1}})$
surj.
is nothing but the canonical projection $(E_n \oplus F_n) \oplus (E_{n+1} \oplus F_{n+1}) +$
$+ E_n \oplus F_{n+1} \oplus E_{n+1} \oplus F_n$ (see notations). Then, the map

$K_q(\overline{M_{n+1}}) \oplus K_q(\overline{M_n})$ injects $F_r$ onto a direct summand and projects
$F_r$ onto $F_n$, for $r \to \infty$. But these image generated $K_q(\overline{M_n})$
hence $K_q(\overline{M_n}) \cong E \oplus F_n$, as in §8. Once we know that, we can
do surgery as follows: 1°) make $E$ free by trivial surgery,
2°) each basis element $e_i \in E$ is in $E_r$ for $r \to \infty$, but
moreover, we saw at the end of §6, that $K_q(\overline{M_n})$ is in the
image of $K_q(\mathfrak{g}M_r)$ for large $r$, hence $e$ can be represented
by a sequence of maps $(D^{q+1}, S^q) \to (\mathfrak{g}M_r)$ for $r \to \infty$.
Now, we know that the intersection form between elements of $E$
vanesishes with $\pi_1 M_n$-coefficients. Hence it already vanishes
with some $\pi_1(\overline{M_n - M_s}) = \pi_1 M_n$-coefficients, because the group
functor $L_n$ commutes with direct limits.

By modifying $\mathfrak{g}M_r$ inside $\overline{M_n}$ with 1 and 2-handles we
can assume that $\pi_1 \mathfrak{g}M_r \cong \pi_1 M_n$ (see [8]). Now, the intersecton
between the e's vanish with $\pi_1 \mathfrak{M}_r$ coefficients, so we can do 
a sequence of surgeries on $\alpha_r$.

To each $e_i$ is substituted in $K_q(M_n)$ a free module generated 
by the $e_i, r$ and a corresponding free module over $f_i, r$ appears 
as $K_{q+1}(M_n)$. So the new K-systems look like

\[
\begin{align*}
K_q(M'_n) &= F_n \oplus E'_n & K_{q+1}(M'_n) &= F'_n \\
\text{surj.} & \quad \uparrow & \text{inj.} & \quad \uparrow \\
K_q(M'_{n+1})^\# &= F'_{n+1} \oplus E'_{n+1} & K_{q+1}(M'_{n+1})^\# &= F'_{n+1}
\end{align*}
\]

where the injection of free modules are of the form $A^{\text{st comp}} \oplus A \oplus B$, 
with $B$ free of finite rank. Hence $K_q(M'_n)$ is free, and the 
cokernel of $K_q(M'_{n+1})^\# \rightarrow K_q(M'_n)$ is free of finite rank.
In other words, one can write $K_q(M'_n) = \bigoplus A_n$ where each $A_n$ is free of finite rank. Observe that $K_q(M'_n) \cong K_q(M'_n, M''_n)$ now.

3°) Represent each basis element of $A_n$ by an element in $K_q(M'_n)$, and do surgery on it. This gives a cobordism $W \rightarrow X$ from $M' \rightarrow X$ to $M'' \rightarrow X$, such that $K_{q+1}(W_n, M'_n) \xrightarrow{\partial} K_q(M'_n)$ is an isomorphism for each $n$, and $K_k(W_n, M'_n) = 0$ for $k \neq q+1$.

The exact sequence

$$0 \rightarrow K_{q+1}(M'_n) \rightarrow K_{q+1}(W'_n) \rightarrow K_{q+1}(W'_n, M'_n) \xrightarrow{\partial} K_q(M'_n) \rightarrow K_q(W'_n) \rightarrow 0$$

shows that $K_q(W'_n) = 0$, and $K_{q+1}(W'_n) \cong K_{q+1}(M'_n)$. On the other side, we get the exact sequence

$$0 \rightarrow K_{q+1}(M''_n) \rightarrow K_{q+1}(W'_n) \rightarrow K_{q+1}(W'_n, M''_n) \rightarrow K_q(M'_n) \rightarrow 0.$$

Claim: The middle map is an equivalence of inverse systems. We can take $W'_n = M'_n \times I$, hence $K_{q+1}(W'_n, M''_n) \cong K_{q+1}(W'_n, W'_n \cup M''_n)$.

As $K_{q+1}(W'_n) \cong K_{q+1}(M'_n) \cong K_{q+1}(M'_n, M''_n)$, we have the commutative square

$$\begin{array}{ccc}
K_{q+1}(W'_n) & \rightarrow & K_{q+1}(W'_n, M''_n) \\
\text{equiv.} \quad \psi & & \text{equiv.} \\
K_q(M'_n) & \rightarrow & K_{q+1}(W'_n, M''_n).
\end{array}$$
By construction, $K_{q+1}(W_r, M_r') \rightarrow K_q(M_r')$ is an isomorphism, \( \forall r \), so is its dual, hence we get by direct limit over \( r \) an isomorphism $K^q_e(M_n') \overset{\delta}{\rightarrow} K^q_{e+1}(W_n, M_n')$. Then the diagram

\[
\begin{array}{cccccc}
0 & \rightarrow & K^q_e(M_n') & \rightarrow & K^q_c(M_n') & \rightarrow & K^q(M_n) & \rightarrow & K^q_{e+1}(M_n') \\
& \cong & & \delta & & \cong & & \downarrow & \\
0 & \rightarrow & K^q_{e+1}(W_n, M_n') & \rightarrow & K^q_{c+1}(W_n, M_n') & \rightarrow & K^q_{e+1}(W_n, M_n') & \rightarrow & 0
\end{array}
\]

where $K^q_{e+1}(M_n') = \lim_{\rightarrow} F_r^*$ shows that the middle map $\delta$ is an isomorphism, and this implies the assertion. Hence the inverse systems $\{K_q(M_n')\}$ and $\{K_{q+1}(M_n')\}$ are equivalent to 0, which implies that $M'' \xrightarrow{f} X$ is a proper homotopy equivalence at $\infty$. This achieves the proof of the theorem. A more refined formulation of Theorem 9 is given by the following result.

11. Corollary. Let $L_{2q+1}(\varepsilon X)$ be the obstruction group for our problem, i.e., to each surgery data $(M, 2q+1, \partial M)^f \rightarrow (X, \partial X)$ rel. boundary ($f|\partial M$ is already a proper homotopy equivalence at $\infty$) is associated $\sigma(f) \in L_{2q+1}(\varepsilon X)$ which vanishes iff $f$ is cobordant rel. $\partial M$ to a proper homotopy equivalence at $\infty$, and each element of $L_{2q+1}(\varepsilon X)$ is equal to $\sigma(f')$ for some surgery data rel. boundary $(M, \partial M') \xrightarrow{f'} (X, \partial X')$, where $\{\pi_1 X_n\}$ is conjugate equivalent to $\varepsilon X$ in a specific way. Then we have an exact sequence.
\[ 0 \rightarrow \lim_+^1 L_{2q+1}(\pi_1 X_n) \rightarrow L_{2q+1}(\varepsilon X) \rightarrow \lim_+^1 L_{2q}(\pi_1^+ X_n) \rightarrow L_{2q}(\pi_1 X). \]

Sketch of proof: For the first map, take an open 2q-manifold \( N \) such that \( \{\pi_1^+ N_n\} \) is conjugate equivalent to \( \{\pi_1^+ X_n\} \).

Then, following [11], do surgery on \( N \xrightarrow{id} N \) to kill enough trivial \( (q-1) \)-spheres in each \( N_n^+ \rightarrow N_{n+1}^+ \) \( (r_n^+ \) say, if the Lagrangian plane \( \mathscr{L}_n^+ \) is in the free hyperbolic module of rank \( r_n^+ \). Let \( N' \xrightarrow{f} N \) be the result of this surgery. Then \( K_q(N_n^+ \rightarrow N_{n+1}^+) \) is free of finite rank, and \( K_q(N_n^+) \cong \oplus_{r \geq 0} K_q(N_{n+r}^+ \rightarrow N_{n+r+1}^+) \) is free of countable rank. By definition, \( \mathscr{L}_n \) is a Lagrangian plane in \( K_q(N_n^+ \rightarrow N_{n+1}^+) \), so we can do surgery on \( N' \xrightarrow{f} N \) killing a finite set of generators of \( \mathscr{L}_n \). The result \( N'' \xrightarrow{f''} N \) of this surgery is a proper homotopy equivalence (see end of IV.4).

If \( M^{2q+1} \xrightarrow{f''} N \times I \) is the cobordism so obtained between \( N \xrightarrow{id} N \) and \( N'' \xrightarrow{f''} N \), \( M \xrightarrow{f} N \times I \) provides the surgery data \( (M, \varnothing M) \rightarrow (X, \varnothing X) \) we are looking for in \( L_{2q+1}(\varepsilon X) \). For the second map: if \( (M', \varnothing M') \rightarrow (X', \varnothing X') \) is a surgery data, we take the sequence of quadratic forms \( \mathcal{Q}_n \in \lim_+ L_{2q}(\pi_1^+ X_n) \approx \lim_+ L_{2q}(\pi_1 X_n) \). Observe that in this case with boundary, where the map on the boundary is already a proper homotopy equivalence, everything looks like if \( M' \) were open. The composition of the two first maps is \( 0 \) by construction. The composition of the two last maps is \( 0 \), by the argument proving that \( \mathcal{Q}_n \# = \mathcal{Q}_n \) (case \( M_n = \varnothing, \) see proof of Theorem 10). For the exactness at \( \lim_+ L_{2q}(\pi_1 X_n) \), note that any element of this limit can be represented by a free (singular) quadratic module. Take \( N \) as above, and by [11] again, do surgery.
on each identity map $\mathcal{N}_n \to \mathcal{N}_n$ to some map (which would be a homotopy equivalence iff the quadratic form on the free module were nonsingular) so that the cobordism map $\mathcal{M}_n \to \mathcal{N}_n \times I$ has obstruction $\varphi_n$

The condition $\varphi_{n+1}' = \varphi_n'$ and $\varphi_1' = 0$ in $\mathbb{L}_{2q}(\pi_1 X)$ allows to do surgery by strips on the other side $N'$, rel $\mathcal{N}_n$, to get a proper homotopy equivalence $N'' \to N$. This construction provides a cobordism $M \to X$ between $N \xrightarrow{id} N$ and some proper homotopy equivalence $N'' \to N$

This shows that $(\varphi_n')$ comes from a surgery data $(M, \partial M) \to (X, \partial X)$. For the exactness at $\mathbb{L}_{2q+1}(\pi_1 X)$ Theorem 10 gives an injective retraction of the map $\lim_+ \mathbb{L}_{2q+1}(\pi_1 X)$ to $\text{Ker} \varphi'$, hence the latter is an isomorphism.
12. **Globalization.** If $L_{2q+1}(X)$ denotes the formal obstruction group for surgering maps to proper homotopy equivalences then we have an exact sequence

$$\lim_{\leftarrow n} L_{2q+1}(\pi_{1}X_{n}) \longrightarrow L_{2q+1}(\pi_{1}X) \longrightarrow L_{2q+1}(X) \longrightarrow L_{2q+1}(cX) \longrightarrow 0$$

where $L_{2q+1}(\pi_{1}X) \longrightarrow L_{2q+1}(X)$ is the usual realization map, and the composition of the first two maps is 0 by the "altered sequence" trick. Moreover, if one takes care of $X_{0} = X$ in constructing the sequence of Corollary 11, then one gets the sequence

$$\Pi_{2q+1}^{1-S} \longrightarrow \longrightarrow L_{2q+1}(\pi_{1}X) \otimes \Pi_{2q+1}^{1-S} \longrightarrow L_{2q+1}(X) \longrightarrow \Pi_{2q} \longrightarrow L_{2q}(\pi_{1}X) \otimes \Pi_{2q}$$

where $\Pi_{*} = \prod_{n \rightarrow 1} L_{*}(\pi_{1}X)$ and

$$(1-S)(a_{1}, a_{2}, a_{3}, \ldots) = (-a_{1}, a_{1} - a_{2}, a_{2} - a_{3}, \ldots) .$$

This sequence is exact by virtue of the previous exact sequence and 11.
CHAPTER IV. THE OPEN EVEN DIMENSIONAL CASE

1. If in the data of Chapter III, 1 one lets \( m = 2q+2 \geq 6 \), then one can also do preliminary surgery to make \( \mathcal{M}_n \xrightarrow{f} X_n \)\( q \)-connected and \( \mathcal{M}_n \xrightarrow{f} X_n \)\( (q+1) \)-connected. Then \( f \) is bijective on ends. spaces, and each map \( M_n + X_n \) is \( (q+1) \)-connected: \( K_k(M_n) = 0 \) for \( k \leq q \). This implies \( K_k(M_n, \mathcal{M}_n) = 0 \) for \( k \leq q \) (because \( K_k(\mathcal{M}_n) = 0 \) for \( k \leq q-1 \)), and \( K_k(M_n, \mathcal{M}_n \cup \mathcal{M}_r) = 0 \) for \( k \leq q \). Hence \( K_k(M_n, \mathcal{M}_n) = 0 \) for \( k \in q \), and the duality equivalence shows that \( \{K_{q+1}(M_n)\} \) is the only inverse system not equivalent to \( 0 \). Similarly, \( \{K_{q+1}(M_r, \mathcal{M}_r)\} \) is the only direct system not equivalent to \( 0 \). Now, the data \( M \xrightarrow{f} X \) can be decomposed into two cobordisms with common boundary. By infinite simple homotopy type theory, \( X \) is simply homotopy equivalent to a CW-complex of the form \( X^0 \cup H \), where \( H \) is a locally finite \( m \)-handlebody of \( 0 \) and \( 1 \)-handles (see [10]):
(ΘH is collared in X0). As nbd of ∞ in H, we can take subhandlebodies Hn, with relative frontier Hn a disjoint union of 2q+1-discs. Denote X0 ∪ H by X again, and choose nbd of ∞ X0 in X, with finite subcomplexes Xn containing the frontier. Then X0 ∪ Hn is a nbd of ∞ Xn in X, and X0 ∪ Hn a finite subcomplex Xn containing the frontier of Xn. By using a collar, we can assume that Xn resp. X0 meets X0 and H along X0 and Hn resp. X0 and Hn and that X0 is bicolled in X, Xn bicolled in X.

After preliminary surgery on M→F as above, we meet Kq(ΘMn).

Represent each generator by an embedded q-sphere S° C ΘMn (nullhomotopic in Xn). By the argument of [11, lemma 8.1], these spheres bound immersed (right) q+1-discs in M°-Mn, that one can assume to generate Kq+1(Mn-1-Mn, ΘMn). Similarly, they bound immersed (left) q+1-discs in M°-Mn+1, that one can assume to generate Kq+1(Mn-1-Mn+1, ΘMn). The immersed left and right discs which coincide along their boundary S° form an immersion S°+1 → M°-Mn+1 that we pipe to ∞.
getting an immersion $\mathbb{R}^{q+1} \rightarrow M_{n-1}$. We also connect by paths all the $S^q$ contained in a connected component of $\partial M_n$. Let $V$ be a regular ngbd of this connected union of images of immersions, and let $M^0 \equiv \overline{M-V}$. The ngbd of $\partial$ in $V$ are $V_n \equiv V \cap M_n$, with frontier $U_n \equiv V \cap \partial M_n$ (connected union of $S^q \times D^{q+1}$ and those in $\partial V$ are $\partial V_n \equiv \partial V \cap M_n$. Observe that the regular ngbd of the left and right $q+1$-discs is a handlebody on $\partial M_n$, with only 1 and $q+1$-handles. One sees that by taking the preimage $x' \cup x''$ in $D^{q+1}$ of a self intersection point $x$, disj.

joining each of them to $S^q$ by a path and taking a regular ngbd $N', N''$ of

![Diagram](attachment:image.png)

each path. In the image in $M$, a regular ngbd of $N'$ matches with a regular ngbd of $N''$ around $x$, to form a 1-handle.
As $D_{q+1-n} \cup N$ is an embedded disc, its regular nbd forms a $q+1$-handle attached to $M_n \cup 1$-handle. As a result, by using standard geometrical arguments like in [7], one can arrange $f$ to induce proper maps of degree 1 $V \xrightarrow{f} H$, $V \xrightarrow{f} H$, $M^0 \xrightarrow{f} X^0$

($H$ may be smaller now). We want to apply III. 9 to $M^0 \xrightarrow{f} X^0$, by considering it as a cobordism. But first of all, what is the connectivity of the maps $\partial V \xrightarrow{f} \partial H$, $V \xrightarrow{f} H$ and $M^0 \xrightarrow{f} X^0$? The map $U_n \xrightarrow{f} H_n$ is obviously $q$-connected. The map $V_n \xrightarrow{f} H_n \xrightarrow{f} H_{n+1}$ is only $q$-connected, but it satisfies at least $K_q(V_n;V_{n+1}) = 0$. The map $\partial U_n \xrightarrow{f} \partial H_n$ is $q$-connected, because $\partial U_n$ is a union of $S^q \times S^q$, and $\partial H_n$ a union of $S^{2q}$. The map

$$\partial V_n \xrightarrow{f} \partial H_n \xrightarrow{f} \partial V_{n+1}$$

is $q$-connected ($\partial H_n = \partial H \cap X_n$), because by general position, the connectivity in this range is the same as for $V_n \xrightarrow{f} H_n \xrightarrow{f} H_{n+1}$. But moreover $K_q(\partial V_n;\partial V_{n+1}) = 0$ because, on one hand, a transverse $q$-sphere to $S^q \subset \partial M_n$ can be translated across the left disc and along the pipe to $\infty$, and on the other hand, the equatorial $S^q \subset \partial M_n$ itself is homotop over the left disc to a "slice" of the pipe, which can be translated to $\infty$. So the map $\partial V \xrightarrow{f} \partial H$, has the required connectivity (see III. 1). Now as for the connectivity of $M^0 \xrightarrow{f} X^0$, note that $\pi_1(\partial U_n) = \pi_1(U_n) = \{e\}$ and

$$\pi_1(\partial V_n \xrightarrow{f} \partial V_{n+1}) \cong \pi_1(V_n \xrightarrow{f} V_{n+1})$$

hence by van Kampen, we have

$$\pi_1(X_n) = \pi_1(\partial M_n) \quad \text{and} \quad \pi_1(M_n \xrightarrow{f} M_{n+1}) \cong \pi_1(M_n \xrightarrow{f} M_{n+1})$$. Similarly,

$$\pi_1(X_n) = \pi_1(M_n) \quad \text{and} \quad \pi_1(X_n \xrightarrow{f} X_{n+1}) \cong \pi_1(X_n \xrightarrow{f} X_{n+1})$$. Hence the maps
\[ M_n \xrightarrow{f} X_n \] and \[ \overline{M}_n \xrightarrow{M^0} M_n^{0} \xrightarrow{X_n^0} X_n^{0} \] are \( \pi_1 \)-isomorphisms. The exact sequence

\[ \text{surj.} \quad K_q(U_n) \rightarrow K_q(\partial M_n) + K_q(\partial M_n, U_n) + K_{q-1}U_n + \cdots \]

shows, together with excision, that \( K_k(M_n^0, \partial U_n) = 0 \) for \( k \leq q \), hence also \( K_k(M_n^0) = 0 \) for \( k \leq q-1 \). Next, the exact sequence of \( (\overline{M}_n^{0}, M_n^{0}) \) shows similarly that \( K_k(M_n^{0} - M_n^{0}) = 0 \) but only for \( k \leq q-1 \), while for \( k = q \) we get

\[ K_{q+1}(\overline{M}_n^{0} - M_n^{0}) + \quad K_{q+1}(\overline{V}_n - V_n) + \quad K_{q+1}(\overline{V}_n - V_n) + \quad \overline{\partial} \quad K_q(M_n^{0} - M_n^{0}) \rightarrow 0 \]

where the first map is non-trivial in general, for intersection reason. We can do surgery in the interior of \( M_n^{0} - M_n^{0} \) to kill \( K_q(M_n^{0} - M_n^{0}) \) without altering anything on \( M_n^0 \).

2. **Proposition.** If \( M^0 U V \xrightarrow{f} X^0 \cup H \) is a Mayer-Vietoris decomposition of \( M \xrightarrow{f} X \) as above, and if \( M^0 \xrightarrow{f} X \) is made q-connected, then for some convenient \( X_n^0 \), \( K_q(M_n^0) \# \) is a projective Lagrangian plane in \( K_q(\partial U_n) \# \).

**Proof.** First note that, as \( \partial V \) and \( \partial H \) are both manifolds, the canonical equivalences \( \psi: K_q(\partial^r V_n) + K_{q+1}(\partial^r V_n, \partial U_n) \) and \( \overline{\psi}: K_q(\partial^r V_n, \partial U_n) + K_{q+1}(\partial^r V_n) \) are actually isomorphisms. The induced quadratic module \( K_q(\partial U_n) \) is clearly free hyperbolic, because of the exact sequence
\[ 0 \to K_{q+1}(U_n, \partial U_n) \to K_q(\partial U_n) \to K_q(U_n) \to 0 \]

and the duality isomorphisms between the extreme terms. We apply III. 9 to modify \(X_n \to X_n^0 \to \partial \mathcal{H}\), and get a split exact sequence

\[ 0 \to E \to K_q(\partial U_n)^\# \to K_q(M_n^0)^\# \to 0 \]

exhibiting \(K_q(M_n^0)^\#\) as a Lagrangian plane in a standard hyperbolic module (\(E\) is the image of \(K_{q+1}(M_n^0, \partial U_n)^\#\)).

3. **Cobordism invariance.** Let \(F: W^{2q+3} \to Y\) be a cobordism between \(f^+: M^+ \to X^+\) and \(f^-: M^- \to X^-\) (\(Y\) has a \(2q+3\)-fundamental class \(\mod X^+ \cup X^-\) at \(\infty\), and the inclusions \(X^\pm \subseteq Y\) are simple homotopy equivalences). **Claim:** the Mayer-Vietoris decompositions of \(f^+\) (see §1) extends to \(F\). One can assume that \(X^\pm\) is already decomposed into \(X^0 \cup H^\pm\). By infinite simple homotopy type theory, \(Y\) is simply homotopy equivalent to \(\rel X^+ \cup X^-\) to a CW-complex of the form \(Y^0 \cup H\), where \(H\) is a locally finite 2q+3-handlebody on \(H^+ \cup H^-\) composed of 1 and 2-handles.
Moreover, $H$ is a cobordism on $H^+ \cup H^-$ resulting by surgery on (trivial) 0-spheres in $\mathbb{H}_n^+ \cup \mathbb{H}_n^-$ and then by surgery on 1-spheres in $\mathbb{H}_n^+ - \mathbb{H}_n^+ \cup \mathbb{H}_n^- - \mathbb{H}_n^- (\text{see Chapter II.}).$ Let $\partial^i H = \partial H^+ \cup H^-.$

Denote by $H_n$ the cobordism so obtained on $H_n^+ \cup H_n^-,$ which is a ngbd of $\ast,$ with frontier $\mathbb{H}_n = \text{cobordism obtained on } \mathbb{H}_n^+ \cup \mathbb{H}_n^-.$ Choose subcomplexes $Y^0_n$ in $Y^0$ (ngbd of $\ast$) and finite subcomplexes $\mathbb{Y}^0_n$ containing the frontier of $Y^0_n.$ By using a collar along $\partial^i H,$ we can assume that the subcomplexes $Y_n = Y^0_n \cup H_n,$ $\mathbb{Y}^0_n \cup \mathbb{H}_n$ meet $H$ actually along $H_n$ and $\mathbb{H}_n.$ Moreover, that $\mathbb{Y}^0_n$ is bicoloured in $Y^0$ (hence also $Y_n$ and $Y$). At this stage, we do all the necessary preliminary surgeries, first on $f^:\mathbb{M} \rightarrow X^,$ then on $F: W \rightarrow Y$ rel $\mathbb{M}^+ \cup \mathbb{M}^-.$ In particular, one can kill $K_{q+1}(\overline{W_n-W_n^+} \cup \overline{M_n^+-M_n^-})$ by representing each generator by an embedded $q+1$-sphere inside $\overline{W_n-W_n^+}$ piped to $\overline{M_n^+-M_n^-}$ and subtracting them. This preserves $K_q(M_n^0 \cup \overline{M_n^0}) = 0,$ and $V^\pm$ is unaltered. Also, as in III. 1, we can kill $K_{q+1}(W_n \cup W_n^+) rel \mathbb{M}_n^+ \cup \mathbb{M}_n^-.$ Then we decompose $f^\pm$ into $\mathbb{M}_n^0 \cup V^\pm \rightarrow X^0 \cup \mathbb{H}^+.$ To extend this, consider the
q-spheres $S^q \subset \partial M_n$. They bound immersed $q+1$-discs in $\tilde{W}_n$, which bound, together with the left and right $q+1$-discs in $M^\pm$, immersed $q+2$-discs in $\tilde{W}_n - \tilde{W}_{n+1}$, because $K_{q+1}(\tilde{W}_n - \tilde{W}_{n+1}, M^+_{n-} - M^-_{n+1}) = 0$ and $K_{q+1}(\tilde{W}_n, \tilde{W}_{n+1}) = 0$.

Moreover, one can assume that the $q+1$-discs in $\tilde{W}_n$ generate $K_{q+1}(\tilde{W}_n, \partial M^+_n \cup \partial M^-_n)$ (see lemma 8.1 of [11]). Next we pipe the lower $q+2$-discs to $M^+$ (see figure), connect the $S^q$'s contained in $\partial M^+_n$ and take a regular nbhd $V$ of this connected union of immersions $D^{q+2} + \tilde{W}$. Let $V_n = V \cap \tilde{W}_n$, $U_n = V \cap \tilde{W}_n$, $\partial^r V = \partial V - \partial V^+ \cup \partial V^-$ and $\partial^r U_n = \partial^r V \cap \tilde{W}_n = \partial U_n - \partial U^+_n \cup \partial U^-_n$. 
Now, $\overline{V_n-V_{n+1}}$ is a handlebody on $A_n \equiv U_n \cup \overline{V_n-V_{n+1}} \cup U_{n+1}$ formed by 1, 2 and $q+2$-handles as follows. The self intersections of $D^{q+2}$ are arcs with both ends in $A_n$ and circles. One can exchange the circles into arcs by joining a point of the circle to $A_n$ by one path in each of the two branches crossing through the point, getting an arc with ends in $A_n$, and then isotoping everything along a 2-disc bounded by the arc mod $A_n$.

The preimage of an arc $a$ in $D^{q+2}$ is the disjoint union of two arcs $a', a''$ in $D^{q+2}$, with both ends in $\partial D^{q+2}$. 
Let \( N', N'' \) be regular ngbd of \( \alpha', \alpha'' \) respectively. Then in \( W \), a regular ngbd of \( N' \) coincides with a regular ngbd of \( N'' \) to form a 1-handle with core \( \alpha \).

Next, \( \alpha', \alpha'' \) bound 2-discs \( \Delta', \Delta'' \) mod \( \partial D^{q+2} \), the image of \( \Delta' \) being in one of the two branches through \( \alpha \) and the image of \( \Delta'' \), in the other. Let \( A', A'' \) be regular ngbd of \( \Delta', \Delta'' \) in \( D^{q+2} \), embedded in \( W \). Then a regular ngbd of \( A' \) and a regular ngbd of \( A'' \) in \( W \) matches along the 1-handle with core \( \alpha \), and form a 2-handle attached to \( V^+ \cup \overline{W_n} \cup V^- \).

Observe that \( A' \cup N' \) is a \( q+2 \)-ball attached to \( \partial D^{q+2} \) along a hemisphere, and similarly for \( A'' \cup N'' \). Hence \( D^{q+2} - A' \cup A'' \cup N' \cup N'' \) is a \( q+2 \)-disc, which embeds in \( W \), forming the core of a \( q+2 \)-handle of \( V \). This handlebody structure of \( V \) allows us to arrange \( F, \text{ rel. } M^+ \cup M^- \) so that it induces proper maps of degree 1 \( V \to H, \partial V \to \partial H, W^0 \to Y^0 \), where \( W^0 = \overline{W-V} \) (\( H \) may be smaller). This is the required Mayer-Vietoris decomposition of \( F \). **Claim:** The operations III. 9 which provide the Lagrangian planes \( K_q(M^0_n) \) extend to \( Y^0 \),
rel. V. The problem is to see if the square survives. For the first operation, which enlarge $X_n^{0\pm}$ and $X_n^{0\pm}$ by a piece of $M_n^{0\pm}$, it suffices to enlarge $Y_n^0$ by a corresponding piece of $W_n^0$. We still get a square by taking the old one with extended coefficients. The second operation kills the kernel of $\bar{\psi}^\pm: K_{q+1}(M_n^{0\pm}, \mathcal{M}_n^{0\pm}) \rightarrow K_{q+1}(M_n^{0\pm}, \mathcal{M}_n^{0\pm})$ by adding cells $e^{q+2}$ to $X_n^{0\pm} \cup M_n^{0\pm}$ inside $X_n^{0\pm}$. If we enlarge $Y$ correspondingly (with $e^{q+2}$) then $\bar{\psi}'$ passes to the quotient, because of the commutative diagram.
Then the diagram

\[
0 \to \Phi_{q+1}(M^0_n) \to K^q_c(W_n^0, M_n^0 \cup M_n^0) \to K^q_c(W_n^0) \to 0
\]

provides an induced dotted map, which by duality gives a \( \psi' \).

For the last operation, which enlarges both \( X_n^{0\pm} \) and \( Y_n^{0\pm} \) by the same cells \( c_n^{q+2} \) representing generators of the kernel of 
\( \psi': K_{q+1}(W_n^0) \to K_{q+1}(M_n^0, M_n^0) \), it suffices to enlarge \( Y_n \) correspondingly. The verification that the above square survives runs as above. Now, we can do the operations III.5

\[
W_0 \to Y_n^0 \text{ rel. } \partial W_0.
\]

This will provide the diagram

\[
0 \to K^{q+1}(W_n^0) \to K^q_c(W_n^0) \to K^q_c(W_n) \to 0
\]

which implies that the image \( t \) of the left bottom map is dual to \( K_{q+1}(W_n^0) \). The dual diagram gives a reciprocal duality 
\( K_{q+1}(W_n^0) \cong t \). **Claim**: Via the Lagrangian transformation associated to the canonical maps \( \Phi: K_{q+1}(W_n^0) \to t \) and

\[
\gamma: \Phi_{q+1}(U_n^+) \to \Phi_{q+1}(W_n^+, U_n^+) \to t, \text{ where } t \text{ is } \Phi_{q+1}(W_n^+, U_n^+) \to K_{q+1}(W_n^+, U_n^+ \cup U_n^+) \cong K_{q+1}(U_n, 3U_n)
\]

the Lagrangian plane
\[ K_q(\mathfrak{e}U_n)^\# \text{ in } (\langle K_q(\mathfrak{e}U_n^+) \oplus K_q(\mathfrak{e}U^-) \rangle')^\# \text{ is trivial. The dual } \gamma^* \text{ is such that the composition} \]

\[ \varphi^* \]

\[ \begin{array}{ccc}
\varphi \colon & K_{q+1}(M_n^+) & \longrightarrow & K_{q+1}(U_n) \\
\gamma & & & \gamma^* \\
\end{array} \]

is, via excision, the first map of the exact sequence

\[ K_{q+1}(M_n^+) # + K_{q+1}(M_n^+ M_n^0) # + K_{q+1}(M_n^+) # + K_{q+1}(M_n^-) #. \]

Note that \( \varphi \) vanishes on the image of \( \varphi K_{q+1}(M_n^+) \rightarrow K_{q+1}(U_n) \). The composition

\[ K_{q+1}(\mathfrak{e}U_n, \mathfrak{e}U_n^+ \cup \mathfrak{e}U_n^-) # \longrightarrow \varphi K_{q+1}(U_n^+ # \rangle \longrightarrow \varphi K_{q+1}(U_n^-) # \langle t \rangle \]

vanishes because of the exact sequence

\[ K_{q+2}(W_n, \mathfrak{e}) # \oplus K_{q+2}(U_n, \mathfrak{e}) # + K_{q+1}(\mathfrak{e}U_n, \mathfrak{e}U_n^+ \cup \mathfrak{e}U_n^-) # + K_{q+1}(W_n, M_n \cup M_n^-) #. \]

We consider the Lagrangian plane \( K_{q+1}(\mathfrak{e}U_n, \mathfrak{e}U_n^+ \cup \mathfrak{e}U_n^-) # \oplus t \) in \( (\langle K_q(\mathfrak{e}U_n^+) \oplus K_q(\mathfrak{e}U_n^-) \rangle')^\# \oplus \gamma^* \). Let us parametrize

\[ \varphi K_{q+1}(U_n^+) # \text{ by } x, \varphi K_{q+1}(U_n^+ \mathfrak{e}U_n^-) # \text{ by } p, t \text{ by } t, \text{ and } t^* \text{ by } h. \]

Our Lagrangian plane becomes \( \{(x(u), p(u), t, 0)\} \), where \( u \) describes \( K_{q+1}(\mathfrak{e}U_n, \mathfrak{e}U_n^+ \cup \mathfrak{e}U_n^-) #. \) It projects along \( l_1 \equiv \{(x, t^* y, t, -\gamma x - \phi t)\} \) to a plane which projects along \( \{(x, 0, 0, 0)\} \) isomorphically to \( l_0 \equiv \{(0, p, 0, h)\} \) by diagram chasing as follows. If \( u = 0 \) and \( t \) comes form \( K_{q+1}(M_n^+) #, \) then \( \phi t = 0 \) and we hit \( (0, y^*(K_{q+1}(M_n^+) #, 0, 0). \) By the latter exact sequence, the obstruction to hit all values of \( (0, p, 0, 0) \)
is then the kernel of \( \Phi_K(\mathfrak{Z}^0_+)^\# \rightarrow \Phi_K(\mathfrak{Z}^+_n)^\# \). But we shall see that \( K_{q+1}(\partial^U_n,\partial^U_+ \cup \partial^U_-)^\# \) projects onto \( \Phi_K(\mathfrak{Z}^0_+)^\# \) hence we can hit all \((0,p,0,0)\). Next, a section of \( \Phi_K(\mathfrak{Z}^+_n)^\# \rightarrow \Phi_K(\mathfrak{Z}^+_n)^\# \) provides a map \( K_{q+1}(W_n,\mathfrak{Z}^+_n \cup \mathfrak{Z}^-_n)^\# \rightarrow K_q(\mathfrak{Z}^+_n)^\# \) such that \( h^{-\gamma}(\phi h) \) is of the form \( \phi t \). So we hit all \((\phi h,p,0,h)\), i.e. the graph of \((p,0,h) \mapsto \phi h\). This projects along \(\{(x,0,0,0)\}\) isomorphically to \(\mathcal{H}_0\). **Claim:** In \(\langle K_q(\partial^U_+)^\# \rangle \oplus \langle K_q(\partial^U_-)^\# \rangle\), the Lagrangian plane \(K_q(\mathfrak{Z}^0_+)^\#\) projects along \(K_{q+1}(\partial^U_n,\partial^U_+ \cup \partial^U_-)^\#\) to 0, while the Lagrangian plane \(K_{q+1}(\mathfrak{Z}^0_-,\partial^U_n)^\#\) projects injectively onto a direct summand of \(K_q(\partial^U_n)^\#\). First, the composition \(K_{q+1}(\partial^U_n,\partial^U_+ \cup \partial^U_-)^\# \rightarrow K_q(\partial^U_+ \cup \partial^U_-)^\# \rightarrow K_q(\mathfrak{Z}^0_+)^\#\) is surjective, because in the exact sequence

\[
K_{q+1}(\partial^U_n,\partial^U_+ \cup \partial^U_-)^\# \rightarrow K_q(\partial^U_+ \cup \partial^U_-)^\# \rightarrow K_q(\mathfrak{Z}^0_+)^\# \rightarrow K_q(\mathfrak{Z}^0_-,\partial^U_n)^\#
\]

the right term vanishes by Mayer-Vietoris argument:

\[
\begin{CD}
K_q(\partial^U_+)^\# @>>> K_q(\mathfrak{Z}^0_+)^\# \oplus K_q(\partial^U_n,\partial^U_-)^\# @>>> K_q(\mathfrak{Z}^0_+ \cup \partial^U_+,\partial^U_-)^\# @>>> 0
\end{CD}
\]

Hence a section of \(K_q(\partial^U_+ \cup \partial^U_-)^\# \rightarrow K_q(\mathfrak{Z}^0_+)^\#\) can be obtained by using a section of \(K_{q+1}(\partial^U_n,\partial^U_+ \cup \partial^U_-)^\# \rightarrow K_q(\mathfrak{Z}^0_+)^\#\). Then \(K_q(\mathfrak{Z}^0_+)^\#\) projects to 0 along \(K_{q+1}(\partial^U_n,\partial^U_+ \cup \partial^U_-)^\#\). Note that the same argument would apply to \(K_q(\mathfrak{Z}^0_-)^\#\). Next, by replacing \(K_{q+1}(\mathfrak{Z}^0_-,\partial^U_n)^\#\) by \(K_{q+1}(\mathfrak{Z}^0_- \cup \partial^U_n,\partial^U_n)^\#\), the exact sequence \(K_{q+1}(\mathfrak{Z}^0_- \cup \partial^U_n)^\# \rightarrow K_{q+1}(\mathfrak{Z}^0_- \cup \partial^U_n,\partial^U_n)^\# \rightarrow K_q(\partial^U_n)^\#\)
starts with $0$ by the Mayer-Vietoris argument

$$K_{q+1}(\mathcal{M}_n^0, \mathcal{U}_n^-) \to K_{q+1}(\mathcal{M}_n^0, \mathcal{U}_n^+) \to K_{q+1}(\mathcal{M}_n^0, \mathcal{U}_n^-) \to K_{q+1}(\mathcal{U}_n^-) \to K_{q+1}(\mathcal{U}_n^+) \to K_{q+1}(\mathcal{U}_n^-) \to K_{q+1}(\mathcal{U}_n^+) \to K_{q+1}(\mathcal{U}_n^-) \to K_{q+1}(\mathcal{U}_n^+) \to K_{q+1}(\mathcal{U}_n^-)$$

where $K_{q+1}(\mathcal{U}_n^-) = K_{q+1}(\mathcal{U}_n^+) = 0$ because of the pipes induced in $\mathcal{U}_n$. As a result, the Lagrangian plane

$$K_q(\mathcal{M}_n^0) \oplus K_q(\mathcal{M}_n^0, \mathcal{U}_n^-)$$

in $\langle K_q(\mathcal{U}_n^+) \rangle \oplus \langle K_q(\mathcal{U}_n^-) \rangle$ is trivial (see former claim about $K_q(\mathcal{U}_n^+)$. As $K_{q+1}(\mathcal{M}_n^0, \mathcal{U}_n^-) = K_{q+1}(\mathcal{M}_n^0)$, this means that $K_q(\mathcal{M}_n^0)$ and $K_q(\mathcal{M}_n^0)$ are equivalent.

4. **Theorem.** Let $M$ be an open manifold of dim $2q + 2 \geq 6$ and $f: M \to X$ be a proper normal map of degree 1. Then, to the cobordism class $[f]$ $f$ are associated a sequence

$$\lambda_n \in \lim_{+} L^{q+1}(\pi_1 X)$$

and, if all $\lambda_n$ vanish, an element

$$\sigma_n \in \lim_{+} L^{q+2}(\pi_1 X)$$

such that $[f]$ contains a proper homotopy equivalence at $\infty$ iff all $\lambda_n = 0$ and $\sigma_n = 0$.

**Proof.** With the notations of Proposition 2, define $\lambda_n$ by the Lagrangian plane $K_q(\mathcal{M}_n^0)$ in $K_q(\mathcal{U}_n)$, considering the latter as a standard free hyperbolic form by the exact sequence

$$0 \to K_{q+1}(\mathcal{U}_n^+, \mathcal{U}_n^-) \to K_{q+1}(\mathcal{U}_n^-) + K_q(\mathcal{U}_n) + K_q(\mathcal{U}_n^-) + 0.$$

A "canonical" equivalence between $\lambda_{n+1}^1$ and $\lambda_n$ is obtained similarly as in §3 above, as follows in two steps: first, the Lagrangian plane $K_q(\mathcal{U}_n^-) \mathcal{U}_{n+1})$ in $\langle K_q(\mathcal{U}_n^+) \rangle \oplus \langle K_q(\mathcal{U}_n^-) \rangle$. 


is trivialized by a "canonical" Lagrangian transformation, secondly the Lagrangian plane \((K_q(\mathbf{M}_n^0) \oplus K_q(\mathbf{M}_{n+1}^0))\) in the above hyperbolic module projects along \(K_q(\vartheta^2 V_n - \vartheta^2 V_{n+1})\) onto a direct summand of \(K_q(\vartheta^2 V_n - \vartheta^2 V_{n+1})\). From the diagram

\[
\begin{array}{ccc}
0 & \rightarrow & K^{q+1}(M_n^1, M_n \cup M_{n+1})^\# \\
\uparrow & & \uparrow \approx \\
K^{q+1}(M_n^1 - M_{n+1})^\# & \rightarrow & K^{q+1}(M_n) \rightarrow K^{q+1}(M_{n+1}) \\
\end{array}
\]

and its dual, we get a reciprocal duality between \(t = K^{q+1}(M_n^1 - M_{n+1})^\#\) and the image \(t^*\) of \(K^{q+1}(M_n^1 - M_{n+1})^\#\). Let \(\phi: t^* \rightarrow t\) be the canonical map, and \(\gamma: K_q(U_n^\#) \oplus K_q(U_{n+1}^\#) \rightarrow t\) be a lifting of the canonical map \(K_q(U_n^\#) \oplus K_q(U_{n+1}^\#) \rightarrow K_q(M_n^1) \oplus K_q(M_{n+1}^1)\), which exists because \(t \rightarrow K_q(M_n^1) \oplus K_q(M_{n+1}^1)\) is surjective. Now the whole argument of §3 goes through, with \(\vartheta U_n\) instead of \(\vartheta U_n^+\) and \(\vartheta U_{n+1}\) instead of \(\vartheta U_{n+1}^-\). The above Lagrangian transformation is canonical in the sense that \(\phi\) is canonical. By §3, the element \((\ell_n) \in \lim_{n} L_2 q+1(q_{n+1} X_n)\) is independent of all choices and invariant by cobordism. Suppose that all \(\ell_n = 0\), i.e. for each \(n\) there is a Lagrangian transformation \(a_n\) of \(K_q(\vartheta U_n)^\#\) which trivializes \(K_q(\mathbf{M}_n^0)^\#\). By superposition, we get
a Lagrangian transformation $\beta_0(\alpha_n \oplus \alpha_{n+1}^{-1})$ of
$\left( k_q(\partial U_n)^# \right) \oplus \left( k_q(\partial U_{n+1})^# \right)'$, where $\beta$ is the above canonical transformation. As $k_q(M_n)^# \oplus k_q(M_{n+1})^#$ projects onto a direct summand of $k_q(\partial^* V_n - \partial^* V_{n+1})^#$, the above composite Lagrangian transformation carries the standard Lagrangian plane $k_q(U_n)^# \oplus k_q(U_{n+1})^#$ to a trivial plane. According to [5], this transformation determines a non-singular quadratic module $Q_n \in L_{2q+2}(\pi_1 X_n)$. In the case $\alpha = id$ this is nothing but the non-singular part of $C$, a direct summand of $C$ on which the intersection pairing is non-singular. If one changes the choice of the Lagrangian transformations $\alpha_n$, then the sequence $(Q_n)$ is altered by a sequence in the image of $S$ (see III.10 for definition). The same is true if one replaces $f: M \to X$ by a cobordant map. Actually, if $F: W^{2q+3} \to Y$ is a cobordism between $f^+: M^+ \to X^+$, then as in Chapter III one produces singular quadratic modules $k_{q+1}(W_n)^#$ whose non-singular part determines an element $\omega_n \in L_{2q+2}(\pi_1 X_n)$

\[
\begin{array}{c|c}
M_n^+ & M_{n+1}^+ \\
\hline
W_n & W_{n+1} \\
\hline
M_n^- & M_{n+1}^-
\end{array}
\]

Then from the exact sequence

$K_{q+2}(\overline{W_n - W_{n+1}}, \partial)^# \to K_{q+1}(W_n)^# \oplus K_{q+1}(M_n^+ - M_{n+1}^+) \oplus K_{q+1}(W_{n+1})^# \to K_{q+1}(\overline{W_n - W_{n+1}})^# \to 0$
one deduces a trivialization of the non-singular part of the middle quadratic module, i.e. \( \sigma_n^+ - \sigma_n^- = \omega_n^+ - \omega_{n+1}^\# \). In this way we get a well-defined element \( (\sigma_n^+ \in \lim_{+} L_2 q+2 (\pi_1 X_n) \). Note that by [11] one can arrange to get \( \alpha_n = \text{id} \). Suppose then that \( (\sigma_n^+ = 0 \). This means that the intersection pairing on \( K_{q+1} (M_n - \bar{M}_{n+1}) \) has the following property:

\[ K_{q+1} (M_n - \bar{M}_{n+1}) = \ker \phi \otimes H_n, \] where the form \( H_n \) is hyperbolic.

By Mayer-Vietoris argument, the \( H_r \) for \( r > n \) do not match up in \( K_{q+1} (M_n) \), so one sees that there is a subsystem

\[ Q_n = \bigoplus_{r > n} H_r \subset K_{q+1} (M_n), \]

such that the inclusion is an equivalence and \( Q_n \) is a projective hyperbolic form which can be assumed free of countable rank: \( Q_n \equiv U_n \oplus (U_n)^\# \) (the second factor is the dual with compact support). Now, each basis element \( u \) can be represented by an embedded sphere \( S^{q+1} \subset M_n^{2q+2} \) (because \( \langle u, u \rangle = 0 \)). By piping each \( S^{q+1} \) to \( \infty \) and carving out the result (as in Chapter II), one verifies easily that \( Q_n \) is killed, and the new inverse system \( \{ K_{q+1} (M_n) \} \) becomes equivalent to \( 0 \). In other words, we have found a cobordism to a proper homotopy equivalence at \( \infty \).

5. **Corollary** (see III.11). We have an exact sequence

\[ 0 \rightarrow \lim_{+} L_2 q+2 (\pi_1 X_n) \rightarrow L_2 q+2 (eX) \rightarrow \lim_{+} L_2 q+1 (\pi_1 X_n) \rightarrow L_2 q+1 (\pi_1 X). \]

The proof is analog to III. 11. This can also be globalized as in III.12, to form an exact sequence.
\[ \pi_{2q+2} \xrightarrow{1-\Sigma} L_{2q+2}(\pi_1 X) \oplus \pi_{2q+2} + L_{2q+2}(X) \xrightarrow{\pi_{2q+1} \rightarrow 1} L_{2q+1}(\pi_1 X) \oplus \pi_{2q+1}. \]

Together with III. 12, this provides a long exact sequence.
1. An inverse system of groups $\{G_n\}$ is a sequence of homomorphisms $G_1 \leftarrow G_2 \leftarrow \ldots$ and an inverse system of modules $\{A_n\}$, where $A_n$ is a $G_n$-module, is a sequence of pseudo-linear maps $A_1 \leftarrow A_2 \leftarrow \ldots$. A morphism $(\alpha): \{A_n\} \to \{A'_n\}$ is a class of compatible pseudo-linear maps $A_r \to A'_r$, for some subsequences $r_n, r'_n$ where $[\alpha] \sim [\beta]$ if the diagram

\[\begin{array}{ccc}
A_{r_n} & \xrightarrow{\alpha} & A'_{r_n} \\
\downarrow & & \downarrow \\
A_{u_n} & \xrightarrow{\beta} & A'_{u_n}
\end{array}\]

commutes for some subsequences $u_n, u'_n$. Two morphisms $\{A_n\} \to \{A'_n\}$, $\{A''_n\} \to \{A''''_n\}$ may be composed in a well-defined class. In particular, there are defined canonical isomorphisms $\{A_n\} \to \{A^\#_n\}$ and $\{A^\#_{n+1}\} \to \{A_n\}$. By reversing all the arrows, we get the notion of a direct system. The following progressive assertions are easy to prove (for both direct and inverse systems).

2. A system $\{A_n\}$ is equivalent to $0$ iff, for some subsequence $r_n'$, the maps $A_{r_{n+1}} \to A_{r_n}$ are $0$.

3. Let $\alpha: \{A_n\} \to \{B_n\}$ be an equivalence of systems given by $\alpha_n: A_n \to B_n$. Then the systems $\{\ker \alpha_n\}$ and $\{\coker \alpha_n\}$ are equivalent to $\{0\}$.

4. Let $0 \to \{A_n\} \in \alpha \{B_n\} \xrightarrow{\beta} \{C_n\} \to 0$ be an exact sequence of systems, i.e. for some subsequence $r_n \leq s_n \leq t_n < r_{n+1}$ the sequences $0 \to A^\#_{r_n} \alpha_n \to B^\#_{s_n} \xrightarrow{\beta_n} C_t \to 0$ are exact. Then $\alpha$,
res. β, is an equivalence iff \( \{C_n\} \), resp. \( \{A_n\} \), is equivalent to \( \{0\} \).

5. A morphism of systems \( \alpha: \{A_n\} \rightarrow \{B_n\} \) is an equivalence iff the systems \( \{\ker \alpha_n\} \) and \( \{\coker \alpha_n\} \) are equivalent to \( \{0\} \).

6. Let \( \xymatrix{ \{A_n\} \ar[r]^-{\alpha} \ar[d]^-{\phi} & \{B_n\} \ar[d]^-{\phi'} \cr \{A'_n\} \ar[r]^-{\alpha'} & \{B'_n\} } \) be a commutative square of systems, i.e. for some subsequences \( r_n, s_n, t_n, u_n \), the squares \( \xymatrix{ A_r \ar[r]^-{\alpha_n} \ar[d]^-{\phi_n} & B_r \ar[d]^-{\phi'_n} \cr A_t \ar[r]^-{\alpha'_n} & B_t } \) are commutative. If \( \phi \) and \( \phi' \) are equivalences, then so are the induced morphisms \( \{\ker \alpha_n\} \) \( \rightarrow \) \( \{\ker \alpha_n\} \) and \( \{\coker \alpha_n\} \) \( \rightarrow \) \( \{\coker \alpha_n\} \).

7. Let \( \xymatrix{ \{0\} \ar[r]^-{} \ar[d]^-{\alpha} & \{A_n\} \ar[r]^-{\beta} \ar[d]^-{\gamma} & \{B_n\} \ar[d]^-{} \ar[r]^-{\gamma'} & \{C_n\} \ar[r]^-{} & \{0\} } \) be a commutative exact ladder of systems. Then, if two of the morphisms \( \alpha, \beta, \gamma \) are equivalences so is the third.
8. The five lemma holds for systems.

9. **Proposition.** Let \{C(n)\} be a system of chain complexes. Assume that each \( C(n) \) has the form

\[
0 \to C_L(n) \to \ldots \to C_1(n) \to C_0(n) \to 0
\]

where \( L > 0 \) is independent of \( n \), each \( C_k(n) \) is free of countable (resp. finite) rank, moreover, that the associated homology system \{H_k(n)\} are equivalent to \( \{0\} \) for \( k < L \). Then \{H_L(n)\} is equivalent by injections \( H_L(n) \to P_n \) to a system of countably (resp. finitely) generated projective module \( P_n \).

**Proof.** To fix the idea, suppose the system is inverse.

By induction on \( r \leq L \), we can factorize \( C(n) \to C(n-r) \) through a free chain complex \( E(n) \) of the above form, such that \( H_k E(n) = 0 \) for \( k < r \). For \( r = 0 \), take \( E(n) = C(n) \). Suppose we are done for \( r-1 \). By the folding trick (see [6]), \( E(n) \) is chain homotopy equivalent to a similar chain complex null in dimension \( < r-1 \). Hence \( H_{r-1} E(n) \) is countably (resp. finitely) generated. Let \( \{z_1\} \) be a countable (resp. finite) set of \( (r-1) \)-cycles in \( E_{r-1}(n) \) generating \( H_{r-1} E(n) \), and \( F \) the free module on \( \{z_1\} \). Define a chain complex \( \Phi(n) \) by

\[
0 \to E_L(n) \to \ldots \to E_{r+1}(n) \to E_r(n) \to E_{r-1}(n) \to \ldots \to E_0(n) \to 0
\]

\[
\Phi \quad \Phi \quad \Phi
\]

\[
0 \to E_L(n) \to \ldots \to E_{r+1}(n) \to E_r(n) \to E_{r-1}(n) \to \ldots \to E_0(n) \to 0
\]
where \( \alpha_p(z_i) = z_i \). We can assume that the chain map
\[ C(n-r+1) \longrightarrow C(n-r) \] induces 0 on homology in dimensions < \( L \), hence so does the composite map \( E(n) \longrightarrow C(n-r+1) \longrightarrow C(n-r) \).
This implies that the map \( F \xrightarrow{\alpha_p} E_{r-1}(n) \longrightarrow C_{r-1}(n-r) \) has its image in \( \partial C_r(n-r) \), so can be lifted to \( C_r(n-r) \). This provides a factorization
\[ C(n) \longrightarrow E(n) \xrightarrow{\text{incl.}} \overline{E}(n) \longrightarrow C(n-r) \]
where \( H_k \overline{E}(n) = 0 \) for \( k < r \). When we reach \( r = L \), \( E(n) \) has homology only in the top dimension \( L \), hence \( H_L E(n) \) is a direct summand \( P_n \) of \( E_L(n) \) (ibid.) Finally, the injections \( C(n) \longrightarrow E(n) \) induce the equivalence \( H_L(n) \longrightarrow P_n \).

10. Addendum. There is a system of projective modules \( P_n' \), such that the image of \( P_{n+1} \longrightarrow P_n \) is a retract (in particular projective), and an equivalence \( H_L(n) \longrightarrow P_n \) which is injective for all \( n \).

**Proof:** We can replace \( \{P_n\} \) by the inverse system

\[
\begin{align*}
P_1 & \xleftarrow{I} P_2 & \xleftarrow{I} P_3 & \xleftarrow{I} P_4 & \ldots \\
P & \xleftarrow{I} P_2 & \xleftarrow{I} P_3 & \xleftarrow{I} P_4 & \ldots
\end{align*}
\]

which contains \( \{P_n\} \) as an equivalent retract. This can also be done at chain level.

11. Addendum. If all \( \{H_k(n)\} \) are equivalent to \( \{0\} \), then \( C(n) \longrightarrow C(n-L-i) \) is chain homotopic to 0.

**Proof:** As in the proof of Proposition 9, we can factorize this map through a projective acyclic chain complex.
12. Corollary. Let \( \alpha: \{A(n)\} \rightarrow \{B(n)\} \) be a map of free chain systems (each \( A(n), B(n) \) is free and of finite dimension \( \leq L \) independent of \( n \)) inducing an equivalence on the associated homology systems. Then so does the dual map \( \alpha^*: \{B^*(n)\} \rightarrow \{A^*(n)\} \).

Proof. By applying the above addendum to the mapping cylinders \( M(n) \) of \( A(n) \rightarrow B(n) \), we see that \( \{M(n)\} \) is equivalent to a system of free acyclic chain complexes. Hence so is the dual system \( \{M^*(n)\} \).
REFERENCES

[1] W.Browder  Surgery on simply-connected manifolds
               Springer (1972)

               Algebraic torsion for infinite simple homotopy types

       ibid., 474 - 501 (1972)

               Pac. J. Math. 12, 337 - 341 (1962)

               analogues of K-theory for rings with involution, from
               the point of view of the hamiltonian formalism. Some
               applications to differential topology and the theory
               of characteristic classes
               Izv. Akad. Nauk SSSR ser. mat. 34, I. 253 - 288,
               II. 478 - 500 (1970)

               Torsion et type simple d'homotopie
               Springer Lecture Notes 48 (1967)

               even-dimensional case
               Ann. of Maths. 98, 187 - 209 (1973)

[8] L.Siebenmann
               The obstruction to finding a boundary for an open
               manifold of dimension greater than five
               Princeton Ph.D. thesis (1965)

[9]  Infinite simple homotopy types
[10] L.R.Taylor  
Surgery on paracompact manifolds  
Berkeley Ph.D. thesis (1972)

Surgery on compact manifolds  

[12] S.Maumary  
Proper surgery groups and Wall-Novikov groups  
Springer Lecture Notes 343 (1973)

Wall's surgery obstruction groups for $G \times \mathbb{Z}$  
Ann. of Maths. 90, 296 - 334 (1969)

[14] A.A.Ranicki  
Algebraic L-theory I. Foundations, II. Laurent extensions  

Projective surgery theory  
Topology 19, 239 - 254 (1980)