Annals of Mathematics
Vol. 56, No. 2, September, 1952
Printed in U.S.A.

EXACT COUPLES IN ALGEBRAIC TOPOLOGY

(Parts I and II)

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(Received December 11, 1951)

Introduction[1]

The main purpose of this paper is to introduce a new algebraic object int topology. This new algebraic structure is called an *exact couple of groups* (or (modules, or of vector spaces, etc.). It apparently has many applications to prob lems of current interest in topology. In the present paper it is shown how exa(couples apply to the following three problems: (a) To determine relations betwee the homology groups of a space X, the Hurewicz homotopy groups of X, and ce tain additional topological invariants of X; (b) To determine relations betwee the cohomology groups of a space X, the cohomotopy groups of X, and certai additional topological invariants of X; (c) To determine relations between th homology (or cohomology) groups of the base space, the bundle space, and th fibre in a fibre bundle.

In each of these problems, the final result is expressed by means of a Lera Koszul sequence. The notion of a Leray-Koszul sequence (also called a spectr homology sequence or spectral cohomology sequence) has been introduced an exploited by topologists of the French school. It is already apparent as a resu of their work that the solution to many important problems of topology is be expressed by means of such a sequence. With the introduction of exact couples, seems that the list of problems, for which the final answer is expressed by mea of a Leray-Koszul sequence, is extended still further.

This paper is divided into five parts. The first part gives the purely algebra aspects of the idea of an exact couple. The remaining four parts give applicatio of the algebraic machinery developed in part one to topological problems. Pa two shows how exact couples may be applied to express relations between th homology and homotopy groups of a space, and certain new groups which a topological invariants of the space. Part three treats what may be called the du situation. Exact couples are applied to obtain relations between the cohomolog and cohomotopy groups of a space. In part four the duality which exists betwee the applications in parts two and three is given a precise formulation, and co nection is made with the usual duality between the homology and cohomolog groups of a space. In both parts two and three the relations involved are express by means of a Leray-Koszul sequence, and it seems rather unlikely that the Leray-Koszul sequences could be obtained from a differential-filtered group.

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[1] An abstract of the principal results of this paper was submitted to the American Math matical Society in March, 1951; cf. Bull. Amer. Math. Soc., 57 (1951), pp. 281–282.

The fifth and last part applies the theory of exact couples to the study of the cohomology structure of a fibre bundle for which the base space is a cell complex. A Leray-Koszul sequence is obtained which gives relations between the cohomology groups of the base space, bundle space, and fibre. The results obtained in this case are included among those obtained by Leray [11] under more general hypotheses. Furthermore, we have not included the multiplicative structure of the cohomology ring into the Leray-Koszul sequence obtained, and thus our results are not as complete as those of Leray. This application of exact couples to obtain the Leray-Koszul sequence of a fibre bundle is published in spite of these shortcomings mainly because it is our belief that the methods we use are closer to the usual methods of algebraic topology, and hence can be understood by most topologists with less effort than the methods of Leray. They should serve as an introduction to the important papers of Leray.

The notations, definitions, and conventions used for homology, cohomology, homotopy, and cohomotopy groups are collected together for ready reference in the appendix. Here also are contained the explicit statements of some lemmas from homotopy theory which are needed.

Parts I and II are published in the present issue of these ANNALS; parts III, IV, and V will appear in these ANNALS in the near future.

PART I. GENERAL ALGEBRAIC THEORY

1. Differential Groups

The principle algebraic objects with which we shall be concerned in this paper are abelian groups with certain additional elements of structure, together with certain homomorphisms of these abelian groups.

Let A be an abelian group. An endomorphism $d: A \to A$ is called a *differential operator* if $d^2 = 0$ (i.e., $d[d(a)] = 0$ for any $a \in A$). A *differential group* is a pair (A, d) consisting of an abelian group A and a differential operator d. If (A, d) is a differential group, we will denote by $\mathcal{Z}(A)$ the kernel of d, and by $\mathcal{B}(A)$ the image, $d(A)$. Both are subgroups of A, and from $d^2 = 0$, it follows that $\mathcal{B}(A) \subset \mathcal{Z}(A)$. The factor group $\mathcal{Z}(A)/\mathcal{B}(A)$ will be denoted by $\mathcal{H}(A)$, and called the *derived group*.

Let (A, d) and (A', d') be differential groups; a homomorphism $f: A \to A'$ is called *allowable* if the commutativity relation $d' \circ f = f \circ d$ holds. Such an allowable homomorphism f has the property that $f[\mathcal{Z}(A)] \subset \mathcal{Z}(A')$ and $f[\mathcal{B}(A)] \subset \mathcal{B}(A')$, and hence f *induces* a homomorphism $f^*: \mathcal{H}(A) \to \mathcal{H}(A')$. This operation of assigning to each allowable homomorphism the induced homomorphism of the derived groups has the following two obvious, but important, properties: (1) The identity homomorphism $i: A \to A$ is allowable, and the induced homomorphism $i^*: \mathcal{H}(A) \to \mathcal{H}(A)$ is also the identity. (2) Let (A, d), (A', d'), and (A'', d'') be differential groups, and let $f: A \to A'$, $g: A' \to A''$ be allowable homomorphisms. Then the composition $g \circ f: A \to A''$ is also allowable, and $(g \circ f)^* = g^* \circ f^*$. In the language of Eilenberg and MacLane [3], these two facts may be conveniently expressed by saying that the set of all differential groups and

allowable homomorphisms constitutes a category, and the operation of assigning to each differential group its derived group and to each homomorphism its induced homomorphism is a covariant functor.

Let (A, d) and (A', d') be differential groups, and $f, g: A \rightarrow A'$ allowable homomorphisms. Then f and g are said to be *algebraically homotopic* (notation: $f \simeq g$) if there exists a homomorphism $\xi: A \rightarrow A'$ which satisfies the following condition:

$$f - g = d' \circ \xi + \xi \circ d.$$

It is readily verified that this relation is an equivalence relation. Furthermore, if f and g are algebraically homotopic, then the induced homomorphisms $f^*, g^*: \mathcal{H}(A) \rightarrow \mathcal{H}(A')$ are the same.

A subgroup B of a group A with differential operator d is said to be *allowable* if $d(B) \subset B$; if this is the case, then d defines differential operators on B and on the factor group A/B in an obvious fashion; furthermore, the inclusion homomorphism $B \rightarrow A$ and the natural projection of A onto the factor group A/B are both allowable homomorphisms in the above mentioned sense.

2. Graded and Bigraded Groups

An abelian group A is said to be *graded*, or to have a *graded structure*, if there is prescribed a sequence of subgroups, A_n, $n = 0, \pm 1, \pm 2, \cdots$, such that A can be expressed as a direct sum,

$$A = \sum_{-\infty}^{+\infty} A_n .$$

An abelian group A is said to be *bigraded*, or to have a *bigraded structure*, if there is prescribed a double sequence of subgroups, A_{mn}, $m, n = 0, \pm 1, \pm 2, \cdots$, such that $A = \sum_{m,n} A_{m,n}$. In case of a graded group, $A = \sum_m A_m$, the elements of the subgroup A_p are said to be *homogeneous of degree* p; in the case of a bigraded group, $A = \sum_{m,n} A_{m,n}$, the elements of the subgroup $A_{p,q}$ are said to be *homogeneous of degree* (p, q).

When dealing with graded or bigraded groups, only a certain limited class of homormorphisms are of interest, the so-called homogeneous homomorphisms. If $A = \sum A_m$ and $B = \sum B_m$ are graded groups, then a homomorphism $f: A \rightarrow B$ is said to be *homogeneous of degree* p provided $f(A_m) \subset B_{m+p}$ for all values of m. If $A = \sum A_{m,n}$ and $B = \sum B_{m,n}$ are bigraded, then a homomorphism $f: A \rightarrow B$ is *homogeneous of degree* (p, q) provided $f(A_{m,n}) \subset B_{m+p,n+q}$ for all pairs (m, n). Note that the identity homomorphism of a graded or bigraded group onto itself is homogeneous, and that the composition of two homogeneous homomorphisms is again homogeneous.

Let $A = \sum A_{m,n}$ be a bigraded group; a subgroup $B \subset A$ is said to be *allowable* (with respect to the bigraded structure) in case $B = \sum (B \cap A_{m,n})$. If this is true, then B has a bigraded structure defined by $B = \sum B_{m,n}$, where $B_{m,n} = B \cap A_{m,n}$. We will express this fact by saying that the allowable subgroup B *inherits* a bigraded structure from A. It is readily verified that the factor group A/B is isomorphic to the direct sum $\sum A_{m,n}/B_{m,n}$; moreover, this isomorphism

is natural in the sense of Eilenberg and MacLane [3]. We will agree to identify these naturally isomorphic groups. This representation of A/B as a direct sum defines a bigraded structure on A/B. This bigraded structure will also be referred to as the bigraded structure that A/B *inherits* from A.

It is clear how to define in an analogous way the concept of an allowable subgroup of a graded group, etc. Note that the kernel and image of a homogeneous homomorphism are always allowable subgroups.

Often in algebraic topology we have to deal with differential groups which also have a graded (or bigraded) structure, and for which the differential operator is homogeneous. In this case the derived group inherits a graded (or bigraded) structure from the given group.

If A and G are abelian groups, then we will use the notation $A \otimes G$ to denote their tensor product (see Whitney, [24]). If (A, d) is a differential group, and G is an arbitrary abelian group, then we define a differential operator $d': A \otimes G \rightarrow A \otimes G$ in the obvious way:

$$d'(a \otimes g) = (da) \otimes g$$

for any $a \in A$ and $g \in G$. If $A = \sum A^p$ is a graded group, then the direct sum decomposition

$$A \otimes G = \sum (A^p \otimes G),$$

defines a graded structure on the tensor product $A \otimes G$. An analogous definition is applicable in case A is a bigraded group. If (A, d) is a differential group, and G is an abelian group, then for the sake of convenience we will use the notation $\mathfrak{H}(A, G)$ for the derived group of $(A \otimes G, d')$; i.e., $\mathfrak{H}(A, G) = \mathfrak{H}(A \otimes G)$.

3. Definition of a Leray-Koszul Sequence

A sequence of differential groups, (A^n, d^n), where the index n ranges over all integers larger than some given integer N, is called a *Leray-Koszul sequence* in case each group in the sequence is the derived group of the preceding:

$$A^{n+1} = \mathfrak{H}(A^n).$$

In a Leray-Koszul sequence there exist natural homomorphisms

$$\kappa_n : \mathcal{Z}(A^n) \rightarrow A^{n+1}.$$

κ_n is defined by assigning to each element of the subroup $\mathcal{Z}(A^n)$ its coset modulo $\mathfrak{B}(A^n)$. Thus κ_n is a homomorphism of a subgroup of A^n onto A^{n+1}. We will define a homomorphism κ_n^p of a subgroup of A^n onto A^{n+p} by the formula

$$\kappa_n^p = \kappa_{n+p-1} \circ \kappa_{n+p-2} \circ \cdots \circ \kappa_n .$$

Then $\kappa_n^1 = \kappa_n$. The precise definition of the domain of definition of κ_n^p is left to the reader. Let $\bar{A}^n$ denote the subgroup of A^n consisting of these elements $a \in A^n$ such that $\kappa_n^p(a)$ is defined for all values of p. Define $\bar{\kappa}_n : \bar{A}^n \rightarrow \bar{A}^{n+1}$ to be the restriction of κ_n to the subgroup $\bar{A}^n$. Then the sequence of groups $\{\bar{A}^n\}$ and homo-

morphisms $\{\bar{\kappa}_n\}$ constitutes a direct sequence of groups in the usual sense (see, for example, [10, ch. VIII, definition VIII 12]). The limit group of this direct sequence of groups will be called the *limit group* of the given Leray-Koszul sequence.

In most cases we shall have to deal with Leray-Koszul sequences (A^n, d^n) for which each of the groups A^n is bigraded, $A^n = \sum A^n_{p,q}$, each of the differential operators d^n is homogeneous, and A^{n+1} inherits its bigraded structure from A^n. In this case each of the homomorphisms κ_n is homogeneous of degree (0, 0), and there is determined a bigraded structure on the limit group in a natural way. Also, it will usually be true that for each pair of integers (p, q) there exists an integer N such that if $n > N$, then κ_n maps $A^n_{p,q}$ isomorphically onto $A^{n+1}_{p,q}$. This makes it possible to determine any homogeneous component of the limit group by an essentially finite process.

4. Definition of an Exact Couple; The Derived Couple

An *exact couple* of abelian groups consists of a pair of abelian groups, A and C, and three homomorphisms:

$$f:A \to A,$$

$$g:A \to C,$$

$$h:C \to A.$$

These homomorphisms are required to satisfy the following "exactness" conditions:

$$\text{image } f = \text{kernel } g,$$

$$\text{image } g = \text{kernel } h,$$

$$\text{image } h = \text{kernel } f.$$

These three conditions can be easily remembered if one makes the following triangular diagram,

$$\begin{tikzcd} A \arrow[rr, "f"] & & A \arrow[dl, "g"] \\ & C \arrow[ul, "h"] & \end{tikzcd}$$

and observes that the kernel of each homomorphism is required to be the image of the preceding homomorphism. We shall denote such an exact couple by the notation $\langle A, C; f, g, h\rangle$. When there is no danger of confusion, we shall often abbreviate this to $\langle A, C\rangle$.

There is an important operation which assigns to an exact couple $\langle A, C; f, g, h\rangle$ another exact couple, $\langle A', C'; f', g', h'\rangle$, called the *derived* exact couple. This derived exact couple is defined as follows.

Define an endomorphism $d:C \to C$ by $d = g \circ h$. Then $d^2 = d \circ d = g \circ h \circ g \circ h = 0$, since $h \circ g = 0$ by exactness. Therefore d is a differential operator. Let $C' = \mathfrak{H}(C)$, the derived group of the differential group (C, d). Let $A' = f(A) =$ image $f =$ kernel g. Define $f':A' \to A'$ by $f' = f \mid A'$, the restriction of f to the subgroup A'. The homomorphism $h':C' \to A'$ is induced by h: it is readily verified that $h[\mathcal{Z}(C)] \subset A'$, and $h[\mathcal{B}(C)] = 0$, hence h induces a homomorphism of the factor group $C' = \mathcal{Z}(C)/\mathcal{B}(C)$ into A'. The definition of $g':A' \to C'$ is more complicated. Let $a \in A'$; choose an element $b \in A$ such that $f(b) = a$. Then $g(b) \in \mathcal{Z}(C)$, and $g'(a)$ is defined to be the coset of $g(b)$ modulo $\mathcal{B}(C)$. It is easily verified that this definition is independent of the choice made of the element $b \in A$, and that g' is actually a homomorphism.

Of course, it is necessary to verify that the homomorphisms f', g', and h' satisfy the exactness condition of an exact couple. This verification is straightforward, and is left to the reader.

It is clear that this process of derivation can be applied to the derived exact couple $\langle A', C'; f', g', h' \rangle$ to obtain another exact couple $\langle A'', C''; f'', g'', h'' \rangle$, called the *second derived couple*, and so on. In general, we shall denote the n^{th} derived couple by $\langle A^{(n)}, C^{(n)}; f^{(n)}, g^{(n)}, h^{(n)} \rangle$.

5. Maps of Exact Couples

Let $\langle A, C; f, g, h \rangle$ and $\langle A_0, C_0; f_0, g_0, h_0 \rangle$ be two exact couples; a *map*,

$$(\phi, \psi): \langle A, C; f, g, h \rangle \to \langle A_0, C_0; f_0, g_0, h_0 \rangle,$$

consists of a pair of homomorphisms,

$$\phi : A \to A_0,$$
$$\psi : C \to C_0,$$

which satisfy the following three commutativity conditions:

$$\phi \circ f = f_0 \circ \phi$$
$$\psi \circ g = g_0 \circ \phi$$
$$\phi \circ h = h_0 \circ \psi.$$

If $d = g \circ h:C \to C$ and $d_0 = g_0 \circ h_0:C_0 \to C_0$ denote the differential operators on C and C_0 respectively, then our definitions imply the following commutativity relation:

$$\psi \circ d = d_0 \circ \psi.$$

Therefore ψ is an allowable homomorphism, in the sense defined in the preceding section, and hence induces a homomorphism

$$\psi':C' \to C_0'$$

of the corresponding derived groups. Also, it is clear that $\phi(A') \subset A_0'$; therefore ϕ defines a homomorphism

$$\phi':A' \to A_0'.$$

It can now be verified without difficulty that the pair of homomorphisms (ϕ', ψ') constitute a map of the first derived exact couples,

$$(\phi', \psi'):\langle A', C'\rangle \rightarrow \langle A_0', C_0'\rangle$$

in the sense just defined. We will say that the map (ϕ', ψ') is *induced* by (ϕ, ψ). By iterating this process, one obtains a map $(\phi^{(n)}, \psi^{(n)}):\langle A^{(n)}, C^{(n)}\rangle \rightarrow \langle A_0^{(n)}, C_0^{(n)}\rangle$ which is induced by the given map (ϕ, ψ).

The set of all exact couples and maps of exact couples constitutes a category in the sense of Eilenberg and MacLane [3], and the operation of derivation is a covariant functor.

Let (ϕ_0, ψ_0) and $(\phi_1, \psi_1):\langle A, C; f, g, h\rangle \rightarrow \langle A_0, C_0; f_0, g_0, h_0\rangle$ be two maps of exact couples in the sense we have just defined. The maps (ϕ_0, ψ_0) and (ϕ_1, ψ_1) are said to be *algebraically homotopic*[2] (notation: $(\phi_0, \psi_0) \simeq (\phi_1, \psi_1)$) if there exists a homomorphism $\xi: C \rightarrow C_0$ such that for any element $c \in C$,

$$\psi_1(c) - \psi_0(c) = \xi[d(c)] + d_0[\xi(c)],$$

and for any $a \in A$,

$$\phi_1(a) - \phi_0(a) = h_0\, \xi g(a).$$

It is readily verified that the relation so defined is reflexive, transitive, and symmetric, and hence is an equivalence relation. The main reason for the importance of this concept is the following proposition:

THEROEM 5.1. *If the maps*

$$(\phi_0, \psi_0), (\phi_1, \psi_1):\langle A, C; f, g, h\rangle \rightarrow \langle A_0 C_0; f_0, g_0, h_0\rangle$$

are algebraically homotopic, then the induced maps (ϕ_0', ψ_0') *and* (ϕ_1', ψ_1') *of the derived couples are the same.*

The proof is entirely trivial. It follows that the induced maps of the n^{th} derived couples, $(\phi_0^{(n)}, \psi_0^{(n)})$ and $(\phi_1^{(n)}, \psi_1^{(n)})$ are also the same.

6. Bigraded Exact Couples; The Associated Leray-Koszul Sequence

In the applications later on it will usually be true that groups occurring in the exact couples with which we are concerned will be bigraded groups, and that all the homomorphisms involved will be homogeneous homomorphisms. Then the groups of the successive derived couples will inherit a bigraded structure from the original groups, and the homomorphisms in the successive derived couples will also be homogeneous. To be precise, if $\langle A, C; f, g, h\rangle$ is a bigraded exact couple, and $\langle A', C'; f', g', h'\rangle$ denotes the first derived couple, then f' and f have the same degree of homogeneity, as do h' and h; however, the degree of homogeneity of g' is that of g minus that of f.

Let $\langle A, C; f, g, h\rangle$ be an exact couple, and let $\langle A^{(n)}, C^{(n)}; f^{(n)}, g^{(n)}, h^{(n)}\rangle$, $n = 1, 2, \cdots$ denote the successive derived couples. Let $d^{(n)} = g^{(n)} \circ h^{(n)}: C^{(n)} \rightarrow C^{(n)}$ denote the differential operator of $C^{(n)}$. Then the sequence of differential groups

[2] This definition is patterned after a similar one given by J. H. C. Whitehead, [20].

$(C^{(n)}, d^{(n)})$ is a Leray-Koszul sequence. It will be referred to as *the Leray-Koszul sequence associated with the exact couple* $\langle A, C\rangle$.

PART II. THE HOMOTOPY EXACT COUPLE OF A COMPLEX

7. Basic Definitions

In this part we shall be concerned with the Hurewicz homotopy groups. The notations and conventions we shall use are described in the appendix.

Let K be a connected CW-complex (for the definition of a CW-complex, see J. H. C. Whitehead, [21]). We shall define an exact bi-graded couple associated with the complex K, using various homotopy groups in the definition. The definition of this exact couple will depend on the cellular decomposition we have chosen for the space K. However, we shall prove that the first derived couple is independent of the choice of the cellular decomposition, and is in fact an invariant of the homotopy type of K. It then follows that all the successive derived couples are also homotopy type invariants. The Leray-Koszul sequence associated with this exact couple is of particular interest, because it turns out that the first term of the sequence is a bigraded group which is closely connected with the homology groups of the space, and the limit group is related to the homotopy groups of the space.

Let K^n, $n = 0, 1, 2, \ldots$, denote the n-skeleton of K, and let x be a vertex. Let $i:\pi_q(K^p) \to \pi_q(K^{p+1})$ and $j:\pi_q(K^p) \to \pi_q(K^p, K^{p-1})$ be injections (i.e., homomorphisms induced by inclusion maps) and let ∂: $\pi_q(K^p, K^{p-1}) \to \pi_{q-1}(K^{p-1})$ be the homotopy boundary operator. Define

$$A_{p,q}(K, x) = \begin{cases} \pi_{p+q}(K^p, x) & \text{if } p \geqq 0 \text{ and } p + q \geqq 2, \\ \{0\} & \text{for all other values of } p \text{ and } q, \end{cases}$$

$$C_{p,q}(K, x) = \begin{cases} \pi_{p+q}(K^p, K^{p-1}, x) & \text{if } p \geqq 1 \text{ and } p + q \geqq 3, \\ j[\pi_{p+q}(K^p, x)] & \text{if } p \geqq 1 \text{ and } p + q = 2, \\ \{0\} & \text{for all other values of } p \text{ and } q. \end{cases}$$

Thus the groups $C_{p,q}(K, x)$ and $A_{p,q}(K, x)$ are defined for all integral values of p and q, positive or negative. Note particularly that $A_{p,q}(K)$ is *not* defined to be $\pi_{p+q}(K^p)$ if $p + q = 1$, and that $C_{2,0}(K)$ will, in general, be different from $\pi_2(K^2, K^1)$. Also, note that the groups $C_{p,q}(K)$ and $A_{p,q}(K)$ are all abelian. The homomorphisms i, j and ∂ defined above may be extended in a unique way to define homomorphisms

$$i:A_{p,q}(K, x) \to A_{p+1,q-1}(K, x),$$

$$j:A_{p,q}(K, x) \to C_{p,q}(K, x),$$

$$\partial:C_{p,q}(K, x) \to A_{p-1,q}(K, x),$$

for all values of p and q, positive or negative. This collection of groups and homomorphisms may be conveniently exhibited together on one diagram, as

indicated in figure 1. This diagram extends to infinity in all directions. From the fact that the homotopy sequence of a pair such as (K^p, K^{p-1}) is exact, and from the way we have defined the groups $A_{p,q}(K)$ and $C_{p,q}(K)$ for $p + q \leqq 2$ or $p \leqq 0$, it follows that if we follow any zig-zag path in the above diagram, always moving downward and to the right, we obtain an exact sequence of groups and homomorphisms. Stated precisely, this means that at each stage,

$$\text{image } i = \text{kernel } j,$$

$$\text{image } j = \text{kernel } \partial,$$

$$\text{image } \partial = \text{kernel } i.$$

Next, we define two bigraded groups, $A(K, x)$ and $C(K, x)$ as follows:

$$A(K, x) = \sum A_{p,q}(K, x)$$
$$C(K, x) = \sum C_{p,q}(K, x).$$

$$\begin{tikzcd}
 & \vdots \arrow[d, "i"] & & \vdots \arrow[d, "i"] & & \\
\cdots \arrow[r, "\partial"] & A_{p,q} \arrow[r, "j"] \arrow[d, "i"] & C_{p,q} \arrow[r, "\partial"] & A_{p-1,q} \arrow[r, "j"] \arrow[d, "i"] & C_{p-1,q} \arrow[r, "\partial"] & \cdots \\
\cdots \arrow[r, "\partial"] & A_{p+1,q-1} \arrow[r, "j"] \arrow[d, "i"] & C_{p+1,q-1} \arrow[r, "\partial"] & A_{p,q-1} \arrow[r, "j"] \arrow[d, "i"] & C_{p,q-1} \arrow[r, "\partial"] & \cdots \\
 & \vdots & & \vdots & &
\end{tikzcd}$$

FIG. 1

The set of homomorphisms i, j, and ∂ above define homogeneous homomorphisms, which we will denote by the same letters:

$$i: A(K, x) \to A(K, x),$$

$$j: A(K, x) \to C(K, x),$$

$$\partial: C(K, x) \to A(K, x).$$

The homomorphism i is homogeneous of degree $(+1, -1)$, j is homogeneous of degree $(0, 0)$, and ∂ is homogeneous of degree $(-1, 0)$. The exactness property of the diagram in figure 1 translates into the exactness property contained in the definition of an exact couple, and thus we see that $\langle A(K, x), C(K, x); i, j, \partial \rangle$ is an exact couple. To be consistent with our earlier notation, we should denote the first derived couple by $\langle A'(K, x), C'(K, x); i', j', \partial' \rangle$, where $A'(K)$ and $C'(K)$ are bigraded groups; however, we will instead write

$$\Gamma(K, x) = A'(K, x),$$

$$\mathcal{H}(K, x) = C'(K, x),$$

where $\Gamma(K) = \sum \Gamma_{p,q}(K)$ and $\mathcal{H}(K) = \sum \mathcal{H}_{p,q}(K)$ are bigraded groups. Note that $\Gamma_{p,q}(K)$ is a subgroup of $A_{p,q}(K)$ and $\mathcal{H}_{p,q}(K)$ is a factor group of a subgroup of $C_{p,q}(K)$. Thus we have the exact bigraded couple $\langle \Gamma(K), \mathcal{H}(K) \rangle$ and the successive derived couples $\langle \Gamma'(K), \mathcal{H}'(K) \rangle$, $\langle \Gamma''(K), \mathcal{H}''(K) \rangle$, etc.

The following facts about the groups $A_{p,q}(K)$ and $C_{p,q}(K)$ are here recorded for later reference:

LEMMA 7.1. *$A_{p,q}(K) = 0$ if $p < 2$ or $p + q < 2$; $A_{p,q}(K)$ is isomorphic to $\pi_{p+q}(K)$ if $q < 0$.*

LEMMA 7.2. *$C_{p,q}(K) = 0$ if $p < 2$ or if $q < 0$ or if $p + q < 2$.*

The proofs follow directly from the definitions and from elementary properties of the homotopy groups.

8. Dependence on the Base Point

Let x_0 and x_1 be any two vertices of the complex K, and let $\eta: I \to K^1$ (where I denotes the closed unit interval, [0, 1]) be a continuous path from x_0 to x_1 in the 1-skeleton of K (i.e., $\eta(0) = x_0$ and $\eta(1) = x_1$). As is well known, the path η induces isomorphisms onto,

$$\bar{\eta}: \pi_{p+q}(K^p, x_1) \to \pi_{p+q}(K^p, x_0), \qquad (p \geqq 1),$$

$$\tilde{\eta}: \pi_{p+q}(K^p, K^{p-1}, x_1) \to \pi_{p+q}(K^p, K^{p-1}, x_0), \qquad (p \geqq 2).$$

The definitions of these isomorphisms can be extended in a unique way so there are defined isomorphisms onto,

$$\bar{\eta}: A_{p,q}(K, x_1) \to A_{p,q}(K, x_0),$$

$$\tilde{\eta}: C_{p,q}(K, x_1) \to C_{p,q}(K, x_0),$$

for all values of p and q, positive or negative. Furthermore, the isomorphisms $\bar{\eta}$ and $\tilde{\eta}$ obviously commute with the homomorphisms i, j, and ∂, and hence define a map

$$(\bar{\eta}, \tilde{\eta}): \langle A(K, x_1), C(K, x_1) \rangle \to \langle A(K, x_0), C(K, x_0) \rangle$$

in the sense defined in section 5, and this map is an isomorphism of the one couple onto the other.

Next, suppose we have two paths η_0, $\eta_1 : I \to K^1$ joining the vertices x_0 and x_1, and assume that these two paths are homotopic, with endpoints fixed, in K. Obviously, we may assume that the homotopy takes place in K^2, the 2-skeleton. It can now be proved that the two maps

$$(\bar{\eta}_0, \tilde{\eta}_0), (\bar{\eta}_1, \tilde{\eta}_1): \langle A(K, x_1), C(K, x_1) \rangle \to \langle A(K, x_0), C(K, x_0) \rangle$$

are the same. The proof makes use of the fact that we have defined $A_{p,q}(K) = 0$ if $p + q = 1$, and that we have defined $C_{2,0}(K) = j[A_{2,0}(K)]$.

We can express these facts concisely by saying that the set of exact couples $\langle A(K, x), C(K, x) \rangle$ for x a vertex of K constitutes a local system of exact couples in K (cf. Steenrod, [15]). This implies that the fundamental group $\pi_1(K, x)$

operates as a group of automorphisms of $\langle A(K, x), C(K, x)\rangle$. Ovbiously, the same remarks apply to the first derived couple, $\langle \Gamma(K, x), H(K, x)\rangle$, and to all the higher derived couples.

9. The Map Induced by a Cellular Map

Let K and L be connected CW-complexes, let x and y be vertices of K and L respectively, and let $f:K \to L$ be a cellular map having the property that $f(x) = y$; we will often express this latter fact by writing $f:(K, x) \to (L, y)$. Then, as is well known, the map f induces homomorphisms,

$$f_*:\pi_{p+q}(K^p, x) \qquad \to \pi_{p+q}(L^p, y), \qquad (p \geqq 1),$$

$$f_\#:\pi_{p+q}(K^p, K^{p-1}, x) \to \pi_{p+q}(L^p, L^{p-1}, y), \qquad (p \geqq 2).$$

The definition of f_* and $f_\#$ can be extended in a unique way so as to define homomorphisms

$$f_*:A_{p,q}(K, x) \to A_{p,q}(L, y),$$

$$f_\#:C_{p,q}(K, x) \to C_{p,q}(L, y),$$

for all values of p and q, positive or negative. It is clear that the homomorphisms f_* and $f_\#$ commute with the homomorphisms i, j, and ∂. The homomorphisms f_* and $f_\#$ above define homomorphisms

$$f_*:A(K, x) \to A(L, y),$$

$$f_\#:C(K, x) \to C(L, y),$$

which are homogeneous of degree (0, 0), and the pair $(f_*, f_\#)$ constitutes a map,

$$(f_*, f_\#):\langle A(K, x), C(K, x)\rangle \to \langle A(L, y), C(L, y)\rangle.$$

Let $\mathfrak{K}_0$ denote the category in which the objects are all pairs (K, x), where K is a connected CW-complex and x is a vertex of K, and in which the maps are all cellular maps $(K, x) \to (K', x')$ which map the prescribed vertex of the first complex onto the prescribed vertex of the second. Let $\mathfrak{E}$ denote the category in which the objects are bigraded exact couples with homogeneous homomorphisms, and the maps consist of pairs of homomorphisms, both homogeneous of degree (0, 0). Then it is clear that the operation of assigning to each pair (K, x) the exact couple $\langle A(K, x), C(K, x)\rangle$ and to each cellular map f the induced map $(f_*, f_\#)$ is a covariant functor from $\mathfrak{K}_0$ to $\mathfrak{E}$.

In what follows, we will make use of the following two properties of cellular maps. Let K and L be connected CW-complexes, and $x \in K$ and $y \in L$ be vertices. Then:

(a) Any continuous map $K \to L$ is homotopic to a cellular map $(K, x) \to (L, y)$.

(b) If two cellular maps f_0, $f_1 : K \to L$ are homotopic, then there exists a cellular homotopy $f_t:K \to L$, $0 \leqq t \leqq 1$, between them, i.e., one such that $f_t(K^n) \subset L^{n+1}$, $n = 0, 1, 2, \cdots$, and $0 \leqq t \leqq 1$.

For the proof of these two properties, see [21].

10. The Algebraic Homotopy Induced by a Celluar Homotopy; Proof of Invariance

Let K and L be connected CW-complexes, x a vertex of K, y_0 and y_1 vertices of L and

$$f_0:(K, x) \to (L, y_0), f_1 :(K, x) \to (L, y_1),$$

cellular maps. Let

$$f_t:K \to L, \qquad 0 \leqq t \leqq 1,$$

be a cellular homotopy between f_0 and f_1, and let

$$\eta:I \to L^1$$

be the path from y_0 to y_1, defined by

$$\eta(t) = f_t(x), \qquad 0 \leqq t \leqq 1.$$

We then have defined the following maps of exact couples:

$$(f_{0*}, f_{0\#}):\langle A(K, x), C(K, x)\rangle \to \langle A(L, y_0), C(L, y_0)\rangle,$$
$$(f_{1*}, f_{1\#}):\langle A(K, x), C(K, x)\rangle \to \langle A(L, y_1), C(L, y_1)\rangle,$$
$$(\bar{\eta}, \bar{\eta}) \quad :\langle A(L, y_1), C(L, y_1)\rangle \to \langle A(L, y_0), C(L, y_0)\rangle.$$

PROPOSITION 10.1. *The map $(f_{0*}, f_{0\#})$ is algebraically homotopic to the composite map*

$$(\bar{\eta}, \bar{\eta}) \circ (f_{1*}, f_{1\#}) = (\bar{\eta} \circ f_{1*}, \bar{\eta} \circ f_{1\#}).$$

The proof requires the construction of an algebraic homotopy between these two maps, which we proceed to make. Obviously, we will have to make use of the homotopy f_t.

First, we define a homomorphism

$$\xi:\pi_m(K^n, K^{n-1}, x) \to \pi_{m+1}(L^{n+1}, L^n, y_0) \qquad (m > 1, n > 0)$$

as follows. Let $\alpha \in \pi_m(K^n, K^{n-1}, x)$, and let

$$a:(E^m, S^{m-1}, p_0) \to (K^n, K^{n-1}, x), \qquad (m > 1, n > 0),$$

be a map representing α; here E^m is an oriented m-cell, S^{m-1} is its bounding sphere, and $p_0 \in S^{m-1}$ is the base-point. Define a map[3]

$$b:(I \times E^m, (I \times E^m)^{\cdot}, 0 \times p_0) \to (L^{n+1}, L^n, y_0)$$

by

$$b(t, s) = f_t[a(s)], \qquad t \in I, \ s \in E^m.$$

[3] If X is an n-cell, then the notation $\dot{X}$ or $(X)^{\cdot}$ will be used to denote the bounding $(n-1)$-sphere of X.

Orient the cell $I \times E^m$ as follows: let $w^m \in H_m(E^m, S^{m-1})$ denote the orientation of E^m, and let $v^1 \in H_1(I, \dot{I})$ denote the orientation of I as defined on p. 166 of [2]. Then $v^1 \times w^m \in H_{m+1}[I \times E^m, (I \times E^m)^{\cdot}]$ shall be the orientation of $I \times E^m$. With this convention, the map b determines an element $\xi(\alpha) \in \pi_{m+1}(L^{n+1}, L^n, y_0)$. It is readily seen that this definition is independent of the choice of the map a in the homotopy class α, and that ξ is a homomorphism.

LEMMA 10.1. *For any element $\alpha \in \pi_m(K^n, K^{n-1}, x)$, $(m > 1, n > 0)$,*

$$\tilde{\eta} f_{1\#}(\alpha) - f_{0\#}(\alpha) = \xi j \partial(\alpha) + j\partial\xi(\alpha).$$

Here j and ∂ are the injection and the homotopy boundary operator.

PROOF. We will actually prove the following equivalent equation:

$$j\partial\xi(\alpha) = [\tilde{\eta} f_{1\#}(\alpha) - f_{0\#}(\alpha)] - [\xi\partial j(\alpha)]$$

Let

$$a:(E^m, S^{m-1}, p_0) \to (K^n, K^{n-1}, x)$$

be a map representing α; in order to simplify the proof, we will assume that a is chosen so that $a(E_-^{m-1}) = x$, which is clearly possible. Let

$$a' :(E_+^{m-1}, S^{m-2}) \quad \to (K^{n-1}, x),$$
$$a'':(E_+^{m-1}, S^{m-2}, p_0) \to (K^{n-1}, K^{n-2}, x),$$

be maps defined by a. Then a' represents $\partial(\alpha)$ and a'' represents $j\partial(\alpha)$. Let

$$\bar{a}:(I \times E_+^{m-1},(I \times E_+^{m-1})^{\cdot}, 0 \times p_0) \to (L^n, L^{n-1}, y_0)$$

be defined by

$$\bar{a}(t, s) = f_t[a''(s)], \qquad t \in I,\ s \in E_+^{m-1}.$$

Then $\bar{a}$ represents $\xi j\partial(\alpha)$. Next, define

$$b:(I \times E^m,(I \times E^m)^{\cdot}, 0 \times p_0) \to (L^{n+1}, L^n, y_0)$$

by

$$b(t, s) = f_t[a(s)], \qquad t \in I,\ s \in E^m.$$

Then b represents $\xi(\alpha)$. Let

$$b':[(I \times E^m)^{\cdot}, 0 \times p_0) \to (L^n, y_0)$$

be the map defined by b; then b' represents $\partial\xi(\alpha) \in \pi_m(L^n, y_0)$. The proof is now completed by applying Lemma 22.1 to the map b'; it is clear from our definitions that $b' \mid I \times E_+^{m-1} = \bar{a}$; also, the map b' restricted to $(I \times E_-^{m-1}) \cup (0 \times E^m) \cup (1 \times E^m)$ represents the element $\tilde{\eta} f_{1\#}(\alpha) - f_{0\#}(\alpha)$, by Lemma 22.2. It is necessary to check that all orientations are chosen correctly in order to be able to apply these lemmas, but that is not difficult, if one makes use of the basic properties of cross products.

LEMMA 10.2. *For any element* $\beta \in \pi_m(K^n, x)$, $(m > 1, n > 0)$

$$\bar{\eta} f_{1*}(\beta) - f_{0*}(\beta) = \partial \xi j(\beta).$$

PROOF. This is a direct application of Lemma 22.3.

Having established these two fundamental properties of the homomorphism ξ, we are ready to complete the proof of Proposition 10.1. The homomorphism ξ defines a homomorphism

$$\xi: C_{p,q}(K, x) \to C_{p+1,q}(L, y_0)$$

for $p > 0$ and $p + q > 1$; we extend the definition of ξ to all other values of p, q in the obvious way. ξ then induces a homomorphism

$$\xi: C(K, x) \to C(L, y_0)$$

which is homogeneous of degree (1, 0). From Lemma 10.1 it follows that for any element $\alpha \in C_{p,q}(K, x)$,

$$\bar{\eta} f_{1\#}(\alpha) - f_{0\#}(\alpha) = \xi j \partial(\alpha) + j \partial \xi(\alpha)$$

for all values of p and q (a little extra trouble is necessary to check the validity of this formula for the case $p = 2$ or 3 and $q = 0$, on account of the way $C_{2,0}(K)$ is defined). From Lemma 10.2 it follows that for any element $\beta \in A_{p,q}(K, x)$

$$\bar{\eta} f_{1*}(\beta) - f_{0*}(\beta) = \partial \xi j(\beta).$$

These two formulas prove that ξ is an algebraic homotopy between the maps $(f_{0*}, f_{0\#})$ and $(\bar{\eta}, \bar{\eta}) \circ (f_{1*}, f_{1\#})$, as desired.

Let

$$\begin{aligned}
(f'_{0*}, f'_{0\#}) &: \langle \Gamma(K, x), \mathcal{H}(K, x) \rangle \to \langle \Gamma(L, y_0), \mathcal{H}(L, y_0) \rangle \\
(f'_{1*}, f'_{1\#}) &: \langle \Gamma(K, x), \mathcal{H}(K, x) \rangle \to \langle \Gamma(L, y_1), \mathcal{H}(L, y_1) \rangle \\
(\bar{\eta}', \bar{\eta}') &: \langle \Gamma(L, y_1), \mathcal{H}(L, y_1) \rangle \to \langle \Gamma(L, y_0), \mathcal{H}(L, y_0) \rangle
\end{aligned}$$

denote the maps induced on the derived couples by the maps $(f_{0*}, f_{0\#})$, $(f_{1*}, f_{1\#})$, and $(\bar{\eta}, \bar{\eta})$ respectively. Then we have

THEOREM 10.1. *The map* $(f'_{0*}, f'_{0\#})$ *is the same as the composite map* $(\bar{\eta}', \bar{\eta}') \circ (f'_{1*}, f'_{1\#}) = (\bar{\eta}' \circ f'_{1*}, \bar{\eta}' \circ f'_{1\#})$.

This follows directly from Proposition 10.1 and Theorem 5.1.

From Theorem 10.1 it now follows readily that the exact couple $\langle \Gamma(K), \mathcal{H}(K) \rangle$ is an invariant of the homotopy type of K. (Compare, for example, the proof of Theorem 7.6 of [7], or section 11 of [20].)

11. Some Trivial Results about the Groups $\mathcal{H}_{p,q}(K)$ and $\Gamma_{p,q}(K)$

We immediately obtain the following facts about the groups $\mathcal{H}_{p,q}(K)$: *if* $p < 2$, *or if* $q < 0$, *or if* $p + q < 2$, *then*

(11.1) $$\mathcal{H}_{p,q}(K) = 0.$$

Similarly, we see that *if* $p < 3$, *or if* $p + q < 2$, *then*

$$\Gamma_{p,q}(K) = 0. \tag{11.2}$$

If $p + q > 1$ *and* $q < 0$, *then*

$$\Gamma_{p,q}(K) \approx \pi_{p+q}(K). \tag{11.3}$$

Next, we note that the exact couple $\langle \Gamma(K), \mathfrak{H}(K); i, j, \partial \rangle$ contains the following exact sequence:

$$\cdots \xrightarrow{j} \mathfrak{H}_{p,0} \xrightarrow{\partial} \Gamma_{p-1,0} \xrightarrow{i} \Gamma_{p,-1} \xrightarrow{j} \mathfrak{H}_{p-1,0} \xrightarrow{\partial} \cdots$$

which terminates as follows:

$$\cdots \xrightarrow{j} \mathfrak{H}_{3,0} \xrightarrow{\partial} \Gamma_{2,0} \xrightarrow{i} \Gamma_{3,-1} \xrightarrow{j} \mathfrak{H}_{2,0} \xrightarrow{\partial} 0.$$

This is the exact sequence considered by J. H. C. Whitehead in [20], as is obvious when one compares our definitions with those used by Whitehead. Whitehead proves that

$$\mathfrak{H}_{p,0}(K) \approx H_p(\tilde{K}) \tag{11.4}$$

where $\tilde{K}$ denotes the universal covering complex of K, and H_p denotes the p-dimensional integral homology group.

The relation between the homotopy exact couple of a complex and a covering complex is very simple. Let K be a connected CW-complex, $\tilde{K}$ a covering complex, and $P : \tilde{K} \to K$ the covering map. Then it follows (cf. Theorem 8.1 of [7]) that the induced homomorphisms

$$P_* : \pi_q(\tilde{K}^p, \tilde{K}^{p-1}) \to \pi_q(K^p, K^{p-1}),$$
$$P_* : \pi_q(\tilde{K}^p) \to \pi_q(K^p)$$

are isomorphisms onto for $q \geqq 2$. From this it follows readily that the exact couples $\langle A(\tilde{K}), C(\tilde{K}); i, j, \partial \rangle$ and $\langle A(K), C(K); i, j, \partial \rangle$ are naturally isomorphic. The isomorphism is induced by the covering map P. It also follows that the successive derived exact couples are isomorphic.

In view of these facts, there is no essential lack of generality if we assume the complex K is simply connected when discussing the properties of the homotopy exact couple associated with K. For, we may as well replace K by its universal covering complex.

Next, assume that K is r-connected, $r \geqq 1$ (i.e., the homotopy groups $\pi_i(K)$ vanish for $i \leqq r$). Then we have the

PROPOSITION 11.1. *If K is r-connected, then* $\Gamma_{p,q}(K) = 0$ *for* $p \leqq r + 1$ *and* $\mathfrak{H}_{p,q}(K) = 0$ *for* $p \leqq r$.

PROOF. Since K is r-connected, it follows that the n-skeleton, K^n, is also r-connected for $n \geqq r + 1$, while K^n is $(n - 1)$-connected if $n \leqq r$. Hence the inclusion map $K^{n-1} \to K^n$ is inessential (i.e., homotopic to a constant map) if

$n \leqq r + 1$, and therefore the injection $\pi_{p+q}(K^{p-1}) \to \pi_{p+q}(K^p)$ is the zero homomorphism if $p \leqq r + 1$. From this fact, and the definition of $\Gamma_{p,q}(K)$, the first conclusion follows. To prove the second conclusion, consider the following groups and homomorphisms:

$$\Gamma_{p+1,\, q-1}(K) \xrightarrow{j} \mathcal{H}_{p,q}(K) \xrightarrow{\partial} \Gamma_{p-1,q}(K).$$

If $p \leqq r$, then $\Gamma_{p+1,q-1}(K) = 0$ and $\Gamma_{p-1,q}(K) = 0$ by what we have just proved. Since image j = kernel ∂, the desired conclusion follows.

Finally, we note that if K is n-dimensional, then $C_{p,q}(K) = 0$ for $p > n$; hence, it follows that in this case $\mathcal{H}_{p,q}(K) = 0$ for $p > n$.

12. Definition of The Groups $\mathcal{G}_m$

The symbol E^n will denote the unit n-cell in Cartesian n-space, and S^{n-1} is its bounding $(n - 1)$-sphere (see appendix, Section 21). These cells and spheres all are assumed to have definite orientations, as outlined, for example, in [2], Section 1.1. For each integer n, choose a definite map

$$\phi_n : E^n \to S^n$$

such that $\phi_n(S^{n-1}) = p_0$, the base point, and such that ϕ_n preserves orientations, i.e., has degree $+1$. We then have the following sequence of homomorphisms $(m \geqq 0)$:

$$\phi_n : \pi_{m+n}(E^n, S^{n-1}) \to \pi_{m+n}(S^n),$$
$$\partial_n : \pi_{m+n}(E^n, S^{n-1}) \to \pi_{m+n-1}(S^{n-1}).$$

The homotopy boundary operator ∂_n is obviously an isomorphism onto for all values of n; the homomorphism ϕ_n is equivalent to the suspension homomorphism (see [18]) and is an isomorphism onto if $n > m + 2$; it is a homomorphism onto if $n = m + 2$. These statements are a direct consequence of the Freudenthal theorems, [6]. For the purposes of this section, we will identify all the groups $\pi_{m+n}(S^n)$ for $n \geqq m + 2$, and the groups $\pi_{m+n}(E^n, S^{n-1})$ for $n > m + 2$, using the isomorphisms ϕ_n and ∂_n to perform the identifications. The group obtained as a result of this identification will be denoted by $\mathcal{G}_m$. Then $\mathcal{G}_0$ is infinite cyclic and $\mathcal{G}_1$ and $\mathcal{G}_2$ are cyclic of order 2. Finally, we will also identify $\mathcal{G}_m$ with the factor group of $\pi_{2m+2}(E^{m+2}, S^{m+1})$ modulo the kernel of the homomorphism

$$\phi_{m+2} : \pi_{2m+2}(E^{m+2}, S^{m+1}) \to \pi_{2m+2}(S^{m+2}),$$

using the homomorphism ϕ_{m+2} for this purpose.

13. The Exact Couples $\langle A(K, m), C(K, m)\rangle$

It is the purpose of this section to define a sequence of bigraded exact couples, $\langle A(K, m), C(K, m); i_m, j_m, \partial_m\rangle$, $m = 0, 1, 2, \cdots$, such that $A(K, 0) = A(K)$, $C(K, 0) = C(K)$, $A(K, m + 1) \subset A(K, m)$, $C(K, m + 1) \subset C(K, m)$, and the homomorphisms i_m, j_m, and ∂_m are the restrictions of i, j, and ∂ to the subgroups $A(K, m)$ *and* $C(K, m)$.

The groups $A(K, m)$ and $C(K, m)$ are defined by the following conditions $(m \geqq 0)$:

$$A_{p,q}(K, m) = \begin{cases} A_{p,q}(K) & \text{if } p + q \geqq m + 2, \\ \{0\} & \text{if } p + q < m + 2. \end{cases}$$

$$C_{p,q}(K, m) = \begin{cases} C_{p,q}(K) & \text{if } p + q > m + 2, \\ j[A_{p,q}(K)] & \text{if } p + q = m + 2, \\ \{0\} & \text{if } p + q < m + 2. \end{cases}$$

$$A(K, m) = \sum A_{p,q}(K, m),$$
$$C(K, m) = \sum C_{p,q}(K, m).$$

Also, we define the homomorphisms

$$i_m : A_{p,q}(K, m) \to A_{p+1,q-1}(K, m),$$
$$j_m : A_{p,q}(K, m) \to C_{p,q}(K, m),$$
$$\partial_m : C_{p,q}(K, m) \to A_{p-1,q}(K, m),$$

by $i_m = i \mid A_{p,q}(K, m)$, $j_m = j \mid A_{p,q}(K, m)$, and $\partial_m = \partial \mid C_{p,q}(K, m)$. These homomorphisms define homogeneous homomorphisms,

$$i_m : A(K, m) \to A(K, m),$$
$$j_m : A(K, m) \to C(K, m),$$
$$\partial_m : C(K, m) \to A(K, m),$$

as usual, and it is clear that the necessary exactness conditions hold, so that $\langle A(K, m), C(K, m); i_m, j_m, \partial_m \rangle$ is an exact couple. We will denote the successive derived couples by $\langle \Gamma(K, m), \mathfrak{K}(K, m) \rangle$, $\langle \Gamma'(K, m), \mathfrak{K}'(K, m) \rangle$, etc.

If $f : (K, x) \to (L, y)$ is a cellular map of one complex into another, then it is clear that the induced homomorphisms $f_* : A(K) \to A(L)$ and $f_\# : C(K) \to C(L)$ have the property that $f_*[A(K, m)] \subset A(L, m)$ and $f_\#[C(K, m)] \subset C(L, m)$; hence there are defined homomorphisms

$$f_* : A(K, m) \to A(L, m),$$
$$f_\# : C(K, m) \to C(L, m),$$

and the pair $(f_*, f_\#)$ is obviously a map of couples, $\langle A(K, m), C(K, m) \rangle \to \langle A(L, m), C(L, m) \rangle$. If $f_0, f_1 : K \to L$ are two cellular maps which are homotopic, then the algebraic homotopy ξ: $C(K) \to C(L)$ which was defined in section 10 between the couple maps $(f_{0*}, f_{0\#})$ and $(\bar{\eta}, \tilde{\eta})\circ(f_{1*}, f_{1\#})$ obviously has the property that $\xi[C(K, m)] \subset C(L, m)$. Hence the couple maps $(f_{0*}, f_{0\#})$ and $(\bar{\eta}, \tilde{\eta})\circ(f_{1*}, f_{1\#})$: $\langle A(K, m), C(K, m) \rangle \to \langle A(L, m), C(L, m) \rangle$ are also algebraically homotopic. Making use of these facts, one can prove exactly as before that *the exact*

couple $\langle \Gamma(K, m)\, \mathcal{H}(K, m)\rangle$ *is an invariant of the homotopy type of* K; naturally the successive derived couples are also homotopy type invariants.

For future reference, we note that if $p + q > m + 2$, then the injection $\mathcal{H}_{p,q}(K, m) \rightarrow \mathcal{H}_{p,q}(K)$ is an isomorphism onto, while if $p + q < m + 2$, then $\mathcal{H}_{p,q}(K, m) = 0$.

14. Application of the Composition Operation

It is the purpose of this section to define a sequence of allowable homogeneous homomorphisms

$$\Xi_m : C(K, m) \otimes \mathcal{G}_m \rightarrow C(K), \qquad (m \geqq 0).$$

The homomorphisms Ξ_m are defined in a natural manner, using the so-called "composition operation" (see section 23), and are important in identifying the groups $\mathcal{H}_{p,q}(K)$ for certain values of p and q.

We define the homomorphisms Ξ_m as follows. First, suppose that $p + q > m + 2$. Then a homomorphism

$$\Xi_m : C_{p,q}(K, m) \otimes \pi_{p+q+m}(E^{p+q}, S^{p+q-1}) \rightarrow C_{p,q+m}(K)$$

is defined by

$$\Xi_m(\beta \otimes \alpha) = \beta \circ \alpha$$

for any $\alpha \in \pi_{p+q+m}(E^{p+q}, S^{p+q-1})$ and $\beta \in C_{p,q}(K, m) = \pi_{p+q}(K^p, K^{p-1})$. Next, suppose $p + q = m + 2$. Let $\alpha \in \pi_{p+q+m}(E^{p+q}, S^{p+q-1})$ and $\beta \in C_{p,q}(K, m)$. Then there exists an element $\beta' \in A_{p,q}(K)$ such that

$$j(\beta') = \beta.$$

Define

$$\Xi_m(\beta \otimes \alpha) = j[\beta' \circ (\phi\alpha)]$$

where $\phi = \phi_{p+q}: \pi_{p+q+m}(E^{p+q}, S^{p+q-1}) \rightarrow \pi_{p+q+m}(S^{p+q})$ is defined above. It follows from equation (23.11) that this definition is independent of the choice made for β'. Also if $\alpha' \in \pi_{p+q+m}(E^{p+q}, S^{p+q-1})$, and $\phi\alpha' = \phi\alpha$, then $\Xi_m(\beta \otimes \alpha') = \Xi_m(\beta \otimes \alpha)$, so Ξ_m is actually defined for $\alpha \in \mathcal{G}_m$, a factor group of $\pi_{p+q+m}(E^{p+q}, S^{p+q-1})$. Finally, if $p + q < m + 2$, then the definition of Ξ_m is obvious.

LEMMA 14.1. *The homomorphism* Ξ_m *thus defined is allowable, i.e., it commutes with the differential operator* d.

PROOF. Let $\alpha \in \pi_{p+q+m+1}(E^{p+q+1}, S^{p+q})$, $\beta \in C_{p+1,q}$, $p + q \geqq m + 2$, and choose an element $\alpha' \in \pi_{p+q+m}(E^{p+q}, S^{p+q-1})$ such that $\phi(\alpha') = \partial(\alpha)$. Then it follows from (23.10) and (23.11) that

$$\begin{aligned} j\partial(\beta \circ \alpha) &= j[(\partial\beta) \circ (\partial\alpha)] = j[(\partial\beta) \circ (\phi\alpha')] \\ &= (j\partial\beta) \circ \alpha' \end{aligned}$$

and from this result the lemma follows easily.

From this result, it follows that Ξ_m induces a natural homomorphism,

$$\Xi'_m : \mathfrak{H}(K, m, \mathcal{G}_m) \to \mathfrak{H}(K),$$

which is homogeneous of degree $(0, m)$. We will prove that under appropriate conditions this homomorphism maps some of the homogeneous components of $\mathfrak{H}(K, m, \mathcal{G}_m)$ isomorphically.

LEMMA 14.2. *If K is r-connected, $r > 0$, $p > r + 1$, and $m < r$, then the homomorphism*

$$\Xi_m : C_{p,0}(K, m) \otimes \mathcal{G}_m \to C_{p,m}(K)$$

is an isomorphism onto.

PROOF. Since $p > r + 1$ and $r > m$, it follows that $p > m + 2$, and hence

$$C_{p,0}(K, m) = C_{p,0}(K) = \pi_p(K^p, K^{p-1}).$$

Since $p - 1 \geqq r + 1$, it follows that K^{p-1} is also r-connected. Now $\mathcal{G}_m \approx \pi_{p+m}(E^p, S^{p-1})$, and if $\alpha \in \pi_{p+m}(E^p, S^{p-1})$, $\beta \in \pi_p(K^p, K^{p-1})$, then $\Xi_m(\beta \otimes \alpha) = \beta \circ \alpha$. The lemma now follows directly from theorem 23.1.

LEMMA 14.3. *If K is r-connected, $r > 0$, then for any $m < r$ the homomorphism*

$$\Xi'_m : \mathfrak{H}_{p,0}(K, m, \mathcal{G}_m) \to \mathfrak{H}_{p,m}(K)$$

is an isomorphism onto.

PROOF. For $p > r + 2$, the fact that Ξ'_m is an isomorphism onto is a direct consequence of the preceding lemma. For $p \leqq r$, it is also obvious that Ξ'_m is an isomorphism onto, because then $\mathfrak{H}_{p,0}(K, m, \mathcal{G}_m) = 0$ and $\mathfrak{H}_{p,m}(K) = 0$. Hence only the cases $p = r + 1$ and $p = r + 2$ remain unproven. To complete the proof, let L be a complex of the same homotopy type of K having the property that the r-skeleton, L^r, consists of a single vertex; that such a complex exists follows from a theorem of J. H. C. Whitehead, [21]. For this complex, it is readily verified that the allowable homomorphism $\Xi_m : C_{p,0}(L, m) \otimes \mathcal{G}_m \to C_{p,m}(L)$ is an isomorphism onto for all values of p, and from this the theorem follows.

For the statement of the next lemma, let $k_m : C(K, m) \to C(K)$ denote the inclusion map, and let $k'_m : \mathfrak{H}(K, m, G) \to \mathfrak{H}(K, G)$ denote the induced homomorphism, where G is any abelian group.

LEMMA 14.4. *If K is r-connected, $r > 0$, then for any $m < r$ the homomorphism $k'_m : \mathfrak{H}(K, m, G) \to \mathfrak{H}(K, G)$ is an isomorphism onto.*

PROOF. Again we make use of the fact that there exists a complex L of the same homotopy type as K, and such that L^r is a single vertex. For this complex L, it follows from the definitions $k_m : C(L, m) \to C(L)$ is an isomorphism onto for $m < r$, and therefore $k'_m : \mathfrak{H}(L, m, G) \to \mathfrak{H}(L, G)$ is also an isomorphism onto. The conclusion of the lemma now follows, since K and L have the same homotopy type.

LEMMA 14.5. *If K is r-connected, $r > 0$, then for any $m < r$ there exists a natural isomorphism $\mathfrak{H}_{p,m}(K) \approx \mathfrak{H}_{p,0}(K, \mathcal{G}_m)$.*

This isomorphism is obtained by combining the isomorphisms of the two preceding lemmas.

For a simply connected complex K, we may identify the group $C_{p,0}(K)$ for $p > 2$ with the integral homology group $H_p(K^p, K^{p-1})$, by using the Hurewicz isomorphism: $\rho_p : C_{p,0}(K) \to H_p(K^p, K^{p-1})$. But the group $H_p(K^p, K^{p-1})$ is the group of integral p-chains of K. Hence we may identify $C_{p,0}(K)$ and the group of integral p-chains, and therefore $\mathcal{H}_{p,0}(K, \mathcal{G}_m)$ is isomorphic to the homology group $H_p(K, \mathcal{G}_m)$. Hence:

THEOREM 14.1. *If K is r-connected, $r > 0$, then for any $m < r$ the group $\mathcal{H}_{p,m}(K)$ is naturally isomorphic to the homology group $H_p(K, \mathcal{G}_m)$.*

15. The Subgroups $C(K/m)$

For use in the next section, we define a sequence of allowable subgroups of $C(K)$,

$$C(K) = C(K/1) \supset C(K/2) \supset C(K/3) \supset \cdots,$$

as follows (we omit the base point from the notation):

$$\begin{aligned} C_{p,q}(K/m) &= C_{p,q}(K) && \text{if } p > m + 1, \\ &= j[A_{p,q}(K)] && \text{if } p = m + 1, \\ &= 0 && \text{if } p < m + 1. \end{aligned}$$

It is clear that $C(K/m) = \sum C_{p,q}(K/m)$ is an allowable subgroup of $C(K)$ with respect to both the differential and the bigraded structure. The following facts are readily verified:

(a) If $f : K \to L$ is a cellular map, then the induced homomorphism $f_{\#} : C(K) \to C(L)$ has the property that $f_{\#}[C(K/m)] \subset C(L/m)$ for all $m \geqq 1$.

(b) If $f_0, f_1 : K \to L$ are cellular maps, $f_t : K \to L$ is a cellular homotopy between f_0 and f_1, and $\xi : C(K) \to C(L)$ is the algebraic homotopy between $f_{0\#}$ and $f_{1\#}$ induced by f_t, then $\xi[C(K/m)] \subset C(L/m)$ for all $m \geqq 1$.

These two facts suffice to prove that the derived group $\mathcal{H}(K/m) = \sum \mathcal{H}_{p,q}(K/m)$ is an invariant of the homotopy type of K. Note that

$$\begin{aligned} \mathcal{H}_{p,q}(K/m) &= \mathcal{H}_{p,q}(K) && \text{if } p > m + 1 \\ &= \{0\} && \text{if } p < m + 1 \end{aligned}$$

Also, if K is m-connected, then

$$\mathcal{H}_{p,q}(K/m) = \mathcal{H}_{p,q}(K)$$

for all pairs (p, q). This is easily proved by using the fact that there exists a complex L of the same homotopy type as K and such that L^m is a single vertex.

16. Application of The Generalized Whitehead Products

We now define a sequence of homomorphisms

$$\Upsilon_m : C_{p,q}(K/m) \otimes \pi_m(K) \to C_{p,q+m-1}(K)$$

as follows. First, assume $p > m + 1$, $\alpha \in C_{p,q}(K/m) = C_{p,q}(K) = \pi_{p+q}(K^p, K^{p-1})$ and $\beta \in \pi_m(K)$. The injection $\pi_m(K^{p-1}) \to \pi_m(K)$ is an isomorphism onto. Let β' be the unique element of $\pi_m(K^{p-1})$ which maps onto β. Define

$$\Upsilon_m(\alpha \otimes \beta) = (-1)^p[\alpha, \beta'],$$

where the square brackets mean "Generalized Whitehead Product" (see Section 24). Next, assume $p < m + 1$; then $C_{p,q}(K/m) = 0$, and Υ_m is defined to be trivial. Finally, assume $p = m + 1$. Let $\alpha \in C_{m+1,q}(K/m)$ and $\beta \in \pi_m(K)$. Choose an element $\alpha' \in \pi_{m+q+1}(K^{m+1}) = A_{m+1,q}(K)$ such that $j(\alpha') = \alpha$; choose an element $\beta' \in \pi_m(K^m)$ such that β' maps onto β under the injection $\pi_m(K^m) \to \pi_m(K)$. Then it follows from [13, section 7, property (c)] that

$$j[\alpha', i\beta'] = [\alpha, \beta'].$$

$$\begin{tikzcd}
\pi_{p+q+1}(K^{p+1}, K^p) \arrow[d, "\partial"] & \otimes & \pi_m(K^p) \arrow[r] \arrow[d, leftrightarrow] & \pi_{p+q+m}(K^{p+1}, K^p) \arrow[d, "\partial"] \\
\pi_{p+q}(K^p) \arrow[d, "j"] & \otimes & \pi_m(K^p) \arrow[r] & \pi_{p+q+m-1}(K^p) \arrow[d, "j"] \\
\pi_{p+q}(K^p, K^{p-1}) & \otimes & \pi_m(K^{p-i}) \arrow[r] \arrow[u, "i"] & \pi_{p+q+m-1}(K^p, K^{p-1})
\end{tikzcd}$$

FIG. 2

Since $i\beta'$ is a unique element of $\pi_m(K^{m+1}) \approx \pi_m(K)$ which is independent of the choice made for β', it follows that both sides of the above equation are independent of the choices made for α' and β'. Define

$$\Upsilon_m(\alpha \otimes \beta) = (-1)^{m+1}j[\alpha', i\beta'] = (-1)^{m+1}[\alpha, \beta'].$$

LEMMA 16.1. *The homomorphism Υ_m thus defined is allowable*, i.e., *it commutes with the differential operator d.*

PROOF. Consider the diagram in figure 2. We will only give the proof in case $p > m + 1$, leaving modifications necessary in the other cases to the reader. Let $\alpha \in C_{p+1,q}(K/m) = C_{p+1,q}(K) = \pi_{p+q+1}(K^{p+1}, K^p)$ and $\beta \in \pi_m(K)$. We must prove that

$$\Upsilon_m(d\alpha \otimes \beta) = d\Upsilon_m(\alpha \otimes \beta).$$

Choose $\beta' \in \pi_m(K^p)$ and $\beta'' \in \pi_m(K^{p-1})$ so that β' and β'' correspond to β under the injection isomorphisms $\pi_m(K^p) \to \pi_m(K)$ and $\pi_m(K^{p-1}) \to \pi_m(K)$ respectively. Then from the definitions, we have,

$$\Upsilon_m(d\alpha \otimes \beta) = (-1)^p[j\partial\alpha, \beta''],$$

$$d\Upsilon_m(\alpha \otimes \beta) = (-1)^{p+1}j\partial[\alpha, \beta'].$$

But from [13, section 7, property (a)],

$$\partial[\alpha, \beta'] = -[\partial\alpha, \beta'],$$

and from [13, section 7, property (c)],

$$j[\partial\alpha, \beta'] = [j\partial\alpha, \beta''].$$

From these relations, the proof follows.

The main reason for the importance of the homomorphisms Υ_m is contained in the following lemma and theorem:

LEMMA 16.2. *Assume the complex K is r-connected, $r > 0$. Then the homomorphisms*

$$\Xi_r : C_{p,0}(K, r) \otimes \mathcal{G}_r \rightarrow C_{p,r}(K)$$

$$\Upsilon_{r+1} : C_{p,0}(K/r + 1) \otimes \pi_{r+1}(K) \rightarrow C_{p,r}(K)$$

are isomorphisms into for $p \geqq r + 3$, and $C_{p,r}(K)$ is the direct sum of the two image subgroups.

The proof of this lemma is a direct application of Theorem 24.1 of the appendix.

THEOREM 16.1. *If K is an r-connected complex, then for $p > r + 3$,*

$$\mathcal{H}_{p,r}(K) \approx H_p(K, \mathcal{G}_r) + H_p(K, \pi_{r+1}(K)).$$

PROOF. From the preceding lemma, it follows that

$$\mathcal{H}_{p,r}(K) \approx \mathcal{H}_{p,0}(K, r, \mathcal{G}_r) + \mathcal{H}_{p,0}(K/r + 1, \pi_{r+1}(K)).$$

But $\mathcal{H}_{p,0}(K/r + 1) = \mathcal{H}_{p,0}(K)$ for $p > r + 2$, and $\mathcal{H}_{p,0}(K, r) = \mathcal{H}_{p,0}(K)$ if $p > r + 2$. Hence the result follows.

Note that it follows readily from the above proof that for $p = r + 3$, $\mathcal{H}_{p,r}(K)$ is isomorphic to a subgroup of the direct sum $H_p(K, \mathcal{G}_r) + H_p(K, \pi_{r+1}(K))$. The groups $\mathcal{H}_{p,r}(K)$ for $p = r + 2$ and $p = r + 1$ are unknown in general. Obviously, $\mathcal{H}_{p,r}(K) = 0$ if $p \leqq r$, because K is r-connected.

17. Dependence of $\mathcal{H}_{p,q}(K)$ and $\Gamma_{p,q}(K)$ on the n-Type of K

In this section we make use of the concept of the *n-type* of a complex as introduced by J. H. C. Whitehead, [21]. Our main purpose is to determine for what values of n the groups $\mathcal{H}_{p,q}(K)$ and $\Gamma_{p,q}(K)$ depend on the n-type of K.

LEMMA 17.1. *The injection $\mathcal{H}_{p,q}(K^n) \rightarrow \mathcal{H}_{p,q}(K)$ is an isomorphism onto if $p < n$ and is a homomorphism onto if $p = n$. The injection $\Gamma_{p,q}(K^n) \rightarrow \Gamma_{p,q}(K)$ is an isomorphism onto if $p \leqq n$.*

The proof is a direct consequence of the definitions of $\mathcal{H}_{p,q}(K)$ and $\Gamma_{p,q}(K)$.

THEOREM 17.1. *If the complexes K and L are of the same n-type, then for all $p \leqq n - 1$,*

$$\mathcal{H}_{p,q}(K) \approx \mathcal{H}_{p,q}(L),$$

$$\Gamma_{p,q}(K) \approx \Gamma_{p,q}(L).$$

PROOF. We will give the proof of the first assertion only; the proof of the second assertion is similar. Let

$$\phi : K^n \to L^n,$$
$$\psi : L^n \to K^n,$$

be n-homotopy equivalences between K and L; let $k : K^{n-1} \to K^n$, $l : L^{n-1} \to L^n$ be inclusion maps. Then $\psi\phi \mid K^{n-1}$ and k are homotopic, and $\phi\psi \mid L^{n-1}$ and l are homotopic. Consider the following diagram:

$$\begin{tikzcd}
& \mathcal{H}_{p,q}(K^{n-1}) \arrow[dl, "k'"] \arrow[dr, "k'"] & \\
\mathcal{H}_{p,q}(K^n) \arrow[r, "\phi'"'] & \mathcal{H}_{p,q}(L^n) \arrow[r, "\psi'"'] & \mathcal{H}_{p,q}(K^n)
\end{tikzcd}$$

Here k', ϕ', and ψ' are homomorphisms induced by k, ϕ, and ψ respectively. For $p < n - 1$, k' is an isomorphism onto, and for $p = n - 1$, k' is a homomorphism onto. It is clear that

$$\psi' \circ \phi' \circ k' = k'.$$

From this it readily follows that for $p \leqq n - 1$, ϕ' is an isomorphism into, and ψ' is a homomorphism onto. Interchanging the roles of K and L, we see that ϕ' is onto, and ψ' is an isomorphism. The proof can now be completed by making use of the lemma above.

As an example of the application of this result, we will determine the group $\mathcal{H}_{2,1}(K)$. If Π is any abelian group and n is a positive integer, then $H_p(\Pi, n)$ will denote the p-dimensional homology groups of the abstract complex $K(\Pi, n)$; cf. Eilenberg and MacLane, [4].

THEOREM 17.2. *For any connected CW-complex* K, $\mathcal{H}_{2,1}(K) \approx H_4(\pi_2(K), 2)$.

PROOF. According to the preceding theorem, $\mathcal{H}_{2,1}(K)$ depends only on the 3-type of K; let $\tilde{K}$ denote the universal covering complex of K; then $\mathcal{H}_{2,1}(\tilde{K}) \approx \mathcal{H}_{2,1}(K)$. Also, the 3-type of a simply-connected complex depends only on the 2-dimensional homotopy group. This result can either be proved directly, or is a special case of a more general result of MacLane and Whitehead, [12]. Since $\pi_2(\tilde{K}) \approx \pi_2(K)$, it follows that $\mathcal{H}_{2,1}(K)$ depends only on $\pi_2(K)$, i.e., if $\pi_2(K) \approx \pi_2(L)$, then $\mathcal{H}_{2,1}(K) \approx \mathcal{H}_{2,1}(L)$. Let L be a complex such that $\pi_2(L) \approx \pi_2(K)$, and $\pi_p(L) = 0$ for $p \neq 2$; that such a complex exists, follows from a theorem of J. H. C. Whitehead, [23]. Consider the exact couple, $\langle \Gamma(L), \mathcal{H}(L) \rangle$. Since $\Gamma_{1,1}(L) = 0$ and $\Gamma_{2,1}(L) = 0$ by Proposition 11.1, it follows that $j' : \Gamma_{3,0}(L) \to \mathcal{H}_{2,1}(L)$ is an isomorphism onto. Also, $\Gamma_{5,-1}(L) \approx \pi_4(L) = 0$, and $\Gamma_{4,-1}(L) \approx \pi_3(L) = 0$ by (11.3). Therefore $\partial' : \mathcal{H}_{4,0}(L) \to \Gamma_{3,0}(L)$ is also an isomorphism onto. Com-

bining these two isomorphisms, we see that $\mathcal{H}_{2,1}(L) \approx \mathcal{H}_{4,0}(L)$. By (11.4), $\mathcal{H}_{4,0}(L) \approx H_4(L)$, and by Theorem I^m of Eilenberg and MacLane, [4], $H_4(L) \approx H_4(\pi_2(K), 2)$. This completes the proof.

It should be noted that $H_4(\Pi, 2)$ is isomorphic to the group $\Gamma(\Pi)$ described in [20].

18. The Groups $\pi_{n,m}(K)$

As before, let K be a connected CW-complex, let x be a vertex of K, and let

$$k_{n,m} : \pi_n(K^m, x) \to \pi_n(K, x) \qquad (n \geqq 1, \; m \geqq 1)$$

denote the injection. For $n \geqq 1$ and $m \geqq 1$ define

$$\pi_{n,m}(K, x) = k_{n,m}[\pi_n(K^m, x)].$$

Then $\pi_{n,m}(K, x)$ is a subgroup of $\pi_n(K, x)$. The following facts about these subgroups are obvious:

(18.1). If $p < q$, then $\pi_{n,p}(K) \subset \pi_{n,q}(K)$.

(18.2) If $m \geqq n$, then $\pi_{n,m}(K) = \pi_n(K)$.

(18.3) If $n > 1$, then $\pi_{n,1}(K) = 0$.

If $f : (K, x) \to (L, y)$ is a cellular map of one complex into another, and $f_n : \pi_n(K, x) \to \pi_n(K, y)$ denotes the induced homomorphism, then $f_n[\pi_{n,m}(K)] \subset \pi_{n,m}(L)$, and therefore f_n defines a homomorphism

$$f_{n,m} : \pi_{n,m}(K, x) \to \pi_{n,m}(L, y).$$

Furthermore, if $f, g : (K, x) \to (L, y)$ are two cellular maps which are homotopic, then the induced homomorphisms

$$f_{n,m}, g_{n,m} : \pi_{n,m}(K, x) \to \pi_{n,m}(L, y)$$

differ at most by an operator from the fundamental group, $\pi_1(L, y)$. From these two facts it follows by standard arguments that the groups $\pi_{n,m}(K, x)$ are invariants of the homotopy type of the complex K.

Let $\langle\Gamma(K), \mathcal{H}(K)\rangle$, $\langle\Gamma'(K), \mathcal{H}'(K)\rangle$, $\Gamma''(K), \mathcal{H}''(K)\rangle$, $\cdots$ denote the homotopy exact couple and successive derived couples as usual; each of the groups $\Gamma^{(r)}(K)$ is bigraded, $\Gamma^{(r)}(K) = \sum \Gamma^{(r)}_{p,q}(K)$. The following proposition expresses a relationship between the groups $\pi_{n,m}(K)$ and $\Gamma^{(r)}_{p,q}(K)$.

Propositon 18.1. *If $q < 0$, $p + q > 1$, and $r \geqq 0$, then*

$$\Gamma^{(r)}_{p,q}(K) \approx \pi_{p+q,p-r-1}(K).$$

This result follows readily from the definition of the groups $\Gamma^{(r)}_{p,q}(K)$ and $\pi_{m,n}(K)$. The details are left to the reader.

19. The Associated Leray-Koszul Sequence and its Limit Group

Let K be a connected CW-complex; associated with the sequence of successive derived exact couples, $\langle\Gamma(K), \mathcal{H}(K)\rangle$, $\langle\Gamma'(K), \mathcal{H}'(K)\rangle$, $\langle\Gamma''(K), \mathcal{H}''(K)\rangle$, $\cdots$,

is a Leray-Koszul sequence $(\mathcal{H}(K), d)$, $(\mathcal{H}'(K), d')$, $(\mathcal{H}''(K), d'')$, $\cdots$, as explained in Section 6. It is the purpose of this section to discuss the properties of this sequence. First, we list the following facts:

(a) Each of the groups $\mathcal{H}^{(n)}(K)$ is bigraded, $\mathcal{H}^{(n)}(K) = \sum \mathcal{H}^{(n)}_{p,q}(K)$.

(b) $\mathcal{H}^{(n)}_{p,q}(K) = 0$ if $p + q < 2$, or if $p < 2$, or if $q < 0$.

(c) The differential operator $d^{(n)} : \mathcal{H}^{(n)}(K) \to \mathcal{H}^{(n)}(K)$ is homogeneous of degree $(-n-2, n+1)$.

Associated with this Leray-Koszul sequence is a limit group, $\mathcal{H}^{\infty}(K) = \sum \mathcal{H}^{\infty}_{p,q}(K)$. Note that in this case, any homogeneous component of the limit group is determined in a finite number of steps. For, consider any fixed pair of integers (p, q) and the following groups:

$$\mathcal{H}^{(n)}_{p+n+2,q-n-1} \xrightarrow{d^{(n)}} \mathcal{H}^{(n)}_{p,q} \xrightarrow{d^{(n)}} \mathcal{H}^{(n)}_{p-n-2,q+n+1}$$

If $p - n - 2 < 2$, i.e., if $n > p - 4$, then $\mathcal{H}^{(n)}_{p-n-2,q+n+1} = 0$. If $q - n - 1 < 0$, i.e., if $n > q - 1$, then $\mathcal{H}^{(n)}_{p+n+2,q-n-1} = 0$. Therefore if $n > \max(p - 4, q - 1)$, it follows that $\mathcal{H}^{(n+1)}_{p,q} = \mathcal{H}^{(n)}_{p,q}$, and hence $\mathcal{H}^{(n)}_{p,q}(K) = \mathcal{H}^{\infty}_{p,q}(K)$.

The relationship of the limit group, $\mathcal{H}^{\infty}(K)$, to the homotopy groups of K is given by the following important result:

THEOREM 19.1. *The group $\mathcal{H}^{\infty}_{p,q}(K)$ is isomorphic to the factor group*

$$\pi_{p+q,p}(K)/\pi_{p+q,p-1}(K).$$

PROOF. The proof is best made by drawing a sequence of diagrams which indicate clearly the bigraded structure of the successive derived couples

$$\langle \Gamma(K),\ \mathcal{H}(K) \rangle,\ \ \langle \Gamma'(K),\ \mathcal{H}'(K) \rangle,\ \text{etc.},$$

and also indicate clearly which of the groups $\Gamma^{(r)}_{p,q}$, $\mathcal{H}^{(r)}_{p,q}$ are trivial. A portion of such a diagram is indicated for the exact couple $\langle \Gamma(K), \mathcal{H}(K) \rangle$ in figure 3. It is suggested that the reader make a similar "ladder-like" or "lattice-like" diagram for the couples $\langle \Gamma'(K), \mathcal{H}'(K) \rangle$, $\langle \Gamma''(K), \mathcal{H}''(K) \rangle$, etc. Figure 4 shows the portion of such a diagram for the couple $\langle \Gamma^{(n-2)}(K), \mathcal{H}^{(n-2)}(K) \rangle$ which is particularly relevant for the proof of this theorem. The diagram in figure 5 is obtained from figure 4 by making use of Proposition 18.1 and the statement above. The theorem now follows from the exactness relations which hold in figure 5.

The reader may wonder why there should be any interest in the Leray-Koszul sequence: $(\mathcal{H}(K), d)$, $(\mathcal{H}'(K), d')$, $\cdots$. Since this Leray-Koszul sequence is completely determined by the exact couple $\langle \Gamma(K), \mathcal{H}(K) \rangle$, why not concentrate our attention on this exact couple, and forget about the Leray-Koszul sequence? The answer is that it seems difficult or impossible to relate the groups $\Gamma_{p,q}(K)$ and the homomorphisms involved in the homotopy exact couple to any calculable invariants of the complex K. This is not true of the groups $\mathcal{H}_{p,q}(K)$; as we have seen in Sections 14 and 16 above, many of the groups $\mathcal{H}_{p,q}(K)$ are calculable, and their definition is such that there seems to be more possibility of determining these groups than there does of determining the

$$
\begin{tikzcd}
{} & {} & \vdots \arrow[d] \\
{} \arrow[r] & 0 \arrow[r] & 0 \arrow[d] \\
{} \arrow[r] & \mathcal{H}_{2,4} \arrow[r] & 0 \arrow[d] & & \vdots \arrow[d] \\
{} \arrow[r] & \mathcal{H}_{3,3} \arrow[r] & 0 \arrow[d] \arrow[r] & 0 \arrow[r] & 0 \arrow[d] \\
{} \arrow[r] & \mathcal{H}_{4,2} \arrow[r] & \Gamma_{3,2} \arrow[d] \arrow[r] & \mathcal{H}_{2,3} \arrow[r] & 0 \arrow[d] & & \vdots \arrow[d] \\
{} \arrow[r] & \mathcal{H}_{5,1} \arrow[r] & \Gamma_{4,1} \arrow[d] \arrow[r] & \mathcal{H}_{3,2} \arrow[r] & 0 \arrow[d] \arrow[r] & 0 \arrow[r] & 0 \arrow[d] \\
{} \arrow[r] & \mathcal{H}_{6,0} \arrow[r] & \Gamma_{5,0} \arrow[d] \arrow[r] & \mathcal{H}_{4,1} \arrow[r] & \Gamma_{3,1} \arrow[d] \arrow[r] & \mathcal{H}_{2,2} \arrow[r] & 0 \arrow[d] & & \vdots \arrow[d] \\
{} \arrow[r] & 0 \arrow[r] & \Gamma_{6,-1} \arrow[d] \arrow[r] & \mathcal{H}_{5,0} \arrow[r] & \Gamma_{4,0} \arrow[d] \arrow[r] & \mathcal{H}_{3,1} \arrow[r] & 0 \arrow[d] \arrow[r] & 0 \arrow[r] & 0 \arrow[d] \\
& & \Gamma_{7,-2} \arrow[d] \arrow[r] & 0 \arrow[r] & \Gamma_{5,-1} \arrow[d] \arrow[r] & \mathcal{H}_{4,0} \arrow[r] & \Gamma_{3,0} \arrow[d] \arrow[r] & \mathcal{H}_{2,1} \arrow[r] & 0 \arrow[d] & & \vdots \arrow[d] \\
& & \vdots & & \Gamma_{6,-2} \arrow[d] \arrow[r] & 0 \arrow[r] & \Gamma_{4,-1} \arrow[d] \arrow[r] & \mathcal{H}_{3,0} \arrow[r] & 0 \arrow[d] \arrow[r] & 0 \arrow[r] & 0 \arrow[d] \arrow[r] & \cdots \\
& & & & \vdots & & \Gamma_{5,-2} \arrow[d] \arrow[r] & 0 \arrow[r] & \Gamma_{3,-1} \arrow[d] \arrow[r] & \mathcal{H}_{2,0} \arrow[r] & 0 \arrow[d] \arrow[r] & \cdots \\
& & & & & & \vdots & & \Gamma_{4,-2} \arrow[d] \arrow[r] & 0 \arrow[r] & 0 \arrow[d] \arrow[r] & \cdots \\
& & & & & & & & \vdots & & \vdots
\end{tikzcd}
$$

Fig. 3

$$
\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[d] \arrow[r] & 0 \arrow[r] & 0 \arrow[d] \arrow[r] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] & \Gamma^{(n-2)}_{n+1,-1} \arrow[d] \arrow[r] & \mathcal{H}^{(n-2)}_{2,n-2} \arrow[r] & 0 \arrow[d] \arrow[r] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] & \Gamma^{(n-2)}_{n+2,-2} \arrow[d] \arrow[r] & \mathcal{H}^{(n-2)}_{3,n-3} \arrow[r] & 0 \arrow[d] \arrow[r] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] & \Gamma^{(n-2)}_{n+3,-3} \arrow[d] \arrow[r] & \mathcal{H}^{(n-2)}_{4,n-4} \arrow[r] & 0 \arrow[d] \arrow[r] & \cdots \\
& & \vdots \arrow[d] & & \vdots \arrow[d] & \\
\cdots \arrow[r] & 0 \arrow[r] & \Gamma^{(n-2)}_{2n-2,-n+2} \arrow[d] \arrow[r] & \mathcal{H}^{(n-2)}_{n-1,1} \arrow[r] & 0 \arrow[d] \arrow[r] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] & \Gamma^{(n-2)}_{2n-1,-n+1} \arrow[d] \arrow[r] & \mathcal{H}^{(n-2)}_{n,0} \arrow[r] & 0 \arrow[d] \arrow[r] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] & \Gamma^{(n-2)}_{2n,-n} \arrow[d] \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \cdots \\
& & \vdots & & &
\end{tikzcd}
$$

Fig. 4

groups $\Gamma_{p,q}(K)$. Furthermore, as we shall see later, the differential operator $d : \mathcal{H}(K) \to \mathcal{H}(K)$ is related to certain known topological invariants.

$$
\begin{tikzcd}
0 \arrow[r] & \pi_{n,1} \arrow[r] \arrow[d] & \mathcal{H}^{\infty}_{1,n-1} \arrow[r] & 0 \arrow[d] \\
0 \arrow[r] & \pi_{n,2} \arrow[r] \arrow[d] & \mathcal{H}^{\infty}_{2,n-2} \arrow[r] & 0 \arrow[d] \\
0 \arrow[r] & \pi_{n,3} \arrow[r] \arrow[d] & \mathcal{H}^{\infty}_{3,n-3} \arrow[r] & 0 \arrow[d] \\
0 \arrow[r] & \pi_{n,4} \arrow[r] \arrow[d] & \mathcal{H}^{\infty}_{4,n-4} \arrow[r] & 0 \arrow[d] \\
 & \vdots \arrow[d] & & \vdots \arrow[d] \\
0 \arrow[r] & \pi_{n,n-1} \arrow[r] \arrow[d] & \mathcal{H}^{\infty}_{n-1,1} \arrow[r] & 0 \arrow[d] \\
0 \arrow[r] & \pi_{n,n} \arrow[r] \arrow[d] & \mathcal{H}^{\infty}_{n,1} \arrow[r] & 0 \\
 & \pi_{n,n+1} & &
\end{tikzcd}
$$

FIG. 5

APPENDIX

20. Conventions Used Regarding Homology and Cohomology Groups

In general, we will use the definitions and notations for homology and cohomology groups that are followed in the forthcoming book of Eilenberg and Steenrod, [5]. If X is a topological space, A a subspace, and G an abelian group, then $H_p(X, A, G)$ denotes the p-dimensional homology group of X modulo A with coefficients in G, and $H^p(X, A, G)$ denotes the corresponding p-dimensional cohomology group. For brevity, we will often abbreviate these to $H_p(X, A)$ and $H^p(X, A)$ respectively, particularly when using integral coefficients. Since in most cases we are dealing with spaces which are complexes, it usually does not matter what kind of homology or cohomology theory we use—singular, Čech, etc. However, in the few cases in which we do consider more general spaces, it is always to be understood that we use the singular theory for homology, and the Čech theory for cohomology. We will only consider cohomology groups of compact spaces.

If K is a cell complex, finite or infinite, then the group of p-dimensional

integral chains may be identified with the integral homology group $H_p(K^p, K^{p-1})$, where K^n denotes the n-skeleton of K. It is a free abelian group. The group of p-chains with coefficients in an arbitrary abelian group G, $C_p(K, G) = C_p(K)\otimes G$ (tensor product), may be identified with $H_p(K^p, K^{p-1}, G)$. The boundary operator for chains, $\partial : C_p(K) \to C_{p-1}(K)$ is then to be identified with the boundary operator of the triple, (K^p, K^{p-1}, K^{p-2}); viz., $\partial : H_p(K^p, K^{p-1}) \to H_{p-1}(K^{p-1}, K^{p-2})$. Analogous conventions hold for cohomology groups. If K is a finite cell complex, then the group of integral p-dimensional cochains, $C^p(K)$, may be identified with $H^p(K^p, K^{p-1})$, and the group of p-dimensional cochains with coefficients in G, $C^p(K, G) = C^p(K)\otimes G$, may be identified with $H^p(K^p, K^{p-1}, G)$. The coboundary operator for cochains, $\delta : C^p(K) \to C^{p+1}(K)$, is to be identified with the coboundary operator of the triple (K^{p+1}, K^p, K^{p-1}), $\delta : H^p(K^p, K^{p-1}) \to H^{p+1}(K^{p+1}, K^p)$.

21. Basic Conventions and Notations Regarding Homotopy Groups

We will make free use of the notation, terminology, and definitions which are given in Part 1 of [2]. In particular, E^n will denote the closed unit n-cell and S^{n-1} its bounding $n - 1$-sphere in Cartesian n-space. E_+^{n-1} and E_-^{n-1} denote the "upper" and "lower" hemispheres of S^{n-1}, and $p_0 \in E_+^{n-1} \cap E_-^{n-1}$ denotes the point $(1, 0, \cdots, 0)$. If X is any topological space, and $x \in X$, then an element of the absolute homotopy group $\pi_n(X, x)$ is represented by a map $(S^n, p_0) \to (X, x)$ or by a map $(E^n, S^{n-1}) \to (X, x)$, provided a choice of orientation is made for the sphere S^n or the cell E^n. Similarly, if A is a subspace of X, and $x \in A$, then an element of the relative homotopy group $\pi_n(X, A, x)$ may be represented by a map $(E^n, S^{n-1}, p_0) \to (X, A, x)$ or by a map $(E^n, S^{n-1}, E_+^{n-1}) \to (X, A, x)$, provided, of course, an orientation has been chosen for E^n. If x_0 and x_1 are points of A, and ω denotes a homotopy class of paths in A *from* x_0 *to* x_1, then ω induces isomorphisms onto, $\pi_n(X, A, x_1) \to \pi_n(X, A, x_0)$, $n = 1, 2, \cdots$,; similarly for operators on the absolute homotopy groups. We will consistently use the notation

$$\rho_n : \pi_n(X, x_0) \to H_n(X), \qquad n \geqq 1,$$

$$\rho_n : \pi_n(X, A, x_0) \to H_n(X, A), \qquad n \geqq 2,$$

to denote the natural homomorphisms of the homotopy groups into the integral singular homology groups. According to the Hurewicz equivalence theorem, if X is $(n - 1)$-connected, $n \geqq 2$, then $\rho_n : \pi_n(X) \to H_n(X)$ is an isomorphism onto; if (X, A) is $(n - 1)$-connected and n-simple, $n \geqq 3$, then $\rho_n : \pi_n(X, A) \to H_n(X, A)$ is an isomorphism onto.

22. Some Lemmas from Homotopy Theory

In this section we collect together four lemmas from homotopy theory which we need.

(a) Let the n-sphere, S^n, be oriented, $n > 1$, let E_+^n, E_-^n, S^{n-1}, and p_0 have

the meaning described in section 21. Let (X, A) be a pair with base point $x_0 \in A$, and

$$f : (S^n, p_0) \rightarrow (X, x_0)$$

a map such that $f(S^{n-1}) \subset A$. Then f defines maps

$$f_1 : (E_+^n, S^{n-1}, p_0) \rightarrow (X, A, x_0),$$

$$f_2 : (E_-^n, S^{n-1}, p_0) \rightarrow (X, A, x_0).$$

Give E_+^n and E_-^n the orientation induced by the given orientation of S^n. Then f represents an element $\alpha \in \pi_n(X, x_0)$ and f_1, f_2 represent elements $\alpha_1, \alpha_2 \in \pi_n(X, A, x_0)$. Let $j : \pi_n(X, x_0) \rightarrow \pi_n(X, A, x_0)$ denote the injection.

LEMMA 22.1. $j(\alpha) = \alpha_1 + \alpha_2$.

$$\begin{tikzcd}
 & H_n(S^n, E_+^n) & & H_n(E_-^n, S^{n-1}) \arrow[ll] \arrow[dl, "k_2"] \\
H_n(S^n) \arrow[ur] \arrow[dr] \arrow[rr, "i"] & & H_n(S^n, S^{n-1}) \arrow[ul] \arrow[dl] & \\
 & H_n(S^n, E_-^n) & & H_n(E_+^n, S^{n-1}) \arrow[ll] \arrow[ul, "k_1"]
\end{tikzcd}$$

FIG. 6

$$\begin{tikzcd}
 & \pi_n(S^n, E_+^n) & & \pi_n(E_-^n, S^{n-1}) \arrow[ll] \arrow[dl, "k_2"] \\
\pi_n(S^n) \arrow[ur] \arrow[dr] \arrow[rr, "i"] & & \pi_n(S^n, S^{n-1}) \arrow[ul] \arrow[dl] & \\
 & \pi_n(S^n, E_-^n) & & \pi^n(E_+^n, S^{n-1}) \arrow[ll] \arrow[ul, "k_1"]
\end{tikzcd}$$

FIG. 7

PROOF. Consider the diagrams in figures 6 and 7. All homomorphisms in these two diagrams are induced by inclusion maps. The Hurewicz equivalence theorem furnishes a natural isomorphism between these two diagrams, provided $n > 2$; the modifications necessary when $n = 2$ are left to the reader. Let $u \in H_n(S^n)$, $u_1 \in H_n(E_+^n, S^{n-1})$, $u_2 \in H_n(E_-^n, S^{n-1})$ denote the elements corresponding to the orientations of the sphere and cells. Then in figure 6 we have the relation

$$k_1(u_1) + k_2(u_2) = i(u).$$

Let β_1, β_2, and β correspond to u_1, u_2 and u under the Hurewicz isomorphism:

$$\beta_1 = \rho_n(u_1), \qquad \beta_2 = \rho_n(u_2), \qquad \beta = \rho_n(u).$$

Then in figure 7 the relation

$$k_1(\beta_1) + k_2(\beta_2) = i(\beta) \tag{22.1}$$

holds. Next, consider the diagrams in figures 8 and 9. Here $f_3 : (S^n, S^{n-1}, p_0) \to (X, A, x_0)$ is a map induced by f. Then the following relations obviously hold:

$$f_{1*}(\beta_1) = \alpha_1, \quad f_{2*}(\beta_2) = \alpha_2, \tag{22.2}$$
$$f_*(\beta) = \alpha.$$

Combining the relations (22.1) and (22.2) with the fact that commutativity holds in figures 8 and 9, we obtain the desired result.

$$\begin{tikzcd}
\pi_n(E^n_-, S^{n-1}) \arrow[d, "k_2"] \arrow[drr, "f_{2*}"] & & \\
\pi_n(S^n, S^{n-1}) \arrow[rr, "f_{3*}"] & & \pi_n(X, A) \\
\pi_n(E^n_+, S^{n-1}) \arrow[u, "k_1"] \arrow[urr, "f_{1*}"'] & &
\end{tikzcd}$$

FIG. 8

$$\begin{tikzcd}
\pi_n(S^n) \arrow[r, "i"] \arrow[d, "f_*"] & \pi_n(S^n, S^{n-1}) \arrow[d, "f_{3*}"] \\
\pi_n(X) \arrow[r, "j"] & \pi_n(X, A)
\end{tikzcd}$$

FIG. 9

(b) Let $w^n \in H_n(E^n, S^{n-1})$, $(n > 1)$, and $v^1 \in H_1(I, \dot{I})$ be the orientations of E^n and I defined in Section (1.1) of [2]. Let $f_0 : (E^n, S^{n-1}, E_-^{n-1}) \to (X, A, x_0)$ and $f_1 : (E^n, S^{n-1}, E_-^{n-1}) \to (X, A, x_1)$, represent elements $\alpha_0 \in \pi_n(X, A, x_0)$ and $\alpha_1 \in \pi_n(X, A, x_1)$ respectively. Let $g : I \to A$ represent a class of paths, ω, in A from x_0 to x_1. The cross product, $v^1 \times w^n$, defines an orientation of the $(n + 1)$-cell $I \times E^n$. The n-cell $\mathcal{E}^n = (I \times E_-^{n-1}) \cup (0 \times E^n) \cup (1 \times E^n)$ is a face of $I \times E^n$; the orientation chosen for $I \times E^n$ determines in a natural way an orientation of the face $\mathcal{E}^n$. Define a map

$$h : (\mathcal{E}^n, \dot{\mathcal{E}}^n, 0 \times p_0) \to (X, A, x_0),$$

(where $\dot{\mathcal{E}}^n$ denotes the boundary of $\mathcal{E}^n$) by

$$h(t, x) = \begin{cases} f_0(x), & t = 0, x \in E^n, \\ g(t), & 0 \leqq t \leqq 1, x \in E_-^{n-1}, \\ f_1(t), & t = 1, x \in E^n. \end{cases}$$

LEMMA 22.2. *The map h is a representative of the element* $(\omega \cdot \alpha_1) - \alpha_0 \in \pi_n(X, A, x_0)$.

The proof is left to the reader.

(c) Let the cells E^n, I, and $I \times E^n$ have the orientations described above. Let $f_0 : (E^n, S^{n-1}) \to (X, x_0)$ and $f_1 : (E^n, S^{n-1}) \to (X, x_1)$ represent elements $\alpha_0 \in \pi_n(X, x_0)$ and $\alpha_1 \in \pi_n(X, x_1)$ respectively, let $g : I \to X$ represent a class of paths, ω, from x_0 to x_1. Let $\mathcal{S}^n$ denote the bounding n-sphere of the cell $I \times E^n$:

$$\mathcal{S}^n = (\dot{I} \times E^n) \cup (I \times \dot{E}^n).$$

Give $\mathcal{S}^n$ the orientation determined in the natural way by the orientation of $I \times E^n$. Define a map $h : (\mathcal{S}^n, 0 \times p_0) \to (X, x_0)$ by

$$h(t, x) = \begin{cases} f_0(x), & t = 0, x \in E^n, \\ g(t), & 0 \leqq t \leqq 1, x \in \dot{E}^n, \\ f_1(x), & t = 1, x \in E^n. \end{cases}$$

LEMMA 22.3. *The map h represents the element* $(\omega \cdot \alpha_1 - \alpha_0) \in \pi_n(X, x_0)$.

The proof is left to the reader.

(d) Let $f : E^n \to X$ in such a way that $f(S^{n-2}) \subset A$, where $A \subset X$, and $f(p_0) = x_0 \in A$. Then f defines maps

$$f_1 : (E_+^{n-1}, S^{n-2}, p_0) \to (X, A, x_0),$$

$$f_2 : (E_-^{n-1}, S^{n-2}, p_0) \to (X, A, x_0).$$

Let E_+^{n-1} and E_-^{n-1} be oriented so that they induce the *same* orientations of S^{n-2}. Then f_1 and f_2 represents elements α_1 and α_2 of $\pi_{n-1}(X, A, x_0)$.

LEMMA 22.4. $\alpha_1 = \alpha_2$.

The proof is left to the reader.

23. The Composition Operation for Homotopy Groups

If $\alpha \in \pi_n(S^r)$ and $\beta \in \pi_r(X)$, then $\beta \circ \alpha \in \pi_n(X)$. For definitions, see [18, p. 202]. The following properties are known to hold for this operation:

(23.1) $$\beta \circ (\alpha_1 + \alpha_2) = \beta \circ \alpha_1 + \beta \circ \alpha_2, \qquad \alpha_1, \alpha_2 \in \pi_n(S^r), \beta \in \pi_r(X).$$

(23.2) $$\gamma \circ (\beta \circ \alpha) = (\gamma \circ \beta) \circ \alpha, \qquad \alpha \in \pi_n(S^r), \beta \in \pi_r(S^s), \gamma \in \pi_s(X)$$

(23.3) $$\gamma \circ [\alpha, \beta] = [\gamma \circ \alpha, \gamma \circ \beta], \qquad \alpha \in \pi_p(S^r), \beta \in \pi_q(S^r), \gamma \in \pi_r(X).$$

If $\alpha \in \pi_{n-1}(S^{r-1})$ and $\beta_1, \beta_2 \in \pi_r(X)$, then

(23.4) $$(\beta_1 + \beta_2) \circ E(\alpha) = \beta_1 \circ E(\alpha) + \beta_2 \circ E(\alpha),$$

where E denotes the suspension homomorphism. If $\alpha \in \pi_p(S^r)$, $\beta \in \pi_r(S^s)$, then

(23.5) $$E(\beta \circ \alpha) = E(\beta) \circ E(\alpha).$$

If $\alpha \in \pi_n(S^r)$, $\beta_1, \beta_2 \in \pi_r(X)$, and $n < 3r - 2$,

$$(23.6) \qquad (\beta_1 + \beta_2) \circ \alpha = \beta_1 \circ \alpha + \beta_2 \circ \alpha + [\beta_1, \beta_2] \circ H(\alpha)$$

where H denotes the generalized Hopf homomorphism. If $f : (X, x_0) \to (Y, y_0)$ is a continuous map, $\alpha \in \pi_n(S^r)$, and $\beta \in \pi_r(X)$, then

$$(23.7) \qquad f_*(\beta \circ \alpha) = \beta \circ (f_*\alpha).$$

For proofs see the paper of G. W. Whitehead referred to above.

In a similar manner, if $\alpha \in \pi_n(E^r, S^{r-1})$ and $\beta \in \pi_r(X, A)$, then one can define $\beta \circ \alpha \in \pi_n(X, A)$ by composition of functions representing β and α. The following two properties are obvious:

$$(23.8) \quad \beta \circ (\alpha_1 + \alpha_2) = \beta \circ \alpha_1 + \beta \circ \alpha_2, \qquad \alpha_1, \alpha_2 \in \pi_n(E^r, S^{r-1}), \beta \in \pi_r(X, A).$$

$$(23.9) \quad (\gamma \circ \beta) \circ \alpha = \gamma \circ (\beta \circ \alpha),$$

$$\alpha \in \pi_n(E^r, S^{r-1}), \beta \in \pi_r(E^s, S^{s-1}), \gamma \in \pi_s(X, A).$$

Let us agree to use the symbol ∂ to represent the homotopy boundary operator:

$$\partial : \pi_n(X, A) \to \pi_{n-1}(A),$$

$$\partial : \pi_n(E^r, S^{r-1}) \to \pi_{n-1}(S^{r-1}).$$

Then we obviously have

$$(23.10) \qquad \partial(\beta \circ \alpha) = (\partial\beta) \circ (\partial\alpha), \qquad \alpha \in \pi_n(E^r, S^{r-1}), \beta \in \pi_r(X, A).$$

Next, let $j_n : \pi_n(X) \to \pi_n(X, A)$, $n = 2, 3, \cdots$, denote an injection and

$$\phi : \pi_n(E^r, S^{r-1}) \to \pi_n(S^r, p_0)$$

the homomorphism induced by a map $(E^r, S^{r-1}) \to (S^r, p_0)$ of degree $+1$. Then we have

$$(23.11) \qquad j_n[\beta \circ \phi(\alpha)] = j_r(\beta) \circ \alpha, \qquad \alpha \in \pi_n(E^r, S^{r-1}), \beta \in \pi_r(X).$$

The proof is easy. Note that the homomorphism ϕ is equivalent to the suspension.

If $f : (X, A, x_0) \to (Y, B, y_0)$ is a continuous map, then

$$(23.12) \qquad f_*(\beta \circ \alpha) = \beta \circ (f_*\alpha), \qquad \alpha \in \pi_n(E^r, S^{r-1}), \beta \in \pi_r(X, A, x_0).$$

Let $E : \pi_{n-1}(E^{r-1}, S^{r-2}) \to \pi_n(E^r, S^{r-1})$ be a homomorphism which is equivalent to the suspension, as explained in [18, p. 200]. Then we have

$$(23.13) \qquad (\beta_1 + \beta_2) \circ E(\alpha) = \beta_1 \circ E(\alpha) + \beta_2 \circ E(\alpha),$$

where $\alpha \in \pi_{n-1}(E^{r-1}, S^{r-2})$ and β_1 and $\beta_2 \in \pi_r(X, A)$. The proof is the exact analogue of the proof of (23.4) above.

Next, we have the exact analogue of equation (23.6). It is an easy matter to define a "generalized Hopf Homomorphism"

$$H : \pi_r(E^n, S^{n-1}) \to \pi_r(E^{2n-2}, S^{2n-3}),$$

for $r < 3n - 4$ which is equivalent to the generalized Hopf homomorphism defined by G. W. Whitehead. Then we have

$$(\beta_1 + \beta_2) \circ \alpha = \beta_1 \circ \alpha + \beta_2 \circ \alpha + [\beta_1, \beta_2] \circ H(\alpha)$$

where $\alpha \in \pi_n(E^r, S^{r-1})$, $\beta_1, \beta_2 \in \pi_r(X, A)$, and $n < 3r - 4$. The Whitehead product $[\beta_1, \beta_2]$ is that defined by S. T. Hu [9].

The important cases in the above formulas are those in which the composition operation is bilinear. These cases should be noted carefully.

For the statement of the next result, we assume X^* is obtained from the space X by the adjunction of n-cells, $\mathcal{E}_1^n$, $\mathcal{E}_2^n$, $\mathcal{E}_3^n$, $\cdots$, as described in section (4.1) of [2]. Assume that $n > 2$, and that X is m-connected, $0 < m < n - 1$. Under these conditions the composition operation,

$$(\beta, \alpha) \to \beta \circ \alpha$$

for $\alpha \in \pi_p(E^n, S^{n-1})$ and $\beta \in \pi_n(X^*, X)$ is bilinear if $p < 2n - 2$, and hence defines a homomorphism of the tensor product,

$$\psi : \pi_n(X^*, X) \otimes \pi_p(E^n, \dot{E}^n) \to \pi_p(X^*, X)$$

for $p < 2n - 2$.

THEOREM 23.1. *If $p < m + n$, then this homomorphism ψ is an isomorphism onto.*

This theorem is an easy consequence of Theorems I and II of [2]; cf. also Section 9 of [13]. This theorem can also be deduced from Theorem 1 of [19] together with Theorem II of [2].

24. Remarks on Generalized Whitehead Products

If (X, A) is an arbitrary pair, $x_0 \in A$ a basepoint, $\alpha \in \pi_p(A, x_0)$, $\beta \in \pi_q(X, A, x_0)$, then the *Whitehead products of α and β*, denoted by $[\alpha, \beta]$ and $[\beta, \alpha]$, are defined, and are elements of $\pi_r(X, A, x_0)$, where $r = p + q - 1$. The definition and principal properties of this product are briefly sketched (without proofs) in [13]; complete proofs will appear in a forthcoming paper.

Our main use of generalized Whitehead Products is for the following theorem. Assume that X^* is obtained from X by the adjunction of n-cells, $\mathcal{E}_1^n$, $\cdots$, $\mathcal{E}_k^n$ as described in Section (4.1) of [2]. Assume that $n > 2$, and that X is m-connected, $0 < m < n - 2$. Then the composition operation defines a homomorphism

$$\psi_p : \pi_n(X^*, X) \otimes \pi_p(E^n, S^{n-1}) \to \pi_p(X^*, X)$$

as described in the preceding section, provided $p < 2n - 2$. The generalized Whitehead Product defines a homomorphism

$$\zeta: \pi_{m+1}(X) \otimes \pi_n(X^*, X) \rightarrow \pi_{m+n}(X^*, X)$$

by $\zeta(\alpha \otimes \beta) = [\alpha, \beta]$ for $\alpha \in \pi_{m+1}(X)$ and $\beta \in \pi_n(X^*, X)$.

THEOREM 24.1. *Under the above hypotheses, both ψ_{m+n} and ζ are isomorphisms into, and $\psi_{m+n}(X^*, X)$ is the direct sum of the two image subgroups.*

This theorem is a direct application of the result stated in section 9 of [13]. A complete proof will appear in a forthcoming paper.

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