ALGEBRAIC L-THEORY, I: FOUNDATIONS

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Introduction

Where algebraic K-theory deals with modules, L-theory considers modules with quadratic forms. The L-groups are of interest to topologists because they are the surgery obstruction groups, as described by Wall ([**6**]). Although isomorphism groups of quadratic forms have been studied before, by Witt and others, the topological applications require new algebraic methods (cf. [**5**], [**7**]).

The L-groups $L_n(\pi)$ were obtained in [**6**] as the solutions to a specific topological problem, leaving open the question of the algebraic framework best suited for an 'L-theory'. In [**1**] Novikov used algebraic K-theory and the formalism of hamiltonian physics to provide such machinery, though not as coherently as might be desired.

In Part I of this paper we shall use the ideas of [**1**] to give the foundations of L-theory over a ring with involution, A. We shall define L-groups $U_n(A)$, $V_n(A)$, and $W_n(A)$ as stable isomorphism groups of '$\pm$ forms' and '$\pm$ formations' involving finitely generated (f.g.) projective, stably f.g. free, and based A-modules respectively, depending on n (mod 4) only.

Part II, [**2**], will be devoted to a detailed study of the L-groups of the Laurent extension ring $A_z = A[z, z^{-1}]$, with involution $z \mapsto z^{-1}$. It will be shown that there exist natural direct sum decompositions

$$W_n(A_z) = W_n(A) \oplus V_{n-1}(A),$$

$$V_n(A_z) = V_n(A) \oplus U_{n-1}(A).$$

Part III, [**3**], deals with the L-theory of twisted Laurent extensions, using intermediate L-groups, in which all the torsions (resp. projective classes) lie in a prescribed subgroup of $\tilde{K}_1(A)$ (resp. $\tilde{K}_0(A)$). The intermediate L-theories are also studied by Wall in [**7**], which offers a more K-theoretic approach to L-theory.

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0. Conventions

Let A be an associative ring with 1, and with an *involution*, that is a function

$$^{-}: A \to A; \quad a \mapsto \bar{a}$$

such that

$$\begin{aligned} &\text{(i)}\ \bar{1} = 1, \\ &\text{(ii)}\ \overline{a+b} = \bar{a} + \bar{b}, \\ &\text{(iii)}\ \overline{ab} = \bar{b}.\bar{a}, \\ &\text{(iv)}\ \bar{\bar{a}} = a \end{aligned}$$

for all $a, b \in A$.

It is further required that f.g. free A-modules have well-defined dimension. (This condition is not essential. Its use, in Theorems 4.2, 5.7 below, can be avoided by considering f.g. free A-modules with prescribed dimension, as is done in [**3**].)

Example 0.1. The group ring $\mathbf{Z}[\pi]$ of a multiplicative group π, with the involution

$$^{-}: \mathbf{Z}[\pi] \to \mathbf{Z}[\pi]; \ \sum_{g \in \pi} n_g g \mapsto \sum_{g \in \pi} w(g) n_g g^{-1} \quad (n_g \in \mathbf{Z})$$

defined by a morphism of groups

$$w: \pi \to \mathbf{Z}_2 = \{1, -1\}$$

satisfies these conditions.

(This is the ground ring occurring in topology, with π the fundamental group $\pi_1(M)$ of a compact manifold M, and $w: \pi_1(M) \to \mathbf{Z}_2$ the first Stiefel–Whitney class, cf. [**6**].)

We shall be dealing with left A-modules, $M, N, P, Q, \ldots$.

Denote by $\mathrm{Hom}_A(M, N)$ the additive group of A-module morphisms $f: M \to N$.

Given M, define the *dual* A-module, M^*, to be $\mathrm{Hom}_A(M, A)$ with A acting by

$$A \times \mathrm{Hom}_A(M, A) \to \mathrm{Hom}_A(M, A); \quad (a, f) \mapsto (x \mapsto f(x).\bar{a}).$$

Accordingly, given $f \in \mathrm{Hom}_A(M, N)$ define the *dual* morphism

$$f^*: N^* \to M^*; \quad g \mapsto (x \mapsto g(f(x)))$$

in $\mathrm{Hom}_A(N^*, M^*)$.

Morphisms in $\mathrm{Hom}_A(M\oplus N,\ P\oplus Q)$ can be displayed as matrices

$$f=\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}: M\oplus N\to P\oplus Q;\quad (x,y)\mapsto(\alpha(x)+\beta(y),\ \gamma(x)+\delta(y))$$

with

$$\alpha\in \mathrm{Hom}_A(M,P),\quad \beta\in\mathrm{Hom}_A(N,P),\quad \gamma\in\mathrm{Hom}_A(M,Q),$$

and

$$\delta\in\mathrm{Hom}_A(N,Q).$$

Composition of such morphisms corresponds to right multiplication of the matrices. The morphism dual to f (as above) has matrix

$$f^*=\begin{pmatrix}a^* & \gamma^*\\ \beta^* & \delta^*\end{pmatrix}: P^*\oplus Q^*\to M^*\oplus N^*,$$

identifying $(M\oplus N)^*$ with $M^*\oplus N^*$ in the obvious way.

If Q is a f.g. free A-module, with base $b=(b_1,b_2,\ldots,b_m)$, then Q^* is a f.g. free A-module of the same dimension, with *dual* base $b^*=(b_1^*,b_2^*,\ldots,b_m^*)$ given by

$$b_i^*(b_j)=\begin{cases}1 & \text{if } i=j\\ 0 & \text{otherwise.}\end{cases}$$

It follows that if P is a f.g. projective A-module, so is P^*, though P and P^* are not in general isomorphic. However, the natural map

$$P\to P^{**};\quad x\mapsto(f\mapsto\overline{f(x)})$$

is an isomorphism. It is used to identify P^{**} with P, whenever P is f.g. projective. In particular, given a morphism $f\in\mathrm{Hom}_A(Q,P^*)$ with P f.g. projective, we can write

$$f^*: P\to Q^*;\quad x\mapsto(y\mapsto\overline{f(y)(x)}).$$

1. Forms

A *hermitian product* (over A) is an A-module morphism

$$\theta: P\to P^*$$

defined on a f.g. projective A-module P, such that

$$\theta^*=\varepsilon\theta\in\mathrm{Hom}_A(P,P^*)$$

where $\varepsilon\ (=+1$ or $-1)$ is the *sign* of the product. It can be expressed in the usual way, as a sesquilinear function

$$\langle\ \rangle: P\times P\to A;\quad (x,y)\mapsto\langle x,y\rangle\equiv\theta(x)(y)$$

such that

(i) $\langle y,x\rangle=\varepsilon\overline{\langle x,y\rangle}$,

(ii) $\langle x,ay\rangle=a\langle x,y\rangle$ (or, equivalently, $\langle ax,y\rangle=\langle x,y\rangle\bar{a}$)

for all $x,y\in P$, $a\in A$.

A $\pm$ *product* is a hermitian product of the sign indicated.

A *quadratic form* (over A), (Q,φ), is a pair consisting of a f.g. projective A-module Q and a morphism $\varphi \in \mathrm{Hom}_A(Q,Q^*)$. The *associated* $\pm$ product

$$\varphi \pm \varphi^* : Q \to Q^*$$

corresponds to the sesquilinear function

$$\langle\,\rangle_\varphi : Q \times Q \to A\,; \quad (x,y) \mapsto \langle x,y\rangle_\varphi \equiv \varphi(x)(y) \pm \overline{\varphi(y)(x)}.$$

A $\pm$ *form* is a quadratic form, together with a choice of sign for the associated hermitian product.

A $\pm$ product $\theta \in \mathrm{Hom}_A(P,P^*)$ is *even* if

$$\theta = \varphi \pm \varphi^* : P \to P^*$$

for some $\pm$ form(P,φ).

(It is easily verified that a $\pm$ product $\theta \in \mathrm{Hom}_A(P,P^*)$ is even if and only if for each $x \in P$

$$\theta(x)(x) = a \pm \bar{a} \in A$$

for some $a \in A$.)

Let (P,θ), (Q,φ), (R,ψ) be $\pm$ forms over A.

A *morphism* of $\pm$ forms

$$f : (P,\theta) \to (Q,\varphi)$$

is defined by $f \in \mathrm{Hom}_A(P,Q)$ such that

$$(f^*\varphi f - \theta) : P \to P^*$$

is an even $\mp$ product. We shall write such morphisms as

$$(f,\chi) : (P,\theta) \to (Q,\varphi),$$

where (P,χ) is some $\mp$ form such that

$$f^*\varphi f - \theta = \chi \mp \chi^* : P \to P^*$$

but the choice of χ is not officially a part of the morphism structure.

A morphism of $\pm$ forms

$$(f,\chi) : (P,\theta) \to (Q,\varphi)$$

preserves the associated $\pm$ products, in that

$$f^*(\varphi \pm \varphi^*)f = (\theta \pm \theta^*) : P \to P^*,$$

which can also be expressed as

$$\langle f(x), f(y)\rangle_\varphi = \langle x,y\rangle_\theta \in A \quad (x,y \in P).$$

The composite of $\pm$ form morphisms

$$(f,\chi) : (P,\theta) \to (Q,\varphi), \quad (g,\nu) : (Q,\varphi) \to (R,\psi)$$

is defined by

$$(g,\nu)(f,\chi) = (gf, \chi + f^*\nu f) : (P,\theta) \to (R,\psi).$$

We thus have a category of $\pm$ forms over A (or rather two categories, one for each choice of sign). In general, we shall be interested in $\pm$ forms up to equivalence only, noting that

(i) a morphism of $\pm$ forms

$$(f,\chi)\colon (P,\theta) \to (Q,\varphi)$$

is an equivalence precisely when $f \in \operatorname{Hom}_A(P,Q)$ is an isomorphism;

(ii) for any $\pm$ form (P,θ),

$$(1,\chi)\colon (P,\theta) \to (P,\theta+\chi\mp\chi^*)$$

is an equivalence of $\pm$ forms for every $\mp$ form (P,χ).

The category has a direct sum operation

$$(P,\theta)\oplus(Q,\varphi) = (P\oplus Q,\ \theta\oplus\varphi)$$

with $(0,0)$ as zero.

A $\pm$ form (P,θ) is *non-singular* if the associated $\pm$ product

$$\theta\pm\theta^*\colon P \to P^*$$

is an A-module isomorphism. Non-singularity is a $\pm$ form equivalence invariant.

Given a non-singular $\pm$ form (Q,φ) and a direct summand L of Q, define the *annihilator* of L in (Q,φ),

$$L^\perp = \{x \in Q \mid \langle L,x\rangle_\varphi = \{0\} \subseteq A\},$$

a submodule of Q isomorphic to $(Q/L)^*$ by

$$L^\perp \to (Q/L)^*;\quad x \mapsto (L+y \mapsto \langle x,y\rangle_\varphi).$$

For any direct complement L_1 to L in Q, we have

$$Q = L\oplus L_1 = (\varphi\pm\varphi^*)^{-1}(Q^*) = (\varphi\pm\varphi^*)^{-1}(L^*\oplus L_1^*) = L_1^\perp\oplus L^\perp,$$

so that $L^\perp$ is a direct summand of Q as well. Call L *self-orthogonal* if

$$L \subseteq L^\perp$$

(or, equivalently, $\langle L,L\rangle_\varphi = 0$) in which case L is a direct summand of $L^\perp$ (with direct complement $L_1\cap L^\perp$). The restriction of φ to L,

$$\varphi\colon L \to L^*;\quad g \mapsto (h \mapsto \varphi(g)(h)),$$

is hermitian, the *hessian* $\mp$ product on L in (Q,φ).

A *sublagrangian* of a non-singular $\pm$ form (Q,φ) is a self-orthogonal direct summand L of Q such that the hessian $\mp$ product on L is even. A $\mp$ form (L,λ) such that

$$\varphi = \lambda\mp\lambda^*\colon L \to L^*$$

is a *hessian* $\mp$ form on L in (Q,φ).

An equivalence of non-singular $\pm$ forms

$$(f,\chi)\colon (P,\theta)\to(Q,\varphi)$$

sends a sublagrangian L of (P,θ), with hessian $\mp$ form (L,λ), to the sublagrangian $f(L)$ of (Q,φ), with hessian $\mp$ form $(f(L),\,(f^{-1})^*(\lambda+\chi)f^{-1})$. Thus sublagrangians are $\pm$ form equivalence invariant.

A sublagrangian L of a non-singular $\pm$ form (Q,φ) such that

$$L^{\perp}=L$$

is a *lagrangian* of (Q,φ). These, too, are $\pm$ form equivalence invariant.

Lagrangians are maximal sublagrangians, in the sense that if a sublagrangian M contains a lagrangian L, then $L=M$, as

$$L\subseteq M\subseteq M^{\perp}\subseteq L^{\perp}=L.$$

Given a f.g. projective A-module P, define the *hamiltonian* $\pm$ *form* on P,

$$H_{\pm}(P)=\left(P\oplus P^*,\begin{pmatrix}0&1\\0&0\end{pmatrix}\colon P\oplus P^*\to P^*\oplus P\right).$$

The associated $\pm$ product of $H_{\pm}(P)$ is the isomorphism

$$\begin{pmatrix}0&1\\\pm1&0\end{pmatrix}\colon P\oplus P^*\to P^*\oplus P,$$

corresponding to the sesquilinear function

$$\langle\,\rangle\colon (P\oplus P^*)\times(P\oplus P^*)\to A;\quad ((x,g),\,(y,h))\mapsto g(y)\pm\overline{h(x)},$$

so that $H_{\pm}(P)$ is non-singular.

A $\pm$ form is *trivial* if it is equivalent to a hamiltonian $\pm$ form.

THEOREM 1.1. *A* $\pm$ *form is trivial if and only if it admits a lagrangian.*

Proof. For any f.g. projective P, P is a lagrangian of $H_{\pm}(P)$.

Conversely, let (Q,φ) be a non-singular $\pm$ form with lagrangian L. Choosing a direct complement L_1 to L in Q, express φ as

$$\varphi=\begin{pmatrix}\lambda\mp\lambda^*&\gamma\\\delta&\theta\end{pmatrix}\colon L\oplus L_1\to L^*\oplus L_1^*,$$

for some hessian $\mp$ form (L,λ). The associated $\pm$ product

$$\varphi\pm\varphi^*=\begin{pmatrix}0&\gamma\pm\delta^*\\\delta\pm\gamma^*&\theta\pm\theta^*\end{pmatrix}\colon L\oplus L_1\to L^*\oplus L_1^*$$

is an isomorphism. Thus

$$\gamma\pm\delta^*\colon L_1\to L^*$$

is onto, and as L is lagrangian

$$L^{\perp}=L\oplus\ker(\gamma\pm\delta^*\colon L_1\to L^*)=L,$$

it is one-to-one as well, and so an isomorphism.

Define

$$\alpha = (\gamma \pm \delta^*)^{-1} \colon L^* \to L_1, \quad \chi = \begin{pmatrix} \lambda & 0 \\ \delta & 0 \end{pmatrix} \colon L \oplus L_1 \to L^* \oplus L_1^*,$$

$$f = \begin{pmatrix} 1 & -\alpha^*\theta\alpha \\ 0 & \alpha \end{pmatrix} \colon L \oplus L^* \to L \oplus L_1 = Q.$$

Then

$$(f, f^*\chi f) \colon H_\pm(L) \to (Q, \varphi)$$

is an equivalence of $\pm$ forms, and so (Q, φ) is trivial.

Given the inclusion $j \colon L \to Q$ of a lagrangian of a non-singular $\pm$ form (Q, φ), we have a morphism of $\pm$ forms

$$(j, \lambda) \colon (L, 0) \to (Q, \varphi),$$

which can be extended to an equivalence of $\pm$ forms

$$(f, \chi) \colon H_\pm(L) \to (Q, \varphi)$$

by the proof of Theorem 1.1. (This is the usual statement of Witt's theorem in the classical theory of quadratic forms.)

COROLLARY 1.2. *A f.g. projective A-module L is a sublagrangian of $H_\pm(L) \oplus (P, \theta)$, for any non-singular $\pm$ form (P, θ).*

Conversely, the inclusion of a sublagrangian

$$(j, \lambda) \colon (L, 0) \to (Q, \varphi)$$

may be extended to an equivalence of $\pm$ forms

$$(f, \chi) \colon (L^\perp/L, \hat{\varphi}) \oplus H_\pm(L) \to (Q, \varphi),$$

where $(L^\perp/L, \hat{\varphi})$ is the non-singular $\pm$ form to which $\varphi \colon Q \to Q^*$ restricts on a direct complement to L in $L^\perp$.

Proof. The first part is obvious.

Given a sublagrangian L in (Q, φ), choose a direct complement L_1 to $L^\perp$ in Q. Then $\varphi \colon Q \to Q^*$ has an expression as

$$\varphi = \begin{pmatrix} \hat{\varphi} & \alpha & \beta \\ \mp\alpha^* & \lambda \mp \lambda^* & \gamma \\ \mp\beta^* & \delta & \theta \end{pmatrix} \colon (L^\perp/L) \oplus L \oplus L_1 \to (L^\perp/L)^* \oplus L^* \oplus L_1^*.$$

Adding the appropriate even $\mp$ product to φ, it is clear that we may set $\alpha = 0$, $\beta = 0$. Now L is a lagrangian of

$$\left(L \oplus L_1, \begin{pmatrix} \lambda \mp \lambda^* & \gamma \\ \delta & \theta \end{pmatrix}\right).$$

Applying Theorem 1.1, the desired equivalence follows. (It should be noted that a different choice of direct complement to L in $L^\perp$ gives an equivalent $\pm$ form $(L^\perp/L, \hat{\varphi})$.)

A *subhamiltonian complement* to a sublagrangian L of (Q,φ) is a sublagrangian M such that

$$Q = L \oplus M^\perp \text{ (or, equivalently, } Q = L^\perp \oplus M\text{)}.$$

For example, L, L^* are subhamiltonian complements in $H_\pm(L) \oplus (P,\theta)$ for any f.g. projective L and non-singular $\pm$ form (P,θ). Form equivalences preserve subhamiltonian complements so that, by Corollary 1.2, every sublagrangian has a subhamiltonian complement.

Given subhamiltonian complements L, M in a non-singular $\pm$ form (Q,φ), we can identify M with L^* via the isomorphism

$$M \to L^*; \quad x \mapsto (y \mapsto \langle x,y\rangle_\varphi).$$

Then $\varphi\colon Q \to Q^*$ can be expressed as

$$\varphi = \begin{pmatrix} \hat{\varphi} & \alpha & \beta \\ \mp\alpha^* & \lambda\mp\lambda^* & \gamma \\ \mp\beta^* & \delta & \lambda_1\mp\lambda_1^* \end{pmatrix} : (L^\perp/L)\oplus L\oplus L^* \to (L^\perp/L)^*\oplus L^*\oplus L$$

with $\gamma \pm \delta^* = 1\colon L^* \to L^*$.

The subhamiltonian complements of lagrangians are also lagrangians, in which case they are called *hamiltonian complements*.

Given a lagrangian L in a non-singular $\pm$ form (Q,φ) we shall in general identify with L^* any one hamiltonian complement to L, but having chosen one such, reserve the notation L^* for it alone. A choice of hamiltonian complement to L is given by a morphism of $\pm$ forms

$$(j_1,\lambda_1)\colon (L^*,0) \to (Q,\varphi)$$

such that

$$\langle j_1(g),x\rangle_\varphi = g(x) \in A \quad (x \in L, g \in L^*).$$

There is one hamiltonian complement to L for every even $\mp$ product on L^*; this is proved in

LEMMA 1.3. *The hamiltonian complements to P^* in $H_\pm(P)$ are the* graphs

$$\Gamma_{(P,\theta)} = \{(x,(\theta\mp\theta^*)(x)) \in P\oplus P^* \mid x \in P\}$$

of $\mp$ forms (P,θ).

Proof. The direct complements to P^* in $P\oplus P^*$ are just the graphs

$$\Gamma_h = \{(x,h(x)) \in P\oplus P^* \mid x \in P\}$$

of morphisms $h \in \mathrm{Hom}_A(P, P^*)$. Such a graph is self-orthogonal if and only if h is a hermitian $\mp$ product, with h as hessian $\mp$ product (up to isomorphism).

The next result, corresponding to Theorem 3 in [5], is used by Wall to help justify the sort of definition of quadratic form adopted above.

LEMMA 1.4. *The* diagonal *of a non-singular* $\pm$ *form* (Q,φ)

$$\Delta_{(Q,\varphi)} = \{(x,x) \in Q\oplus Q \,|\, x \in Q\}$$

is a lagrangian of $(Q\oplus Q,\ \varphi\oplus -\varphi)$, *with hamiltonian complements*

$$\Delta^*_{(Q^*,\psi)} = \{(\psi y,\ \mp\psi^*(y)) \in Q\oplus Q \,|\, y \in Q^*\}$$

classified by $\pm$ *forms* (Q^*,ψ) *such that*

$$\psi \pm \psi^* = (\varphi \pm \varphi^*)^{-1} \colon Q^* \to Q$$

with ψ *differing from* $(\varphi\pm\varphi^*)^{-1}\varphi(\varphi\pm\varphi^*)^{-1}$ *by an even* $\mp$ *product.*

In particular, the diagonal $\Delta_{(Q,\varphi)}$ of a trivial $\pm$ form (Q,φ) is a hamiltonian complement in $(Q\oplus Q,\ \varphi\oplus -\varphi)$ to $F\oplus F^*$, for any hamiltonian complements F, F^* in (Q,φ).

2. Formations

A $\pm$ *formation* (over A), $(Q,\varphi;\ F,G)$, is a triple consisting of

(i) a trivial $\pm$ form (Q,φ),

(ii) a lagrangian F of (Q,φ),

(iii) a sublagrangian G of (Q,φ).

An *equivalence* of $\pm$ formations

$$(h,\nu)\colon (Q,\varphi;\ F,G) \to (Q',\varphi';\ F',G')$$

is an equivalence of $\pm$ forms

$$(h,\nu)\colon (Q,\varphi) \to (Q',\varphi')$$

which takes F to F' and G to G'.

We thus have a category of $\pm$ formations (with every morphism an equivalence). A direct sum operation $\oplus$ is defined by

$$(Q,\varphi;\ F,G)\oplus(Q',\varphi';\ F',G') = (Q\oplus Q',\varphi\oplus\varphi';\ F\oplus F',G\oplus G'),$$

with $(0,0;\ 0,0)$ as zero.

A $\pm$ formation $(Q,\varphi;\ F,G)$ is *non-singular* if G is a lagrangian. Non-singularity is $\pm$ formation equivalence invariant.

For any f.g. projective A-module P, define the *hamiltonian* $\pm$ *formation* on P, $(H_\pm(P);\ P,P^*)$, clearly non-singular.

A $\pm$ formation is *trivial* if it is equivalent to a hamiltonian $\pm$ formation.

LEMMA 2.1. *A $\pm$ formation $(Q,\varphi;\, F,G)$ is trivial if and only if it is non-singular and F,G are hamiltonian complements in (Q,φ).*

Proof. Given hamiltonian complements F,G in a trivial $\pm$ form (Q,φ), express $\varphi\colon Q\to Q^*$ as

$$\varphi=\begin{pmatrix}\lambda\mp\lambda^* & \gamma\\ \delta & \mu\mp\mu^*\end{pmatrix}\colon F\oplus G\to F^*\oplus G^*.$$

Then

$$\left(\begin{pmatrix}1 & 0\\ 0 & (\gamma\pm\delta^*)^{-1}\end{pmatrix},\begin{pmatrix}\lambda & \pm\delta^*(\gamma\pm\delta^*)^{-1}\\ 0 & \mu\end{pmatrix}\right)\colon (H_\pm(F);\, F,F^*)\to(Q,\varphi;\, F,G)$$

is an equivalence of $\pm$ formations.

For any $\mp$ form (P,θ) define the *graph* $\pm$ formation on (P,θ), $(H_\pm(P);\, P,\Gamma_{(P,\theta)})$, where $\Gamma_{(P,\theta)}$ is as in Lemma 1.3.

A $\pm$ formation is *elementary* if it is equivalent to a graph $\pm$ formation.

LEMMA 2.2. *A $\pm$ formation $(Q,\varphi;\, F,G)$ is elementary if and only if it is non-singular and F,G share a hamiltonian complement in (Q,φ).*

Proof. Let H be a hamiltonian complement in (Q,φ) to lagrangians F,G. By Lemma 2.1, there exists an equivalence

$$(h,\nu)\colon (H_\pm(F);\, F,F^*)\to(Q,\varphi;\, F,H).$$

By Lemma 1.3, (h,ν) must send some $\Gamma_{(F,\theta)}$ to G, so that

$$(h,\nu)\colon (H_\pm(F);\, F,\Gamma_{(F,\theta)})\to(Q,\varphi;\, F,G)$$

is also an equivalence of $\pm$ formations.

Lemmas 2.1, 2.2 are the special cases $P=0$, $L=0$ of

THEOREM 2.3. *A $\pm$ formation $(Q,\varphi;\, F,G)$ is equivalent to the direct sum*

$$(H_\pm(P);\, P,\Gamma_{(P,\theta)})\oplus(H_\pm(L);\, L,L^*)$$

of an elementary and a trivial $\pm$ formation if and only if it is non-singular and F has a hamiltonian complement F^* in (Q,φ) such that the projection *on F along F^*,*

$$\pi\colon Q=F\oplus F^*\xrightarrow{(1\,0)}F,$$

sends G onto a direct summand P of F.

The roles played by F and G may be reversed.

Proof. For any $\mp$ form (P,θ) and f.g. projective L, $P^*\oplus L^*$ is a hamiltonian complement to $P\oplus L$ in $H_\pm(P\oplus L)$ such that the projection on $P\oplus L$ along $P^*\oplus L^*$ sends $\Gamma_{(P,\theta)}\oplus L$ onto P.

Conversely, let $(Q, \varphi;\, F, G)$ be a non-singular $\pm$ formation, and F^* a hamiltonian complement to F such that the projection on F along F^*

$$\pi\colon Q = F \oplus F^* \xrightarrow{(1\,0)} F,$$

sends G onto a direct summand P of F, with $F = P \oplus L$, say. Dualizing, we have a direct sum decomposition of the hamiltonian complement to F,

$$F^* = P^* \oplus L^* \subseteq Q$$

and

$$L^\perp = F \oplus P^*, \quad P^\perp = F \oplus L^*$$

with

$$P^* = F^* \cap L^\perp, \quad L^* = F^* \cap P^\perp.$$

Hence

$$\langle P \oplus P^*,\, L \oplus L^* \rangle_\varphi = \{0\} \subseteq A$$

and there exists an equivalence of $\pm$ forms

$$(1, \chi)\colon (Q, \varphi) \to H_\pm(P) \oplus H_\pm(L)$$

for some $\mp$ form (Q, χ).

Now

$$G \subseteq \pi(G) \oplus (1-\pi)(G) \subseteq P \oplus P^* \oplus L^*,$$

so that

$$\langle G, L^* \rangle_\varphi = \langle P \oplus P^* \oplus L^*, L^* \rangle_\varphi = \{0\} \subseteq A$$

and

$$L^* \subseteq G^\perp = G.$$

Denoting $(P \oplus P^*) \cap G$ by M, it follows that

$$G = M \oplus L^*$$

and so

$$(1, \chi)\colon (Q, \varphi;\, F, G) \to (H_\pm(P);\, P, M) \oplus (H_\pm(L);\, L, L^*)$$

is an equivalence of $\pm$ formations.

As π sends M onto $\pi(G) = P$, the projection on P along P^*,

$$(1\,0)\colon P \oplus P^* \to P,$$

a restriction of π, does the same.

Thus M is a hamiltonian complement to P^* in $H_\pm(P)$, necessarily the graph $\Gamma_{(P,\theta)}$ of a $\mp$ form (P, θ), by Lemma 1.3.

Symmetry with respect to F and G follows from that of Lemmas 2.1, 2.2.

A *stable equivalence* of $\pm$ formations

$$[h, \nu]\colon (Q, \varphi;\, F, G) \to (Q', \varphi';\, F', G')$$

is an equivalence of $\pm$ formations

$$(h, \nu)\colon (Q, \varphi;\, F, G) \oplus (H_\pm(P);\, P, P^*) \to (Q', \varphi';\, F', G') \oplus (H_\pm(P');\, P', P^*)$$

defined for some f.g. projective P, P'. In general, we shall be interested in $\pm$ formations up to stable equivalence only.

Note that a $\pm$ formation which is stably equivalent to a trivial $\pm$ formation is itself trivial, by Lemma 2.1.

3. U-theory

Let I be an abelian monoid, and J a submonoid.

Call $i_1, i_2 \in I$ *J-stably equivalent*, $i_1 \sim_J i_2$, if

$$i_1 \oplus j_1 = i_2 \oplus j_2 \in I \quad \text{for some } j_1, j_2 \in J,$$

where $\oplus$ denotes the composition law in I.

The quotient monoid $I/\sim_J$ may be denoted by $I/\bar{J}$, because it depends only on the *stabilization* of J in I, the submonoid

$$\bar{J} = \{i \in I \,|\, i \sim_J 0\}$$

of I, containing J.

Note that $I/\bar{J}$ is an abelian group if and only if, for every $i \in I$, there exists $i' \in I$ such that $i \oplus i' \in J$.

THEOREM 3.1. *For $n \pmod 4$ let $X_n(A)$ be*

$$\text{the abelian monoid of } \begin{cases}\text{equivalence}\\ \text{stable equivalence}\end{cases} \text{classes of } \begin{cases}\pm \text{ forms}\\ \pm \text{formations}\end{cases} \text{over } A,$$

under the direct sum $\oplus$, where

$$n = \begin{cases}2i\\ 2i+1\end{cases} \quad \text{and} \quad \pm = (-)^i.$$

The morphisms

$$\partial\colon X_n(A) \to X_{n-1}(A); \quad \begin{cases}(P,\theta) \mapsto (H_{\mp}(P); P, \Gamma_{(P,\theta)}) & (n = 2i),\\ (Q,\varphi; F, G) \mapsto (G^{\perp}/G, \hat{\varphi}) & (n = 2i+1)\end{cases}$$

are well defined, with $\partial^2 = 0$.

The quotient monoids

$$U_n(A) = \ker(\partial\colon X_n(A) \to X_{n-1}(A))/\overline{\operatorname{im}(\partial\colon X_{n+1}(A) \to X_n(A))}$$

are abelian groups.

Proof. (i) *n even.*

By Lemma 2.1,

$$\begin{aligned}
&\ker(\partial\colon X_{2i}(A) \to X_{2i-1}(A))\\
&\quad = \{(P,\theta) \in X_{2i}(A) \mid (H_{\mp}(P); P, \Gamma_{(P,\theta)}) \text{ trivial}\}\\
&\quad = \{(P,\theta) \in X_{2i}(A) \mid P \oplus \Gamma_{(P,\theta)} = P \oplus P^*\}\\
&\quad = \{(P,\theta) \in X_{2i}(A) \mid (P,\theta) \text{ non-singular}\}.
\end{aligned}$$

By Corollary 1.2,

$$\begin{aligned}
&\mathrm{im}(\partial\colon X_{2i+1}(A) \to X_{2i}(A))\\
&\quad = \{(G^{\perp}/G, \hat{\varphi}) \in X_{2i}(A) \mid (Q, \varphi;\, F, G) \in X_{2i+1}(A)\}\\
&\quad = \{(P, \theta) \in X_{2i}(A) \mid (P, \theta) \oplus H_{\pm}(G) = H_{\pm}(F) \text{ for some f.g.}\\
&\qquad\qquad \text{projective } F, G\}\\
&\quad = \{\overline{(P, \theta) \in X_{2i}(A) \mid (P, \theta) \text{ trivial}}\} \subseteq \ker(\partial\colon X_{2i}(A) \to X_{2i-1}(A)).
\end{aligned}$$

By Theorem 1.1 and Lemma 1.4, for every non-singular $\pm$ form (P, θ)

$$(P, \theta) \oplus (P, -\theta) = 0 \in U_{2i}(A),$$

giving inverses for $U_{2i}(A)$.

(ii) *n odd.*

Here

$$\begin{aligned}
&\ker(\partial\colon X_{2i+1}(A) \to X_{2i}(A))\\
&\quad = \{(Q, \varphi;\, F, G) \in X_{2i+1}(A) \mid (G^{\perp}/G, \hat{\varphi}) = 0 \in X_{2i}(A)\}\\
&\quad = \{(Q, \varphi;\, F, G) \in X_{2i+1}(A) \mid (Q, \varphi;\, F, G) \text{ non-singular}\}
\end{aligned}$$

and by Lemma 2.2

$$\begin{aligned}
&\mathrm{im}(\partial\colon X_{2i+2}(A) \to X_{2i+1}(A))\\
&\quad = \{(H_{\pm}(P);\, P, \Gamma_{(P,\theta)}) \in X_{2i+1}(A) \mid (P, \theta) \in X_{2i+2}(A)\}\\
&\quad = \{(Q, \varphi;\, F, G) \in X_{2i+1}(A) \mid (Q, \varphi;\, F, G) \text{ elementary}\}\\
&\qquad\qquad \subseteq \ker(\partial\colon X_{2i+1}(A) \to X_{2i}(A)).
\end{aligned}$$

For every non-singular $\pm$ formation $(Q, \varphi;\, F, G)$,

$$(Q, \varphi;\, F, G) \oplus (Q, -\varphi;\, F^*, G^*) = 0 \in U_{2i+1}(A)$$

as the diagonal $\Delta_{(Q,\varphi)}$ is a hamiltonian complement in $(Q \oplus Q, \varphi \oplus -\varphi)$ to $F \oplus F^*$ and $G \oplus G^*$ for any hamiltonian complements F^*, G^* to F, G in (Q, φ), by Lemma 1.4; this gives inverses for $U_{2i+1}(A)$.

EXAMPLE 3.2. For the ground ring $\mathbf{Z}[\pi]$ of Example 0.1

$$U_n(\mathbf{Z}[\pi]) = L_n^A(\pi),$$

the surgery obstruction group in the category A of § 17D in [**6**], of Poincaré complexes up to homotopy.

The construction of the groups $U_*(\mathrm{A})$ is similar to that of the groups $\tilde{K}_0(A)$, $\tilde{K}_1(A)$ of algebraic K-theory.

The *projective class group of* A, $\tilde{K}_0(A)$, is the abelian group of isomorphism classes $[P]$ of f.g. projective A-modules P modulo the stably f.g. free ones, under the direct sum $\oplus$. Similarly, $U_{2i}(A)$ is the abelian group of equivalence classes of non-singular $\pm$ forms over A, modulo the stably trivial ones.

The *Whitehead torsion group of A*,

$$\tilde{K}_1(A) = GL(A)/\{E(A), -1\},$$

is the quotient of the general linear group $GL(A)$ of A by -1 and $E(A)$, the subgroup generated by the elementary matrices, those with 1 along the diagonal and at most one other non-zero entry. Whitehead's lemma states that

$$E(A) = [GL(A), GL(A)],$$

the commutator subgroup of $GL(A)$. This allows $\tilde{K}_1(A)$ to be considered as the abelian group of isomorphism classes of triples (Q,f,g) consisting of a f.g. free A-module Q and bases $f = (f_1, f_2, \ldots, f_m)$, $g = (g_1, g_2, \ldots, g_m)$, under the direct sum

$$(Q,f,g) \oplus (Q',f',g') = (Q \oplus Q', f \oplus f', g \oplus g'),$$

modulo the elementary triples

(i) $(Q, (f_1, \ldots, f_m), (f_1, \ldots, f_{j-1}, f_j + af_k, f_{j+1}, \ldots, f_m))$
$(1 \leqslant j,\ k \leqslant m,\ j \neq k,\ a \in A)$,

(ii) $(Q,f,g) \oplus (Q,g,h) \oplus (Q,h,f)$,

(iii) $(Q,f,-f)$.

Similarly, $U_{2i+1}(A)$ is the abelian group of stable equivalence classes of non-singular $\pm$ formations modulo the elementary ones. Although it is not possible to identify elements of $U_{2i+1}(A)$ as the 'torsions' of self-equivalences of a trivial $\pm$ form, they have the formal properties of such. In particular, we have the following sum formula.

LEMMA 3.3. $(Q,\varphi;\ F,G) \oplus (Q,\varphi;\ G,H) = (Q,\varphi;\ F,H) \in U_{2i+1}(A)$.

Proof. Consider first the special case when F and G have a common hamiltonian complement in (Q,φ), L say. Then

$$(Q,\varphi;\ F,G) = 0 \in U_{2i+1}(A) \text{ and}$$

$$(Q,\varphi;\ F,H) = -(Q,-\varphi;\ L,H^*) = (Q,\varphi;\ G,H) \in U_{2i+1}(A).$$

For general $(Q,\varphi;\ F,G) \in U_{2i+1}(A)$,

$$\begin{aligned}
&(Q,\varphi;\ F,G) \oplus (Q,\varphi;\ G,H)\\
&\quad = (Q,\varphi;\ F,G) \oplus (Q \oplus Q, \varphi \oplus -\varphi;\ G \oplus G^*, H \oplus G^*)\\
&\quad = (Q,\varphi;\ F,G) \oplus (Q \oplus Q, \varphi \oplus -\varphi;\ F \oplus F^*, H \oplus G^*)\\
&\qquad\qquad \text{(by special case and Lemma 1.4)}\\
&\quad = (Q \oplus Q, \varphi \oplus -\varphi;\ F \oplus F^*, G \oplus G^*) \oplus (Q,\varphi;\ F,H)\\
&\quad = (Q,\varphi;\ F,H) \in U_{2i+1}(A).
\end{aligned}$$

4. V-theory

V-theory is the analogue of U-theory obtained by considering $\pm$ forms and $\pm$ formations on stably f.g. free A-modules rather than f.g. projective ones. All the U-theory done above has an obvious V-theory version. In particular, we can define abelian monoids $Y_n(A)$ for $n \pmod 4$, and morphisms $\partial\colon Y_n(A) \to Y_{n-1}(A)$, exactly as for $X_n(A)$, to obtain quotient groups

$$V_n(A) = \ker(\partial\colon Y_n(A) \to Y_{n-1}(A))/\overline{\operatorname{im}(\partial\colon Y_{n+1}(A) \to Y_n(A))}.$$

EXAMPLE 4.1. For the ground ring $\mathbf{Z}[\pi]$ of Example 0.1

$$V_n(\mathbf{Z}[\pi]) = L_n^{\mathrm{B}}(\pi),$$

the surgery obstruction group in the category B of § 17D in [6], of finite Poincaré complexes up to homotopy.

The odd-dimensional groups $V_{2i+1}(A)$ will now be identified as stable unitary groups.

Define for $m \geqslant 1$ the unitary group $\mathscr{U}_\pm(A, m)$ of self-equivalences of $H_\pm(mA)$, where mA is the free A-module on m generators.

The function

$$\pi_m\colon \mathscr{U}_\pm(A,m) \to V_{2i+1}(A);\quad (f,\chi) \mapsto (H_\pm(mA);\, mA, f(mA))$$

is a morphism of groups; for given (f,χ), $(g,\nu) \in \mathscr{U}_\pm(A,m)$,

$$\begin{aligned}\pi_m((g,\nu)(f,\chi)) &= (H_\pm(mA);\, mA, gf(mA))\\ &= (H_\pm(mA);\, mA, g(mA)) \oplus (H_\pm(mA);\, g(mA), gf(mA))\\ &\qquad\text{(by the sum formula of Lemma 3.3 for } V\text{-theory)}\\ &= (H_\pm(mA);\, mA, g(mA)) \oplus (H_\pm(mA);\, mA, f(mA))\\ &= \pi_m(g,\nu) \oplus \pi_m(f,\chi) \in V_{2i+1}(A).\end{aligned}$$

Defining inclusions

$$\mathscr{U}_\pm(A,m) \to \mathscr{U}_\pm(A,m+1);\quad (f,\chi) \mapsto (f,\chi) \oplus (1,0),$$

let

$$\mathscr{U}_\pm(A) = \bigcup_{m=1}^{\infty} \mathscr{U}_\pm(A,m),$$

with the obvious multiplicative group structure. There is induced a morphism of groups

$$\pi\colon \mathscr{U}_\pm(A) \to V_{2i+1}(A)$$

which agrees with π_m on each $\mathscr{U}_\pm(A,m)$.

Denote the kernel of π by $\mathscr{H}_\pm(A)$, calling its elements the *hamiltonian transformations.*

THEOREM 4.2. *The morphism*

$$\pi\colon \mathscr{U}_{\pm}(A) \to V_{2i+1}(A)$$

is onto, inducing an isomorphism

$$\mathscr{U}_{\pm}(A)/\mathscr{H}_{\pm}(A) \cong V_{2i+1}(A)$$

of abelian groups.

$\mathscr{H}_{\pm}(A)$ contains the commutator subgroup $[\mathscr{U}_{\pm}(A), \mathscr{U}_{\pm}(A)]$ of $\mathscr{U}_{\pm}(A)$ and it is generated by the elementary hamiltonian transformations:

(i) $\left(\begin{pmatrix} 1 & 0 \\ \theta \mp \theta^* & 1 \end{pmatrix}, \begin{pmatrix} \theta & 0 \\ 0 & 0 \end{pmatrix}\right) \in \mathscr{U}_{\pm}(A, m)$ *for any* $\mp$ *form* (mA, θ),

(ii) $\left(\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{*-1} \end{pmatrix}, 0\right) \in \mathscr{U}_{\pm}(A, m)$ *for any automorphism* $\alpha\colon mA \to mA$,

(iii) $\sigma \oplus \sigma \oplus \ldots \oplus \sigma \in \mathscr{U}_{\pm}(A, m)$ *involving m copies of*

$$\sigma = \left(\begin{pmatrix} 0 & \pm\gamma^{-1} \\ \gamma & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}\right) \in \mathscr{U}_{\pm}(A, 1),$$

where

$$\gamma\colon A \to A^*; \quad a \mapsto (b \mapsto b\bar{a}).$$

Proof. It is sufficient to verify that π is onto.

Every $(Q, \varphi; F, G) \in V_{2i+1}(A)$ may be represented by a non-singular $\pm$ formation $(H_{\pm}(mA); mA, G)$ with G free, of dimension

$$\dim_A G = \tfrac{1}{2}\dim_A(G \oplus G^*) = \tfrac{1}{2}\dim_A(mA \oplus (mA)^*) = m.$$

Choosing a hamiltonian complement G^* to G in $H_{\pm}(mA)$, and an isomorphism $h \in \mathrm{Hom}_A(mA, G)$, note that the isomorphism

$$\begin{pmatrix} h & 0 \\ 0 & h^{*-1} \end{pmatrix}\colon mA \oplus (mA)^* \to G \oplus G^*$$

defines a self-equivalence of $H_{\pm}(mA)$ which takes mA to G. Thus

$$\pi\begin{pmatrix} h & 0 \\ 0 & h^{*-1} \end{pmatrix} = (Q, \varphi; F, G) \in V_{2i+1}(A).$$

V-theory differs from U-theory in 2-torsion only.

THEOREM 4.3. *There is an exact sequence*

$$\begin{array}{l} \ldots \to \Sigma_{+}(A) \xrightarrow{\text{(I)}} V_{2i+1}(A) \xrightarrow{\text{(II)}} U_{2i+1}(A) \xrightarrow{\text{(III)}} \Sigma_{-}(A) \\ \qquad \xrightarrow{\text{(IV)}} V_{2i}(A) \xrightarrow{\text{(V)}} U_{2i}(A) \xrightarrow{\text{(VI)}} \Sigma_{+}(A) \to V_{2i-1}(A) \to \ldots \end{array}$$

of abelian groups and morphisms, defined for $i \pmod 2$.

The groups $\Sigma_{\pm}(A)$ *are defined by*

$$\Sigma_{\pm}(A) = \{[P] \in \tilde{K}_0(A) \mid [P^*] = \pm [P]\}/\{[Q] \pm [Q^*] \mid [Q] \in \tilde{K}_0(A)\}$$

and are of exponent 2.

The morphisms

$$V_n(A) \to U_n(A)$$

are induced by the obvious inclusions of $Y_n(A)$ *in* $X_n(A)$. *The others are given by:*

$$U_{2i}(A) \to \Sigma_+(A);\quad (P,\theta) \mapsto [P],$$

$$U_{2i+1}(A) \to \Sigma_-(A);\quad (Q,\varphi;\, F, G) \mapsto [G]-[F^*],$$

$$\Sigma_-(A) \to V_{2i}(A);\quad [P] \mapsto H_{\pm}(P),$$

$$\Sigma_+(A) \to V_{2i-1}(A);\quad [P] \mapsto (H_{\mp}(P\oplus -P);\ \ P\oplus -P,\ P\oplus -P^*)$$

for any representative $-P$ *of* $-[P] \in \tilde{K}_0(A)$.

Proof. It is easy to verify that the given morphisms are well defined, except perhaps $\Sigma_+(A) \to V_{2i-1}(A)$. This sends $[P\oplus P^*]$ $(= 0 \in \Sigma_+(A))$ to $(H_{\mp}(P\oplus P^*\oplus -P\oplus -P^*);\ P\oplus P^*\oplus -P\oplus -P^*,\ P\oplus P^*\oplus(-P\oplus -P^*)^*)$ which vanishes in $V_{2i-1}(A)$ because

$$\{(0,0,x,y,z,w,\ \pm y,x) \in P\oplus P^*\oplus -P\oplus -P^*\oplus(P\oplus P^*\oplus -P\oplus -P^*)^* \mid x \in -P,\ y \in -P^*,\ z \in P,\ w \in P^*\}$$

is a common hamiltonian complement.

Further, it is not difficult to see that the composition of successive morphisms is 0, except perhaps at (III) and (VI):

at (III) note that every $(Q,\varphi;\, F,G) \in U_{2i+1}(A)$ has a representative $\pm$ formation with G free, so that

$$H_{\pm}(F) = (Q,\varphi) = H_{\pm}(G) = 0 \in V_{2i}(A);$$

at (VI) the composite $U_{2i}(A) \to V_{2i-1}(A)$ sends $(Q,\varphi) \in U_{2i}(A)$ to

$$\begin{aligned}(H_{\mp}(Q\oplus -Q);\, Q\oplus -Q,\, Q\oplus -Q^*)\\ = (H_{\mp}(Q\oplus -Q);\, Q\oplus -Q,\, Q^*\oplus -Q) \in V_{2i-1}(A)\end{aligned}$$

(by Lemma 3.3 for V-theory)

and $\Gamma_{(Q,\varphi)}\oplus -Q^*$ is a hamiltonian complement to $Q\oplus -Q$ and $Q^*\oplus -Q$ in $H_{\mp}(Q\oplus -Q)$, as (Q,φ) is non-singular. We now verify exactness at each point of the sequence.

(I) $\Sigma_+(A) \to V_{2i+1}(A) \to U_{2i+1}(A)$.

Every $(Q,\varphi;\, F,G) \in \ker(V_{2i+1}(A) \to U_{2i+1}(A))$ can be represented as

$$(H_{\pm}(P\oplus L);\, P\oplus L,\ \Gamma_{(P,\theta)}\oplus L^*)$$

for some $\mp$ form (P,θ) and f.g. projective L, such that $P\oplus L$, $P\oplus L^*$ are free.

Applying the sum formula of Lemma 3.3 for V-theory

$$\begin{aligned}(Q,\varphi;\, F,G) &= (H_\pm(P\oplus L);\, P\oplus L,\, \Gamma_{(P,\theta)}\oplus L^*)\\ &= (H_\pm(P\oplus L);\, P\oplus L,\, P^*\oplus L) \in \operatorname{im}(\Sigma_+(A)\to V_{2i+1}(A)).\end{aligned}$$

(II) $V_{2i+1}(A)\to U_{2i+1}(A)\to \Sigma_-(A)$.

Let $(Q,\varphi;\, F,G)\in \ker(U_{2i+1}(A)\to\Sigma_-(A))$, so that

$$[G]-[F^*] = [P^*]-[P]\in \tilde{K}_0(A)$$

for some f.g. projective P.

Denote by M a f.g. projective A-module such that

$$[M] = -[G^*\oplus P^*] = -[F\oplus P]\in \tilde{K}_0(A).$$

Then

$$\begin{aligned}(Q,\varphi;\, F,G) &= ((Q,\varphi)\oplus H_\pm(P\oplus M);\, F\oplus P\oplus M,\, G\oplus P\oplus M^*)\\ &\qquad\in \operatorname{im}(V_{2i+1}(A)\to U_{2i+1}(A)).\end{aligned}$$

(III) $U_{2i+1}(A)\to\Sigma_-(A)\to V_{2i}(A)$.

If $[P]\in\ker(\Sigma_-(A)\to V_{2i}(A))$, it may be assumed that $H_\pm(P)$ has a free lagrangian, L say. Then $U_{2i+1}(A)\to\Sigma_-(A)$ sends $(H_\pm(P);\, L,P)$ to $[P]\in\Sigma_-(A)$.

(IV) $\Sigma_-(A)\to V_{2i}(A)\to U_{2i}(A)$.

If $(Q,\varphi)\in\ker(V_{2i}(A)\to U_{2i}(A))$, it may be assumed that (Q,φ) has a (projective) lagrangian, P say. Then

$$[P]+[P^*] = [Q] = 0\in\tilde{K}_0(A)$$

and $(Q,\varphi) = H_\pm(P)\in\operatorname{im}(\Sigma_-(A)\to V_{2i}(A))$.

(V) $V_{2i}(A)\to U_{2i}(A)\to\Sigma_+(A)$.

If $(Q,\varphi)\in\ker(U_{2i}(A)\to\Sigma_+(A))$, then

$$[Q] = [P]+[P^*]\in\tilde{K}_0(A)$$

for some f.g. projective P and

$$(Q,\varphi) = (Q,\varphi)\oplus H_\pm(-P)\in\operatorname{im}(V_{2i}(A)\to U_{2i}(A)).$$

(VI) $U_{2i}(A)\to\Sigma_+(A)\to V_{2i-1}(A)$.

Given $[P]\in\ker(\Sigma_+(A)\to V_{2i-1}(A))$ it may be assumed that, up to equivalence of $\mp$ formations,

$$\begin{aligned}(H_\mp(P\oplus -P);\, P\oplus -P,\, P\oplus -P^*)\oplus(H_\mp(L);\, L,\, \Gamma_{(L,\lambda)})\\ = (H_\mp(M);\, M,\Gamma_{(M,\mu)})\end{aligned}$$

for some $\pm$ forms (L,λ), (M,μ) defined on f.g. free L, M. Now

$$(Q,\varphi;\, F,G) = (H_{\mp}(M);\, M, \Gamma_{(M,\mu)})$$

is an elementary $\mp$ formation, with $H = M^*$ a hamiltonian complement to both F and G in (Q,φ). Moreover, $F^* = P^* \oplus -P^* \oplus L^*$ is a hamiltonian complement to $F = P \oplus -P \oplus L$ in (Q,φ) such that the projection on F along F^*,

$$\pi : Q = F \oplus F^* \xrightarrow{(1\;0)} F$$

sends $G = P \oplus -P^* \oplus \Gamma_{(L,\lambda)}$ onto the direct summand $\pi(G) = P \oplus L$ of F. Thus $F \oplus F^*$ is a hamiltonian complement to $\Delta_{(Q,\varphi)}$ in $(Q \oplus Q,\, \varphi \oplus -\varphi)$ such that the projection on $\Delta_{(Q,\varphi)}$ along $F \oplus F^*$,

$$Q \oplus Q \to \Delta_{(Q,\varphi)};\; (x,y) \mapsto (\pi(x) + (1-\pi)(y), \pi(x) + (1-\pi)(y)),$$

sends $G \oplus H$ onto the submodule

$$N = \{(x,x) \in \Delta_{(Q,\varphi)} \mid x \in \pi(G) \oplus (1-\pi)(H)\} \subseteq \Delta_{(Q,\varphi)}.$$

As H is a hamiltonian complement to F in (Q,φ), $(1-\pi)(H) = F^*$, and N is a direct summand of $\Delta_{(Q,\varphi)}$ isomorphic to $P \oplus L \oplus F^*$, with direct complement isomorphic to $-P$.

Applying Theorem 2.3, we have, up to equivalence,

$$(Q \oplus Q,\, \varphi \oplus -\varphi,\, \Delta_{(Q,\varphi)},\, G \oplus H) = (H_{\mp}(N);\, N,\, \Gamma_{(N,\nu)}) \oplus (H_{\mp}(-P);\, -P,\, -P^*)$$

for some $\pm$ form (N,ν), which must be non-singular, as

$$(Q \oplus Q,\, \varphi \oplus -\varphi;\, \Delta_{(Q,\varphi)},\, G \oplus H)$$

is a trivial $\mp$ formation (H being a hamiltonian complement to G in (Q,φ)). Thus

$$[N] = [P \oplus L \oplus F^*] = [P] \in \operatorname{im}(U_{2i}(A) \to \Sigma_{+}(A)).$$

5. W-theory

A *based* A-module, Q, is a f.g. free A-module Q together with a base $b = (b_1, b_2, \ldots, b_m)$. The *dual* based A-module Q^* is defined, with base $b^* = (b_1^*, b_2^*, \ldots, b_m^*)$, where

$$b_j^*(b_k) = \begin{cases} 1 & \text{if } j = k, \\ 0 & \text{if } j \neq k, \end{cases}$$

and Q^{**} is identified with Q for based A-modules as well.

The matrix representation in $\mathrm{GL}(A)$ of an isomorphism $f \in \operatorname{Hom}_A(P,Q)$ of based A-modules P,Q defines the *torsion* of f, $\tau(f) \in \tilde{K}_1(A)$. The isomorphism is *simple* if

$$\tau(f) = 0 \in \tilde{K}_1(A).$$

A *based* $\pm$ *form* (over A), (Q,φ), is a $\pm$ form defined on a based A-module Q. An equivalence of based $\pm$ forms

$$(f,\chi)\colon (P,\theta)\to(Q,\varphi)$$

is *simple* if the isomorphism

$$f\colon P\to Q$$

is simple.

W-theory deals with the simple equivalence properties of based $\pm$ forms, just as U-theory considers the equivalence of $\pm$ forms, and V-theory that of $\pm$ forms on stably f.g. free modules.

Define the *hamiltonian based* $\pm$ *form* on a based A-module P to be $H_{\pm}(P)$ with base $b\oplus b^*$ if b is the given A-base of P. A based $\pm$ form is *trivial* if it is simply equivalent to a based hamiltonian one.

Let L be a free lagrangian of a trivial $\pm$ form (Q,φ) (not based yet). A base b of L, together with dual b^* on a hamiltonian complement L^* define a *hamiltonian base* of (Q,φ).

A different choice of hamiltonian complement L^* alters this hamiltonian base by a simple automorphism

$$\begin{pmatrix} 1 & \theta\mp\theta^* \\ 0 & 1 \end{pmatrix}\colon L\oplus L^*\to L\oplus L^*$$

of Q, for some $\mp$ form (L^*,θ), by Lemma 1.3. Thus every base of L can be extended to a hamiltonian base of (Q,φ) uniquely up to simple equivalence.

A *based lagrangian* of a based $\pm$ form (Q,φ) is a free lagrangian L together with a base b for L such that the given base of (Q,φ) differs from a hamiltonian base extending b by a simple equivalence.

By analogy with Theorem 1.1 we have

THEOREM 5.1. *A based* $\pm$ *form is trivial if and only if it admits a based lagrangian.*

Define the *torsion* of a non-singular based $\pm$ form (Q,φ) over A,

$$\tau(Q,\varphi)=\tau((\varphi\pm\varphi^*)\colon Q\to Q^*)\in\tilde{K}_1(A).$$

Torsion is a simple $\pm$ form equivalence invariant, and as based hamiltonian $\pm$ forms have zero torsion, so do all trivial based $\pm$ forms.

A *based sublagrangian* of a based $\pm$ form (Q,φ) is a free sublagrangian L of (Q,φ) such that $L^{\perp}/L$ is free, together with bases L, $L^{\perp}/L$ such that the *subhamiltonian base* these determine on (Q,φ) agrees with the given base up to simple equivalence.

By analogy with Corollary 1.2 we have

COROLLARY 5.2. *The inclusion of a based sublagrangian L in a based $\pm$ form (Q, φ) can be extended to a simple equivalence*

$$(L^{\perp}/L, \hat{\varphi}) \oplus H_{\pm}(L) \to (Q, \varphi)$$

and $\tau(Q, \varphi) = \tau(L^{\perp}/L, \hat{\varphi}) \in \tilde{K}_1(A)$.

A *based hamiltonian complement* to a based lagrangian F of a non-singular based $\pm$ form (Q, φ) is a based lagrangian G such that

$$G \to F^*; \quad x \mapsto (y \mapsto \langle x, y \rangle_{\varphi})$$

is a simple isomorphism, in which case G may be identified with F^*. Lemma 1.3 has based version

LEMMA 5.3. *Let P be a based A-module, with base b. The based hamiltonian complements to P^* in $H_{\pm}(P)$ are the graphs $\Gamma_{(P,\theta)}$ of $\mp$ forms (P, θ), based by $(b, (\theta \mp \theta^*)b) \subseteq P \oplus P^*$ up to simple changes.*

Lemma 1.4 has based version

LEMMA 5.4. *Let (Q, φ) be a non-singular based $\pm$ form, with Q based by b. If*

$$\tau(Q, \varphi) = 0 \in \tilde{K}_1(A)$$

then the diagonal $\Delta_{(Q,\varphi)}$, based by $(b, b) \subseteq Q \oplus Q$, is a based lagrangian of $(Q \oplus Q,\ \varphi \oplus -\varphi)$, with based hamiltonian complements $\Delta^*_{(Q^*,\psi)}$ (defined as in Lemma 1.4), based by $(\psi b,\ \mp \psi^* b) \subseteq Q \oplus Q$ up to simple changes.

In particular, the based diagonal $\Delta_{(Q,\varphi)}$ of a trivial based $\pm$ form (Q, φ) is a based hamiltonian complement in $(Q \oplus Q,\ \varphi \oplus -\varphi)$ to $F \oplus F^*$, for any based hamiltonian complements F, F^* in (Q, φ).

A *based $\pm$ formation* $(Q, \varphi; F, G)$ is a triple consisting of
(i) a trivial based $\pm$ form (Q, φ),
(ii) a based lagrangian F of (Q, φ),
(iii) a based sublagrangian G of (Q, φ).
An equivalence of based $\pm$ formations

$$(h, \nu)\colon (Q, \varphi;\ F, G) \to (Q', \varphi';\ F', G')$$

is *simple* if it is defined by a simple equivalence of $\pm$ forms

$$(h, \nu)\colon (Q, \varphi) \to (Q', \varphi')$$

which restricts to simple isomorphisms

$$F \to F',\ G \to G',\ G^{\perp}/G \to G'^{\perp}/G'.$$

The definitions and propositions of §§ 2, 3 have obvious based analogues. In particular, we have

THEOREM 5.5. *For n (mod 4), let $Z_n(A)$ be*

$$\text{the abelian monoid of} \begin{cases} \text{simple equivalence} \\ \text{stable simple equivalence} \end{cases} \text{classes of} \begin{cases} \text{based} \pm \text{forms} \\ \text{based} \pm \text{formations} \end{cases} \text{over } A,$$

under the direct sum $\oplus$, where $\pm$ means $(-)^i$ if $n = \begin{cases} 2i, \\ 2i+1. \end{cases}$
The monoid morphisms

$$\partial\colon Z_n(A) \to Z_{n-1}(A); \quad \begin{cases} (P,\theta) \mapsto (H_{\mp}(P); P, \Gamma_{(P,\theta)}) & (n = 2i) \\ (Q,\varphi; F,G) \mapsto (G^{\perp}/G, \hat{\varphi}) & (n = 2i+1) \end{cases}$$

are well defined, with $\partial^2 = 0$. The quotient monoids

$$W_n(A) = \ker(\partial\colon Z_n(A) \to Z_{n-1}(A))/\overline{\operatorname{im}(\partial\colon Z_{n+1}(A) \to Z_n(A))}$$

are groups.

As in the proof of Theorem 3.1, we can identify:

$\ker(\partial\colon Z_{2i}(A) \to Z_{2i-1}(A)) = \{(P,\theta) \in Z_{2i}(A) \mid (P,\theta)$ non-singular and $\tau(P,\theta) = 0 \in \tilde{K}_1(A)\}$,

$\operatorname{im}(\partial\colon Z_{2i+1}(A) \to Z_{2i}(A)) = \overline{\{(Q,\varphi) \in Z_{2i}(A) \mid (Q,\varphi) \text{ trivial}\}}$,

$\ker(\partial\colon Z_{2i+1}(A) \to Z_{2i}(A)) = \{(Q,\varphi; F,G) \in Z_{2i+1}(A) \mid (Q,\varphi; F,G)$ non-singular$\}$,

$\operatorname{im}(\partial\colon Z_{2i+2}(A) \to Z_{2i+1}(A)) = \{(Q,\varphi; F,G) \in Z_{2i+1}(A) \mid (Q,\varphi; F,G)$ elementary$\}$.

EXAMPLE 5.6. For the ground ring $\mathbf{Z}[\pi]$ of Example 0.1,

$$W_n(\mathbf{Z}[\pi]) = L_n^{\mathrm{E}}(\pi),$$

the surgery obstruction group in the category E of § 17D of [**6**], of simple Poincaré complexes up to simple homotopy.

(This is slightly bowdlerized: in the geometrical case, simple equivalence of bases is measured not in $\tilde{K}_1(\mathbf{Z}[\pi])$, but in the *Whitehead group of* π,

$$\mathrm{Wh}(\pi) = \tilde{K}_1(\mathbf{Z}[\pi])/\operatorname{im}(\pi \to U(\mathbf{Z}[\pi]) \to \tilde{K}_1(\mathbf{Z}[\pi])),$$

where $U(\mathbf{Z}[\pi])$ is the multiplicative group of units of $\mathbf{Z}[\pi]$, regarded as a subgroup of $GL(\mathbf{Z}[\pi])$ in the obvious way.)

The odd-dimensional groups $W_{2i+1}(A)$ will now be identified as stable special unitary groups, by analogy with Theorem 4.2 for V-theory.

Define, for $m \geqslant 1$, the special unitary group $\mathscr{SU}_{\pm}(A,m)$ of simple self-equivalences of $H_{\pm}(mA)$, where mA is the based A-module on m generators.

The functions

$$\pi'_m\colon \mathscr{S}\mathscr{U}_\pm(A,m) \to W_{2i+1}(A);\quad (f,\chi) \mapsto (H_\pm(mA);\, mA, f(mA))$$

are group morphisms (by Lemma 3.3 for W-theory).

Defining inclusions

$$\mathscr{S}\mathscr{U}_\pm(A,m) \to \mathscr{S}\mathscr{U}_\pm(A,m+1);\quad (f,\chi) \mapsto (f,\chi)\oplus(1,0),$$

there is induced a group morphism

$$\pi'\colon \mathscr{S}\mathscr{U}_\pm(A) = \bigcup_{m=1}^{\infty} \mathscr{S}\mathscr{U}_\pm(A,m) \to W_{2i+1}(A),$$

agreeing with π'_m on each $\mathscr{S}\mathscr{U}_\pm(A,m)$.

Denote the kernel of π' by $\mathscr{S}\mathscr{H}_\pm(A)$, calling its elements the *special hamiltonian transformations*.

THEOREM 5.6. *The morphism*

$$\pi'\colon \mathscr{S}\mathscr{U}_\pm(A) \to W_{2i+1}(A)$$

is onto, inducing an isomorphism

$$\mathscr{S}\mathscr{U}_\pm(A)/\mathscr{S}\mathscr{H}_\pm(A) \cong W_{2i+1}(A)$$

of abelian groups.

Moreover, $\mathscr{S}\mathscr{H}_\pm(A)$ is generated by the elementary special hamiltonian transformations:

(i) $\left(\begin{pmatrix} 1 & 0 \\ \theta \mp \theta^* & 1 \end{pmatrix}, \begin{pmatrix} \theta & 0 \\ 0 & 0 \end{pmatrix}\right) \in \mathscr{S}\mathscr{U}_\pm(A,m)$ *for any based $\mp$ form (mA,θ),*

(ii) $\left(\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{*-1} \end{pmatrix}, 0\right) \in \mathscr{S}\mathscr{U}_\pm(A,m)$ *for any simple automorphism $\alpha\colon mA \to mA$,*

(iii) $\sigma\oplus\sigma\oplus\ldots\oplus\sigma \in \mathscr{S}\mathscr{U}_\pm(A,m)$ *involving m copies of*

$$\sigma = \left(\begin{pmatrix} 0 & \pm\gamma^{-1} \\ \gamma & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}\right) \in \mathscr{S}\mathscr{U}_\pm(A,1),$$

where

$$\gamma\colon A \to A^*;\quad a \mapsto (b \mapsto b\bar{a}).$$

Theorem 6.3 in [**6**] states:

> *$\mathscr{S}\mathscr{H}_\pm(A)$ contains the commutator subgroup $[\mathscr{S}\mathscr{U}_\pm(A), \mathscr{S}\mathscr{U}_\pm(A)]$; the quotient $\mathscr{S}\mathscr{H}_\pm(A)/[\mathscr{S}\mathscr{U}_\pm(A), \mathscr{S}\mathscr{U}_\pm(A)]$ is generated by σ, so has order at most 2.*

Using Theorem 5.6, it is possible to prove this without the complicated identity of Lemma 6.2 in [**6**] (but see also [**7**]).

W-theory differs from V-theory in 2-torsion only.

THEOREM 5.7. *There is an exact sequence*

$$\ldots \to \Omega_{(-)^{n+1}}(A) \to W_n(A) \to V_n(A) \to \Omega_{(-)^n}(A) \to \ldots$$

of abelian groups and morphisms, defined for n (mod 4).

The groups

$$\Omega_{(-)^n}(A) = \{\tau \in \tilde{K}_1(A) \mid \tau^* = (-)^n\tau \in \tilde{K}_1(A)\}/\{\omega + (-)^n\omega^* \mid \omega \in \tilde{K}_1(A)\}$$

are of exponent 2.

The morphisms $W_n(A) \to V_n(A)$ *are induced by the monoid morphisms* $Z_n(A) \to Y_n(A)$ *which forget bases. The others are given by:*

$V_{2i}(A) \to \Omega_+(A)$; $(P,\theta) \mapsto \tau(P,\theta)$, *for any base of P, assumed free,*

$V_{2i+1}(A) \to \Omega_-(A)$; $(Q,\varphi;\, F, G) \mapsto \tau(\pi^{-1}(Q,\varphi;\, F,G))$, π *as in Theorem* 4.2,

$\Omega_-(A) \to W_{2i}(A)$; $\tau(\alpha\colon P \to P) \mapsto \left(P \oplus P^*, \begin{pmatrix} 0 & 0 \\ \alpha & 0 \end{pmatrix}\right)$,

$\Omega_+(A) \to W_{2i-1}(A)$; $\tau(\alpha\colon P \to P) \mapsto (H_\mp(P);\, P, \alpha(P))$.

Proof. This is by analogy with that of Theorem 4.3, with torsions of automorphisms in $\Omega_\pm(A)$ replacing projective classes in $\Sigma_\pm(A)$.

This is the exact sequence of Proposition 4.1 in [**4**] (in the geometrically realizable case, as in Example 0.1, with π finitely presented). A geometrical interpretation of this and of the exact sequence of Theorem 4.3 may be found in § 17D of [**6**]. Theorem 3 of [**7**] establishes this exact sequence in the wider context of L-theories which lie between V- and W-theory (cf. Theorems 2.3 and 3.3 in [**3**]).

6. Functoriality

All our constructions are functorial on the ground ring A.

Let

$$f\colon A \to B$$

be a morphism of ground rings (preserving 1 and the involutions). Give B a (B, A)-bimodule structure by

$$B \times B \times A \to B; \quad (b, x, a) \mapsto b.x.f(a).$$

Given a f.g. projective left A-module P, let fP denote the f.g. projective left B-module $B \otimes_A P$, identifying $(fP)^*$ with $f(P^*)$. A morphism $\varphi \in \mathrm{Hom}_A(P, Q)$ induces

$$f\varphi = (1 \otimes \varphi\colon B \otimes_A P \to B \otimes_A Q) \in \mathrm{Hom}_B(fP, fQ).$$

Given a

$$\left\{\begin{array}{l}\pm \text{ form } (Q,\varphi)\\ \pm \text{ formation } (Q,\varphi;\,F,G)\end{array}\right. \text{ over } A, \text{ there is defined a}$$

$$\left\{\begin{array}{l}\pm \text{ form } f(Q,\varphi)=(fQ,f\varphi)\\ \pm \text{ formation } f(Q,\varphi;\,F,G)=(f(Q,\varphi);\,fF,fG)\end{array}\right. \text{ over } B,$$

and similarly for morphisms.

The induced monoid morphisms

$$f\colon X_n(A)\to X_n(B)$$

are such that the squares

$$\begin{tikzcd} X_n(A) \arrow[r, "f"] \arrow[d, "\partial"'] & X_n(B) \arrow[d, "\partial"] \\ X_{n-1}(A) \arrow[r, "f"] & X_{n-1}(B) \end{tikzcd}$$

commute, inducing morphisms

$$f\colon U_n(A)\to U_n(B).$$

Similar procedure applies to V-, W-theories.

The isomorphisms of Theorems 4.2, 5.6 and the exact sequences of Theorems 4.3, 5.7 are natural on A.

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