

THE QUOTIENT SPACE OF THE COMPLEX
PROJECTIVE PLANE UNDER CONJUGATION
IS A 4-SPHERE

The purpose of this paper is to outline a proof of the following:

THEOREM. *Suppose we form the quotient space of the complex projective plane by identifying two points if and only if their (homogeneous) coordinates are complex conjugates of each other. Then the resulting space is a 4-sphere.*

It is not difficult to verify that the quotient space is a closed 4-dimensional manifold which is simply connected and has the homology groups of the 4-sphere. If the 4-dimensional Poincaré Conjecture were known to be true, this would suffice to prove the theorem. In the paragraphs that follow, we will give a direct proof without invoking the Poincaré Conjecture. In the course of the proof, we will use the following notation, which is more or less standard:

- $S^n =$ unit n -sphere in R^{n+1} , euclidean $(n+1)$ -space.
- $RP^n =$ real projective n -space (the quotient space of S^n under the identification of antipodal points).
- $CP^n =$ complex projective n -space.
- $SP^n(X) =$ n -fold symmetric product of the space X (the quotient space of the n -fold cartesian product under the obvious action of the symmetric group of degree n).

We will make use of the following two known facts:

$$CP^1 = S^2, \quad \text{and}$$

$$SP^n(S^2) = SP^n(CP^1) = CP^n.$$

(This last fact depends essentially on the algebraic closure of the field of complex numbers).

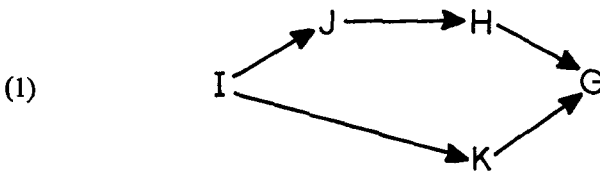
Let G denote the group of self-homeomorphisms of the product space $S^2 \times S^2$ generated by interchanging the two coordinates of any point and by the antipodal maps on either factor. To be precise, G is a group of order 8, consisting of the identity I and the seven other homeomorphisms of $S^2 \times S^2$ which send the point $(x, y) \in S^2 \times S^2$ into the following seven points: (y, x) , $(-x, y)$, $(x, -y)$, $(-x, -y)$, $(-y, x)$, $(y, -x)$, and $(-y,$

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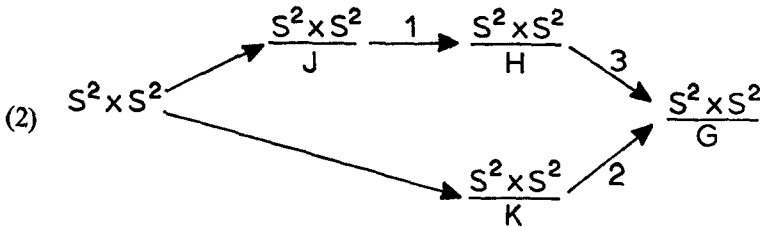
$-x$). It is readily seen that G is isomorphic to the dihedral group, i.e. the group of symmetries of the square in the (x, y) -plane with vertices $(\pm 1, \pm 1)$. We also need to consider the following three subgroups of G :

$$\begin{aligned}
 J &= \{I, (x, y) \rightarrow (y, x)\} \\
 K &= \{I, (x, y) \rightarrow (-x, y), (x, y) \rightarrow (x, -y), \\
 &\quad (x, y) \rightarrow (-x, -y)\}, \\
 H &= \{I, (x, y) \rightarrow (y, x), (x, y) \rightarrow (-x, -y), \\
 &\quad (x, y) \rightarrow (-y, -x)\}.
 \end{aligned}$$

The inclusion relations between these subgroups are shown by the following diagram:



We also have the corresponding diagram of quotient spaces and natural maps:



Note the following facts about diagram (2):

- (a) $(S^2 \times S^2)/J = SP^2(S^2) = CP^2$.
- (b) $(S^2 \times S^2)/K = RP^2 \times RP^2$.
- (c) To pass from $(S^2 \times S^2)/J = CP^2$ to $(S^2 \times S^2)/H$

(arrow No. 1 in the diagram), form the quotient space of CP^2 under the action of the group H/J , which is cyclic of order 2. One can prove by an explicit calculation (which we will not reproduce) that the action of the generator of H/J on CP^2 is equivalent to complex conjugation; *therefore, to prove our theorem, it suffices to prove that $(S^2 \times S^2)/H = S^4$.*

(d) To pass from $(S^2 \times S^2)/K = RP^2 \times RP^2$ to $(S^2 \times S^2)/G$ (arrow No. 2), form quotient space of $RP^2 \times RP^2$ under the action of the group G/K , which is cyclic of order 2. It is clear that a generator of G/K acts on $RP^2 \times RP^2$ by interchanging coordinates. Therefore

$$(S^2 \times S^2)/G = SP^2(RP^2).$$

(e) To pass from $(S^2 \times S^2)/H$ to $(S^2 \times S^2)/G$ (arrow No. 3), form the quotient space of $(S^2 \times S^2)/H$ by the group G/H , which is cyclic of order 2. It is easily verified that the generator of G/H acts on $(S^2 \times S^2)/H$ without fixed points. Therefore $(S^2 \times S^2)/H$ is a 2-sheeted unbranched covering space of $(S^2 \times S^2)/G = SP^2(RP^2)$.

To complete the proof, it suffices to prove that $SP^2(RP^2) = RP^4$; it will then follow from the preceding sentence that $(S^2 \times S^2)/H = S^4$, as required. Actually, we will prove the following more general statement*:

LEMMA 1. $SP^n(RP^2) = RP^{2n}$.

This lemma, in turn, is an easy consequence of the following lemma:

LEMMA 2. *Let X be a compact Hausdorff space with a fixed-point free involution $T: X \rightarrow X$. Let $T': SP^{2n}(X) \rightarrow SP^{2n}(X)$ denote the induced involution of the $2n$ -fold symmetric product. Then the fixed point set of T' is naturally homeomorphic to $SP^n(X/T)$.*

The proof of this lemma may be left to the reader.

To derive Lemma 1 from Lemma 2, take $X = S^2$ and $T: S^2 \rightarrow S^2$ the antipodal map. One needs to check that the induced involution $T': CP^{2n} \rightarrow CP^{2n}$ is equivalent to complex conjugation.

This completes the proof of the theorem.

Remarks. (1) It is a curious fact that the closed 4-manifolds of most common occurrence are among the quotient spaces of $S^2 \times S^2$ under the action of various subgroups of the group G . In addition to those which occur in the above proof, the Grassmannian manifold of all (unoriented) 2-planes through the origin in R^4 also occurs, as does $RP^2 \times S^2$.

(2) Define an action of the group $SO(3)$ of all 3×3 orthogonal matrices of determinant +1 on $S^2 \times S^2$ as follows:

$$r(x, y) = (rx, ry)$$

for any $(x, y) \in S^2 \times S^2$ and $r \in SO(3)$. Then the action of $SO(3)$ on $S^2 \times S^2$ commutes with the action of G on $S^2 \times S^2$, and hence defines an action of $SO(3)$ on each of the quotient spaces. This suggests another way of proving our main theorem, as follows: There is an obvious way to define an action of $SO(3)$ on CP^2 , in terms of the homogeneous coordinates. This action commutes with complex conjugation, and hence defines an action of $SO(3)$ on the quotient space. There is also a non-trivial action of $SO(3)$ on S^4 defined as follows: The group $SO(3)$ acts on the space of all 3×3 , real

* I believe this lemma and its proof are originally due to Dennis Sullivan. A more general result is given in the appendix to a recent paper of J. L. Dupont and G. Lusztig, 'On Manifolds Satisfying $w_1^2 = 0$ ', *Topology* 10 (1971).

symmetric matrices of trace 0 by conjugation. Give this 5-dimensional vector space a positive definite, $SO(3)$ -invariant inner product. The restriction of the action to the unit sphere is the desired action of $SO(3)$ on S^4 . Then by analyzing the orbit structure of the action of $SO(3)$ on the quotient space of CP^2 and on S^4 , one would hope to prove that the two manifolds are homeomorphic (by an $SO(3)$ -equivariant map).

(3) The dihedral group G contains a unique cyclic subgroup of order 4. We leave it to the reader to verify that this subgroup acts freely on $S^2 \times S^2$. Note also that the subgroup K acts freely on $S^2 \times S^2$. Thus both of the possible groups of order 4 can act freely on $S^2 \times S^2$. On the other hand, by using the theory of covering spaces and the fact that the Euler characteristic of $S^2 \times S^2$ is 4, one can prove that only groups of order 2 or 4 can act freely on $S^2 \times S^2$.

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