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EXACT COUPLES IN ALGEBRAIC TOPOLOGY

(Parts III, IV, and V)

By W. S. Massey

(Received January 29, 1952)

Introduction

This is a continuation of a previous paper.¹ For a statement of the purpose of this paper and an outline of the organization of the two papers combined, the reader is referred to the introduction of the previous paper. It is assumed that the reader is familiar with all of Part I; in order to understand Part IV of this paper, an understanding of Part II is also necessary.

PART III. THE COHOMOTOPY EXACT COUPLE OF A COMPLEX

1. Definition of the Cohomotopy Exact Couple

In this part we shall be concerned with the Borsuk-Spanier cohomotopy groups. For the conventions and notations used, see the appendix.

Let $K$ be finite² cell complex (cf. J. H. C. Whitehead, [21]; note that a finite complex is automatically a CW-complex). As usual, let $K^n$ denote the $n$-skeleton of $K$, with the understanding that $K^n$ denotes the empty set for $n < 0$. Define

$$A^{p,q}(K) = \pi^{p,q}(K^p),$$

$$C^{p,q}(K) = \pi^{p-1,q}(K^p, K^{p-1}).$$

Define $i: A^{p,q}(K) \to A^{p-1,q+1}(K)$ and $j: C^{p,q}(K) \to A^{p,q}(K)$ to be injections, and $\delta: A^{p,q}(K) \to C^{p+1,q}(K)$ to be the cohomotopy coboundary operator. The groups $A^{p,q}(K)$ and $C^{p,q}(K)$ and the homomorphisms $i$, $j$, and $\delta$ may be conveniently displayed on one diagram, as shown in figure 1. Note that $A^{p,q}(K) = C^{p,q}(K) = 0$ if $q < 0$. Also, note that exactness holds along any path in this diagram which continually moves downward and to the right in a zig-zag direction; to be precise, at each stage,

(1.1) \quad \text{image } i = \text{kernel } \delta,

(1.2) \quad \text{image } \delta = \text{kernel } j,

(1.3) \quad \text{image } j = \text{kernel } i.

Our immediate object is to extend our definitions so that $A^{p,q}(K)$ and $C^{p,q}(K)$

¹ "Exact Couples in Algebraic Topology," (Parts I and II), Ann. of Math., 56 (1952), 364-396. References to this previous paper will be made simply as "Part I," or "Part II," as the case may be.

² The finiteness condition is imposed to insure that $K$ will be compact. It seems probable that the assumption of compactness could be relaxed considerably; however, we shall not try to do this.

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and the homomorphisms $i, j, \text{and } \delta$ are defined for all values of $p$ and $q$, and so that the three exactness conditions listed above still hold. To this end, let $D^q(K)$ be the subgroup of $C^{2q+2,q}(K)$ which is the kernel of $j:C^{2q+2,q}(K) \to A^{2q+2,q}(K)$

$$
\begin{align*}
C_{q,2} & \to \ldots \\
C_{q,1} & \to A_{q,1} \to C_{q,1} \to \ldots \\
C_{q,0} & \to A_{q,0} \to C_{q,0} \to A_{q,0} \to C_{q,0} \to \ldots \\
C_{q,-1} & \to A_{q,-1} \to C_{q,-1} \to A_{q,-1} \to C_{q,-1} \to \ldots \\
C_{q,-2} & \to A_{q,-2} \to C_{q,-2} \to A_{q,-2} \to C_{q,-2} \to \ldots \\
C_{q,-3} & \to A_{q,-3} \to C_{q,-3} \to A_{q,-3} \to \ldots \\
C_{q,-4} & \to \ldots
\end{align*}
$$

Fig. 1

and let $E^q(K)$ be the subgroup of $A^{2q,q-1}(K)$ which is the kernel of $\delta:A^{2q,q-1}(K) \to C^{2q+1,q-1}(K)$, where $q \geq 0$ in both cases. Define

$$
A^{2q+1,q}(K) = D^q(K) \oplus E^q(K)
$$

for $q \geq 0$. Define

$$
\begin{align*}
\delta &: A^{2q+1,q}(K) \to C^{2q+2,q}(K) , \\
i &: A^{2q+1,q}(K) \to A^{2q,q+1}(K)
\end{align*}
$$

to be the projections of $A^{2q+1,q}$ onto its direct summands, $D^q(K)$ and $E^q(K)$ respectively. Define

$$
\begin{align*}
C^{2q+1,q}(K) &= D^q(K), \\
A^{2q+2,q+1}(K) &= E^q(K),
\end{align*}
$$

and let

$$
\begin{align*}
j &: C^{2q+1,q} \to A^{2q+1,q} , \\
i &: A^{2q+2,q+1} \to A^{2q+1,q}
\end{align*}
$$

be inclusion maps. Define

$$
\delta : A^{2q+2,q+1} \to C^{2q+3,q+1}
$$

to be the trivial map. Define $C_{p,q}(K) = 0$ for all values of $p$ and $q$ for which $C_{p,q}(K)$ has not already been defined,

$$
A^{p,q}(K) = A^{2(p-q),p-q}(K)
$$

if $p > 2q$ and $p - q > 0$, and $A^{p,q} = 0$ for all other values of $p$ and $q$ for which $A^{p,q}$ has not already been defined. Define the homomorphisms $i, j, \text{and } \delta$ in the
remaining cases in the obvious way—isomorphisms onto, or trivial homomorphisms.

It can be readily verified that all these definitions have been made so that the three exactness conditions above, (1.1), (1.2), and (1.3), still remain true. Let

\[ A^*(K) = \sum A_{p,q}^*(K), \]
\[ C^*(K) = \sum C_{p,q}^*(K). \]

The homomorphisms \( i, j, \) and \( \delta \) define homomorphisms

\[ i : A^*(K) \to A^*(K), \]
\[ j : C^*(K) \to A^*(K), \]
\[ \delta : A^*(K) \to C^*(K), \]

which are homogeneous of degrees \((-1, -1), (0, 0), \) and \((1, 0)\) respectively. These groups and homomorphisms constitute an exact couple,

\[ \langle A^*(K), C^*(K) ; i, \delta, j \rangle, \]
called the cohomotopy exact couple of \( K. \) The successive derived couples will be denoted by \( \langle \Gamma^*_r(K), 3c_r^*(K) \rangle, \langle \Gamma^*_1(K), 3c_1(K) \rangle, \langle \Gamma^*_2(K), 3c_2(K) \rangle, \ldots, \) etc. Each of these groups is bigraded:

\[ \Gamma^*_r(K) = \sum \Gamma_{r,p}^*(K), \]
\[ 3c^*_r(K) = \sum 3c_{r,p}^*(K). \]

2. The Map Induced by a Cellular Map

Let \( K \) and \( L \) be finite cell complexes, and \( f : K \to L \) a cellular map. The map \( f \) induces homomorphisms

\[ f^* : \pi^{p,q}(L^p) \to \pi^{p,q}(K^p), \]
\[ f^* : \pi^{p,q}(L, L^{-1}) \to \pi^{p,q}(K, K^{-1}) \]

provided \( p > 2q + 1, p \geq 0; \) furthermore, the homomorphisms \( f^* \) and \( f^* \) commute with the homomorphisms \( i, j, \) and \( \delta. \) From this, it readily follows that

\[ f^*[D^q(L)] \subset D^q(K), \]
\[ f^*[E^q(L)] \subset E^q(K), \]

for any \( q \geq 0. \) Therefore \( f^* \) defines a homomorphism \( D^q(L) \to D^q(K), \) and \( f^* \) defines a homomorphism \( E^q(L) \to E^q(K). \) Using this fact, it is possible to extend the definition of \( f^* \) and \( f^* \), so that we have homomorphisms

\[ f^* : A_{p,q}^*(L) \to A_{p,q}^*(K), \]
\[ f^* : C_{p,q}^*(L) \to C_{p,q}^*(K), \]
defined for all values of \( p \) and \( q \), and so that these homomorphisms commute with the homomorphisms \( i, j, \) and \( \delta \) defined for the complexes \( K \) and \( L \). Thus we have defined a map of exact couples,

\[
(f^*, f^*): \langle A^*(L), C^*(L) \rangle \to \langle A^*(K), C^*(K) \rangle,
\]

and both \( f^* \) and \( f^* \) are homogeneous of degree \((0, 0)\).

Let \( \mathfrak{S} \) denote the category in which the objects are all finite cell complexes, and the maps are all cellular maps, and let \( \mathfrak{E} \) denote the category in which the objects are bigraded exact couples with homogeneous homomorphisms, and the maps consist of pairs of homomorphisms, both of which are homogeneous of degree \((0, 0)\). Then it is clear that the operation of assigning to each finite cell complex \( K \) the exact couple \( \langle A^*(K), C^*(K) \rangle \) and to each cellular map \( f \) the induced map \((f^*, f^*)\) is a contravariant functor from \( \mathfrak{S} \) to \( \mathfrak{E} \).

3. Identification of Some of the Groups in the First Derived Couple

Since \( \mathfrak{C}^{p,q}(K) = 0 \) if \( p > 2q + 1 \), or if \( q < 0 \), or if \( p - q < 1 \), or if \( p > \dim K \), it follows that \( \mathfrak{C}^{p,q}(K) = 0 \) also if any of these conditions obtain. It also follows directly that \( \mathfrak{C}^{p,q}(K) = 0 \) if \( p = 2q + 1 \), because the differential operator \( \delta \circ j = d: C^{2q+1,q} \to C^{2q+2,q} \) is an isomorphism into, by definition. Since \( A^{p,q}(K) = 0 \) if \( q < 0 \) or \( p + q < 1 \), it follows that \( \Gamma^{p,q}(K) = 0 \) under either of these conditions. In case \( p \geq \dim K \) and \( \dim K < 2(p - q) - 1 \), \( \Lambda^{p,q}(K) = \pi^{p,q}(K) \), and the homomorphism \( i:A^{p+1,q+1}(K) \to A^{p,q}(K) \) is an isomorphism onto; from this it follows that if \( 2(p - q) - 1 > \dim K \) and \( p \geq \dim K \), then \( \Gamma^{p,q}(K) \approx \pi^{p,q}(K) \).

For future reference, we note that the homomorphism

\[
\delta': \Gamma^{2q+1,q}(K) \to \mathfrak{C}^{2q+4,q+1}(K)
\]

is trivial; this follows directly from the definitions. From this, the fact that \( \mathfrak{C}^{p,q}(K) = 0 \) if \( p \geq 2q + 1 \), and exactness, it follows that for any fixed value of \( q \), the groups \( \Gamma^{2q+n+1,q+n}(K) \) are all isomorphic for \( n = 0, 1, 2, \cdots \). To be precise, the homomorphism

\[
j': \Gamma^{2q+2,q+n+1}(K) \to \Gamma^{2q+n+2,q+n}(K)
\]

is an isomorphism onto for \( n \geq 0 \). It also follows from (3.1) and exactness that the homomorphism

\[
f': \mathfrak{C}^{2q+2,q}(K) \to \Gamma^{2q+2,q}(K)
\]

is an isomorphism into.

Next we define isomorphisms

\[
\Xi^q: \mathfrak{S} \otimes C^p(K) \to C^{p,q}(K) \quad (p > 2q + 1)
\]

as follows. Let

\[
\psi: \pi^p(K^p, K^{p-1}) \to C^p(K)
\]

(where \( C^p(K) = H^p(K^p, K^{p-1}) \) is the group of integral \( p \)-chains) denote natural isomorphism defined by Spanier in Section 11 of [14] (cf. also theorem 17.5 of
By means of this isomorphism, we will identify the groups $\pi^p(K^p, K^{p-1}) = C^{p,0}(K)$ and $C^p(K)$. The group $\xi_p$ is defined in Section 12 of Part II. Then for $\alpha \in \xi_p = \pi^{p-q}(S^p, p_0)$ and $\beta \in C^p(K) = \pi^p(K^p, K^{p-1})$, define

$$\Xi^p(\alpha \otimes \beta) = \alpha \circ \beta.$$ 

That $\Xi^p$ is a homomorphism follows from the bilinearity of the composition operation (see the Appendix, Section 21). That $\Xi^p$ is an isomorphism onto follows from theorem 21.4. If $\delta: C^p(K) \rightarrow C^{p+1}(K)$ denotes the coboundary operator, then the relation

$$\Xi^p(\alpha \otimes \delta \beta) = d(\alpha \otimes \beta)$$

follows readily from equation (21.3). Thus $\Xi^p$ is an allowable homomorphism. Denote the induced homomorphism by

$$\Xi^p: H^p(K, \xi_p) \rightarrow \mathcal{C}^{p,q}(K).$$

**Theorem 3.1.** If $p > 2q + 2$, then $\Xi^p$ is an isomorphism onto.

This follows directly from the preceding discussion.

That the isomorphism $\Xi^p$ thus defined is "natural" in the sense of Eilenberg and MacLane, [3], is readily seen.

4. The Algebraic Homotopy Associated With a Given Cellular Homotopy

Let $K$ and $L$ be finite cell complexes, $f_0, f_1: L \rightarrow K$ be cellular maps, and $f_t: L \rightarrow K$, $0 \leq t \leq 1$, a cellular homotopy between $f_0$ and $f_1$. The maps $f_0$ and $f_1$ induce maps

$$(f_0^*, f_0^*), (f_1^*, f_1^*): \langle A^*(K), C^*(K) \rangle \rightarrow \langle A^*(L), C^*(L) \rangle.$$

We should like to prove that the maps $(f_0^*, f_0^*)$ and $(f_1^*, f_1^*)$ are algebraically homotopic, just as in the case of the homotopy exact couple, and thus prove that $(f_0^*, f_0^*)$ and $(f_1^*, f_1^*)$ induce the same maps $\langle \Gamma^*(K), \mathcal{C}^*(K) \rangle \rightarrow \langle \Gamma^*(L), \mathcal{C}^*(L) \rangle$ of the first derived couple. Unfortunately, it does not seem possible to carry out this program in this manner. We will ultimately prove that the maps $(f_0^*, f_0^*)$ and $(f_1^*, f_1^*)$ induce the same homomorphisms of the first derived couples, but the proof will be rather round-about. The first step is the following lemma.

**Lemma 4.1.** For $p \geq 2q + 3$ there exist homomorphisms

$$\xi_{p,q}: C^{p,q}(K) \rightarrow C^{p-1,q}(L)$$

such that in the following two diagrams,

$$
\begin{array}{ccc}
C^{p,q}(K) & \xrightarrow{d} & C^{p+1,q}(K) \\
\downarrow \xi_{p,q} & \nearrow f_0^* - f_1^* & \downarrow \xi_{p+1,q} \\
C^{p-1,q}(L) & \xrightarrow{d} & C^{p,q}(L)
\end{array}
$$

where $(p \geq 2q + 3)$. 

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\[ A^{p-1,q}(K) \xrightarrow{\delta} C^{p,q}(K) \]
\[ \varepsilon_0^* - f_1^* \]
\[ A^{p-1,q}(L) \xleftarrow{j} C^{p-1,q}(L) \]

the following relations hold:

(4.1) \[ \xi_{p+1,q} \circ d + d \circ \xi_{p,q} = f_0^* - f_1^* \]

(4.2) \[ j \circ \xi_{p,q} \circ \delta = f_0^* - f_1^* \]

(Note. Here \( d = \delta \circ j \), the differential operator.)

Proof. We may as well assume \( q \geq 0 \), since otherwise all groups involved are trivial. The proof is divided into 3 parts: the definition of \( \xi_{p,q} \), the proof of (4.1), and the proof of (4.2).

Part one: The definition of \( \xi_{p,q} \). We may consider the homotopy \( f_{t} \) as a map \( F: L \times I \rightarrow K \) having the property that \( F(L^{p-1} \times I \cup L^p \times \hat{I}) \subseteq K^p \) for all \( p \) (here \( \hat{I} \) denotes the endpoints of \( I \)). The map \( F \) induces homomorphisms

\[ F^* : \pi^m(K^{p-1}, K^{p-2}) \rightarrow \pi^{n-1}(L^{p-1} \times I, (L^{p-2} \times I) \cup (L^{p-1} \times \hat{I})). \]

Consider the cohomotopy sequence of the triad \( (L^{p-1} \times I; L^{p-1} \times 0, (L^{p-1} \times 1) \cup (L^{p-2} \times I)); \) the coboundary operator of this triad is

\[ \Delta : \pi^{n-1}(L^{p-1} \times 0, L^{p-2} \times 0) \rightarrow \pi^{n}(L^{p-1} \times I, L^{p-1} \times \hat{I} \cup L^{p-2} \times I). \]

From the facts that the cohomotopy sequence of a triad is exact, and that

\[ \pi^{m}(L^{p-1} \times I, (L^{p-1} \times 1) \cup (L^{p-2} \times I)) = 0 \]

for all values of \( m \) (see Appendix, Section 20) it follows that \( \Delta \) is an isomorphism onto provided \( n \geq \frac{1}{2}(p + 3) \). Let

\[ P : \pi^n(L^{p-1} \times 0, L^{p-2} \times 0) \rightarrow \pi^n(L^{p-1}, L^{p-2}) \]

denote the isomorphism induced by the homeomorphism of \( L \) onto \( L \times 0 \) defined by \( x \rightarrow (x, 0) \) for any \( x \in L \). We now define for all \( p \geq 2q + 3 \),

\[ \xi_{p,q} : \pi^{p-q}(K^p, K^{p-1}) \rightarrow \pi^{p-q}(L^{p-1}, L^{p-2}) \]

by

\[ \xi_{p,q} = P \circ \Delta^{-1} \circ F^*. \]

The proofs of the relations (4.1) and (4.2) are rather tedious. Before starting the proof, we introduce certain notational conventions which will greatly simplify the writing of the many formulas needed. If \( X \) is any topological space, then \( \hat{X} \) denotes \( X \times I, X_0 \) denotes \( X \times 0, X_1 \) denotes \( X \times 1 \), and \( \hat{X} \) denotes \( X \times \hat{I} \). We assume that the integers \( p \) and \( q \) are chosen so that \( p \geq 2q + 3 \), and let \( p - q = n \); then \( n \geq \frac{1}{2}(p + 3) \). Let \( L^p = X, L^{p-1} = A, L^{p-2} = B, K^{p+1} = Y, K^p = C, K^{p-1} = D \). Then \( F \) is a map of triples:

\[ F: (\hat{X}, \hat{A} \cup \hat{X}, \hat{B} \cup \hat{A}) \rightarrow (Y, C, D), \]
and $F$ induces maps

$$f_0, f_1 : (X, A) \to (C, D).$$

We define $\xi$ and $\xi'$ by

$$\xi = \xi_{p,q} : \pi^n(C, D) \to \pi^{n-1}(A, B),$$
$$\xi' = \xi_{p+1,q} : \pi^{n+1}(Y, C) \to \pi^n(X, A).$$

In this notation, our object is to prove that in the following diagram,

$$\begin{array}{c}
\pi^n(C, D) \xrightarrow{d} \pi^{n+1}(Y, C) \\
\downarrow \xi \Downarrow f_0^*, f_1^* \Downarrow \xi' \\
\pi^{n-1}(A, B) \xrightarrow{d} \pi^n(X, A)
\end{array}$$

the relation

$$(4.1') \quad f_0^* - f_1^* = \xi' \circ d + d \circ \xi.$$

We will have to consider a great many homomorphisms, and to save trouble and avoid confusion, we introduce the following conventions. The symbols $i_1$, $i_2$, $i_3$, etc. will denote injections, $\delta_1$, $\delta_2$, $\delta_3$, etc. will denote cohomotopy coboundary operators for pairs and triples, $\Delta_1$, $\Delta_2$, $\Delta_3$, etc. will denote coboundary operators for triads, $F_1$, $F_2$, $F_3$, etc. will denote homomorphisms induced by the map $F$, $P_1$, $P_2$, $P_3$, etc. will denote isomorphisms induced by the homeomorphism of $L$ onto $L \times 0$ defined by $x \mapsto (x, 0)$ for any $x \in L$, and $h_1$, $h_2$, $h_3$, etc. will denote isomorphisms induced by the homeomorphism of $L \times 1$ defined by $(x, 0) \mapsto (x, 1)$ for any $x \in L$.

Part two: the proof of relation (4.1). The first step of the proof is a reduction of the problem. Consider the diagram in figure 2. It follows readily from the definitions that

$$\begin{align*}
\xi &= P_1 \circ \Delta_1^{-1} \circ i_1 \circ F_1, \\
\xi' &= P_2 \circ \Delta_2^{-1} \circ F_2, \\
f_0^* &= P_2 \circ i_2 \circ F_1, \\
f_1^* &= P_2 \circ h_1 \circ i_2 \circ F_1.
\end{align*}$$

Note that the top and bottom square in this diagram have the property that commutativity holds around each. From this it readily follows that to prove (4.1') it suffices to prove that for any element $\alpha \in \pi^n(\bar{A} \cup \bar{X}, \bar{B} \cup \bar{A})$

$$(4.3) \quad i_2(\alpha) - h_1 i_2(\alpha) = \Delta_2^{-1} \delta_1(\alpha) + \delta_2 \Delta_1^{-1} i_1(\alpha).$$

The second step is a direct sum decomposition of $\pi^n(\bar{A} \cup \bar{X}, \bar{B} \cup \bar{A})$. Consider the diagram shown in figure 3. In this diagram, $i_7$ and $i_8$ are isomorphisms onto,
image \( i_5 \) = kernel \( i_4 \), and image \( i_6 \) = kernel \( i_1 \). It follows readily that \( \pi^n(\tilde{A} \cup \tilde{X}, \tilde{B} \cup \tilde{A}) \) splits into a direct sum. Hence there exist unique elements

\[
\pi^n(C, D) \xrightarrow{d} \pi^{n+1}(Y, C)
\]

\[
\pi^n(\tilde{A} \cup \tilde{X}, \tilde{B} \cup \tilde{A}) \xrightarrow{\delta_1} \pi^{n+1}(\tilde{X}, \tilde{A} \cup \tilde{X})
\]

\[
\pi^n(\tilde{A}, \tilde{B} \cup \tilde{A}) \xrightarrow{h_1} \pi^n(X_1, A_1)
\]

\[
\pi^{n-1}(A_0, B_0) \xrightarrow{\delta_2} \pi^n(X_0, A_0)
\]

\[
\pi^{n-1}(A, B) \xrightarrow{d} \pi^n(X, A)
\]

**Fig. 2**

\[
\pi^n(\tilde{A} \cup \tilde{X}, \tilde{B} \cup \tilde{X}) \xrightarrow{i_7} \pi^n(\tilde{A} \cup \tilde{X}, \tilde{B} \cup \tilde{A})
\]

\[
\pi^n(\tilde{A}, \tilde{B} \cup \tilde{A}) \xrightarrow{i_1} \pi^n(\tilde{B} \cup \tilde{X}, \tilde{B} \cup \tilde{A})
\]

**Fig. 3**

\[
\beta'_1 \in \pi^n(\tilde{A} \cup \tilde{X}, \tilde{B} \cup \tilde{X}), \beta_1 \in \pi^n(\tilde{A}, \tilde{B} \cup \tilde{A}), \beta'_2 \in \pi^n(\tilde{A} \cup \tilde{X}, \tilde{A}), \beta_2 \in \pi^n(\tilde{B} \cup \tilde{X}, \tilde{B} \cup \tilde{A}), \alpha_1, \alpha_2 \in \pi^n(\tilde{A} \cup \tilde{X}, \tilde{B} \cup \tilde{A})
\]

such that

\[
\alpha = \alpha_1 + \alpha_2,
\]

\[
\alpha_1 = i_4(\beta'_1), \quad \alpha_2 = i_4(\beta'_2),
\]

\[
(4.4)
\]

\[
\beta_2 = i_4(\beta'_2) = i_4(\alpha_2)
\]

\[
\beta_1 = i_7(\beta'_1) = i_4(\alpha_1), \quad i_1(\alpha_2) = 0,
\]

\[
\beta_1 = i_7(\beta'_1) = i_4(\alpha_1)
\]

\[
\beta_2 = i_4(\beta'_2) = i_4(\alpha_2)
\]

\[
(4.4)
\]

\[
\beta_1 = i_7(\beta'_1) = i_4(\alpha_1)
\]

\[
\beta_2 = i_4(\beta'_2) = i_4(\alpha_2)
\]

\[
(4.4)
\]

Therefore, in place of (4.3), it suffices to prove the following pair of equations:

\[
i_4(\alpha_1) - h_1 i_2(\alpha_1) = \Delta_2^{-1} \delta_1(\alpha_1) + \delta_2 \Delta_1^{-1} i_1(\alpha_1),
\]

\[
i_4(\alpha_2) - h_1 i_2(\alpha_2) = \Delta_2^{-1} \delta_1(\alpha_2) + \delta_2 \Delta_1^{-1} i_1(\alpha_2).
\]

We will actually prove the following stronger relations:

\[
\Delta_2^{-1} \delta_1(\alpha_1) + \delta_2 \Delta_1^{-1} i_1(\alpha_1) = 0,
\]

\[
(4.7)
\]
(4.8) \[ i_3(\alpha_1) = 0, \quad i_2(\alpha_1) = 0, \]
(4.9) \[ \delta_2 \Delta^{-1} i_1(\alpha_2) = 0, \]
(4.10) \[ i_3(\alpha_2) - h_1 i_2(\alpha_2) = \Delta_2^{-1} \delta_1(\alpha_2). \]

It is obvious that (4.7)–(4.10) together imply (4.5) and (4.6). Also, (4.9) is true, since \( i_1(\alpha_2) = 0 \) by (4.4).

The third step is the proof of (4.7). Consider the diagrams shown in figures 4 and 5. From figure 4 and relations (4.4) we see that

\[
\Delta_2^{-1} \delta_1(\alpha_1) = \Delta_2^{-1} \delta_9(\beta_1'),
\]

(4.11) \[ \delta_2 \Delta_1^{-1} i_1(\alpha_1) = \delta_2 \Delta_1^{-1} \delta_9(\beta_1'). \]

In figure 5 we note that the following commutativity relations hold:

\[
\delta_{10} \circ \delta_9 = \delta_2 \circ i_8,
\]

\[
i_{11} \circ \delta_9 = \delta_2 \circ i_8.
\]
and that from the definitions, it follows that,
\[ \Delta_1 = \delta_4 \circ \iota_8^{-1}, \]
\[ \Delta_2 = \delta_5 \circ \iota_9^{-1}. \]

Note also, that we can apply the "hexagonal lemma" (see [5]) to the upper part of this diagram. From these facts, we can conclude that
\[ \Delta_2^{-1}\delta_5(\beta'_1) + \delta_2 \Delta_1^{-1}\iota_7(\beta'_1) = 0. \]

This, together with (4.11), suffices to prove (4.7).

![Diagram](image)

**Fig. 6**

The fourth step is a proof of (4.8). Consider the diagram in figure 6. We see that
\[ i_2 = i_{12} \circ i_4, \]
\[ i_6 = i_{13} \circ i_{14}. \]

Since \( i_4(\alpha_1) = 0 \) by (4.4), it follows that \( i_2(\alpha_1) = 0 \) and \( i_6(\alpha_1) = 0 \), thus proving (4.8).

The fifth step is a proof of (4.10). Consider the diagram in figure 7. By making use of the relations (4.4), and the various commutativity relations which hold in this diagram, we see that
\[ \Delta_2^{-1}\delta_1(\alpha_2) = \Delta_2^{-1}\delta_7(\beta'_2), \]
\[ (4.12) \]
\[ i_9(\alpha_2) = i_{15}(\beta'_2), \]
\[ h_1 i_2(\alpha_2) = h_1 i_{14}(\beta'_2). \]
Next, consider the diagram in figure 8. In this diagram we can apply the hexagonal lemma. Also, \( \Delta_2 = \delta_3 \delta_9^{-1} \). From these facts, we conclude that
\[
\Delta_2^{-1} \delta_7(\beta'_2) = i_{15}(\beta'_2) - h_1 i_{14}(\beta'_2).
\]
This, together with (4.12), suffices for the proof of (4.10). This completes the proof of (4.1).

\[
\begin{align*}
\pi^n(X, \bar{A} \cup \bar{X}) & \quad \delta_8 \\
\pi^n(\bar{A} \cup \bar{X}, \bar{A} \cup X_1) & \quad \delta_7 \\
\pi^n(A \cup X_1, A_1) & \quad i_{14} \\
\pi^n(X_1, A_1) & \quad h_1 \\
\pi^n(X, A) & \quad i_9 \\
\pi^n(C, D) & \quad \delta
\end{align*}
\]

Part three: the proof of relation (4.2). In our abbreviated notation, we must prove that in the following diagram,
\[
\begin{align*}
\pi^{n-1}(D) & \xrightarrow{\delta} \pi^n(C, D) \\
\pi^{n-1}(A, B) & \xleftarrow{j} \pi^{n-1}(A)
\end{align*}
\]
the relation
\[
(4.2') \quad j \circ \xi \circ \delta = f_0^* - f_1^*
\]
holds.

The first step is a reduction of the problem. Consider the diagram in figure 9. In this diagram,
\[
\xi = P_1 \circ \Delta_1^{-1} \circ F_4, \quad f_0^* = P_3 \circ i_{16} \circ F_3, \quad f_1^* = P_3 \circ h_2 \circ i_{17} \circ F_3,
\]
and the commutativity relations
\[
\begin{align*}
F_4 \circ \delta &= \delta_8 \circ F_3, \\
j \circ P_1 &= P_3 \circ i_{18}
\end{align*}
\]

\[\text{Fig. 8}\]
hold. From these facts we readily deduce that in order to prove (4.2'), it suffices to prove the following relation: for any $\alpha \in \pi^{n-1}(B \cup \tilde{A})$,

$$i_{19}(\alpha) - h_2 i_{17}(\alpha) = i_{18} \Delta_1 \delta_9(\alpha).$$

(4.13)

The second step is a direct sum decomposition of $\pi^{n-1}(B \cup \tilde{A})$. Consider the diagram in figure 10. Exactness holds across each diagonal line and $i_{19}$ and $\delta_9$ are both isomorphisms onto; hence the group $\pi^{n-1}(B \cup \tilde{A})$ decomposes naturally into a direct sum of two subgroups. Thus for any element $\alpha \in \pi^{n-1}(B \cup \tilde{A})$, there

$$\pi^{n-1}(D) \xrightarrow{\delta} \pi^{n}(C, D)$$

$$\pi^{n-1}(B \cup \tilde{A}) \xrightarrow{i_{17}} \pi^{n}(\tilde{A}, B \cup \tilde{A})$$

$$\pi^{n-1}(A_0) \xrightarrow{i_{15}} \pi^{n-1}(A_1)$$

$$\pi^{n-1}(A) \xrightarrow{\Delta_1} \pi^{n-1}(A_0, B_0)$$

$$\pi^{n-1}(A) \xleftarrow{\delta} \pi^{n-1}(A, B)$$

(Fig. 9)

exist unique elements $\alpha_1$ and $\alpha_2 \in \pi^{n-1}(B \cup \tilde{A})$, $\beta_1 \in \pi^{n-1}(\tilde{A})$, $\beta_2 \in \pi^{n-1}(B \cup \tilde{A}, B \cup A_1)$, $\beta'_1 \in \pi^{n-1}(B \cup A_1)$, $\beta'_2 \in \pi^{n}(\tilde{A}, B \cup \tilde{A})$ such that the following relations hold:

$$\alpha = \alpha_1 + \alpha_2$$

$$\alpha_1 = i_{21}(\beta_1), \quad \alpha_2 = i_{22}(\beta_2),$$

(4.14)

$$\beta'_1 = i_{20}(\alpha_1) = i_{19}(\beta_1),$$

$$\beta'_2 = \delta_9(\alpha_2) = \delta_9(\beta_2),$$

$$i_{20}(\alpha_2) = 0, \quad \delta_9(\alpha_1) = 0.$$

In view of these relations, to prove (4.13), it suffices to prove the following two equalities:
Because $\delta_0(\alpha_1) = 0$ from (4.14), instead of proving (4.15), it suffices to prove (4.17)

$$i_{16}(\alpha_1) - h_2 i_{17}(\alpha_1) = 0.$$  

We will prove the following two relations which together imply (4.16):

(4.18) $i_{17}(\alpha_2) = 0,$

(4.19) $i_{16}(\alpha_2) = i_{18} \Delta_1^{-1} \delta_0(\alpha_2).$

The third step is a proof of (4.19). Consider the diagram in figure 11. In this diagram, $\Delta_1 = \delta_0 \circ j_{23}^{-1}$, and the commutativity relation $i_{16} \circ i_{22} = i_{18} \circ i_{23}$ holds. This combined with the relations (4.14) suffices to prove equation (4.19).

The fourth step is a proof of (4.18). Consider the following diagram:

$$\begin{align*}
\pi^{-1}(B \cup \tilde{A}) &\xrightarrow{\delta_0} \pi^{-1}(\tilde{A}, B \cup \tilde{A}) \\
i_{18} &\xrightarrow{i_{22}} \pi^{-1}(B \cup \tilde{A}, B \cup A_1) \\
i_{18} &\xrightarrow{i_{23}} \pi^{-1}(B \cup A_1, B) \\
i_{18} &\xrightarrow{i_{20}} \pi^{-1}(A_1) \\
i_{17} &\xrightarrow{i_{24}} \pi^{-1}(B \cup A_1) \\
\end{align*}$$

Fig. 11

Because of the commutativity relation $i_{24} \circ i_{20} = i_{17}$, and the fact that $i_{20}(\alpha_2) = 0$, it follows that $i_{17}(\alpha_2) = 0$ as desired.

The fifth step is a proof of (4.17). This follows from consideration of the following diagram:

$$\begin{align*}
\pi^{-1}(\tilde{A}) &\xrightarrow{i_{21}} \pi^{-1}(\tilde{A} \cup B) \\
i_{18} &\xrightarrow{i_{17}} \pi^{-1}(A_1) \\
h_2 &\xrightarrow{\_} \pi^{-1}(A_0) \\
\end{align*}$$
Here the relation $h_2 \circ i_{17} \circ i_{21} = i_{15} \circ i_{21}$ holds on account of the homotopy axiom. Now it is only necessary to observe that $\alpha_1 = i_{21}(\beta_1)$ in order to verify (4.17).

This completes the proof of Lemma 4.1.

5. Proof of Invariance of the Cohomotopy Exact Couple

Let us denote by $(\tilde{f}_0, \tilde{f}_0)$ and $(\tilde{f}_1, \tilde{f}_1)$ respectively the maps $\langle \Gamma^*(K), \mathcal{C}^*(K) \rangle \to \langle \Gamma^*(L), \mathcal{C}^*(L) \rangle$ induced by $(f^*_0, f^*_0)$ and $(f^*_1, f^*_1)$; let

$$
\tilde{f}_0^{p,q}, \tilde{f}_1^{p,q}: \Gamma^{p,q}(K) \to \Gamma^{p,q}(L),
$$

$$
\tilde{f}_0^{p,q}, \tilde{f}_1^{p,q}: \mathcal{C}^{p,q}(K) \to \mathcal{C}^{p,q}(L),
$$

denote the homomorphisms defined by the maps $(\tilde{f}_0, \tilde{f}_0)$ and $(\tilde{f}_1, \tilde{f}_1)$ respectively. Then we have the following corollaries of the Lemma 4.1.

Corollary 5.1. If $p \geq 2q + 3$, and $q \geq 0$, then

$$
\tilde{f}_0^{p,q} = \tilde{f}_1^{p,q}: \mathcal{C}^{p,q}(K) \to \mathcal{C}^{p,q}(L).
$$

Corollary 5.2. If $p \geq 2q + 2$, and $q \geq 0$, then

$$
\tilde{f}_0^{p,q} = \tilde{f}_1^{p,q}: \Gamma^{p,q}(K) \to \Gamma^{p,q}(L).
$$

Note that $\tilde{f}_0^{p,q} = \tilde{f}_1^{p,q}$ in case $p \leq 2q + 1$, because in this case $\mathcal{C}^{p,q}(K) = 0$ and $\mathcal{C}^{p,q}(L) = 0$. Thus the only cases still undecided are whether or not $\tilde{f}_0^{p,q} = \tilde{f}_1^{p,q}$ for $p = 2q + 2$, and whether or not $\tilde{f}_0^{p,q} = \tilde{f}_1^{p,q}$ in case $p < 2q + 2$. To settle these cases, consider first the following diagram:

![Diagram](image)

Commutativity holds around this square, and by (3.1), both of the homomorphisms labelled $j'$ are isomorphisms into; also, by Corollary 5.2 above, $\tilde{f}_0^{2q+2,q} = \tilde{f}_1^{2q+2,q}$. From this it readily follows that $\tilde{f}_0^{2q+2,q} = \tilde{f}_1^{2q+2,q}$. Next, consider the diagram in figure 12, in which it is assumed that $p = 2q + 2$. Commutativity holds around each square, and it follows from (3.2) that the homomorphisms represented by horizontal arrows (labelled $i'$) are all isomorphisms onto. Furthermore,
it follows from Corollary 5.2 above, that $f_i^{p,q} = f_i^{p,q}$ if $p = 2q + 2$. Putting all these facts together, it is readily seen that $f_i^{p,q} = f_i^{p,q}$ if $p < 2q + 2$.

We may summarize the results we have achieved as follows:

**Theorem 5.1.** Let $K$ and $L$ be finite cell complexes and let $f_0, f_1: L \to K$ be cellular maps which are homotopic. Then the induced maps

$$(f_0, f_0), (f_1, f_1): \langle \Gamma^*(K), \Xi^*(K) \rangle \to \langle \Gamma^*(L), \Xi^*(L) \rangle$$

are the same.

From this theorem, it follows immediately that the exact couple $\langle \Gamma^*(K), \Xi^*(K) \rangle$ is an invariant of the homotopy type of $K$, and hence is a *fortiori* a topological invariant of $K$; therefore it does not depend on the particular cellular decomposition chosen for the space of $K$. Likewise, the successive derived couples are invariants of the homotopy type of $K$.

6. The Groups $\pi^{n,m}(K)$

Let $K$ be a finite cell complex, $N = \dim K$, and $n$ an integer such that $N < 2n - 1$. Let

$$k^{n,m}: \pi^n(K) \to \pi^n(K^m)$$

denote the injection; define $\pi^{n,m}(K)$ to be the kernel of $k^{n,m}$. Then $\pi^{n,m}(K)$ is a sub-group of $\pi^n(K)$. The following facts about these sub-groups are obvious:

1. If $m < n$, then $\pi^{n,m}(K) = \pi^n(K)$.
2. If $m \geq N$, then $\pi^{n,m}(K) = 0$.
3. If $p > q$, then $\pi^{n,p}(K) \subset \pi^{n,q}(K)$.

If $f: K \to L$ is a cellular map of one finite complex into another, and $f^n: \pi^n(L) \to \pi^n(K)$ denotes the induced homomorphism, then $f^n[\pi^{n,m}(L)] \subset \pi^{n,m}(K)$, and hence $f^n$ defines a homomorphism,

$$f^{n,m}: \pi^{n,m}(L) \to \pi^{n,m}(K).$$

Furthermore, if $f, g: K \to L$ are two cellular maps which are homotopic, then the induced homomorphisms, $f^{n,m}$ and $g^{n,m}$, are equal. From this it follows readily that the groups $\pi^{n,m}(K)$ are invariants of the homotopy type of $K$.

Let $\Gamma^p_r(K) = \sum \Gamma^p_r(K)$ be one of the bigraded groups in the exact couple $\langle \Gamma^*_r(K), \Xi^*_r(K) \rangle$. The following proposition gives a connection between the groups $\Gamma^p_r(K)$ and $\pi^{n,m}(K)$:

**Proposition 6.1.** Assume $K$ is a finite cell complex of dimension $N$; then for any non-negative integers $p, q, r$ such that $N < 2(p - q) - 1$ and $r \geq N - p - 1$, there exists a natural isomorphism

$$\Gamma^p_r(K) \cong \pi^{p,q}(K)/\pi^{p,q+r}(K).$$

This follows readily from the definitions of the groups $\Gamma^p_r(K)$ and $\pi^{n,m}(K)$. The details of the proof are left to the reader.
7. The Associated Spectral Sequence and its Limit Group

Let $K$ be a finite cell complex, and let

$$\langle \Gamma^*(K), 3c^*(K) \rangle = \langle \Gamma^0(K), 3c^0(K) \rangle, \langle \Gamma^*(K), 3c^*(K) \rangle, \langle \Gamma^*(K), 3c^0*(K) \rangle, \cdots, \text{etc.}$$

\[ \cdots, \text{etc.} \] denote the successive derived exact cohomotopy couples of $K$. Associated with this sequence of successive derived exact couples is a spectral sequence of bigraded groups $(3c^*(K), d) = (3c^0(K), d_0), (3c^*(K), d_1), \cdots, \text{etc.}$. It is the purpose of this section to discuss the properties of this spectral sequence. First, we list the following facts:

(a) Each of the groups $3c^*(K)$ is bigraded, $3c^*_n(K) = \sum 3c^*_p(K)$.

(b) $3c^*_p(K) = 0$ if $p \geq 2q + 1$ or if $q < 0$, or if $p - q \leq 1$, or if $p > \dim K$.

(c) The differential operator $d_n : 3c^*_n(K) \to 3c^*_n(K)$ is homogeneous of degree $(n + 2, n + 1)$.

Associated with this spectral sequence is a bigraded limit group $3c^*_\infty(K) = \sum 3c^*_p \cdot q(K)$. Note that any homogeneous component of the limit group is determined in a finite number of steps; the argument for this is the same as in the case of the homotopy exact couple.

The relationship of the limit group, $3c^*_\infty(K)$, to the cohomotopy groups of $K$ is given by the following important result:

**Theorem 7.1.** Let $K$ be a finite cell complex of dimension $N$, and $p$ and $q$ non-negative integers such that $N < 2(p - q) - 1$; then the group $3c^*_p \cdot q(K)$ is isomorphic to the factor group $\pi^{p-2q+1}(K)/\pi^{p-2q+2}(K)$.

The proof can be made by drawing a sequence of diagrams and using the same type of argument as in the proof of Theorem 19.1 in Part II about the relation between the limit group $3c^*_p \cdot q(K)$ and the homotopy groups of $K$. The details are left to the reader.

8. A Theorem of J. Adem

The question naturally arises as to the effective computability of the various terms in the spectral sequence $(3c^*(K), d), (3c^*_1(K), d_1), \cdots$. As we have seen above, the groups $3c^*_p \cdot q(K)$ are, with a few exceptions, isomorphic to cohomology groups of $K$ with coefficients in the groups $G_q$; these groups are therefore theoretically computable if we know the groups $G_q$. The question then arises as to computability of the differential operators $d, d_1, d_2, \cdots$. We mention the following theorem (without proof) which settles this question for the differential operator $d$.

For any integers $p, q$ such that $p > 2q - 1$ and $p > 2$ define a homomorphism

$$\lambda_{p,q} : \pi^{p-2q}(S^p) \to \pi^{p-2q+1}(S^{p+2})$$

as follows. Let $\gamma_p$ be the unique non-zero element of $\pi^p(S^p + 1)$, and

$$E : \pi^p(S^m) \to \pi^{p+1}(S^{m+1})$$

the suspension isomorphism. Then for any $\alpha \in \pi^{p-2q}(S^p)$, set

$$\lambda_{p,q}(\alpha) = E(\gamma_p \circ \alpha) = \gamma_{p+1} \circ (E\alpha).$$
Since the composition operation is bilinear, and $2\gamma_p = 0$, it follows that $\lambda(2\alpha) = 0$ for any $\alpha \in \pi^{p-q}(S^p)$. Furthermore, it is readily seen that commutativity holds in the following diagram:

$$
\begin{array}{ccc}
\pi^{p-q}(S^p) & \xrightarrow{\lambda_{p,q}} & \pi^{p-q+1}(S^{p+2}) \\
\downarrow{E} & & \downarrow{E} \\
\pi^{p-q+1}(S^{p+1}) & \xrightarrow{\lambda_{p+1,q}} & \pi^{p-q+2}(S^{p+3})
\end{array}
$$

Therefore the homomorphisms $\lambda_{p,q}$ define homomorphisms

$$\lambda_q : \mathcal{G}_q \to \mathcal{G}_{q+1}.$$  

Since $\lambda_q(2\alpha) = 0$ for any $\alpha \in \mathcal{G}_q$, it follows that it is possible to define a commutative bilinear self-pairing of $\mathcal{G}_q$ with itself to $\mathcal{G}_{q+1}$ in such a way that $\alpha \cdot \alpha = \lambda_q(\alpha)$ for any element $\alpha \in \mathcal{G}_q$. Use such a self-pairing to define the Steenrod square

$$\text{Sq}^2 : H^p(K, \mathcal{G}_q) \to H^{p+2}(K, \mathcal{G}_{q+1}), \ (p > 2).$$

For the definition of the Steenrod square, see [16]. It may be shown that the Steenrod square thus defined is independent of the particular pairing used. With these conventions, the theorem may be stated as follows:

**Theorem 8.1.** Commutativity holds in the following diagram if $p > 2q + 2$:

$$
\begin{array}{ccc}
\mathcal{T}^{p,q}(K) & \xrightarrow{d} & \mathcal{T}^{p+2,q+1}(K) \\
\downarrow{\varepsilon^q} & & \downarrow{\varepsilon^q} \\
H^p(K, \mathcal{G}_q) & \xrightarrow{\text{Sq}^2} & H^{p+2}(K, \mathcal{G}_{q+1})
\end{array}
$$

This theorem is due to José Adem; the proof will appear in a forthcoming paper.

**PART IV. DUALITY BETWEEN THE HOMOTOPY AND COHOMOTOPY EXACT COUPLES**

**9. General Algebraic Theory**

Let $A$, $B$, and $G$ be abelian groups; we say that $A$ and $B$ are paired to $G$ if there exists a bilinear multiplication which associates with any elements $a \in A$ and $b \in B$ an element $a \cdot b \in G$. If $(A, d)$ and $(B, d')$ are differential groups, then we demand in addition that the relation

$$a \cdot (d'b) = (da) \cdot b$$

hold for any elements $a \in A$ and $b \in B$. If this is the case, then the pairing of $A$ and $B$ to $G$ induces a pairing of the derived groups $\mathcal{T}(A)$ and $\mathcal{T}(B)$ to $G$ in a
natural way, as follows. Let \( u \in \mathcal{C}(A) \) and \( v \in \mathcal{C}(B) \); choose a representative \( u' \in \mathcal{Z}(A) \) for \( u \) and \( v' \in \mathcal{Z}(B) \) for \( v \), and define
\[
 u \cdot v = u' \cdot v' \in G.
\]

It is readily verified that this definition is independent of the choices of the representatives \( u' \) and \( v' \), and that the multiplication thus defined is bilinear. If \( A = \sum A^p \) and \( B = \sum B^p \) are graded groups, then for any pairing of \( A \) and \( B \) to \( G \) we shall demand the following additional condition: there exists an integer \( n \) such that for any integers \( p \) and \( q \) and any elements \( a \in A^p \), \( b \in B^q \), if \( p - q = n \) (or alternatively, if \( p + q = n \) then \( a \cdot b = 0 \). In a similar manner, if \( A = \sum A_{p,q} \) and \( B = \sum B_{p,q} \) are bigraded groups, then we impose the following condition on any pairing: there exist integers \( m \) and \( n \) such that for any two pairs of integers \( (p, q) \) and \( (r, s) \) and any elements \( a \in A_{p,q} \), \( b \in B_{r,s} \), if \( p - r = m \) or \( q - s = n \), then \( a \cdot b = 0 \). (Sometimes we will instead have the condition \( q + s = n \).) If \( A \) and \( B \) are graded (or bigraded) groups with homogeneous differential operators, and \( A \) and \( B \) are paired to \( G \), then the induced pairing of the graded (or bigraded) groups \( \mathcal{C}(A) \) and \( \mathcal{C}(B) \) to \( G \) satisfies the conditions we have imposed on a pairing of graded (or bigraded) groups.

Let \( \langle A_0, C_0; f_0, g_0, h_0 \rangle \) and \( \langle A_1, C_1; f_1, g_1, h_1 \rangle \) be exact couples, and let \( G \) be an abelian group. We shall say that these two exact couples are paired to \( G \) if \( A_0 \) and \( A_1 \) are paired to \( G \) and \( C_0 \) and \( C_1 \) are paired to \( G \) in such a way that the following three conditions are satisfied:

\[
\begin{align*}
(9.1) \quad & \text{For any elements } a_0 \in A_0 \text{ and } a_1 \in A_1, \quad (f_0 a_0) \cdot a_1 = a_0 \cdot (f_1 a_1). \\
(9.2) \quad & \text{For any elements } a_0 \in A_0 \text{ and } c_1 \in C_1, \quad (g_0 a_0) \cdot c_1 = a_0 \cdot (h_1 c_1). \\
(9.3) \quad & \text{For any elements } c_0 \in C_0 \text{ and } a_1 \in A_1, \quad (h_0 c_0) \cdot a_1 = c_0 \cdot (g_1 a_1).
\end{align*}
\]

In case the given exact couples are graded (or bigraded) with homogeneous homomorphisms, we will also assume that the pairings satisfy the conditions we have imposed on pairings of graded (or bigraded) groups.

Let \( \langle A_0, C_0; f_0, g_0, h_0 \rangle \) and \( \langle A_1, C_1; f_1, g_1, h_1 \rangle \) be exact couples which are paired to the group \( G \) in accordance with the above definition. This pairing of the given exact couples induces a pairing of the first derived exact couples to \( G \), defined as follows. Let \( d_0 = g_0 \circ h_0 : C_0 \to C_0 \) and \( d_1 = g_1 \circ h_1 : C_1 \to C_1 \) denote the differential operators on the groups \( C_0 \) and \( C_1 \) respectively. Then it follows readily that if \( c_0 \in C_0 \) and \( c_1 \in C_1 \), \( c_0 \cdot (d_1 c_1) = (d_0 c_0) \cdot c_1 \). Hence there is induced a pairing of the derived groups \( C'_0 = \mathcal{C}(C_0) \) and \( C'_1 = \mathcal{C}(C_1) \) to \( G \) as described above. Next, we define a pairing of \( A'_0 \) and \( A'_1 \) to \( G \) as follows. Let \( u \in A'_0 \) and \( v \in A'_1 \); choose elements \( \bar{u} \in A_0 \) and \( \bar{v} \in A_1 \) such that \( u = f_0 \bar{u} \) and \( v = f_1 \bar{v} \). Then
\[
\bar{u} \cdot v = u \cdot \bar{v}
\]
on account of the conditions we have imposed on the pairings involved. Let \( w = \bar{u} \cdot v = u \cdot \bar{v} \); then \( w \) is independent of the choices made for \( \bar{u} \) and \( \bar{v} \). We define \( u \cdot v = w \). It is readily verified that this multiplication is bilinear.
It remains to verify that the pairing of $A'_0$ and $A'_1$ to $G$ and the pairing of $C'_0$ and $C'_1$ to $G$ that we have defined satisfy the conditions (9.1), (9.2) and (9.3) above for a pairing of the exact couples $\langle A'_0, C'_0; j'_0, g'_0, h'_0 \rangle$ and $\langle A'_1, C'_1; j'_1, g'_1, h'_1 \rangle$ to $G$. This verification is purely mechanical and is left to the reader. By iterating this process, we obtain pairings of the successive derived exact couples to $G$.

Let $(C^{(n)}_0, d^{(n)}_0), n = 0, 1, 2, \cdots,$ and $(C^{(n)}_1, d^{(n)}_1), n = 0, 1, 2, \cdots,$ denote the spectral sequences associated with successive derived exact couples of $\langle A_0, C_0 \rangle$ and $\langle A_1, C_1 \rangle$ respectively. Then it is clear that the pairings we have defined of the successive derived exact couples define pairings of the corresponding terms of these two spectral sequences to $G$.

10. Rough Description of the Pairings Defined by the Composition Operation

It is our ultimate purpose to define, using the composition operation of Section 22, for each integer $r \geq 0$ a pairing of the exact couples $\langle A^*(K), C^*(K) \rangle$ and $\langle A(K, 2r), C(K, 2r) \rangle$ to the group $G_r$. This induces pairings of the successive derived exact couples and the spectral sequences associated with these two exact couples. These pairings furnish additional information about the differential operators in these spectral sequences. The precise definition of these pairings is complicated due to the necessity for separate consideration of a great many special cases. Therefore in this section we indicate the basic idea of the definition, which is really quite simple.

Let $K$ be a finite, connected cell complex. Suppose we restrict our attention for the time being to those values of the indices $p, q, r, \cdots$ for which

\[
C^{p,q}_r(K) = \pi^{p+q}(K^p, K^{p-1}),
\]

\[
C_{p,r}(K) = \pi_{p+r}(K^p, K^{p-1}),
\]

\[
A^{p,q}_r(K) = \pi^{p+q}(K^p),
\]

\[
A_{p,r}(K) = \pi_{p+r}(K^p).
\]

Then the composition operation defines a pairing of $C^{p,q}_r(K)$ and $C_{p,r}(K)$ to the group $\pi_{p+r}(S^{p+q})$ and a pairing of $A^{p,q}_r(K)$ and $A_{p,r}(K)$ to $\pi_{p+r}(S^{p+q})$. If $\alpha \in C^{p,q}_r(K)$ and $\beta \in C_{p,r}(K)$, then the composition $\alpha \circ \beta \in \pi_{p+r}(S^{p+q})$; if $\alpha \in A^{p,q}_r(K)$ and $\beta \in A_{p,r}(K)$, then the composition $\alpha \circ \beta \in \pi_{p+r}(S^{p+q})$. It is shown in Section 22 that the multiplications so defined are bilinear. Furthermore, these pairings satisfy the following three conditions. Let

\[
i: A_{p-r+1}(K) \to A_{p,r}(K),
\]

\[
j: A_{p,r}(K) \to C_{p,r}(K),
\]

\[
\partial: C_{p,r}(K) \to A_{p-1,r}(K)
\]

be the homomorphisms which occur in the homotopy exact couple $\langle A(K), C(K) \rangle$, and let

\[
i^*: A^{p,q}_r(K) \to A^{p-1,q-1}(K),
\]
be the homomorphisms that occur in the cohomotopy exact couple \( \langle A^*(K), C^*(K) \rangle \). Then the following three relations hold:

\[(10.1) \quad \text{If } \alpha \in C^{p,q}(K) \text{ and } \beta \in A_{p,r}(K), \text{ then } \alpha \circ (j \beta) = (j^* \alpha) \circ B. \]

\[(10.2) \quad \text{If } \alpha \in A^{p,q}(K) \text{ and } \beta \in A_{p-1,r+1}(K), \text{ then } \alpha \circ (i \beta) = (i^* \alpha) \circ \beta. \]

\[(10.3) \quad \text{If } \alpha \in A^{p-1,q}(K) \text{ and } \beta \in C_{p,r}(K), \text{ then } \alpha \circ (\delta \beta) = (\delta \alpha) \circ \beta. \]

The proof of these relations follows from (22.1) and (22.2) in the appendix.

These pairings induced by the composition operation will be basic for the definition of the various pairings of exact couples that we are going to consider. The main difficulty in our definitions is caused by the fact that the groups \( A^{p,q}(K), C^{p,q}(K), A_{p,r}(K), \) and \( C_{p,r}(K) \) are of necessity defined in a rather artificial manner for those values of \( p, q, \) and \( r \) for which the groups \( \pi^{p-q}(K^p), \pi^{p-q}(K^p, K^{p-1}), \pi_{p+r}(K^p), \text{ and } \pi_{p+r}(K^p, K^{p-1}) \) are not defined.

11. Precise Definition of the Pairings

Consider the bigraded exact couples, \( \langle A(K, 2r), C(K, 2r); i, j, \delta \rangle \) and \( \langle A^*(K), C^*(K); i^*, \delta, j^* \rangle \), associated with a given finite, connected cell complex \( K \). For each integer \( r \geq 0 \) we will define a pairing of these two exact couples to the group \( \mathcal{G}_r \). These pairings will satisfy the following conditions: if \( \alpha \in C^{p,q}(K) \) and \( \beta \in C_{m,n}(K, 2r), \) then \( \alpha \circ \beta = 0 \), unless \( p = m \) and \( q + n = r \); if \( \alpha \in A^{p,q}(K) \) and \( \beta \in A_{m,n}(K, 2r), \) then \( \alpha \circ \beta = 0 \) unless \( p = m \) and \( q + n = r \). Thus our main problem is to define a pairing of \( C^{p,r-q}(K) \) and \( C_{p,q}(K, 2r) \) to \( \mathcal{G}_r \), and a pairing of \( A^{p,r-q}(K) \) and \( A_{p,q}(K, 2r) \) to \( \mathcal{G}_r \), for all values of \( p \) and \( q \). The definitions of these pairings must be broken into several cases, depending on how the groups \( A^{p,r-q}(K) \), etc., are defined. For convenience, we recall the following facts:

\[
A_{p,q}(K, 2r) = \begin{cases} 
0 & \text{if } p + q < 2r + 2, \\
A_{p,q}(K) & \text{if } p + q \geq 2r + 2,
\end{cases}
\]

\[
C_{p,q}(K, 2r) = \begin{cases} 
0 & \text{if } p + q < 2r + 2, \\
f[A_{p,q}(K)] & \text{if } p + q = 2r + 2, \\
C_{p,q}(K) & \text{if } p + q > 2r + 2.
\end{cases}
\]

First case: \( p + q < 2r + 2 \). In this case \( A_{p,q}(K, 2r) = 0 \) and \( C_{p,q}(K, 2r) = 0 \). Hence the pairing of \( A^{p,r-q}(K) \) and \( A_{p,q}(K, 2r) \) to \( \mathcal{G}_r \) must be the trivial pairing; similarly for the pairing of \( C^{p,r-q}(K) \) and \( C_{p,q}(K, 2r) \) to \( \mathcal{G}_r \).

Second case: \( p + q \geq 2r + 2 \) and \( q > r \). In this case \( C^{p,r-q}(K) = 0 \) and \( A^{p,r-q}(K) = 0 \), so again the pairings must be defined to be trivial.

Third case: \( p + q \geq 2r + 2 \), \( q \leq r \), and \( p \geq 2(r - q) + 2 \). In this case
These representative cases also show that the desired desiderata are satisfied. If we consider the operation of pairing by the pairings defined on \( A_{p,q}(K, 2r) \) and in the suspension \( E \), the desired identity \( \pi_{p+q}(A_{p,q}(K, 2r)) \) is readily obtained. This furnishes the desired pairing.

Fourth case: \( p + q \geq 2r + 2 \). In this case we must have \( q < 0 \), hence \( C_{p,q}(K, 2r) = 0 \). Therefore we must define the pairing of \( C_{p,q}(K) \) and \( C_{p,q}(K, 2r) \) to \( \mathbb{G}_r \) to be the trivial pairing. The definition of the pairing of \( A_{p,r}(K) \) and \( A_{p,q}(K, 2r) \) is less simple, on account of the artificial nature of the definition of \( A_{p,r}(K) \) for the values of \( p \) and \( q \) under consideration. As a first step in such a definition, we define for all \( q \geq r + 1 \) a pairing of \( D^q(K) \) and \( \pi_{q+r+1}(K) \) to \( \mathbb{G}_r \) and a pairing of \( B^q(K) \) and \( \pi_{q+r+1}(K) \) to \( \mathbb{G}_r \).

The first pairing is defined to be the trivial pairing: if \( \alpha \in D^q(K) \) and \( \beta \in \pi_{q+r+1}(K) \), then \( \alpha \circ \beta = 0 \). The definition of the second pairing is simplest in case \( q > r + 1 \); in this case the groups \( \pi_{q+r+1}(K^{2q}) \) and \( \pi_{q+r+1}(K^{2q}) \) are paired to \( \pi_{q+r+1}(K^{2q}) \approx \mathbb{G}_r \), by the composition operation. By definition \( E^q(K) \) is a subgroup of \( \pi_{q+r+1}(K^{2q}) \), and the injection \( \pi_{q+r+1}(K^{2q}) \to \pi_{q+r+1}(K) \) is an isomorphism onto. Putting all these together, we obtain the desired pairing. To define the second pairing in the case \( q = r + 1 \), consider the following homomorphisms:

\[
\delta: A^{2q+1, r}(K) \to C^{2q+1, r}(K),
\]

\[
\partial: C_{2q+1,0}(K) \to A_{2q+1,0}(K),
\]

\[
E: \pi_{2q+1}(S^{r+2}) \to \pi_{2q+1}(S^{r+2}).
\]

If \( \alpha \in A^{2q+1, r}(K) \) and \( \beta \in C_{2q+1,0}(K) \), then we know that

\[
E[\alpha \circ (\partial \beta)] = (\delta \alpha) \circ \beta.
\]

Also, the suspension \( E \) is an isomorphism onto. By definition \( E^{r+1}(K) \) is the kernel of \( \delta \). It is readily seen that \( \pi_{2q+1}(K) \) is naturally isomorphic to the factor group of \( A_{2q+2,0}(K) \) modulo \( \partial[C_{2q+2,0}(K)] \). The desired pairing is defined as follows: If \( \alpha \in E^{r+1}(K) \) and \( \beta \in \pi_{2q+2}(K) \), choose an element \( \beta' \in A_{2q+2,0}(K) \) which is a representative of \( \beta \), and set

\[
\alpha \circ \beta = \alpha \circ \beta'.
\]

It is readily seen that this definition is independent of the choice made for \( \beta' \).

With these preliminaries taken care of, it is now a simple matter to define the pairing of \( A^{p,r-q}(K) \) and \( A_{p,q}(K, 2r) \) to \( \mathbb{G}_r \). Since in the cases under consideration, \( q < 0 \), it follows that \( A_{p,q}(K, 2r) \) is naturally isomorphic to \( \pi_{p+q}(K) \). By definition,

\[
A^{2q+1, r}(K) = D^q(K) + E^q(K),
\]

\[
A^{p,q}(K) = E^{p-q-1}(K) \quad \text{if} \quad p \leq 2q.
\]
Making use of these facts, and the two pairings defined in the preceding paragraph, we define the pairing of $A^{p,r-q}(K)$ and $A_{p,q}(K, 2r)$ to $G_r$ in the obvious way.

This completes the definition of the $r$th pairing, $r = 0, 1, 2, \cdots$.

Having made these definitions, it is now necessary to verify the following three facts for all values of $p$ and $q$:

(11.1) If $\alpha \in C^{p,r-q}(K)$ and $\beta \in A_{p,q}(K, 2r)$, then $\alpha \circ (j\beta) = (j^*\alpha) \circ \beta$.

(11.2) If $\alpha \in A^{p,r-q}(K)$ and $\beta \in A_{p-1,q+1}(K, 2r)$, then $\alpha \circ (i\beta) = (i^*\alpha) \circ \beta$.

(11.3) If $\alpha \in A^{p-1,r-q}(K)$ and $\beta \in C_{p,q}(K, 2r)$, then $\alpha \circ (d\beta) = (\delta\alpha) \circ \beta$.

The verifications involve mechanically checking a great many different cases, corresponding to the different cases occurring in the definition of the pairings involved. The details are left to the reader.

The pairing we have defined of the exact couples $\langle A(K, 2r), C(K, 2r) \rangle$ and $\langle A^*(K), C^*(K) \rangle$ to the group $G_r$ induces a pairing of the first derived couples $\langle \Gamma(K, 2r), 3C(K, 2r) \rangle$ and $\langle \Gamma^*(K), 3C^*(K) \rangle$ to the group $G_r$. It is readily seen that this induced pairing is an invariant of the homotopy type of $K$; the proof of this important fact is left to the reader. Likewise, the spectral sequences associated with these exact couples are paired to the group $G_r$.

12. Computability of These Pairings

We have seen that under certain circumstances the following isomorphisms hold:

$$3C_{p,q}(K) \cong H_p(K, G_q),$$
$$3C^{p,r-q}(K) \cong H^p(K, G_{r-q}).$$

The question then naturally arises, if conditions are such that these isomorphisms hold, can we compute the pairing of $3C^{p,r-q}(K)$ and $3C_{p,q}(K)$ to $G_r$? The answer to this question is provided by the main theorem of this section.

Let $K$ be a finite cell complex, and let $H_p(K, G_1)$, $H^p(K, G_2)$ denote the $p$-dimensional homology group of $K$ with coefficients in $G_1$ and the $q$-dimensional cohomology group of $K$ with coefficients in $G_2$ respectively. Suppose the groups $G_1$ and $G_2$ are paired to another group $G$; then it is well known that there is induced a pairing of $H_p(K, G_1)$ and $H^p(K, G_2)$ to $G$ for all dimensions $p$. If $u \in H_p(K, G_1)$ and $v \in H^p(K, G_2)$, then $u \cdot v \in G$ will denote their product.

Let $\Xi_p : H_p(K, G_m) \to 3C_{p,m}(K)$ denote the isomorphism described in Section 14 of Part II, and let $\Xi^p : H^p(K, G_q) \to 3C^{p,q}(K)$ denote the isomorphism described in Section 3. Define a pairing of the groups $G_{r-q}$ and $G_q$ to $G_r$ as follows. Choose a representative $\pi^{p-r+q}(S^p, p_0)$ for $G_{r-q}$ and a representative $\pi_{p+q}(E^p, E^p)$ for $G_q$; then the composition operation pairs $\pi^{p-r+q}(S^p, p_0)$ and $\pi_{p+q}(E^p, E^p)$ to $\pi_{p+q}(E^{p+r-q}, p_0)$, which is a representative of $G_r$ (for all sufficiently large $p$). Furthermore, the pairing so defined is independent of the choices of representatives. This pairing induces a pairing of $H^p(K, G_{r-q})$ and $H_p(K, G_q)$ to $G_r$. 
THEOREM 12.1. Let \( p, q, \) and \( r \) be integers such that \( 0 \leq q \leq r, \) and \( p > 2(r - q) + 2. \) Let \( K \) be a finite cell complex which is \( (q + 1) \)-connected. For any elements \( u \in H^p(K, S_{r-q}) \) and \( v \in H_p(K, S_q), \) the following relation holds:

\[
(\Xi^{r-q} u) \circ (\Xi^q v) = u \cdot v.
\]

PROOF. As we have done previously, we will identify \( C^p(K, S_{r-q}) \) with \( S_{r-q} \otimes \pi^p(K^p, K^{p-1}) \), and \( C_p(K, S_q) \) with \( \pi_p(K^p, K^{p-1}) \otimes S_q \). Choose \( \pi^{p-r+q}(S^p, p_0) \), and \( \pi_{p+q}(E^p, \dot{E}^p) \) as representatives for \( S_{r-q} \) and \( S_q \) respectively; then

\[
C^p(K, S_{r-q}) = \pi^{p-r+q}(S^p, p_0) \otimes \pi^p(K^p, K^{p-1}),
\]

\[
C_p(K, S_q) = \pi_p(K^p, K^{p-1}) \otimes \pi_{p+q}(E^p, \dot{E}^p).
\]

Let \( e_1, \ldots, e_k \) denote the \( p \)-cells of \( K \), each with a definitely chosen orientation. Let \( \sigma_i \in \pi_p(K^p, K^{p-1}) \), \( i = 1, \ldots, k \) be the element represented by a map \( (E^p, \dot{E}^p) \rightarrow (e_i, e_i) \) of degree +1; let \( \sigma_i^* \in \pi^p(K^p, K^{p-1}) \) be the element represented by a map \( f_i: (K^p, K^{p-1}) \rightarrow (S^p, p_0) \) such that \( f_i(e_i) = p_0 \) for \( i \neq j \). Then \( \sigma_1, \ldots, \sigma_k \) is a basis for the free abelian group \( \pi_p(K^p, K^{p-1}) \) and \( \sigma_1^*, \ldots, \sigma_k^* \) is a basis for \( \pi^p(K^p, K^{p-1}) \). These elements have the property that

\[
\sigma_i^* \circ \sigma_j = \delta_{ij}
\]

where \( \delta_{ij} \in \pi(S^p, p_0) \) is the Kronecker symbol; \( \delta_{ij} = 0 \) if \( i \neq j \), and \( \delta_{ii} \) is the homotopy class represented by the map of degree +1.

Choose a cocycle \( u' \in C^p(K, S_{r-q}) \) representing \( u \), and a cycle \( v' \in C_p(K, S_q) \) representing \( v \). Then \( u' \) and \( v' \) can be written uniquely in the form

\[
u' = \sum_i \alpha_i \otimes \sigma_i^*, \quad \alpha_i \in \pi^{p-r+q}(S^p, p_0),
\]

\[
v' = \sum_i \sigma_j \otimes \beta_j, \quad \beta_j \in \pi_{p+q}(E^p, \dot{E}^p).
\]

Then, by definition,

\[
u \cdot v = \sum_i \alpha_i \circ \beta_i \in \pi_{p+q}(S^{p+r-q}, p_0),
\]

\[
\Xi^{r-q} u' = \sum_i \alpha_i \circ \sigma_i^*,
\]

\[
\Xi q v' = \sum_i \sigma_j \circ \beta_j,
\]

\[
(\Xi^{r-q} u) \circ (\Xi q v) = (\Xi^{r-q} u') \circ (\Xi q v')
\]

\[
= (\sum_i \alpha_i \circ \sigma_i^*) \circ (\sum_i \sigma_j \circ \beta_j)
\]

\[
= \sum_{i,j} \alpha_i \circ \sigma_i^* \circ \sigma_j \circ \beta_j
\]

\[
= \sum_{i,j} \alpha_i \circ \delta_{ij} \circ \beta_j = \sum_i \alpha_i \circ \beta_i,
\]

as was to be proved.
PART V. THE EXACT COUPLE AND SPECTRAL SEQUENCE OF A FIBRE BUNDLE

13. Basic Definitions

In this part we shall make full use of the definitions, terminology, and notations introduced in the recent book of Steenrod on fibre bundles, [17]. When we speak of complexes in this part, we shall always mean finite cell complexes having the property that each q-cell is a homeomorph of $E^q$, as explained on p. 100 of [17]. We shall restrict our attention to fibre bundles in which both the base space and the fibre are finite cell complexes. The main interest will be in bundles for which both the base space and fiber are connected.

Let $\mathfrak{B} = \{B, \pi, X, F\}$ be a fibre bundle with bundle space $B$, projection $\pi: B \to X$, base space $X$, and fibre $F$; the group of the bundle, the coordinate neighborhoods, and the coordinate functions will not concern us in most of what follows. It is assumed that both $X$ and $F$ are finite cell complexes, as explained above. Let $X^p$ denote the $p$-skeleton of $X$, and let $B^p = \pi^{-1}(X^p)$ if $p \geq 0$, $B^p = \emptyset$ set if $p < 0$. Choose a fixed abelian group $\mathfrak{G}$ for coefficient group for cohomology, and define

$$A^{p,q}(\mathfrak{B}) = H^{p+q}(B^p, \mathfrak{G}),$$

$$C^{p,q}(\mathfrak{B}) = H^{p+q}(B^p, B^{p-1}, \mathfrak{G}),$$

for all integral values of $p$ and $q$. Let

$$i: A^{p,q}(\mathfrak{B}) \to A^{p-1,q+1}(\mathfrak{B}),$$

$$j: C^{p,q}(\mathfrak{B}) \to A^{p,q}(\mathfrak{B}),$$

denote injections, and

$$\delta: A^{p,q}(\mathfrak{B}) \to C^{p+1,q}(\mathfrak{B})$$

the coboundary operator of the pair $(B^{p+1}, B^p)$. Define

$$A(\mathfrak{B}) = \sum_{p,q} A^{p,q}(\mathfrak{B}),$$

$$C(\mathfrak{B}) = \sum_{p,q} C^{p,q}(\mathfrak{B}).$$

Then the homomorphisms $i$, $j$, and $\delta$ above define homomorphisms

$$i: A(\mathfrak{B}) \to A(\mathfrak{B}),$$

$$j: A(\mathfrak{B}) \to C(\mathfrak{B}),$$

$$\delta: A(\mathfrak{B}) \to C(\mathfrak{B}),$$

which are homogeneous of degrees $(-1, +1)$, $(0, 0)$, and $(1, 0)$ respectively. Moreover, it is readily seen that the necessary exactness conditions hold, so that $\langle A(\mathfrak{B}), C(\mathfrak{B}); i, \delta, j \rangle$ is an exact couple. We will denote the first derived couple
by \(\langle \Gamma(\mathfrak{B}), \mathfrak{H}(\mathfrak{B}) \rangle\). The groups \(\Gamma(\mathfrak{B})\) and \(\mathfrak{H}(\mathfrak{B})\) inherit bigraded structures from \(A(\mathfrak{B})\) and \(C(\mathfrak{B})\) respectively,

\[
\Gamma(\mathfrak{B}) = \sum \Gamma^{p,q}(\mathfrak{B}), \quad \mathfrak{H}(\mathfrak{B}) = \sum \mathfrak{H}^{p,q}(\mathfrak{B}).
\]

We will indicate the successive derived couples of \(\langle \Gamma(\mathfrak{B}), H(\mathfrak{B}) \rangle\) by the use of subscripts, as was done with the cohomotopy exact couple.

Let \(\mathfrak{B} = \{B, \pi, X, F\}\) and \(\mathfrak{B}' = \{B', \pi', X', F\}\) be two bundles having the same fibre, and such that \(X\) and \(X'\) are both cell complexes. Let \(h: B \to B'\) be a map (for the definition, see p. 9 of [17]) such that the induced map \(h: X \to X'\) is cellular. Such a bundle map will be called a "cellular bundle map". Using the covering homotopy theorem, [17, section 11], it is readily seen that any bundle map \(h: \mathfrak{B} \to \mathfrak{B}'\) is homotopic to a cellular bundle map. Such a map has the property that \(h(B^p) \subseteq (B'^q)\), and hence induces homomorphisms

\[
h^*: A^{p,q}(\mathfrak{B}') \to A^{p,q}(\mathfrak{B}),
\]

\[
h^*: C^{p,q}(\mathfrak{B}') \to C^{p,q}(\mathfrak{B}).
\]

The homomorphisms \(h^*\) and \(h^*\) define homomorphisms

\[
h^*: A(\mathfrak{B}') \to A(\mathfrak{B}),
\]

\[
h^*: C(\mathfrak{B}') \to C(\mathfrak{B}),
\]

both of which are homogeneous of degree \((0, 0)\). It is obvious that the pair \((h^*, h^*)\) is a map \(\langle A(\mathfrak{B}'), C(\mathfrak{B}')\rangle \to \langle A(\mathfrak{B}), C(\mathfrak{B})\rangle\). Also, it is obvious that the operation of assigning to each bundle \(\mathfrak{B}\) (with fibre \(F\) and base space a complex) the exact couple \(\langle A(\mathfrak{B}), C(\mathfrak{B})\rangle\), and to each cellular bundle map \(h: \mathfrak{B} \to \mathfrak{B}'\) the induced map \((h^*, h^*)\) has all the usual functorial properties; cf. the analogous discussion in Sections 9 of Part II and 2 of Part III.

14. The Algebraic Homotopy Induced By a Cellular Homotopy; Invariance

Proof

Let \(\mathfrak{B} = \{B, \pi, X, F\}\) and \(\mathfrak{B}' = \{B', \pi', X', F\}\) be two fibre bundles having the same fibre \(F\), and such that \(X\) and \(X'\) are both cell complexes, and let \(h_0, h_1: \mathfrak{B} \to \mathfrak{B}'\) be two cellular bundle maps which are homotopic (for the definition of a homotopy of bundle maps, see section 11 of [17]). Let \(h: \mathfrak{B} \times I \to \mathfrak{B}'\) denote the homotopy between \(h_0\) and \(h_1\); we assert that we may assume without loss of generality that \(h\) is also a cellular bundle map (with respect to the usual decomposition of the product space \(X \times I\) into a cell complex; cf. Sections 19 and 33 of [17]). For assume we are given an arbitrary homotopy \(h': \mathfrak{B} \times I \to \mathfrak{B}'\), and let \(\tilde{h}' : X \times I \to X'\) be the induced map. Then it is known that there exists a cellular map \(\tilde{h}: X \times I \to X'\) such that \(h\) is homotopic to \(\tilde{h}'\) relative to \((X \times 0) \cup (X \times 1)\); for the proof of this fact, see [21]. Then if we apply the covering homotopy theorem [17, section 11], to the homotopy between \(\tilde{h}'\) and \(\tilde{h}\), we obtain a map \(h: \mathfrak{B} \times I \to \mathfrak{B}'\) having the desired properties.
We shall refer to a homotopy \( h : \mathfrak{B} \times I \to \mathfrak{B} \) having the property that the induced map \( h : X \times I \to X' \) is cellular as a cellular homotopy.

**Theorem 14.1.** Let \( h_0, h_1 : \mathfrak{B} \times I \to \mathfrak{B} \) be two bundle maps which are homotopic. Then the induced maps, \((h^*, h_0^*)\) and \((h^*, h_1^*) : \langle A(\mathfrak{B}), C(\mathfrak{B}) \rangle \to \langle A(\mathfrak{B}), C(\mathfrak{B}) \rangle\) are algebraically homotopic.

**Proof.** Let \( h : \mathfrak{B} \times I \to \mathfrak{B} \) be a cellular bundle homotopy between \( h_0 \) and \( h_1 \); the map \( h \) then has the property that \( h(B^{p-1} \times I) \cup (B^p \times I) \subset B^p \) for \( p = 0, 1, 2, \ldots \), and \( h(b, 0) = h_0(b), h(b, 1) = h_1(b) \) for \( b \in B \). On account of these two facts, one can use the homotopy \( h \) to construct an algebraic homotopy between \((h^*, h_0^*)\) and \((h^*, h_1^*)\) exactly as was done for the cohomotopy exact couple in Section 4. The details of the proof are left to the reader; it is only necessary to replace cohomotopy groups everywhere by cohomology groups.

From this theorem, and our previous remarks about cellular bundle maps and cellular homotopies of bundle maps, it follows by standard arguments that the exact couple \( \langle \Gamma(\mathfrak{B}), \mathcal{X}(\mathfrak{B}) \rangle \) is an invariant of the bundle \( \mathfrak{B} \). It does not depend on the cellular decomposition chosen for the base space.

15. Cross Products

Let \( G_1 \) and \( G_2 \) be two abelian groups which are paired to a third abelian group \( G \), and let \( K_1 \) and \( K_2 \) be cell complexes. Using this given pairing, it is possible to define a pairing of the cochain groups, \( C^*(K_1, G_1) \) and \( C^*(K_2, G_2) \) to \( C^{p+q}(K_1 \times K_2, G) \); for details, see [17, section 33.2]. Let \( L_1 \) and \( L_2 \) be arbitrary sub-complexes of \( K_1 \) and \( K_2 \). Then this pairing of the cochain groups induces a pairing of the relative cohomology groups \( H^p(K_1, L_1, G_1) \) and \( H^q(K_2, L_2, G_2) \) to \( H^{p+q}(K_1 \times K_2, K_1 \times L_2 \cup L_1 \times K_2, G) \) and hence a homomorphism of the tensor product:

\[ \Omega : H^p(K_1, L_1, G_1) \otimes H^q(K_2, L_2, G_2) \to H^{p+q}(K_1 \times K_2, K_1 \times L_2 \cup L_1 \times K_2) \]

This homomorphism \( \Omega \) is a topological invariant of the pairs \((K_1, L_1)\) and \((K_2, L_2)\), i.e., it does not depend on the cellular decomposition chosen for the corresponding spaces.

We need the following lemma about the homomorphism \( \Omega \); it is a special case of more general theorems about the cohomology groups of product complexes.

Let \( G \) be an arbitrary abelian group; pair \( G \) with the additive group of integers to itself in the obvious way. Then we have:

**Lemma 15.1.** Let \( (K_2, L_2) \) be a pair such that the integral cohomology groups \( H^q(K_2, L_2) = 0 \) for \( q \neq n \), and \( H^n(K_2, L_2) \) is infinite cyclic. Then the homomorphism \( \Omega : H^p(K_1, G) \otimes H^n(K_2, L_2) \to H^{p+n}(K_1 \times K_2, K_1 \times L_2, G) \) is an isomorphism onto.

The proof can be made roughly along the lines of the proof of the classical Künneth theorem; cf. [1, Chap. VII, section 3]. The details are left to the reader.

16. Some Trivial Results on the Groups \( \mathcal{X}^{p,q}(\mathfrak{B}) \) and \( \Gamma^{p,q}(\mathfrak{B}) \)

First of all, we note the following fact:

**Lemma 16.1.** For any bundle \( \mathfrak{B} \), \( C^{p,q}(\mathfrak{B}) = 0 \) if \( q < 0 \) or \( q > \dim F \).
Proof. Let \( p \geq 0 \), and \( \sigma_i^p, \cdots, \sigma_k^p \) denote the \( p \)-cells of \( X \), and \( \delta_i^p \) denote the boundary of \( \sigma_i^p \). Define

\[
Y_i^p = \pi^{-1}(\sigma_i^p),
\]
\[
\hat{Y}_i^p = \pi^{-1}(\delta_i^p).
\]

Then it follows readily from the excision property for cohomology groups that

\[
H^{p+q}(B^p, B^{p-1}) \approx H^{p+q}(Y_i^p, \hat{Y}_i^p) + \cdots + H^{p+q}(Y_k^p, \hat{Y}_k^p).
\]

It is readily seen that \( Y_i^p \) is homeomorphic to \( \sigma_i^p \times F \), and \( \hat{Y}_i^p \) is homeomorphic to \( \delta_i^p \times F \); hence by the preceding theorem, we have an isomorphism

\[
\Omega: H^q(F) \otimes H^p(\sigma_i^p, \delta_i^p) \approx H^{p+q}(Y_i^p, \hat{Y}_i^p),
\]

and from this fact the lemma readily follows.

An obvious corollary of this lemma is the following fact:

\[(16.1) \quad \mathcal{C}^p_\ast(q)(\mathcal{B}) = 0 \text{ if } q < 0 \text{ or } q > \text{dim } F.\]

It is obvious that \( C^p,q(\mathcal{B}) = 0 \) if \( p < 0 \) or \( p > \text{dim } X \); hence we have

\[(16.2) \quad \mathcal{C}^p_\ast(q)(\mathcal{B}) = 0 \text{ if } p < 0 \text{ or } p > \text{dim } X.\]

Lemma 16.2. If \( q > \text{dim } F \), then \( A^p_\ast(q)(\mathcal{B}) = 0 \).

This follows from the fact that \( \text{dim } B^p = \text{dim } X^p + \text{dim } F \), and the definition of \( A^p_\ast(q)(\mathcal{B}) \).

As a corollary, we see that

\[(16.3) \quad \Gamma^p_\ast(q)(\mathcal{B}) = 0 \text{ if } q > \text{dim } F.\]

Lemma 16.3. If \( q < 0 \), then \( A^p_\ast(q)(\mathcal{B}) \approx H^{p+q}(B) \).

Proof. Let \( N = \text{dim } X \); then \( X^p = X \) and \( B^p = B \) if \( p \geq N \). Hence \( A^p_\ast(q)(\mathcal{B}) = H^{p+q}(B) \) if \( p \geq N \). From (16.1) it follows that the homomorphism

\[
i: A^p_\ast(q)(\mathcal{B}) \to A^{p-1,q+1}(\mathcal{B})
\]

is an isomorphism onto if \( q < -1 \). Combining these two facts, we obtain the desired result.

As a corollary of this lemma, we have for \( q < 0 \),

\[(16.4) \quad \Gamma^p_\ast(q)(\mathcal{B}) \approx H^{p+q}(B).\]

17. Determination of \( \mathcal{C}^p_\ast(q)(\mathcal{B}) \)

Let \( \mathcal{B} = \{ B, \pi, X, F, G, J, \{ V_j \}, \{ \phi_j \} \} \) be a coordinate bundle; associated with \( \mathcal{B} \) in a natural way is a bundle of coefficients having as fibre the cohomology group \( H^q(F, \mathcal{G}) \). The description of this bundle of coefficients parallels closely Section 30.2 of [17]. The basic definitions are as follows.

The bundle space, \( \Pi \), is the union of the groups \( H^q(F_x, \mathcal{G}) \) for all \( x \in X \), where \( F_x \) is the fibre over \( X \). The projection \( p: \Pi \to X \) is defined by mapping all of \( H^q(F_x, \mathcal{G}) \) onto the point \( x \in X \). Each element \( g \in G \) induces an automorphism
Let $g^* : H^q(F) \to H^q(F)$; let $\Gamma$ denote the group of all such automorphisms. Then $\Gamma$, taken with the discrete topology, is the group of the bundle. Use the same indexing set $J$ and coordinate neighborhoods $V_j$ as for $\mathcal{B}$. The coordinate transformations

$$\psi_j : V_j \times H^q(F, \mathcal{B}) \to \rho^{-1}(V_j)$$

are defined by

$$\psi_j(x, u) = (\phi_{j,x}^{-1})^* u$$

for any $x \in V_j$ and $u \in H^q(F, \mathcal{B})$. We also define

$$\rho_j : \rho^{-1}(V_j) \to H^q(F, \mathcal{B})$$

by

$$\rho_j(u) = \phi_{j,x}^*(u)$$

for $x \in V_j$, $u \in H^q(F_x)$, and an automorphism

$$\gamma_{ji}(x) : H^q(F) \to H^q(F), \quad x \in V_i \cap V_j$$

by

$$\gamma_{ji}(x) = \rho_j \circ \psi_{i,x} = \phi_{j,x}^* \circ (\phi_{i,x}^{-1})^*$$

$$= (\phi_{i,x}^* \circ \phi_{j,x})^* = [g_{ij}(x)]^* = [g_{ij}(x)^{-1}]^*.$$

We will denote the bundle so obtained by $\mathcal{B}(H^q, \mathcal{B})$, or $\mathcal{B}(H^q)$ for brevity.

Using this bundle for coefficient group, we can define the cohomology groups $H^p(X, \mathcal{B}(H^q))$, as described in Section 31 of [17].

**Theorem 17.1.** The group $\mathfrak{C}^{p,q}(\mathcal{B})$ is isomorphic to $H^p(X, \mathcal{B}(H^q))$.

**Proof.** Let $C^p(X, \mathcal{B}(H^q))$ denote the group of $p$-dimensional cochains of the cell complex $X$ with coefficients in the bundle $\mathcal{B}(H^q)$, as defined in Section 31.2 of [17], and let $\delta : C^p(X, \mathcal{B}(H^q)) \to C^{p+1}(X, \mathcal{B}(H^q))$ denote the coboundary operator. To prove this theorem, it clearly suffices to exhibit isomorphisms

$$\eta^{p,q} : C^{p,q}(\mathcal{B}) \cong C^p(X, \mathcal{B}(H^q))$$

such that commutativity holds in the following diagram:

$$
\begin{array}{ccc}
C^{p,q}(\mathcal{B}) & \xrightarrow{d} & C^{p+1,q}(\mathcal{B}) \\
\downarrow \eta^{p,q} & & \downarrow \eta^{p+1,q} \\
C^p(X, \mathcal{B}(H^q)) & \xrightarrow{\delta} & C^{p+1}(X, \mathcal{B}(H^q)).
\end{array}
$$

In our definition of $C^p(X, \mathcal{B}(H^q))$ we will follow very closely the notation of Section 31.2 of [17]. We assume that there has been chosen for each cell $\sigma$ of the complex $X$ a reference point $x_\sigma$, and we denote by $F_\sigma$ the fibre over $x_\sigma$. A $p$-cochain with coefficients in $\mathcal{B}(H^q)$ is a function $c$ which attaches to each oriented...
p-cell \( \sigma \) an element \( c(\sigma) \in H^q(F_\sigma) \) and satisfies the condition \( c(-\sigma) = -c(\sigma) \). If the oriented \( p \)-cell \( \sigma \) is a face of the oriented \( p + 1 \)-cell \( \tau \), then \([\sigma:\tau] = [\tau:\sigma] = \pm 1\) will denote their incidence number. If \( \sigma \) is not a face of \( \tau \), then \([\sigma:\tau] = 0\).

For each pair of incident cells, \( \sigma < \tau \), we assume there has been chosen a curve \( C \) in \( \tau \) from \( x_r \) to \( x_r \), and \( w_\sigma: H^q(F_\sigma) \to H^q(F_\tau) \) denotes the isomorphism \( C^* \) induced by \( C \). The coboundary of a \( p \)-cochain \( c \) is defined by

\[
(\delta c)(\tau) = \sum_\sigma [\sigma:\tau] w_\sigma[c(\sigma)]
\]

where \( \tau \) is any \( p + 1 \)-cell and the summation is over all \( p \)-cells \( \sigma \) which are faces of \( \tau \).

We may assume without loss of generality that the cellular subdivision of \( X \) is so fine that every cell is contained in some one coordinate neighborhood. Choose for each cell \( \sigma \) a coordinate neighborhood \( V_\sigma \) containing it. Let

\[
Y_\sigma = \pi^{-1}(\sigma), \\
\hat{Y}_\sigma = \pi^{-1}(\hat{\sigma}).
\]

Then the coordinate function corresponding to the neighborhood \( V_\sigma \) defines a homeomorphism

\[
\phi_\sigma: \sigma \times F \to Y_\sigma.
\]

For any \( x \in \sigma \), we define a homeomorphism

\[
\phi_{\sigma,x}: F \to F_x
\]

by

\[
\phi_{\sigma,x}(y) = \phi_\sigma(x, y).
\]

If \( \sigma < \tau \), then for any \( x \in \sigma \),

\[
\phi_{\tau,x}^{-1} \circ \phi_{\sigma,x} = g_{r\sigma}(x): F \to F
\]
is an element of \( G \), and \( g_{r\sigma}: \sigma \to G \) is a continuous map. In an analogous manner, the coordinate function of the bundle of coefficients, \( \beta(H^p) \), corresponding to the neighborhood \( V_\sigma \) defines functions \( \psi_\sigma, \psi_{\sigma,x}, \gamma_{r\sigma} \), etc. Note that the function \( \gamma_{r\sigma}: \sigma \to \Gamma \) is constant, since the cell \( \sigma \) is connected, and \( \Gamma \) has the discrete topology. Therefore, we will write \( \gamma_{r\sigma} = \gamma_{r\sigma}(x) \) for any point \( x \in \sigma \).

The group \( C^{p,q}(\mathbb{G}) = H^{p+q}(B^p, B^{p-1}) \) is naturally isomorphic to the direct sum of the groups \( H^{p+q}(Y_\sigma, \hat{Y}_\sigma) \) for all \( p \)-cells \( \sigma \) in \( X \). The homeomorphism \( \phi_\sigma \) induces an isomorphism

\[
\phi_\sigma^*: H^{p+q}(Y_\sigma, \hat{Y}_\sigma) \to H^{p+q}(\sigma \times F, \hat{\sigma} \times F).
\]

We assume that for each cell \( \sigma \) a definite orientation has been chosen, or what amounts to the same thing, a definite generator \( u_\sigma \) has been chosen for the integral cohomology group \( H^p(\sigma, \hat{\sigma}) \). Then by Lemma 15.1, any element of \( H^{p+q}(\sigma \times F, \hat{\sigma} \times F) \) can be expressed uniquely in the form \( u_\sigma \times v \) (cross product) for some element \( v \in H^q(F, \mathbb{G}) \). Let

\[
(u_\sigma \times v) = (\phi_\sigma^*)^{-1}(u_\sigma \times v).
\]
Thus any element of $H^{p+q}(B^p, B^{p-1})$ may be expressed in the form $\sum_\sigma u_\sigma \times v_\sigma$, $v_\sigma \in \tilde{H}^q(F)$, where the sum is over all $p$-cells $\sigma$, and this expression is unique.

We now define $\eta^{p,q}(\sum_\sigma u_\sigma \times v_\sigma) = c$, where $c$ is the $p$-cochain defined by $c(\sigma) = \psi_{\sigma,x}(v_\sigma)$ for any $p$-cell $\sigma$; here $\psi_{\sigma,x}: \tilde{H}^q(F) \to \tilde{H}^q(F_\sigma)$ is defined by a coordinate function of the bundle $\tilde{\omega}(H^n)$. Before completing the proof of the theorem, we will prove some lemmas.

**Lemma 17.1.** For any $p$-cell $u$ and any $v \in H^q(F)$,

$$d(u \times v) = \sum_\tau [\sigma: \tau][u_\sigma \times \gamma_{\sigma\tau}(v)],$$

where the summation is over all $(p+1)$-cells $\tau$.

**Proof of Lemma 17.1.** Let $\sigma$ be an arbitrary $p$-cell and $\tau$ an arbitrary $(p+1)$-cell of $X$. Then $H^{p+q}(Y_\sigma, \tilde{Y}_\sigma)$ is a direct summand of $H^{p+q}(B^p, B^{p-1})$, and $H^{p+q+1}(Y_\tau, \tilde{Y}_\tau)$ is a direct summand of $H^{p+q+1}(B^{p+1}, B^p)$, and we need to consider the isomorphism of each of these direct summands into the whole group.

$$
\begin{array}{ccc}
H^{p+q}(B^p, B^{p-1}) & \xrightarrow{j} & H^{p+q}(B^p) & \xrightarrow{\delta} & H^{p+q+1}(B^{p+1}, B^p) \\
\downarrow i_1 & & \downarrow i_7 & & \downarrow i_6 \\
H^{p+q}(B^p, \text{Cl}(B^p - Y_\sigma)) & \xrightarrow{i_8} & H^{p+q}(Y_\sigma) & \xrightarrow{\delta_1} & H^{p+q+1}(Y_\tau, \tilde{Y}_\tau) \\
\downarrow i_2 & & \downarrow i_5 & & \downarrow i_6 \\
H^{p+q}(Y_\sigma, \tilde{Y}_\sigma) & \xleftarrow{i_4} & H^{p+q}(\tilde{Y}_\tau, \text{Cl}(\tilde{Y}_\tau - Y_\sigma))
\end{array}
$$

**Fig. 13**

$H^{p+q}(Y_\sigma, \tilde{Y}_\sigma) \to H^{p+q}(B^p, B^{p-1})$, $H^{p+q+1}(Y_\tau, \tilde{Y}_\tau) \to H^{p+q+1}(B^{p+1}, B^p)$, and the projections of the whole group onto the direct summands, $H^{p+q}(B^p, B^{p-1}) \to H^{p+q}(Y_\sigma, \tilde{Y}_\sigma)$ and $H^{p+q+1}(B^{p+1}, B^p) \to H^{p+q+1}(Y_\tau, \tilde{Y}_\tau)$. These projections are induced by the corresponding inclusion maps, while the isomorphism into is defined by the following 3 diagram:

$$
\begin{array}{ccc}
H^{p+q}(B^p, \text{Cl}(B^p - Y_\sigma)) & \xrightarrow{i_1} & H^{p+q}(B^p, B^{p-1}) \\
\downarrow i_2 & & \downarrow H^{p+q}(B^p, B^{p-1}) \\
H^{p+q}(Y_\sigma, \tilde{Y}_\sigma)
\end{array}
$$

All homomorphisms are induced by inclusion maps, and $i_2$ is an isomorphism onto (excision property).

Consider now the diagram in figure 13. In this diagram $j$ and $\delta$ have their

---

3 In this and the following diagrams, the notation "Cl(A)" means "closure of A".
usual meanings, $\delta_1$ and $\delta_2$ are coboundary operators, and the other arrows represent injections. To prove Lemma 17.1, it obviously suffices to prove that for $u_\sigma \times v \in H^{p+q}(Y_\sigma, \hat{Y}_\sigma)$,

$$\delta_1 i_3 \hat{\delta}_2^{-1}(u_\sigma \times v) = [\sigma : \tau][u_\tau \times \gamma_{\tau\sigma}(v)].$$

In case $\sigma$ is not a face of $\tau$, $\hat{Y}_\tau \subset \text{Cl}(B^p - Y_\sigma)$, hence $i_3 = 0$. But $[\sigma : \tau] = 0$ also in this case, and hence the formula is proved. In case $\sigma$ is a face of $\tau$, then $i_2, i_3, \ldots, i_6$ are all isomorphisms onto, and commutativity holds around every square and triangle in this diagram. Therefore

$$\delta_1 i_3 \hat{\delta}_2^{-1}(u_\sigma \times v) = \delta_2 \hat{\delta}_4^{-1}(u_\sigma \times v).$$

Now consider the diagram shown in figure 14. Here $\phi^*$ and $\phi_\tau^*$ are isomorphisms induced by the coordinate functions $\phi_\sigma : \sigma \times F \to Y_\sigma$ and $\phi_\tau : \tau \times F \to \hat{Y}_\tau$, and $k_{\tau\sigma} : \sigma \times F \to \tau \times F$ is a homeomorphism into defined by $k_{\tau\sigma}(x, y) = (x, g_{\tau\sigma}(y))$ for any $x \in \sigma$ and $y \in F$. Let $i_5 : H^p(\hat{\tau}, \text{Cl}(\hat{\tau} - \sigma)) \to H^p(\sigma, \hat{\sigma})$ denote the injection,

$$H^{p+q}(Y_\sigma, \hat{Y}_\sigma) \xrightarrow{\phi^*} H^{p+q}(\sigma \times F, \hat{\sigma} \times F)$$

$\downarrow i_4$

$$\downarrow \delta_3$$

$\downarrow \hat{\delta}_2$

$\downarrow \delta_4$

$H^{p+q+1}(Y_\sigma, \hat{Y}_\sigma) \xrightarrow{\phi_\tau^*} H^{p+q+1}(\tau \times F, \hat{\tau} \times F)$

FIG. 14

and $\delta_4 : H^p(\hat{\tau}, \text{Cl}(\hat{\tau} - \sigma)) \to H^{p+1}(\tau, \hat{\tau})$ the coboundary operator. Then it can be verified that if $u \in H^p(\sigma, \hat{\sigma})$ and $v \in H^q(F, \xi)$,

$$(k_{\tau\sigma}^*)^{-1}(u \times v) = (\hat{\delta}_4 i_5 u_\sigma) \times (\gamma_{\tau\sigma} v).$$

Also, it follows from the basic properties of the "cross" product that for $u' \in H^p(\sigma, \text{Cl}(\tau - \sigma)), v' \in H^q(F, \xi)$,

$$\delta_3 (u' \times v') = (\delta_4 u') \times v'.$$

Combining these two equations with the fact that commutativity holds around both squares of figure 14, we see that

$$\delta_2 \hat{\delta}_4^{-1}(u_\sigma \times v) = (\delta_4 i_5^{-1} u_\sigma) \times (\gamma_{\tau\sigma} v).$$

Now, if we make use of the fact that

$$\delta_4 i_5^{-1}(u_\sigma) = [\sigma : \tau] u_\tau,$$

we obtain the desired result.
Lemma 17.2. Let \( r \) be a \( p + 1 \)-cell of \( X \) and \( \sigma \) a \( p \)-cell which is a face of \( r \). Then commutativity holds in the following diagram:

\[
\begin{array}{c}
H^s(F, \mathfrak{g}) \xrightarrow{\gamma_{rs}} H^s(F, \mathfrak{g}) \\
\downarrow \psi_{s,x} \quad \downarrow \psi_{r,x} \\
H^s(F_\sigma) \xrightarrow{w_{sr}} H^s(F_r).
\end{array}
\]

Proof. The bundle \( \mathfrak{g}(H^s) \) is a bundle with a discrete group, hence the results of Section 13.6 of [17] can be applied. The lemma follows directly from this and the definition of the isomorphism \( w_{sr} \).

The proof of Theorem 17.1 now follows directly from the definition of \( \eta^{p,q} \) and Lemmas 17.1 and 17.2.

18. The Groups \( \Delta^{p,q}(\mathfrak{g}, \mathfrak{g}) \)

Let \( \mathfrak{g} = \{B, \pi, X, F\} \) be a fibre bundle with base space a cell complex as before, and let

\[ k^{n,m}: H^n(B, \mathfrak{g}) \to H^n(B^m, \mathfrak{g}) \]

denote the injection. Define \( \Delta^{n,m}(\mathfrak{g}, \mathfrak{g}) \) to be the kernel of \( k^{n,m} \). Then \( \Delta^{n,m}(\mathfrak{g}) \) is a subgroup of \( H^n(B) \), and if \( m < p \), then \( \Delta^{n,p}(\mathfrak{g}) \subseteq \Delta^{n,m}(\mathfrak{g}) \). The following two properties of the groups \( \Delta^{n,m}(\mathfrak{g}) \) are readily proved:

\begin{align*}
(18.1) & \quad \text{If } m \geq n, \text{ then } \Delta^{n,m}(\mathfrak{g}) = 0. \\
(18.2) & \quad \text{If } m < 0, \text{ then } \Delta^{n,m}(\mathfrak{g}) = H^n(B).
\end{align*}

If \( h: \mathfrak{g} \to \mathfrak{g}' \) is a cellular bundle map of one bundle into another, then the induced homomorphism \( h^*: H^n(B') \to H^n(B) \) has the property that \( h^*[\Delta^{n,m}(\mathfrak{g}')] \subseteq \Delta^{n,m}(\mathfrak{g}) \). Therefore \( h^* \) defines a homomorphism

\[ h^{n,m}: \Delta^{n,m}(\mathfrak{g}) \to \Delta^{n,m}(\mathfrak{g}). \]

Furthermore, if \( h_0, h_1: \mathfrak{g} \to \mathfrak{g}' \) are two bundle maps which are homotopic, then the induced homomorphisms, \( h_0^{n,m} \) and \( h_1^{n,m} \), are the same. These facts are enough to enable one to prove that the groups \( \Delta^{n,m}(\mathfrak{g}) \) are topological invariants of \( \mathfrak{g} \), i.e., they are independent of the cellular decomposition chosen for \( X \).

Associated with the bundle \( \mathfrak{g} \) we have the sequence of successive derived exact couples, \( (\Gamma_r(\mathfrak{g}), \mathfrak{F}_r(\mathfrak{g})), r = 0, 1, 2, \ldots \). The group \( \Gamma_r(\mathfrak{g}) \) is bigraded: \( \Gamma_r(\mathfrak{g}) = \sum \Gamma^{p,q}_r(\mathfrak{g}) \). The following proposition establishes a connection between the groups \( \Delta^{n,m}(\mathfrak{g}) \) and the groups \( \Gamma^{p,q}_r(\mathfrak{g}) \).

Proposition 18.1. For all values of \( r \geq q \), \( \Gamma^{p,q}_r(\mathfrak{g}) \) is isomorphic to the factor group, \( H^{p+q}(B)/\Delta^{p+q,p}(\mathfrak{g}) \).

The proof follows readily from the definitions of the groups \( \Gamma^{p,q}_r(\mathfrak{g}) \) and \( \Delta^{p+q,p}(\mathfrak{g}) \); the details are left to the reader.
19. The Spectral Sequence of a Fibre Bundle

Let \( \mathfrak{B} = \{ B, \pi, X, F \} \) be a fibre bundle with base space a cell complex, and let \( \langle \Gamma(\mathfrak{B}), \mathfrak{K}(\mathfrak{B}) \rangle = \langle \Gamma_0(\mathfrak{B}), \mathfrak{K}_0(\mathfrak{B}) \rangle, \langle \Gamma_1(\mathfrak{B}), \mathfrak{K}_1(\mathfrak{B}) \rangle, \langle \Gamma_2(\mathfrak{B}), \mathfrak{K}_2(\mathfrak{B}) \rangle, \cdots \) denote the successive derived exact couples of \( \mathfrak{B} \). Associated with this sequence of successive derived exact couples is the spectral sequence of bigraded groups, (\( \mathfrak{K}(\mathfrak{B}), d \)) = (\( \mathfrak{K}_0(\mathfrak{B}), d_0 \)), (\( \mathfrak{K}_1(\mathfrak{B}), d_1 \)), (\( \mathfrak{K}_2(\mathfrak{B}), d_2 \)), \cdots Each of the groups \( \mathfrak{K}_n(\mathfrak{B}) \) is bigraded,

\[
\mathfrak{K}_n(\mathfrak{B}) = \sum \mathfrak{K}_{p,q}^n(\mathfrak{B}),
\]

the differential operator \( d_n \) is homogeneous of degree \( (n + 2, -n - 1) \), and \( \mathfrak{K}_{p,q}^n(\mathfrak{B}) \cong H^p(X, \mathfrak{B}(H^q)) \). As usual, we will denote the limit group of this spectral sequence by \( \mathfrak{K}_\infty(\mathfrak{B}) \). \( \mathfrak{K}_\infty(\mathfrak{B}) \) has a bigraded structure,

\[
\mathfrak{K}_\infty(\mathfrak{B}) = \sum \mathfrak{K}_{p,q}^\infty(\mathfrak{B}).
\]

**Theorem 19.1.** The group \( \mathfrak{K}_{p,q}^\infty(\mathfrak{B}) \) is isomorphic to the factor group \( \Delta^{p,q} \mathfrak{K}_\infty(\mathfrak{B}) / \Delta^{p,q-1} \mathfrak{K}_\infty(\mathfrak{B}) \).

The proof of this theorem is left to the reader; it is similar to the proofs of the analogous theorems about the homotopy and cohomotopy exact couples.

**APPENDIX**

20. Some Remarks on the Borsuk-Spanier Cohomotopy Groups

The definitions and principal properties of cohomotopy groups are described by Spanier in [14]; in most cases, we will follow the notation and conventions laid down there; it is the purpose of this section to indicate those cases in which we will deviate from Spanier’s conventions.

If \( X \) is any topological space, and \( A \) is any subspace \( (A \) may be void), then the symbol \( \pi^n(X, A) \) will denote the set of all homotopy classes of maps \( (X, A) \to (S^n, p_0) \). Under certain conditions it is possible to define an addition in the set \( \pi^n(X, A) \) so that it becomes a group. Spanier states these conditions in terms of the dimension of \( X - A \). However, dimension is essentially a local property of a space, while the question of whether or not an addition can be defined in \( \pi^n(X, A) \) depends essentially on properties “in the large” of \( (X, A) \), e.g., the homotopy type of \( (X, A) \). We will try to give natural conditions under which it is possible to define an addition in \( \pi^n(X, A) \).

Before discussing this question, we will discuss those properties of the cohomotopy groups which are independent of the existence of the group operation. In all cases, the set \( \pi^n(X, A) \) has an exceptional element, namely the homotopy class of the constant map. This exceptional element will be denoted by the symbol \( O \). Also, the notation “\( \pi^n(X, A) = O \)” will be understood to mean that \( \pi^n(X, A) \) consists of a single element, namely the element \( O \). If \( f: (X, A) \to (Y, B) \) is a continuous map, then \( f \) induces a function, or set transformation, \( f^*: \pi^n(Y, B) \to \pi^n(X, A) \) in the obvious way. If \( (X, A, B) \) is a triple, then there is defined a function, or set transformation,

\[
\Delta: \pi^n(A, B) \to \pi^{n+1}(X, A)
\]
by the rule given in Section 8 of [14]. The functions \(f^*\) and \(\Delta\) have the important properties that \(f^*(O) = O\) and \(\Delta(O) = O\). We will call the subset \(f^*[\pi^n(Y, B)]\) the image of \(f^*\) and the subset \(\Delta[\pi^n(A, B)]\) the image of \(\Delta\); similarly, \(f^{*-1}(O)\) will be called the kernel of \(f^*\), and \(\Delta^{-1}(O)\) will be called the kernel of \(\Delta\). With these definitions, Theorems 7.2, 7.3, 7.4, 7.5, 7.6, and 8.4 of [14] make sense, and are true with only minor assumptions on the topological spaces involved.

The cohomotopy sequence of the triple \((X, A, B)\) is defined to be the following sequence of sets and functions:

\[
\ldots \rightarrow \pi^n(X, A) \xrightarrow{\Delta} \pi^n(X, B) \xrightarrow{j^*} \pi^n(A, B) \xrightarrow{i^*} \pi^n(A, B) \xrightarrow{\Delta} \pi^{n+1}(X, A) \xrightarrow{j^*} \ldots
\]

where \(i: (A, B) \rightarrow (X, B)\) and \(j: (X, B) \rightarrow (X, A)\) are inclusion maps. Since the image and kernel of each of the functions \(\Delta, j^*,\) and \(i^*\) is defined, it makes sense to ask whether or not this sequence is exact in the usual sense. As usual, to prove this sequence exact, one would have to prove the following six statements:

(a) \(\text{image } j^* \subset \text{kernel } i^*\),
(b) \(\text{image } i^* \subset \text{kernel } \Delta\),
(c) \(\text{image } \Delta \subset \text{kernel } j^*\),
(d) \(\text{image } j^* \supset \text{kernel } i^*\),
(e) \(\text{image } i^* \supset \text{kernel } \Delta\),
(f) \(\text{image } \Delta \supset \text{kernel } j^*\).

It is very easy to prove the truth of (a), (b), (c), and (d) for a very general class of triples \((X; A, B)\). However, it is possible to show by examples that (e) and (f) are not always true (see Section 9 of [14]).

**Lemma 20.1.** Let \((X, A, B)\) be a simplicial triple\(^4\) with \(\dim (A - B) < 2n\). Then in the following portion of the exact sequence of this triple,

\[
\pi^n(A, B) \xrightarrow{\Delta} \pi^{n+1}(X, A) \xrightarrow{j^*} \pi^{n+1}(X, B),
\]

the image of \(\Delta\) contains the kernel of \(j^*\).

The proof goes exactly the same as the proof of Lemma 16.1 in [14]. Note that our hypotheses are weaker than those of Spanier.

**Lemma 20.2.** Let \((X, A, B)\) be a simplicial triple with \(\dim (X - A) < 2n\). Then in the following portion of the exact sequence of this triple,

\[
\pi^n(X, B) \xrightarrow{i^*} \pi^n(A, B) \xrightarrow{\Delta} \pi^{n+1}(X, A),
\]

the image of \(i^*\) contains the kernel of \(\Delta\).

The proof is exactly like the proof of Lemma 16.3 of [14]. Note that this is one dimension better than the statement of Lemma 16.3 of [14].

\(^4\) A. L. Blakers has recently communicated to the author a proof of this lemma with the condition \(\dim (A - B) < 2n\) replaced by the weaker condition \(H^p(A, B) = 0\) for \(p \geq 2n\).
Lemma 20.3. Let \((X, A, B)\) be a simplicial triple such that \(H^p(X, A) = 0\) for \(p \geq 2n\). Then in the following portion of the exact sequence of this triple,

\[
\pi^n(X, B) \xrightarrow{i^*} \pi^n(A, B) \xrightarrow{\Delta} \pi^{n+1}(X, A),
\]

the image of \(i^*\) contains the kernel of \(\Delta\).

**Sketch of the proof.** First of all, note that since the integral cohomology groups \(H^p(X, A)\) vanish for \(p \geq 2n\), it follows that the cohomology groups \(H^p(X, A, G)\) vanish for \(p \geq 2n\) and any coefficient group \(G\).

Now let \(\alpha \in \pi^n(A, B)\) be such that \(\Delta(\alpha) = 0\). Let \(f: (A, B) \to (S^n, p_0)\) be a map representing \(\alpha\). To prove the lemma, we must show that \(f\) can be extended over all of \(X\); by applying the preceding lemma, we see that \(f\) can be extended over \(X^{2n-1} \cup A\); by using the theory of obstructions to extensions of continuous maps, we see that our assumption that \(H^p(X, A, \pi_{p-1}(S^n)) = 0\) for \(p > 2n - 1\) implies that \(f\) can be extended over all of \(X\).

Closely allied to the concept of the cohomotopy sequence of a triple is the concept of the cohomotopy sequence of a triad. Let \((X; A, B)\) be a triad; that is, \(X\) is a topological space and \(A\) and \(B\) are subspaces; let \(i: (A, A \cap B) \to (X, B)\), \(j: (X, B) \to (X, A \cup B)\), and \(k: (A, A \cap B) \to (A \cup B, B)\) be inclusion maps, and \(\Delta: \pi^n(A, A \cap B) \to \pi^{n+1}(X, A \cup B)\) the coboundary operator of the triple. Define a function, \(\Delta': \pi^n(A, A \cap B) \to \pi^{n+1}(X, A \cup B)\) by \(\Delta' \circ k^* = \Delta\) (note that \(k^*\) is a 1–1 map). The sequence

\[
\cdots \xrightarrow{\Delta'} \pi^n(X, A \cup B) \xrightarrow{j^*} \pi^n(X, B) \xrightarrow{i^*} \pi^n(A, A \cap B) \xrightarrow{\Delta'} \pi^{n+1}(X, A \cup B) \to \cdots
\]

is called the cohomotopy sequence of the triad \((X; A, B)\). It is readily seen to be isomorphic to the cohomotopy sequence of the triple \((X, A \cup B, B)\), and is an exact sequence if and only if the cohomotopy sequence of this triple is exact.

In order to discuss the question of when it is possible to define a group operation in \(\pi^n(X, A)\), it is convenient to extend the theory of obstructions to deformations of maps which is described briefly in Section 4.4 of [2] to cover the cases of maps of a compact pair \((X, A)\) into a triangulable pair \((Y, B)\). We will briefly outline how such an extension of the theory would go.

First of all, Lemma 4.4.1 of [2] can be proved under the hypotheses that \((X, A)\) is a compact pair (i.e., \(X\) is a compact Hausdorff space, and \(A\) is a closed subset of \(X\)) and that the pair \((Y, B)\) consists of compact ANR, \(Y\), and a closed subset, \(B\), which is also an ANR. The proof is made by repeated application of Borsuk's theorem [10, p. 86].

Now assume that \((X, A)\) is a compact pair, and \((Y, B)\) is a simplicial pair that is 1-connected, for which \(\pi_n(Y, B)\) is abelian, and which is simple in all dimensions. Then by using the standard techniques which enable one to extend the theory of obstructions to homotopies and extensions of continuous maps to the case of maps of a compact space into an ANR (cf. [8] for example), one can extend the theory of obstructions to deformations of mappings described in
section 4.4 of [2] to the case of maps \((X, A) \to (Y, B)\). The details of this extension are left to the reader. Of course it is necessary to use the Čech cohomology theory throughout.

Once this is done, it is possible to prove the following analogue of Theorem 4.4.2 of [2]. Assume \((X, A)\) is a compact pair and \(N = \text{dimension of } X\); (here dimension is that defined by means of the order of coverings of \(X\); cf. Section 2 of [14].) Let \((Y, B)\) be a simplicial pair which is 1-connected, for which the homotopy group \(\pi_q(Y, B)\) is abelian, and which is \(p\)-simple for \(2 \leq p \leq N\). If \(H^q(X, A, \pi_q(Y, B)) = 0\) for \(2 \leq q \leq N\), then any map \(f: (X, A) \to (Y, B)\) is deformable.

Suppose that \((X, A)\) is a pair which satisfies the following two conditions:
(a) Any map \(F_0: (X, A) \to (S^n \times S^n, p_0 \times p_0)\) is homotopic to a map \(F_1: (X, A) \to (S^n \times S^n, p_0 \times p_0)\) such that \(F_1(X) \subset S^n \vee S^n\). (b) Any map
\[
F: (X \times I, A \times I \cup X \times I) \to (S^n \times S^n, S^n \vee S^n)
\]
is deformable rel. \(A \times I \cup X \times I\). It is readily seen that under these hypotheses the method described in Sections 5 and 6 of [14] to define a group operation in \(\pi^n(X, A)\) applies.

**Theorem 20.4.** Assume that \((X, A)\) is a compact pair, and that \(X\) is finite dimensional. If the integral Čech cohomology groups, \(H^p(X, A)\) vanish for all \(p \geq 2n - 1\), then it is possible to define a group operation in \(\pi^n(X, A)\) by the Borsuk-Spanier method.

**Sketch of proof.** It suffices to show that conditions (a) and (b) of the preceding paragraph are satisfied; to do this, one makes use of the theory of obstructions to deformations. First of all, note that by the universal coefficient theorems for Čech cohomology theory, the hypothesis that \(H^p(X, A) = 0\) for \(p \geq 2n - 1\) implies \(H^p(X, A, G) = 0\) for \(p \geq 2n - 1\) and any coefficient group \(G\). Since \(\pi_q(S^n \times S^n, S^n \vee S^n) = 0\) for \(q < 2n\), it follows that any map \((X, A) \to (S^n \times S^n, S^n \vee S^n)\) is deformable, and hence deformable relative to \(A\). Thus condition (a) is satisfied. Next, we note that
\[
H^p(X \times I, A \times I \cup X \times I) = H^{p-1}(X, A)
\]
for all \(p\). Hence \(H^p(X \times I, A \times I \cup X \times I, G) = 0\) for any \(p \geq 2n\) and any coefficient group \(G\). Therefore any map \((X \times I, A \times I \cup X \times I) \to (S^n \times S^n, S^n \vee S^n)\) is deformable, and hence deformable relative to \(A \times I \cup X \times I\). This proves that condition (b) is satisfied, and the proof is complete.

This theorem gives the desired condition under which \(\pi^n(X, A)\) is a group.

**21. Composition Operation for Cohomotopy Groups**

Let \(\alpha \in \pi^p(X, A)\) and \(\beta \in \pi^q(S^p, p_0)\). Then we define an element \(\beta \circ \alpha \in \pi^q(X, A)\) as follows: choose maps
\[
f: (X, A) \to (S^p, p_0),
\]
\[
g: (S^p, p_0) \to (S^q, p_0)
\]
which represent $\alpha$ and $\beta$ respectively. Then $g \circ f:(X, A) \to (S^q, p_0)$ is a representative of $\beta \circ \alpha$.

Assume further that the integers $p$ and $q$ are chosen sufficiently large so that $\pi^p(X, A)$, $\pi^q(S^p, p_0)$, and $\pi^q(X, A)$ are all groups. Then we assert that the mapping $(\beta, \alpha) \to \beta \circ \alpha$ is bilinear. To prove this, first suppose $\beta_1, \beta_2 \in \pi^q(S^p, p_0)$ and $\alpha \in \pi^p(X, A)$. Choose maps $g_1, g_2: (S^p, p_0) \to (S^q, p_0)$ representing $\beta_1$ and $\beta_2$, and $f:(X, A) \to (S^q, p_0)$ representing $\alpha$. Let $h:(S^p, p_0) \to (S^q \times S^q, p_0 \times p_0)$ be a normalization (see [14], Definition 5.1) of $g_1 \times g_2:(S^p, p_0) \to (S^q \times S^q, p_0 \times p_0)$. Then $h \circ f$ is a normalization of $(g_1 \times g_2) \circ f = (g_1 \circ f) \times (g_2 \circ f)$. From this it follows that $(\beta_1 + \beta_2) \circ \alpha = \beta_1 \circ \alpha + \beta_2 \circ \alpha$. Next, let $\beta \in \pi^p(S^p, p_0)$, $\alpha_1, \alpha_2 \in \pi^q(X, A)$, and let $h:(X, A) \to (S^p \vee S^q, p_0 \times p_0)$ be a normalization of $f_1 \times f_2:(X, A) \to (S^p \times S^q, p_0 \times p_0)$. Then $\Omega \circ h:(X, A) \to (S^q, p_0)$ represents $\alpha_1 + \alpha_2$ and $g \circ \Omega \circ h:(X, A) \to (S^q, p_0)$ represents $\beta \circ \alpha_1 + \alpha_2$ (see [14], Section 5, for the definition of $\Omega$). Define a map $\bar{g}:S^p \times S^q \to S^q \times S^q$ by $\bar{g}(x, y) = (gx, gy)$. Then $\bar{g} \circ h:(X, A) \to (S^q \vee S^q, p_0 \times p_0)$ is a normalization of $(g \circ f_1) \times (g \circ f_2):(X, A) \to (S^q \times S^q, p_0 \times p_0)$. Hence the map $\Omega \circ \bar{g} \circ h:(X, A) \to (S^q, p_0)$ represents $\beta \circ \alpha_1 + \beta \circ \alpha_2.$ But

$$\Omega \circ \bar{g} = g \circ \Omega:(S^p \vee S^q, p_0 \times p_0) \to (S^q, p_0)$$

and hence $\Omega \circ \bar{g} \circ h = g \circ \Omega \circ h$. Therefore $\beta \circ \alpha_1 + \beta \circ \alpha_2 = \beta \circ (\alpha_1 + \alpha_2)$, and the assertion is proved.

The following three properties of this composition operation are readily proved:

(Associativity). If $\alpha \in \pi^p(X, A)$, $\beta \in \pi^q(S^p, p_0)$, and $\gamma \in \pi^r(S^q, p_0)$, then

$$\gamma \circ (\beta \circ \alpha) = (\gamma \circ \beta) \circ \alpha. \hfill (21.1)$$

Let $f:(Y, B) \to (X, A)$ be a continuous map, and $\alpha \in \pi^p(X, A)$, $\beta \in \pi^q(S^p, p_0)$. Then

$$f^* (\beta \circ \alpha) = \beta \circ (f^* \alpha). \hfill (21.2)$$

Let $\Delta: \pi^p(A) \to \pi^{p+1}(X, A)$ denote the coboundary operator of the pair $(X, A)$, and $E: \pi^q(S^p, p_0) \to \pi^{q+1}(S^p+1, p_0)$ the suspension homomorphism. Then

$$\Delta (\beta \circ \alpha) = (E \beta) \circ (\Delta \alpha). \hfill (21.3)$$

Finally, let $K$ be a finite cell complex, and let $K^p$ denote the $p$-dimensional skeleton. We define a homomorphism

$$\xi: \pi^p(S^p, p_0) \otimes \pi^p(K^p, K^{p-1}) \to \pi^q(K^p, K^{p-1})$$

for $p < 2q - 1$ by

$$\xi(\beta \otimes \alpha) = \beta \circ \alpha.$$

**Theorem 21.4.** The homomorphism $\xi$ is an isomorphism onto.

**Sketch of Proof.** First, one proves this theorem true for the trivial case in
which $K = K^p = E^p$ and $K^{p-1} = S^{p-1}$. Then in the general case if $\sigma_1, \cdots, \sigma_k$ denote the $p$-cells of $K$, one can prove the direct sum decomposition

$$\pi^r(K^p, K^{p-1}) \approx \sum_{i=1}^k \pi^r(\sigma_i^p, \partial \sigma_i^p)$$

for any $r$ such that $p < 2r - 1$. The excision property is used in proving this. Combining these two facts, and the basic properties of tensor products and the composition operation, the desired result is obtained.

22. Composition Operation between Elements of Homotopy and Cohomotopy Groups

Let $X$ be a topological space. Then if $\alpha \in \pi_p(X, x_0)$ (where $x_0 \in X$) and $\beta \in \pi^q(X)$, we define an element $\beta \circ \alpha \in \pi_p(S^q)$ as follows: Choose maps

$$f: (S^p, p_0) \to (X, x_0),$$
$$g: X \to S^q,$$

representing $\alpha$ and $\beta$ respectively. Then the map $g \circ f: S^p \to S^q$ represents $\beta \circ \alpha$. Obviously the relation

$$\beta \circ (\alpha_1 + \alpha_2) = \beta \circ \alpha_1 + \beta \circ \alpha_2$$

holds for any elements $\alpha_1, \alpha_2 \in \pi_p(X, x_0)$ and $\beta \in \pi^q(X)$. If the integer $q$ is chosen sufficiently large so that $\pi^q(X)$ is a group, then

$$(\beta_1 + \beta_2) \circ \alpha = \beta_1 \circ \alpha + \beta_2 \circ \alpha,$$

for any elements $\alpha \in \pi_p(X, x_0)$ and $\beta_1, \beta_2 \in \pi^q(X)$. To prove this, let $f: (S^p, p_0) \to (X, x_0)$ represent $\alpha$, and $g_1, g_2: X \to S^q$ represent $\beta_1$ and $\beta_2$. Let $h: X \to S^q \vee S^q$ be a normalization of $g_1 \times g_2$. Then $h \circ f: S^p \to S^q \vee S^q$ is a normalization of $(g_1 \circ g_2) \circ f = (g_1 \circ f) \times (g_2 \circ f)$, and from this the assertion follows readily.

In an analogous manner, if $(X, A)$ is a pair, $\alpha \in \pi_p(X, A, x_0)$ and $\beta \in \pi^q(X, A)$, then we define $\beta \circ \alpha \in \pi_p(S^q, p_0)$ by composition of representative functions. Here again the relation $\beta \circ (\alpha_1 + \alpha_2) = \beta \circ \alpha_1 + \beta \circ \alpha_2$ holds in all cases. And if the integer $q$ is sufficiently large so that $\pi^q(X, A)$ is a group, then the other distributive law, $(\beta_1 + \beta_2) \circ \alpha = \beta_1 \circ \alpha + \beta_2 \circ \alpha$, may be proven to hold.

The two principal properties of this law of composition are the following:

Let $f: (X, A, x_0) \to (Y, B, y_0)$ be a continuous map of one pair into another, $\alpha \in \pi_p(X, A, x_0)$ and $\beta \in \pi^q(Y, B)$. Then

$$(f^\ast \beta) \circ \alpha = \beta \circ (f_* \alpha),$$

i.e., the induced homomorphisms $f^\ast$ and $f_*$ are "dual" to each other.

Let $(X, A)$ be a pair, $\Delta: \pi^{q-1}(A) \to \pi^q(X, A)$ the cohomotopy coboundary operator, $\partial: \pi_p(X, A) \to \pi_{p-1}(A)$ the homotopy boundary operator, and $E: \pi_{p-1}(S^{q-1}) \to \pi_p(S^q)$ the suspension homomorphism. Then if $\alpha \in \pi_p(X, A)$ and $\beta \in \pi^{q-1}(A)$,

$$(\Delta \beta) \circ \alpha = E[\beta \circ (\partial \alpha)].$$
Thus in case $E$ is an isomorphism onto, $\Delta$ and $\partial$ can be looked on as dual operators.

**Bibliography**