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III.—On Knots, with a Census for Order Ten. By C. N. Little, Lincoln, Neb.

1. Gauss in 1833* called attention to the importance of the study of the ways in which cords might be linked. Nothing, however, appears to have been written upon the subject until in 1847 Listing published his *Vorstudien zur Topologie.*† In this he briefly but in a masterly way touched upon the subject of knots, established some of the fundamental propositions, and proposed a notation which, as slightly modified by Prof. Tait, furnishes the point of view for the present paper. In a communication‡ to Prof. Tait in 1877, Listing points out the fragmentary character of his own contribution to the subject, and says that the type-symbol used by him is "nichts weiter als ein derartiger Fingerzeig."

It is to Professor Tait, however, that the greater part of our present knowledge of the subject is due. He, independently of Listing, obtained the fundamental propositions and found the knots and their forms for orders from three to seven inclusive.§

In 1884 Kirkman‖ published the forms of knots of orders eight and nine, and immediately Tait, making use of Kirkman's work, extended his census of knots to these orders.¶

2. Professor Tait has shown that any closed plane curve of n crossings divides its plane into $n+2$ compartments; that these compartments are in two groups; that, at the crossings, like compartments are vertically opposite. We shall call these compartments of the plane *parts.* A part is represented by the number equal to the number of double points on its perimeter. The sum of the numbers representing the parts of either group is $2n$, that is, these numbers together constitute a partition of $2n$. The partitions for the two groups together make up Listing's *type-symbol.* As it can lead to

* "Eine Hauptaufgabe aus dem Grenzgebiet der *Geometria Situs* und der *Geometria Magnitudinis* wird die sein, die Umschlingungen zweier geschlossener oder unendlicher Linien zu zählen."—Werke. Göttingen. 1867, vol. v, p. 605.

† Göttingen Studien, 1847. I have been able to see only Tait's apparently full abstract in Proc. Roy. Soc. Edin., vol. ix, pp. 306–309.

‡ Proc. Roy. Soc. Edin., vol. ix, p. 316.

§ On Knots, Trans. Roy. Soc. Edin., xxviii, 145–191, 1876–77

‖ Trans. Roy. Soc. Edin., xxxii, 281–309.

¶ Trans. Roy. Soc. Edin., xxxii, 327–342 no ambiguity we shall also call the number representing a compartment a *part*, and either group of compartments a *partition*.

Since every closed plane curve of n crossings, having double points only, may be read alternately over and under at the crossings, every such curve which gives parts, none greater than n or less than 2, may be taken as a projection of a reduced knot of n crossings. We call such curves *knot-forms* or briefly, *forms*, and regard two forms as distinct if they do not have the same parts similarly arranged.

The first part of the problem is to find all the different knot-forms of any order.

Since the same knot may be transformed so as to be projected into more than one knot-form, the second part of the problem is, from the complete series of knot-forms of n crossings to find all the different n-fold knots. Knots exist for which the law of over and under does not hold; these are not considered in the present paper.

3. It is unnecessary to do more than allude to two very distinct and very ingenious methods devised and used, the one by Tait and the other by Kirkman, for the solution of the first part of this problem. We may perhaps infer from Professor Tait's opinion* that "a full study of 10-fold and 11-fold knottiness seems to be relegated to the somewhat distant future," that they were more laborious than proves to be necessary.

4. A third method, based on Listing's type-symbol, is thus described by Professor Tait at page 168 of his first memoir.

"Write all the partitions of $2n$, in which no one shall be greater than n and no one less than 2. Join each of these sets of numbers into a group, so that each number has as many lines terminating in it as it contains units. Then join the middle points of these lines (which must not intersect one another), by a continuous line which *intersects* itself at these middle points and there only. When this can be done we have the projection of a *knot*. When more continuous lines than one are required we have the projection of a linkage."

On page 160 of the same memoir, he says, speaking of this method: "But we can never be quite sure that we get *all* possible results by a semi-tentative process of this kind. And we have to try an immensely greater number of partitions than there are knots, as the great majority give links of greater or less complexity."

It seems possible however, with the help of some simple theorems to make the "Partition Method" exhaustive, and wholly to do away with the drawing of links.

* In 1884. Trans. R. S. E., xxxii, 328

5. An inspection of form Aa of Plate I will make clear some terms already introduced and others that we shall now require. Regarding the curve Aa as alone in a plane, it divides it into twelve *parts*, two 9-gons, two 3-gons and eight 2-gons. The external 3-gon or *amplexum* differs in no way from the other parts. Of these twelve parts, two 9-gons and one 2-gon form one group—the *leading* partition; the two 3-gons and seven 2-gons form the other group—the *subordinate* partition. The terms leading and subordinate are relative merely, but that partition will be taken as leading which has the smaller number of parts. The *type-symbol* for Aa is $\left\{\begin{matrix} 9^2 2 \\ 3^2 2^7 \end{matrix}\right.$.

6. The double points common to the perimeters of two parts of the same partition will be called *bonds* of those parts, and the parts are said to be bound by these bonds. It is well known that a type-symbol does not determine a form. For this, it is necessary to know the numbers of bonds between the several parts of either partition, together with the arrangement of these parts.

In general the parts of a given partition may be bound in more than one way giving forms that may be projections of either links or knots. Each set of numbers of bonds of the several parts of the given partition is a *clutch* of that partition.

The *class* p of a partition is the number of parts in it. The class of a form is the class of its leading partition. The *order* of any partition is equal to n and is the same as the order of all knot-forms derivable from it. The *deficiency* $\varkappa$ of a partition is its order minus its greatest part.

7. Let the parts of $2n$ be A, B, C, . . . P arranged in order of magnitude, and the numbers of bonds of each part be respectively $\alpha, \beta, \gamma, \ldots \pi$. Let the number of bonds common to any two parts as A and B be (AB). Then

$$\left.\begin{matrix} (AB)+(AC)+ \ldots\ldots\ldots (AP)=\alpha \\ (AB)+(BC)+ \ldots\ldots\ldots (BP)=\beta \\ (AC)+(BC)+ \ldots\ldots\ldots (CP)=\gamma \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ (AP)+(BP)+ \ldots\ldots\ldots (OP)=\pi \end{matrix}\right\}(a)$$

or n equations with $\frac{1}{2}n(n-1)$ unknown quantities which can have only positive integral values. The possible solutions of (a) will evidently give all the clutches for this partition.

8. I. Theorem.—If a part be solely bound to a second part, or if any q parts ($q \not> p-2$) be bound mutually in any way and all free bonds of these parts go to a single part, then this portion of the form constitutes a separate knot (unless there be linkage) and the string concerned in it may be drawn tight without affecting the remainder of the knot-form. Such knots are not considered as belonging to order n.

In particular a 2-gon so bound throws out from consideration a clutch.

9. II. Theorem.—No knot-form of the nth order has as leading partition one whose class exceeds $\varkappa+2$.

Adding equations (a) above, dividing by two, and subtracting the first and any other, say the second, we find

$$-(AB)+(CD)+(CE)+\ \ldots\ (CP)+\ \ldots\ (OP)=n-a-\beta,$$
$$=\varkappa-\beta.$$

Therefore, $(AB) \not< \beta-\varkappa$.

In a similar way

$$(AC) \not< \gamma-\varkappa$$
$$(AD) \not< \delta-\varkappa$$
$$\ldots\ldots$$
$$(AP) \not< \pi-\varkappa$$

Adding $(AB)+(AC)+\ \ldots\ (AP) \not< \beta+\gamma+\ \ldots\ \pi-(p-1)\varkappa$
$$\not< n+\varkappa-(p-1)\varkappa$$
$$\not< n-(p-2)\varkappa,$$

we have then the two conditions

$$\left.\begin{aligned}(AB)+(AC)+\ \ldots\ (AP) &\not< n-(p-2)\varkappa\\ &= n-\varkappa.\end{aligned}\right\}\ (b)$$

Now suppose, if possible, $p=\varkappa+3$

$$(AB)+(AC)+\ \ldots\ (AP)=n-\varkappa \qquad =n-\varkappa$$
$$\not< n-(\varkappa+1)\varkappa \quad \not< n-\varkappa-\varkappa^2.$$

To the minimum values of (AB), (AC), etc., (that is, to $\beta-\varkappa$, $\gamma-\varkappa$, . . .) must be added $\varkappa^2$ in all, and to no one more than $\varkappa-1$, by I. The $\varkappa+2$ smallest parts of $\varkappa$ square are evidently $(\varkappa-1)^4$ $(\varkappa-2)^{\kappa-2}$. By adding these $\varkappa+2$ parts in any way to the minimum values, a clutch will be given in which each of four parts will have a single bond not going to A, and each of $\varkappa-2$ parts will have two. A part of the latter kind cannot have its two free bonds carried to a second part of the same kind, by I. If two parts be joined by a single bond there will be left two free bonds. Ultimately it will be necessary to join two parts of the first kind to a combination of parts having but two free bonds, and I will apply. If any of the $\varkappa+2$ parts of $\varkappa^2$ be diminished by s then will s parts of $2n$ be added to those of the first kind, and however the free bonds may be arranged, ultimately the same result as before will be reached. Therefore p cannot equal $\kappa+3$ and still give knot-forms.

Much less can p be greater than $\kappa+3$.

10. A given clutch of a leading partition does not uniquely determine a form. The following proposition however holds.

III. Theorem.—All or none of the forms determined by any given clutch of a partition are knot-forms of the order considered.

For, all forms to be had from any clutch of a given partition may be obtained by taking all the possible different changes (consistent with the given clutch) of relative position of the various parts. But these can all be effected by successive interchanges of the connections of two parts, whether such connections are direct (by a single bond), by a 2-gon, by a 3-gon, or are more complicated. We may therefore confine the attention to a definite portion of the knot and keep the remainder fixed. Let A and B, (Fig. 1, Plate I), be two parts connected as shown. Two strings, or two parts of a single string, are involved. If there were more all but two would be closed. Let the ends of these strings, or parts of a single string, leave the portion of the form under consideration at a and c on the perimeter of A, and b and d on that of B. Cut at these points and call the ends a and a', b and b', c and c', d and d'; a, b, c and d remain fixed. Now revolve through 180° about the axis AB, and join the free ends.

Before the change there may be three cases. The strings may be $\left\{\begin{matrix} c\,d \\ a\,b \end{matrix}\right.$ $\left\{\begin{matrix} b\,d \\ a\,c \end{matrix}\right.$ or $\left\{\begin{matrix} b\,c \\ a\,d \end{matrix}\right.$ After the change a' is joined to c, and c' to a; b' to d, and d' to b.

$$\begin{array}{lcl}
\left\{\begin{matrix} c\,d \\ a\,b \end{matrix}\right. & \text{become} & \left\{\begin{matrix} c\,a'\,b'\,d \\ a\,c'\,d'\,b \end{matrix}\right. \\
\left\{\begin{matrix} b\,d \\ a\,c \end{matrix}\right. & \text{"} & \left\{\begin{matrix} b\,d'\,b'\,d \\ a\,c'\,a'\,c \end{matrix}\right. \\
\left\{\begin{matrix} b\,c \\ a\,d \end{matrix}\right. & \text{"} & \left\{\begin{matrix} b\,d'\,a'\,c \\ a\,c'\,b'\,d \end{matrix}\right.
\end{array}$$

Therefore coming up to this part of the knot on any string, we must leave on the same string before and after the change. If then the form was a knot before the change it will be one after, and if a link before it will be a link after.

11. *Coils.*—A succession of n 2-gons constitutes an *n-coil*, which may be open or closed. Since at the 3rd or $2n+1$st crossing of a coil the strings have the relative position of the first crossing, if the coil be closed by carrying around the ends to the beginning and

joining them so as to preserve the law of over and under one string will be formed. While if from the $2n$th crossing the strings are carried around, that string over at the 1st is under at the $2n$th and, on joining, there will be two strings.

Hence, as is well known, $\frac{(2n+1)^2}{(2)^{2n+1}}$ is always a knot, while $\frac{(2n)^2}{(2)^{2n}}$ is always a link.

12. For the purpose of distinguishing between clutches giving knots and those giving links, it follows from Theorem III that we may take the direct bonds between any two parts together, and these form open coils of the subordinate partition; and further it is evident from § 11 that any odd open coil (even number of bonds), may be dropped, and any even coil (odd number of bonds) may be replaced by a single bond. If the resulting clutch gives link, so would the original. If the resulting clutch be still too complex for easy recognition of its character, the clutch of the subordinate partition of the resulting form may perhaps be still farther reduced in the same way. If the clutches of lower orders were at hand they also could be used for settling the question.

13. We have the following theorems for throwing out clutches unproductive of knot forms.

IV. Theorem.—If a part be joined to other parts in every case by an even number of bonds, there is linkage. For, the string about this part is closed by Section 12.

V. Theorem.—If two parts are connected by two 2-gons (of the same partition with the parts) there is linking. For they may be put in succession by Theorem III. When this is done there is a 4-gon of the negative partition bound to two other parts in each case by two bonds, and IV applies.

VI. Theorem.—An odd part joined to one part by an odd number of bonds and to other parts in every case by an even number of bonds may be dropped; for, by Theorem III it becomes a loop with a single crossing, and this can have no effect on the question of linking.

In particular a 3-gon joined by one bond to one part and by two bonds to a second part may be dropped.

If two odd parts are joined by an odd number of bonds, and are joined to other parts in every case by an even number of bonds there is linkage.

In particular two 3-gons so joined throw out the clutch.

VII. Theorem.—If two 3-gons, C and D, are themselves joined directly and are joined to A and B in each case by a single bond there is linkage.

For, HCM (fig. 2, Plate I) under we will say at C is over at H and M. LCN is then over at C and under at L and N. LHN is over at L, under at H, and over at N. Its continuation is therefore NML which is over at N, under at M, and over at L.

14. Class III, Order n.—In this class we have

$$(AB)+(AC)+(BC)=n$$

and an unique solution of equations (a), § 9. Therefore $(BC)=\varkappa$, $(AB)=\beta-\varkappa$ and $(AC)=\gamma-\varkappa$. If two or three of these quantities be even, the clutch to which they belong will give a linkage, by Theorem IV. In other cases by § 12 there is a knot.

Suppose n to be odd and α even; then $\varkappa$ is odd, and β and γ are both odd, or both even. In the first case the clutch gives a link, in the second a knot.

Suppose n to be odd, and α odd; then $\varkappa$ is even, and of β and γ one must be odd and the other even. The clutch gives a link.

Suppose n to be even and α even; then $\varkappa$ is even and β, γ are both odd or both even. In the first case a knot, in the second a link.

Suppose n to be even and α odd; then $\varkappa$ is odd, and of β and γ one must be odd and one even, and there is a knot. This proves the following:

VIII. Theorem.—In odd orders only partitions of $2n$ into three even parts, give knots, while in even orders, only these partitions give links.

15. Class IV, Order n. Here

$$-(AB)+(CD)=\varkappa-\beta$$
$$-(AC)+(BD)=\varkappa-\gamma$$
$$-(AD)+(BC)=\varkappa-\delta$$
$$(AB)+(AC)+(AD)\begin{matrix}=n-\varkappa\\ \not< n-2\varkappa\end{matrix}$$

The minimum values of (AB), (AC), (AD), (BC), (BD), (CD), are respectively $\beta-\varkappa$, $\gamma-\varkappa$, $\delta-\varkappa$, 0, 0, 0. To get every clutch we must add in every possible way to the minimum values of (AB), (AC), (AD), all the partitions of $\varkappa$ into not more than three parts, none greater than $\varkappa-1$. But evidently we must increase the minimum values of any quantity of the second set (BC), (BD), (CD), by the same number that we increase the corresponding quantity of the first set. The following scheme which considers in detail every possible case, expresses clearly the propositions for determining whether clutches of partitions of this class furnish knots or links. Let e or o indicate whether a number be even or odd.

n,	κ	Partition.	Add $\kappa=$	$\beta-\kappa$. $\gamma-\kappa$. $\delta-\kappa$.	AB. AC. AD.	BC. BD. CD.	Form.	Props. or Sect. used.
e	e	e e e e	e e e	e e e	e e e	e e e	Link.	IV.
			e o o		e o o	o o e	"	§12, V.
		e e o o	e e e	e o o	e o o	e e e	"	IV.
			e o o		e e e	o o e	"	IV.
			o e o		o o e	o e o	Knot.	§12, VI §11.
	o	o o o o.	o o o	e e e	o o o	o o o	Link.	§12, VII.
			o e e		o e e	e e o	"	§12, VI.
		o o e e.	o o o	e o o	o e e	o o o	Knot.	§12, VI, 11.
			o e e		o o o	e e o	"	§12, VI, 11.
			e o e		e e o	e o e	Link.	IV.
o	e	o o e e	e e e	o e e	o e e	e e e	"	IV.
			e o o		o o o.	o o e	"	§12, V.
			o e o		e e o	o e o	Knot.	§12.
	e	o o o o	e e e	o o o	o o o	e e e	"	§12,
			e o o		o e e	o o e	"	§12.
	o	e e e e	o o o	o o o	e e e	o o o	Link.	IV.
			o e e		e o o	e e o	"	IV.
		e e o o	o o o	o e e	e o o	o o o	"	§12, V.
			o e e		e e e	e e o	"	IV.
			e o e		o o e	e o e	Knot.	§12.

The first line of this scheme says that when n is even and κ even, and $2n$ divided into four even parts, the minimum values of (AB), (AC), (AD), will be even, and that if a partition of κ into three even parts be added to $\beta-\kappa$, $\gamma-\kappa$, $\delta-\kappa$ the numbers constituting the clutch will be even, and the form a link by Prop. IV.

One of the propositions proved in this scheme is worthy of separate enunciation.

IX. Theorem.—In even orders partitions of $2n$ into four even or four odd parts give link-forms only. In odd orders partitions into four even parts give link-forms only.

16. X. Theorem.—In all orders partitions of six even parts give link-forms only.

For, an even number may be divided into five parts, of which four, two or none shall be odd. After the application of §12 the only parts to be found in the leading partition will be 4-gons and 2-gons. Every form under consideration will be reduced so as to have as leading partition one of the following, with a clutch of which every number is 1.

$$4^6,\quad 4^5 2,\quad 4^4 2^2,\quad 4^3 2^3,\quad 4^2 2^4,\quad 4 2^5,\quad 2^6.$$

In the lower orders $4^3 2^3$, $4^2 2^4$, $4 2^5$, and 2^6 have been found to give no knots.

If the form given by 4^6 be drawn it is found in any particular case to consist of four closed curves, and therefore by Theorem III is always a link-form. On drawing $4^4 2^2$ it also proves to be a linkage.

The partition $4^5 2$ with the given clutch can not exist in a plane. This happens in two cases in order 10. On drawing the forms in such cases additional crossings will be found to be necessary. It is therefore true that in all orders, partitions of six even parts give link-forms only.

A synthetic proof of this theorem is possible.

17. Instead of continuing the consideration of the general subject we shall now illustrate the method of determining the knot-forms of any order by a particular consideration of order 10.

The number of partitions of 20 into parts none greater than 10 or less than 2 is 107.* These, arranged in dictionary order, are given in full in Table I.

Of these the single partition of Class II gives a link, § 11; the thirty-seven marked II are cut out by Theorem II. In Class III, Theorem VIII throws out $8^2 4$ and 86^2; in Class IV, Theorem IX the eight partitions marked IX; and in Class X, Theorem X the three marked X. The partitions remaining in Classes III to VI inclusive are alone taken as leading partitions; for, all remaining partitions appear in every possible way as subordinate partitions (§ 2), and can, therefore, furnish no additional knot-forms.

We have then to tabulate the clutches of those partitions still remaining in Classes III to VI. We at once write down Tables II and III in which are omitted all clutches that are thrown out by Sections 14 and 15. In Class V, a table with headings as shown in Table IV is used. We take for illustration the first partition, $7^2 2^3$. Here $\varkappa=3$, and $\beta-\varkappa$, $\gamma-\varkappa$, $\delta-\varkappa$, $\varepsilon-\varkappa$ are 4, -1, -1, -1. In this class we must add in every possible way to the minimum values of (AB), (AC), (AD), (AE) the partitions of $2\varkappa$ into four parts, or fewer, none greater than $\varkappa-1$. The only partitions of 6 meeting these conditions are 2^3 and $2^2 1^2$, which may be added in seven different ways to 4, -1, -1, -1, giving the different clutches of the table. Thus adding 1, 2, 2, 1, to $\beta-\varkappa$, $\gamma-\varkappa$, $\delta-\varkappa$, $\varepsilon-\varkappa$ we have (AB), (AC), (AD), (AE), equal to 5, 1, 1, 0, respectively. Subtracting (AB) from β, etc.

$$\begin{aligned}\Sigma B &= (BC) + (BD) + (BE) = 2\\ \Sigma C &= (BC) + (CD) + (CE) = 1\\ \Sigma D &= (BD) + (CD) + (DE) = 1\\ \Sigma E &= (BE) + (CE) + (DE) = 2.\end{aligned}$$

These quantities are put in the columns headed ΣB, ΣC, ΣD, ΣE,

*See Tait, Trans. Roy. Soc. Edin., vol. xxxii, p. 342.

and all the possible solutions of this set of equations here as throughout the work are written down by inspection. The rest of this table is made out in the same way. In Table V we must, in every possible way, add the partitions of 3κ into four parts or fewer, none greater than $\kappa-1$. In the general case we add the partitions of $(p-3)\kappa$ into $p-1$ parts none greater than $\kappa-1$. In the complete Table IV there are 400 clutches, and in Table V 1000. It has not been thought necessary to publish more than a sample of either table.

18. Having completed the tables of clutches we next cast out in Tables IV and V all clutches unproductive of knot-forms. The theorems already established are sufficient for this purpose. The routine followed will be readily understood from considering its use in the samples given of Tables IV and V.

Partition $7^2 2^3$: in clutch (1) A and B are connected by two 2-gons, and V applies; (4) becomes (3) by interchanging D and E; (7) becomes (6) by interchanging C and D; and in (2) and (5) we have a 2-gon solely bound to a single part, and I applies.

Partition $6^2 42^2$: clutch (1) is thrown out by IV since a 6-gon B is joined to A by four bonds and to C by two; Theorem IV casts out (4) because of C, (14) because of C, (17) because of B, and (20) because of A; in (2), (3), (8), (9), (11), (12), (15) and (18) a 2-gon is solely joined to a single part, and I applies; in (5) B and C form a combination solely bound to A, and I applies; I throws out also (10) where D and E are bound together and to C, and (13) where the same parts are bound to B; interchanging D and E (7) becomes (6); interchanging A and B (16) becomes (7).

Partition $5^3 32$: I throws out (2), (3), (6), (8), (14), (16), (19), (20), (26), (27), (30), (31), (33), (37) and (38); by interchanges of parts

(5)≡(4),	(17)≡(5),	(24)≡(9),	(32)≡(11),
(7)≡(1),	(18)≡(3),	(25)≡(4),	(34)≡(12),
(13)≡(10),	(21)≡(10),	(28)≡(23),	(35)≡(1),
(15)≡(12),	(22)≡(21),	(29)≡(23),	and (36)≡(9).

In (4) D is dropped by VI, and B thus is changed to a 4-gon; then the application of § 12 leaves the two 3-gons A and C joined by two 2-gons, E and the still farther reduced B; V, therefore, shows that the clutch gives links only. In (9), drop C by VI and B becomes a 2-gon and A a 3-gon; the two 3-gons A and D are connected by the two 2-gons E and the reduced B; therefore the clutch gives links, by V. In (11) drop C by VI and the two 3-gons A and B are connected by the two 2-gons E and the reduced D; V applies.

In Table V, clutch No. (1) of 53^5 is thrown out by I. In (2) the two 3-gons E and F are joined together by a single bond and to A in each case by two bonds, and VI applies; or we might use I. Theorem VI throws out (4), since the two 3-gons B and F are joined to each other by a single bond and to other parts in each case by two bonds. In (5), drop F, by VI; B becomes a 2-gon joining D and E, and they are 3-gons connecting A and C. An obvious extension of Theorem VII throws out the clutch. Since in (7) B and C are joined together and to A, I applies.

This process is continued until all clutches which do not give knot-forms have been thrown out, as well as those clutches which are repetitions of clutches previously given. The samples given of Tables IV and V constitute about one-twentieth of the complete tables.

19. We are now ready to draw the forms which the remaining clutches furnish.

Definition.—The number of *Circular arrangements* of n things is the number of distinct ways in which n things can be arranged in a circle, the order whether direct or retrograde being of no consequence.

For illustration of the general method we consider in detail the productive clutches of partition $6^2 42^2$. In clutch (6) the two 6-gons are connected, by three bonds, by a 2 gon, and by a 4-gon which is connected with one 6-gon by two bonds and with the other by a 2-gon and one bond. There will be as many forms from this clutch as there are circular arrangements of a a a b (cd), where c and d may not be separated. These are

a a a b (cd)
a a a b (dc)
a a b a (cd)
a a b a (dc).

The four forms C^2c_1, C^2c_2, C^2c_3, C^2c_4, can at once be drawn. See Plate V, Knot LXIV.

In (19), the only other clutch of this partition that affords forms, a 6-gon is connected with a 4-gon directly, by a 2-gon, and by a 6-gon which is connected with the 4-gon by a single bond and by a 2-gon. There are two circular arrangements of a b (cd), namely a b (cd) and b a (cd); these furnish the two forms C^2d_1, C^2d_2. Knot XXX, Plate III. In practice the operations described in this section and in the preceding are performed simultaneously.

20. In Class VI every productive partition is used as a leading

partition. But since the subordinate partitions here belong to the same class, every form, with certain exceptions, will be found twice; this affords a check on the completeness of this part of the work. The exceptions are the *amphicheiral* knot-forms. These have the same partitions similarly connected both as leading and as subordinate partitions. These therefore appear but once.

In the partition 53^6, which was given as the shortest possible illustration of Table V, clutch (3) furnishes a single form D^2s_2, which already had been obtained under $54^2 32^2$; clutch (6) gives two which had been obtained, Du_1 under $643^2 2^2$ and D^2x_4 under $54^2 32^2$. Clutch (8) gives the only new knot-form from this partition, viz: the amphicheiral D^3o.

From the clutches of Classes III, IV, V and VI 364 ten-fold knot forms are obtained.

21. *The Derivation of Knots from Knot-Forms.*—Prof. Tait has not described the methods which he used in his derivation of the knots of lower orders from the knot-forms. In 1884 he says :* "the treatment to which I have subjected Kirkman's collection of forms, in order to group together mere varieties or transformations of one special form, is undoubtedly still more tentative in its nature; and thus, though I have grouped together many widely different forms, I cannot be *absolutely* certain that all those groups are essentially different from one another."

If a ten-fold knot be placed upon a plane in such a way as to have but ten crossings the eye will project it upon the plane in a form which will be found among the 364 above obtained. If the knot gives more than one form it will be possible to obtain any other of its forms by one or more turnings over of restricted portions of the knot while the remainder is held fixed. Now the string cannot issue from the portion of the knot that is turned at more than four points, for in that case the turning would introduce consecutive overs, and one or more additional crossings; the portion of the knot that is turned must therefore be wholly between two parts of the given knot form and in turning it we untwist two of the strings at one point and twist two at another, the result being simply to change the position of a single bond from one end of the connection to the other. The class of the form is therefore not changed, and all the forms of any knot belong to the same class.

22. Moreover in order 10 and lower orders all the forms coming from any clutch are obtained by changing the position of single

* Trans. Roy. Soc. Edin., vol. xxxii, p. 327.

bonds in the connections of pairs of parts. Therefore in these orders all forms from any clutch are forms of the same knot. The subordinate partitions of these forms are then to be examined and all forms added which are obtained from them by changing the positions of connections of their parts, retaining the given clutch of the subordinate partition. These forms in turn are treated in the same way, but it will usually happen that no new form of the knot is obtained and the complete determination of the knot in all of its forms is finished.

23. We take for illustration Knot I of Class IV, shewn on Plate I. Ba_1 becomes Ba_2 by twisting about a vertical axis the 2-gon connecting the 8-gon and 7-gon. The first crossing below is opened and the strings above are crossed, the rest of the knot remaining fixed. Twisting the 2-gon again Ba_3 is obtained and nothing new is gotten by further changes of the forms. Since the negative partition in every case consists of two parts joined by three symmetrical connections, which have only one circular arrangement, there are no other forms of Knot I.

24. The knots of Class III, order n, are unique; since three things have but one circular arrangement.

25. The knots of Classes III, IV, V will be found with their forms grouped together in Plates I–V. On Plates V–VII are figured the forms of Class VI grouped as they come from the clutches, except that no form is repeated. The knots of this class will be found in Table VI. Every knot-form is the projection of two knots, one of which is the perverted image of the other, and consequently each group of knot-forms belongs to two knots which are in general different. If a series of knot-forms contains any amphicheiral form then it will also contain the perversion of every form of the series not amphicheiral. The series consists of the forms of one knot and not of two.

26. In order 10 I find, counting a knot and its irreconcilable perversion as two:

Class.	Forms.	Knots.	Knots.
III,	6	12	6
IV,	25	30	15
V,	200	128	64
VI,	133	64	39
Totals,	364	234	124

In lower orders Professor Tait has found:

Orders.	Forms.	Knots.	Knots.
3	1	2	1
4	1	1	1
5	2	4	2
6	3	5	3
7	10	14	7
8	27	31	18
9	100	82	41

The fourth column contains the numbers as they are given by Tait, the perversions of knots not being counted.

In so long a labor as is involved in making such a census the opportunities for error are many. Any errors or omissions that may be found in the census are to be attributed to the writer rather than to the method, which is simple and direct.

Table I.—*Partitions of* 20.

Class II.	Cl. IV.—Contin'd	Cl. V.—Contin'd.	Class VI.	Cl. VII.—Conti'd
10^{2} Class III. II. { 1082 1073 1064 105^{2} } $9^{2}2$ 983 974 965 VIII. $8^{2}4$ 875 VIII. 86^{2} $7^{2}6$ Class IV. II. { 1062^{2} 10532 $104^{2}2$ 1043^{2} } II. { 972^{2} 9632 9542 953^{2} $94^{2}3$ }	IX. $8^{2}2^{2}$ 8732 IX. 8642 863^{2} $85^{2}2$ 8543 IX. 84^{3} $7^{2}42$ IX. $7^{2}3^{2}$ 7652 7643 IX. $75^{2}3$ 754^{2} IX. $6^{3}2$ $6^{2}53$ IX. $6^{2}4^{2}$ $65^{2}4$ IX. 5^{4} Class V. II. { 1042^{3} $103^{2}2^{2}$ } II. { 952^{3} 9432^{2} $93^{3}2$ }	II. { 862^{3} 8532^{2} $84^{2}2^{2}$ $843^{2}2$ 83^{4} } $7^{2}2^{3}$ 7632^{2} 7542^{2} $753^{2}2$ $74^{2}32$ 743^{3} $6^{2}42^{2}$ $6^{2}3^{2}2$ $65^{2}2^{2}$ 65432 653^{3} $64^{3}2$ $64^{2}3^{2}$ $5^{3}32$ $5^{2}4^{2}2$ $5^{2}43^{2}$ $54^{3}3$ 4^{5}	II. 102^{5} II. 932^{4} II. { 842^{4} $83^{2}2^{3}$ } II. { 752^{4} 7432^{3} $73^{3}2^{2}$ } X. $6^{2}2^{4}$ 6532^{3} X. $64^{2}2^{3}$ $643^{2}2^{2}$ $63^{4}2$ $5^{2}42^{3}$ $5^{2}3^{2}2^{2}$ $54^{2}32^{2}$ $543^{3}2$ 53^{5} X. $4^{4}2^{2}$ $4^{3}3^{2}2$ $4^{2}3^{4}$ Class VII. II. 82^{6} II. 732^{5}	II. { 642^{5} $63^{2}2^{4}$ } $5^{2}2^{5}$ 5432^{4} $53^{3}2^{3}$ $4^{3}2^{4}$ $4^{2}3^{2}2^{3}$ $43^{4}2^{2}$ $3^{6}2$ Class VIII. II. 62^{7} II. 532^{6} $4^{2}2^{6}$ $43^{2}2^{5}$ $3^{4}2^{4}$ Class XI. II. 42^{8} $3^{2}2^{7}$ Class X. 2^{10}

Table II.—*Clutches for* $p=3$.

Partition.	(AB)	(AC)	(BC)		Partition.	(AB)	(AC)	(BC)		Partition.	(AB)	(AC)	(BC)	
$9^{2}2$	8	1	1	Aa	974	6	3	1	Ac	875	5	3	2	Ad
983	7	2	1	Ab	965	5	4	1	Ae	$7^{2}6$	4	3	3	Af

TABLE III.—*Clutches for $p=4$.*

Partition.	$\beta-\kappa$ $\gamma-\kappa$ $\delta-\kappa$	(AB) (AC) (AD)	(BC) (BD) (CD)	(ΣB) (ΣC) (ΣD)*	
8732	5 1 0	5 2 1	1 1 0	2 1 1	Ba_1, Ba_2, Ba_3.
$\kappa=2$ add 1, 1		6 1 1	1 0 1	1 2 1	Bb.
8633	4 1 1	5 1 2	1 0 1	1 2 1	Bc.
8552	3 3 0	3 4 1	1 1 0	2 1 1	Bd_1, Bd_2.
8543	3 2 1	3 3 2	1 1 0	2 1 1	Be_1, Be_2.
		4 3 1	0 1 1	1 1 2	Bf.
7742	4 1 −1	5 1 1	2 0 1	2 3 1	Bg.
$\kappa=3$ add 2, 1		5 2 0	1 1 1	2 2 2	Same as Bg A:B.†
1, 1, 1					
7652	3 2 −1	3 3 1	2 1 0	3 2 1	Bh_1, Bh_2.
		4 3 0	1 1 1	2 2 2	Bi.
7643	3 1 0	5 1 1	1 0 2	1 3 2	Bj.
		3 3 1	1 2 0	3 1 2	Bk_1, Bk_2.
		4 2 1	1 1 1	2 2 2	Bl.
7544	2 1 1	3 3 1	0 2 1	2 1 3	Bm.
		3 2 2	1 1 1	2 2 2	Bn.
6653	2 1 −1	3 1 2	3 0 1	3 4 1	Bo.
$\kappa=4$ add		3 3 0	1 2 1	3 2 3	Same as Bo A:B.
3, 1		3 2 1	2 1 1	3 3 2	Bp.
2, 1, 1					
2, 2					
6554	1 1 0	1 2 3	3 1 0	4 3 1	Br.
		3 2 1	1 1 2	2 3 3	Bs.
		4 1 1	1 0 3	1 4 3	Bq.

* $\Sigma B=\beta-AB=BC+BD$, etc. † A:B signifies that A and B are to be interchanged.

TABLE IV.—*Clutches for $p=5$.*

Partition.	Add	$\beta-\kappa$ $\gamma-\kappa$ $\delta-\kappa$ $\epsilon-\kappa$	(ΣB) (ΣC) (ΣD) (ΣE)	(AB) (AC) (AD) (AE)	(BC) (BD) (BE)	(CD) (CE) (DE)	No.
$7^2 2^3$	0222	4 −1 −1 −1	3 −1 −1 −1	4 1 1 1	1 1 1	0 0 0	(1) V.
$\kappa=3$ add	2211		1 1 2 2	6 1 0 0	1 0 0	0 0 2	(2) I.
Parti'n of 2κ					0 1 0	0 1 1	(3) Ca.
222					0 0 1	1 0 1	(4) Same as (3) D:E.
2211	1221		2 1 1 2	5 1 1 0	0 0 2	1 0 0	(5) I.
					1 0 1	0 0 1	(6) Cb_1, Cb_2, Cb_3.
					0 1 1	0 1 0	(7) Same as (6) C:D.
*	*	*	*	*	*	*	* * *
$6^2 4 2^2$	2033	2 0 −2 −2	2 4 1 1	4 0 1 1	2 0 0	1 1 0	(1) IV.
$\kappa=4$ add	3023		1 4 2 1	5 0 0 1	1 0 0	2 1 0	(2) I.
332	0323		4 1 2 1	2 3 0 1	1 2 1	0 0 0	(3) I.
3311	0233		4 2 1 1	2 2 1 1	2 1 1	0 0 0	(4) IV.
3221	1133		3 3 1 1	3 1 1 1	3 0 0	0 0 1	(5) I.
2222					2 1 0	0 1 0	(6) $C^2c_1 C^2c_2 C^2c_3 C_2c_4$.
					2 0 1	1 0 0	(7) $\equiv$ (6) D:E.
	3122		1 3 2 2	5 1 0 0	0 0 1	2 1 0	(8) I.
					0 1 0	1 2 0	(9) I.
					1 0 0	1 1 1	(10) I.
	1322		3 1 2 2	3 3 0 0	0 1 2	1 0 0	(11) I.
					0 2 1	0 1 0	(12) I.
					1 1 1	0 0 1	(13) I.
	1223		3 2 2 1	3 2 0 1	2 1 0	0 0 1	(14) IV.
					1 2 0	0 1 0	(15) I.
					1 1 1	1 0 0	(16) $\equiv$ (7) A:B.
	2123		2 3 2 1	4 1 0 1	2 0 0	1 0 1	(17) IV.
					1 0 1	2 0 0	(18) I.
					1 1 0	1 1 0	(19) C^2d_1, C^2d_2.
	2222			4 2 0 0			(20) IV.
*	*	*	*	*	*	*	* * *

TABLE IV.—*Clutches for* $p=5$ (*continued*).

Partition.	Add	$\beta-\kappa$ $\gamma-\kappa$ $\delta-\kappa$ $\epsilon-\kappa$	(ΣB) (ΣC) (ΣD) (ΣE)	(AB) (AC) (AD) (AE)	(BC) (BD) (BE)	(CD) (CE) (DE)	No.
$5^3 32$	0244	0 0 −2 −3	5 3 1 1	0 2 2 1	3 1 1	0 0 0	(1) C^3y.
$\kappa=5$ add	0424		5 1 3 1	0 4 0 1	1 3 1	0 0 0	(2) I.
442	1144		4 4 1 1	1 1 2 1	4 0 0	0 0 1	(3) I ABCA.
4411					3 1 0	0 1 0	(4) VI, §12, V.
433					3 0 1	1 0 0	(5) ≡ (4) B:C.
4321	0433		5 1 2 2	0 4 1 0	1 2 2	0 0 0	(6) I
4222	0334		5 2 2 1	0 3 1 1	2 2 1	0 0 0	(7) ≡ (1) A:B.
3331	0343		5 2 1 2	0 3 2 0	2 1 2	0 0 0	(8) I.
3322	1234		4 3 2 1	1 2 1 1	3 1 0	0 0 1	(9) VI, §12, V.
					2 2 0	0 1 0	(10) C^3z_1, C^3z_2, C^3z_3.
					2 1 1	1 0 0	(11) VI, V.
	1243		4 3 1 2	1 2 2 0	3 0 1	0 0 1	(12) C^4a
					2 1 1	0 1 0	(13) ≡ 10) A:B.
					2 0 2	1 0 0	(14) I.
	1324		4 2 3 1	1 3 0 1	2 2 0	0 0 1	(15) ≡ (12) A:B.
					1 3 0	0 1 0	(16) I.
					1 2 1	1 0 0	(17) ≡ (5) A:B.
	1423		4 1 3 2	1 4 0 0	1 2 1	0 0 1	(18) ≡ (3) A:B.
					0 3 1	0 1 0	(19) I.
					0 2 2	1 0 0	(20) I.
	2224		3 3 3 1	2 2 0 1	1 1 1	2 0 0	(21) ≡ (10) A:B & B:C
					1 2 0	1 1 0	(22) ≡ (21) B:C.
					2 1 0	1 0 1	(23) C^4b.
	1333		4 2 2 2	1 3 1 0	2 1 1	0 0 1	(24) ≡ (9) A:B.
					1 2 1	0 1 0	(25) ≡ (4) A:B.
					1 1 2	1 0 0	(26) I.
	2233		3 3 2 2	2 2 1 0	3 0 0	0 0 2	(27) I.
					2 1 0	0 1 1	(28) ≡ (23) B:C A:B.
					2 0 1	1 0 1	(29) ≡ (23) A:B.
					1 2 0	0 2 0	(30) I.
					1 0 2	2 0 0	(31) I.
					1 1 1	1 1 0	(32) ≡ (11) B:C & A:B
	2323		3 2 3 2	2 3 0 0	0 3 0	0 2 0	(33) I.
					1 2 0	0 1 1	(34) ≡ (12) B:A & B:C
					0 2 1	1 1 0	(35) ≡ (1) A:B & B:C
					1 1 1	1 0 1	(36) ≡ (9) A:B & B:C
					2 1 0	0 0 2	(37) I.
					0 1 2	2 0 0	(38) I.
*	*	*	*	*	*	*	* * *

TABLE V.—*Clutches for* $p=6$.

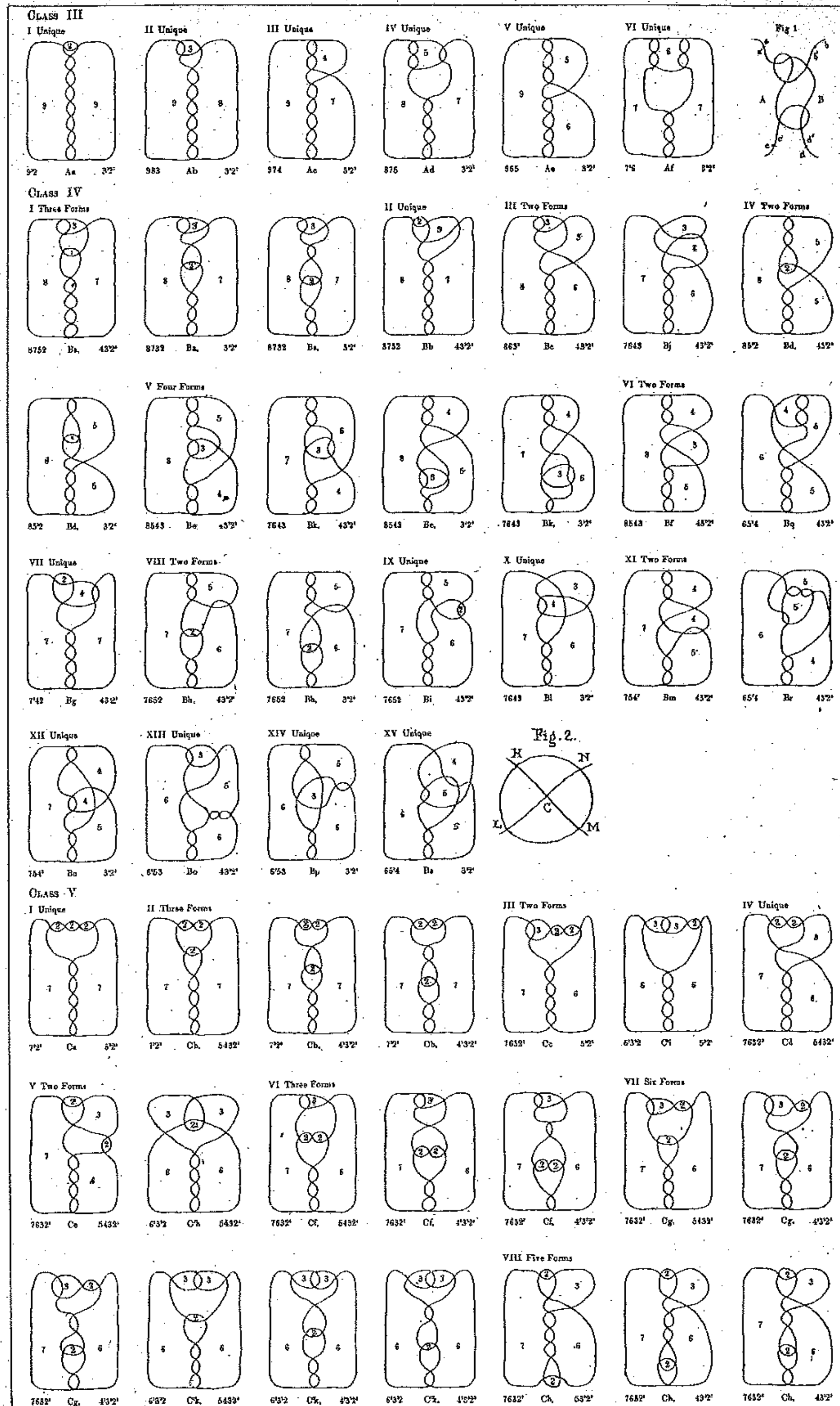
Parti-tion.	Add	$\beta-\kappa$ $\gamma-\kappa$ $\delta-\kappa$ $\epsilon-\kappa$ $\phi-\kappa$	(ΣB) (ΣC) (ΣD) (ΣE) (ΣF)	(AB) (AC) (AD) (AE) (AF)	(BC) (BD) (BE) (BF)	(CD) (CE) (CF)	(DE) (DF) (EF)	
*	*	*	*	*	*	*	*	* * *
53^5 $\kappa=5$	22344	−2 −2 −2 −2 −2	3 3 2 1 1	0 0 1 2 2	3 0 0 0	0 0 0	1 1 0	(1) I.
					2 1 0 0	1 0 0	0 0 1	(2) VI.
					1 1 1 0	1 0 1	0 0 0	(3) Knot D s_2 of $54^2 32^2$.
	23334		3 2 2 2 1	0 1 1 1 2	2 0 0 1	0 0 0	2 0 0	(4) VI.
					0 1 1 1	1 1 0	0 0 0	(5) VII.
					0 2 1 0	0 1 1	0 0 0	(6) Knot Du_1 of $643^2 2^2$, knot D^2x_4 of $54^2 32^2$
	33333		[illegible] 2 2 2	1 1 1 1 1	2 0 0 0	0 0 0	1 1 1	(7) I.
					1 1 0 0	0 1 0	0 1 1	(8) Knot D^2o Amph.
*	*	*	*	*	*	*	*	* * *

TABLE VI.—*Knots of Class VI.*

Knot.	No.	Forms.
I	6	Da_1, Da_2, Da_3, Da_4, Da_5, Da_6.
II	1	Db.
III	4	Dc_1, Dc_2, Dc_3, Dc_4.
IV	4	De_1, De_2, De_3, De_4.
V	2	Df_1, Df_2.
VI	4	Dk_1 Amph, $Dk_2\Vert$,* D^2j Amph.
VII	1	Dl Amph.
VIII	3	Dm, D^2t_1, D^2t_2.
IX	16	Dn_1 Amph, $Dn_2\Vert$, $Dn_3\Vert$, $Dn_4\Vert$, $D^3i_1\Vert$, D^3i_2 Amph, $D^3h_1\Vert$, $D^3h_2\Vert$, D^3h_3 Amph, D^3s Amph.
X	12	Du_1, Du_2, Du_3, Du_4, Du_5, D^2x_1, D^2x_2, D^2x_3, D^2x_4, D^2x_5, Do_1, Do_2.†
XI	12	Dp_1, Dp_2, Dp_3, Dp_4, Dp_5, Dp_6, D^2e_1, D^2e_2, D^2e_3, D^2e_4, D^2e_5, D^2e_6.
XII	9	Dq_1, Dq_2, Dq_3, D^2f_1, D^2f_2, D^2f_3, D^3c, D^2u, D^2h.
XIII	6	Dr_1, Dr_2, Dr_3, D^2q_1, D^2q_2, D^2q_3.
XIV	3	Ds, D^2c_1, D^2c_2.
XV	2	Dt, D^2d.
XVI	9	Dv_1, Dv_2, Dy_1, Dy_2, Dy_3, D^2g_1, D^2g_2, D^2y, D^3j.
XVII	9	$Dw_1\Vert$, Dw_2 Amph, $Dw_3\Vert$, D^2l_1 Amph, $D^2l_2\Vert$, D^3l Amph.
XVIII	6	Dx_1, Dx_2, Dx_3, D^3a_1, D^3a_2, D^2a.
XIX	4	Dz_1 Amph, $Dz_2\Vert$, D^3b Amph.
XX	1	D^2b Amph.
XXI	3	D^2i_1, D^2i_2, D^2i_3.
XXII	1	D^2k.
XXIII	1	D^2m.
XXIV	1	D^2n.
XXV	3	$D^2p_1\Vert$, D^2p_2 Amph.
XXVI	3	D^2r_1 Amph, D^2r_2.
XXVII	3	D^2s_1, D^2s_2, D^2s_3.
XXVIII	2	Dv_1, Dv_2.
XXIX	2	D^2w_1, D^2w_2.
XXX	2	D^2z_1, D^2z_2.
XXXI	1	D^3d Amph.
XXXII	1	D^3e Amph.
XXXIII	4	D^3f Amph, $D^3f_2\Vert$, D^3q Amph.
XXXIV	1	D^3g.
XXXV	1	D^3k.
XXXVI	1	D^3m Amph.
XXXVII	1	D^3o Amph.
XXXVIII	1	D^3p Amph.
XXXIX	1	D^3r.

* The symbol ‖ indicates that a form and its perversion are both included.

† The subordinate partition of Do_2, Plate V. should be 643^22^2.



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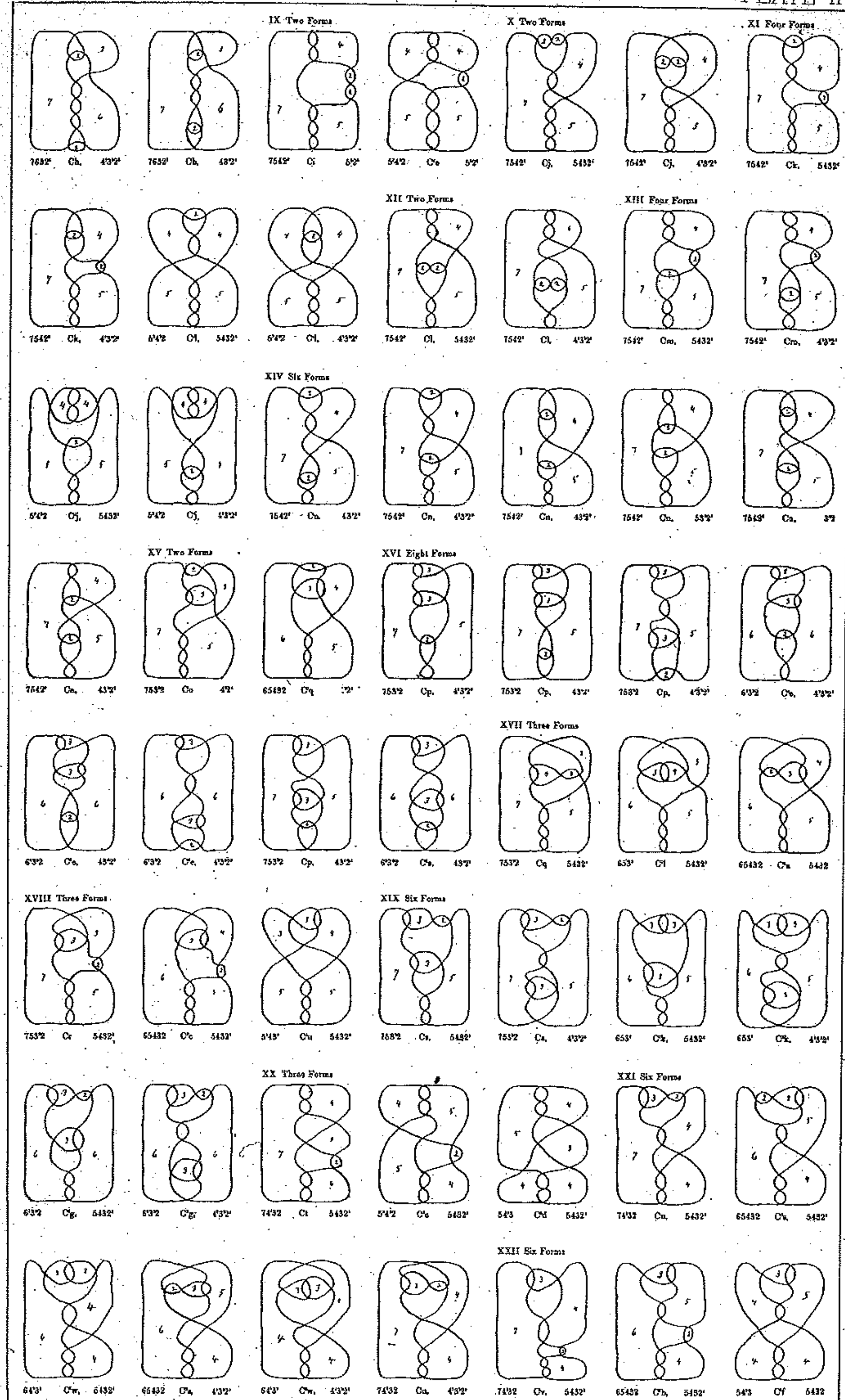

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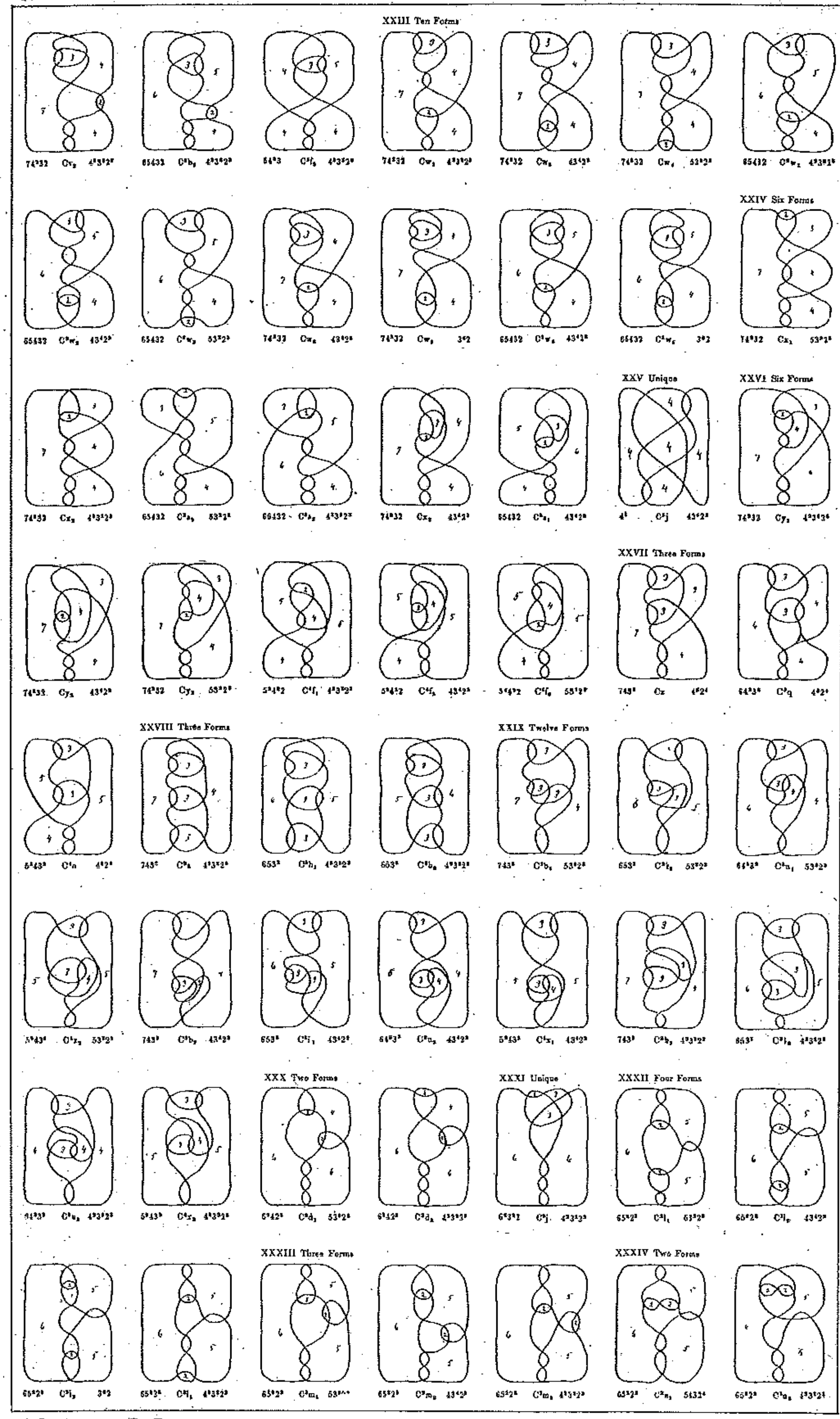

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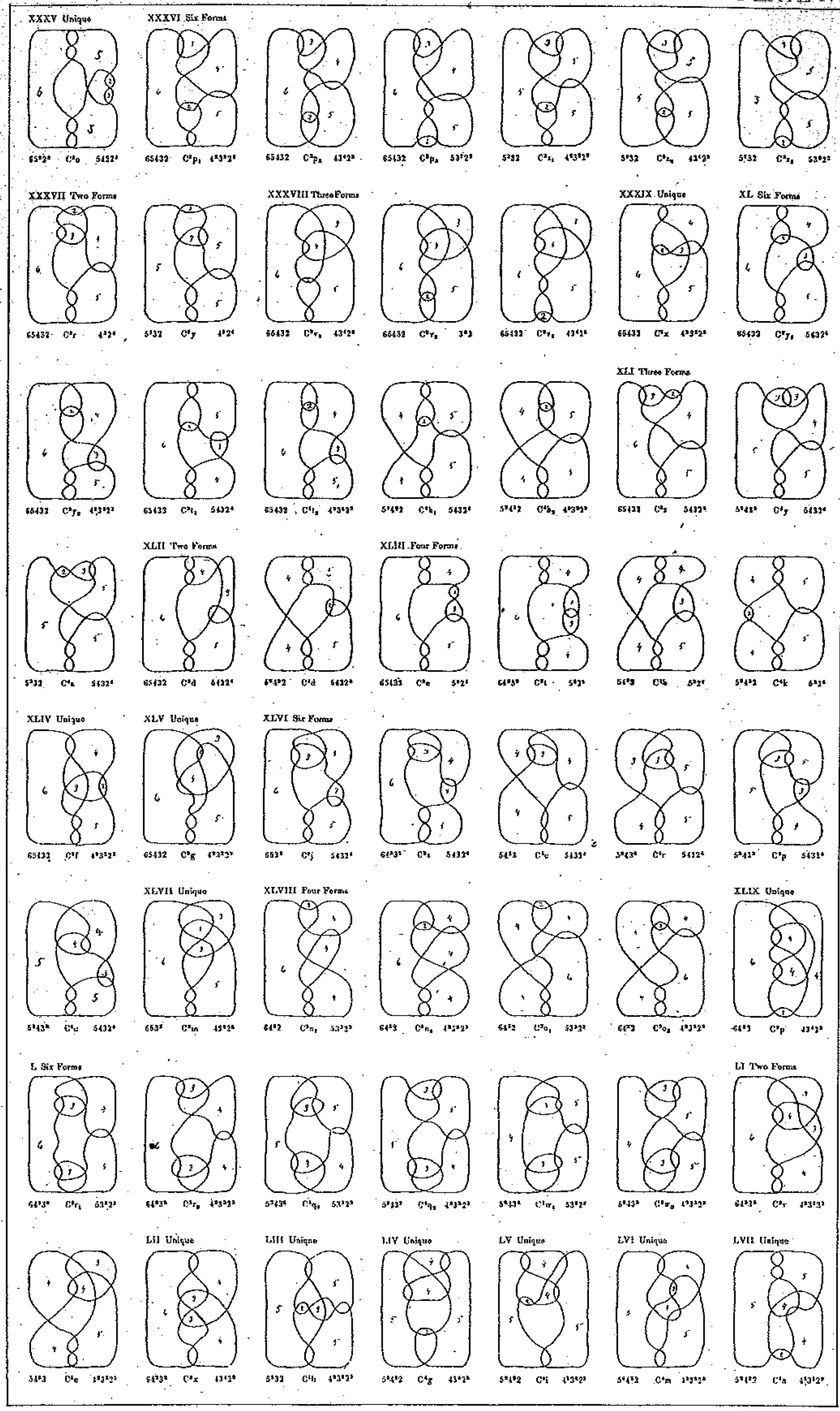

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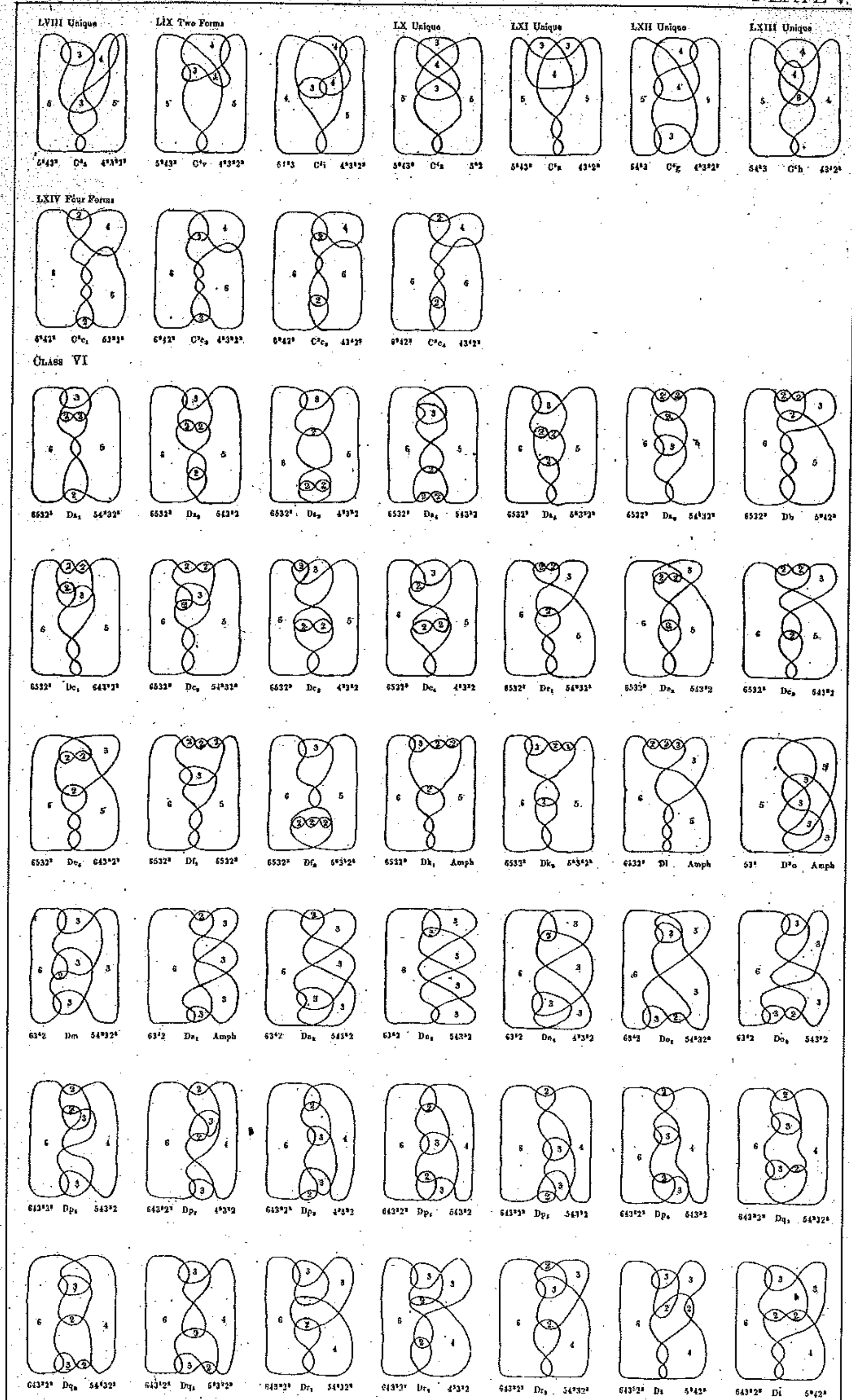

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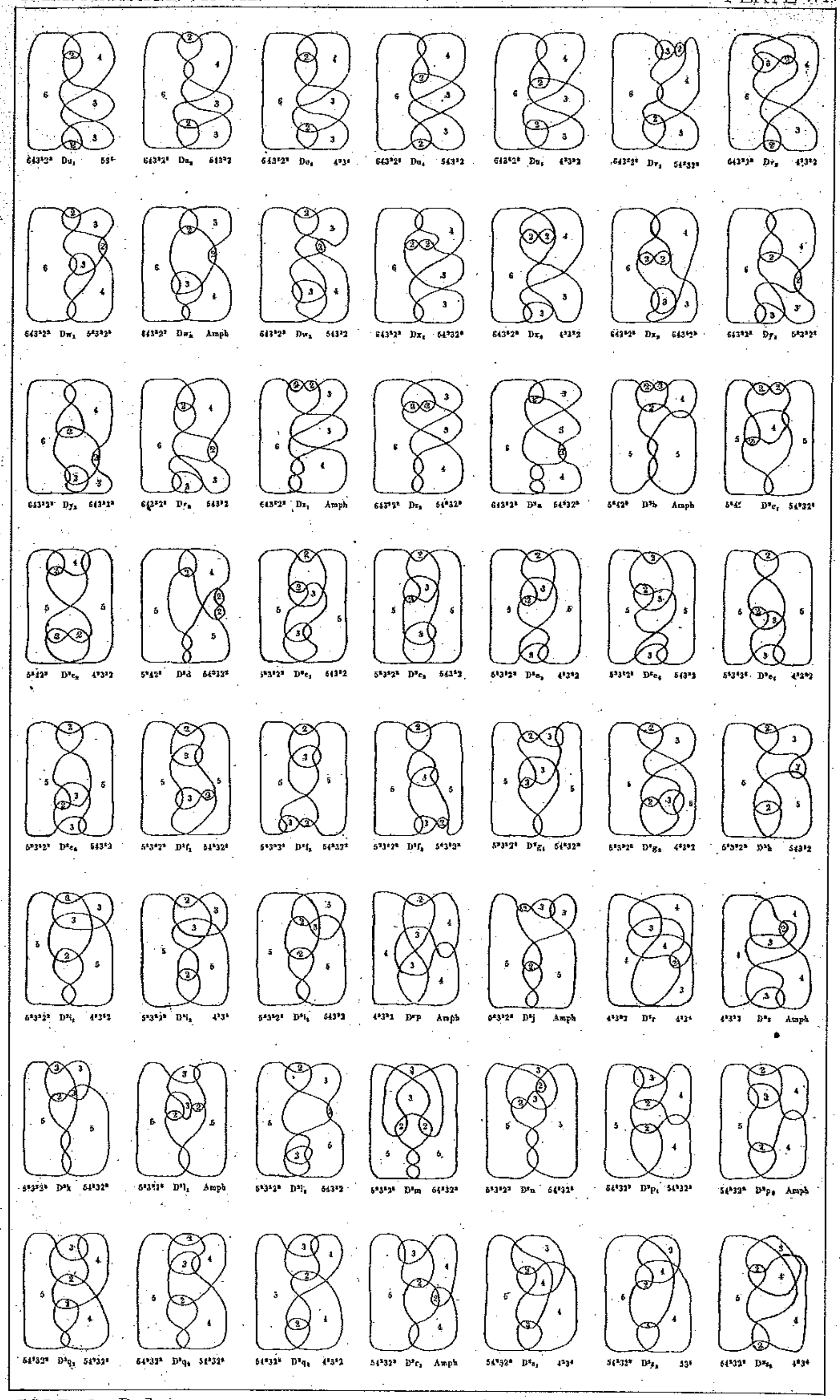

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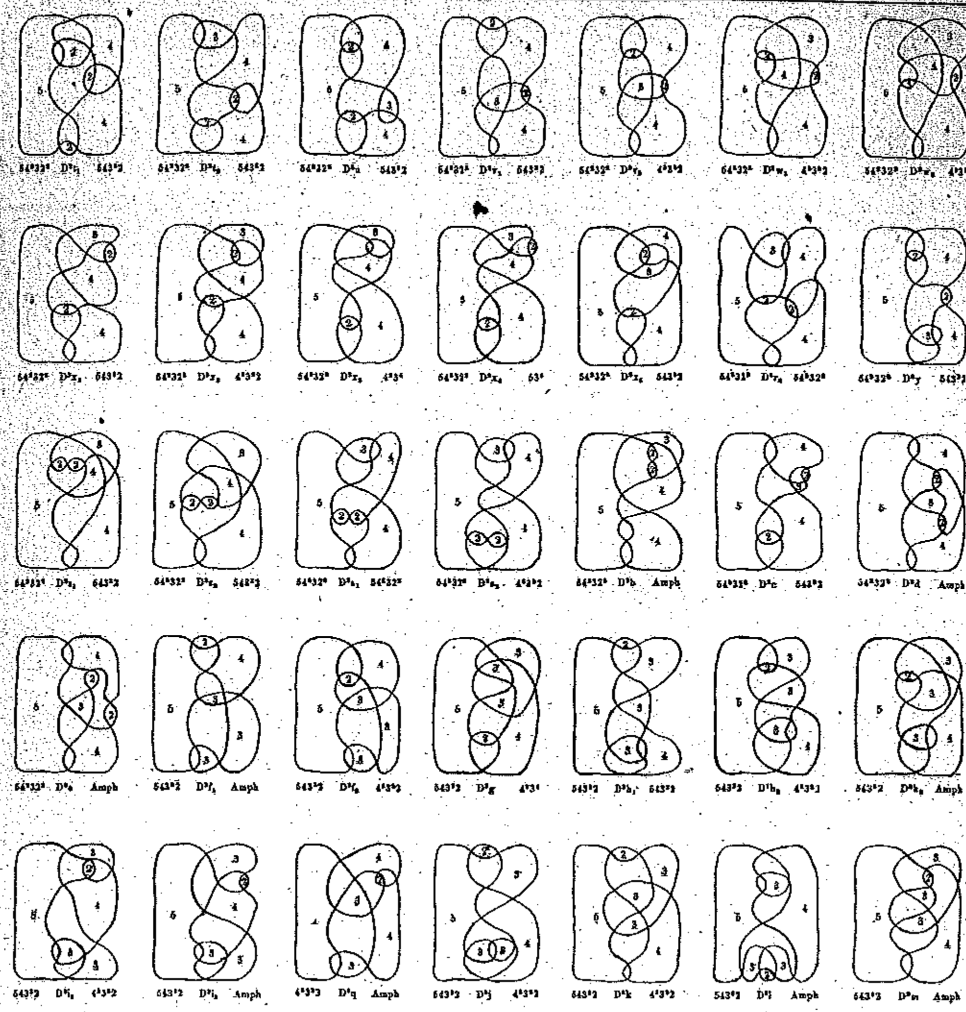

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