Short Notes: Completion of a Symmetric Unitary Matrix
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COMPLETION OF A SYMMETRIC UNITARY MATRIX

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The symmetric unitary matrix has received special attention in physics and electrical engineering because, "The scattering matrix is symmetrical and unitary for a lossless junction" [1]. The present note resulted from an attempt to extend the work of Harold F. Mathis [2] to find a simple procedure for designing a lossless junction, given any number of rows in its scattering matrix.

A given set of orthonormal vectors can be extended to a complete orthonormal set by adjoining arbitrary linearly independent vectors, using the Gram-Schmidt orthogonalization process, and normalizing. If the given set of orthonormal vectors are considered the first rows of a matrix, this method can be used to complete a unitary matrix. If the given rows satisfy a simple minimum condition, the following method yields a symmetric unitary matrix. This method proceeds, as does the Gram-Schmidt process, by constructing one row at a time.

THEOREM. Given a square matrix $A$ and a rectangular matrix $B$ with the same number of rows as $A$, the rectangular matrix $(A:B)$ can be extended to a symmetric unitary matrix $\begin{bmatrix} A & B \\ B^T & X \end{bmatrix}$ if and only if $A = A^T$ and $AA^* + BB^* = I$.

Proof of necessity. From the definition of a unitary matrix,

$$\begin{bmatrix} A & B \\ B^T & X \end{bmatrix} \begin{bmatrix} A^* & B^* \\ B^* & X^* \end{bmatrix} = I.$$

This implies that $AA^* + BB^* = I$. The symmetry of $\begin{bmatrix} A & B \\ B^T & X \end{bmatrix}$ implies $A = A^T$.

Proof of sufficiency. The matrix $X$ to be found must be symmetric. The proof will be by induction on the number of rows of $X$. In particular, the first column of $X$ will be constructed, and it will be shown that when $A$ and $B$ are enlarged using the elements of this column, the resulting incomplete matrix again satisfies the conditions of the theorem.

Equation (1) is expanded to give

$$AA^* + BB^* = I,$$

$$A\bar{B} + BX^* = 0,$$

$$B^TA^* + XB^* = 0,$$

$$B^T\bar{B} + XX^* = I.$$

Equation (4) is the conjugate transpose of (3), so it is only necessary to consider (3) and (5). Let $X_1, B_1$ and $0_1$ denote the first columns of $X, B$ and $0$, respectively.
After taking the conjugate of (3), the corresponding equations for $X_1$ are

$$\bar{A}B_1 + \bar{B}X_1 = 0_1,$$
(6)

$$B_1^T\bar{B}_1 + X_1^T\bar{X}_1 = I.$$  
(7)

First it will be shown that it is always possible to find a column matrix $a$ such that $BB^*a = B_1$. If $BB^*$ is nonsingular, it is obvious that $a$ exists. If $BB^*$ is singular, there exist nonzero column matrices $\lambda$ such that $\lambda^TBB^* = 0_1^T$. For any such $\lambda, \lambda^TBB^*\lambda = 0 = (B_1^T\lambda)^T(B_1^T\lambda)$, which requires $B_1^T\lambda = 0_1$ and $B_1^T\lambda = \lambda^TB_1 = 0$. Consequently, $BB^*$ and the augmented matrix formed by adding $B_1$ as an extra column to $BB^*$ have the same rank. This is a well-known criterion for the existence of $a$.

Finally, let $X_1 = B_1^T(\bar{a} - A^*a) - I_1$, where $I_1$ denotes the first column of $I$. Next, using (2), it is verified that $X_1$ satisfies equations (6) and (7):

$$\bar{A}B_1 + \bar{B}X_1 = \bar{A}B_1 + BB^*(\bar{a} - \bar{A}a) - B_1$$
$$= \bar{A}B_1 + \bar{B}_1 - (I - \bar{A}A)\bar{A}a - \bar{B}_1$$
$$= \bar{A}B_1 - \bar{A}(I - A\bar{A})a$$
$$= \bar{A}B_1 - \bar{A}BB^*a = 0_1,$$

$$B_1^T\bar{B}_1 + X_1^T\bar{X}_1 = B_1^T\bar{B}_1 + (\bar{a}^TB - a^T\bar{A}B - I_1^T)(B^*a - B^*A\bar{a} - I_1)$$
$$= B_1^T\bar{B}_1 + \bar{a}^TB_1 - \bar{a}^T(I - A\bar{A})A\bar{a} - \bar{a}^TB_1$$
$$= a^T\bar{A}B_1 + a^T\bar{A}(I - A\bar{A})A\bar{a} + a^T\bar{A}B_1$$
$$= B_1^T\bar{B}_1 - \bar{a}^TABB^\bar{a} + a^T(I - BB^T)BB^\bar{a}$$
$$= B_1^T\bar{B}_1 - \bar{a}^TABB^\bar{a} + a^T(I - BB^T)BB^\bar{a}$$
$$+ B_1^T\bar{a} + \bar{B}_1^T\bar{A}a + 1$$
$$= B_1^T\bar{B}_1 - (BB^*a)^T\bar{B}_1 + 1 = 1.$$

Let $B = (B_1 \mid B_2)$, let $x_{11}$ be the first element of $X_1$ and let $X_1^T = (x_{11} \mid X_2^T)$. Now $A$ is changed to $\left[ \begin{array}{c} A \\ B_1^T \\ x_{11} \end{array} \right]$ and $B$ is changed to $\left[ \begin{array}{c} B_2^T \\ X_2^T \end{array} \right]$. So (2) becomes

$$\begin{bmatrix} A & B_1^T \\ B_1 & x_{11} \end{bmatrix} \begin{bmatrix} A & B_1^T \\ B_1 & x_{11} \end{bmatrix}^* + \begin{bmatrix} B_2^T \\ X_2^T \end{bmatrix} \begin{bmatrix} B_2^T \\ X_2^T \end{bmatrix}^* = I.$$  
(8)

Equation (8) follows directly from (2), (6) and (7).

Thus the desired column matrix $X_1$ can be constructed. The first row of $X$ is $X_1^T$, and the first column of $X$ is $X_1$. This process can be repeated until the complete matrix $X$ is found. If the first column of $B$ is interchanged with some other column, it is possible to begin constructing $X$ with this column. It is obvious from the nature of the problem and the proof that $X$ need not be unique.

Remark. In some cases the computation of $X_1$ is easier and more direct than in the proof of the theorem. If $B_1 = 0_1$, then $X_1 = I_1$ is a solution. If there exists
a column matrix $b$ such that $B^*b = I_1$, let $X_1 = -B^T \bar{A}b$. It can be verified that this $X_1$ satisfies (6) and (7).

If there exists a column matrix $c$ such that $B^*Bc = I_1$, let $b = Bc$. If $B^*B$ is nonsingular, it is always possible to solve the equation $B^*Bc = I_1$. Consequently, if $B^*B$ is nonsingular, $X_1 = -B^T \bar{A}B(B^*B)^{-1}I_1$ satisfies (6) and (7).

REFERENCES
