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COBORDISM OF SATELLITE KNOTS

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0. INTRODUCTION

In this paper we study the Casson-Gordon invariants of satellite knots. Other cobordism invariants of such knots have been studied by various authors: the (ordinary) signature [23], the Tristram-Levine signatures [15] and the Milnor signatures [10]. In fact, in the last reference the Blanchfield pairing, and so (implicitly) the algebraic cobordism class, of a satellite is determined. See also [17]. The most striking feature to emerge is that the algebraic cobordism class of a satellite depends only on the constituent knots and the winding number. It is intuitively clear that this is not true of the geometric cobordism class, and one motivation for computing the Casson-Gordon invariants is to verify this intuition, which we do in Theorem 3.

We also apply our results to Kawauchi's group of $\tilde{H}$-cobordism classes of homology $S^1 \times S^2$'s [9]. The homomorphism from knot cobordism to algebraic cobordism factors through this group, and we show that the first factor has kernel containing a $\subset \mathbb{Z}^\infty$.

1. TERMINOLOGY, AND AN EXAMPLE

All manifolds will be oriented. Our statements may be interpreted in the PL or the smooth category, according to taste.

Let K be a knot in S^3. By an <u>axis</u> for K of <u>winding number</u> w we mean an unknotted simple closed curve A in $S^3 - K$ having linking number w with K. Let V be a solid torus complementary to a tubular neighborhood of A, with K contained in the interior of V. There is a preferred generator x for $H_1(V)$, specified by the condition $Lk(x,A) = +1$. For any knot C in S^3 there is an untwisted, orientation-preserving embedding $h: V \to S^3$ taking V onto a tubular neighborhood of C so that C represents $h_*(x)$ in $H_1(hV)$. We say that the knot $S = h(K)$ is a <u>satellite</u> of C with <u>orbit</u> K, <u>axis</u> A and <u>winding number</u> w. (In [17], the term "embellishment" is used where we use "orbit".)

The knot S is determined (up to isotopy) by C and the link $K \cup A$. We write $S = \mathcal{S}(K,C;A)$. We also denote the set of all satellites of C with orbit

K and winding number w by $\mathcal{S}_w(K,C)$. Thus we can rephrase the qualitative result on the algebraic cobordism class mentioned in the introduction by saying that (for given K, C and w) the image of $\mathcal{S}_w(K,C)$ in the algebraic cobordism group consists of a single element. We remark that the original examples of non-slice, algebraically slice knots [1] show that there are some $\mathcal{S}_0(K,C)$ containing knots from more than one cobordism class. Take K to be the $n(n+1)$-twist knot, for $n > 1$, and C to be the torus knot of type $(n,n+1)$. Then $\mathcal{S}_0(K,C)$ contains the $n(n+1)$-twist double of C, which is slice (Casson, unpublished; see [16] for a proof). But $\mathcal{S}_0(K,C)$ also contains K itself, by taking a trivial axis A, i.e. one such that $K \cup A$ is a split link. It was proved by Casson and Gordon in [1] that K is not slice. If one disallows trivial satellites, it is still easy to construct an element of $\mathcal{S}_0(K,C)$ cobordant to K, by using for instance the axis shown in Fig. 1.

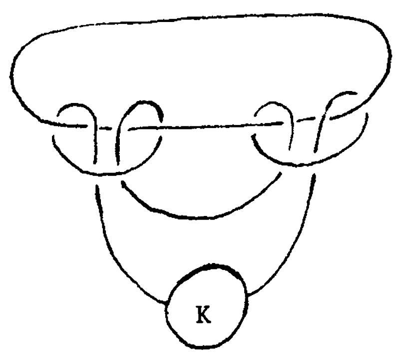

Figure 1

2. ALGEBRAIC COBORDISM

In some cases, our formula for the Casson-Gordon invariants of a satellite involves the algebraic cobordism class (Corollary 2) and for this it is necessary to put the two kinds of invariant on a similar footing. This we shall do in this section by giving a "Casson-Gordon type" definition of the algebraic cobordism class. That this can be done is probably well-known to the experts.

We denote the ring $\mathbb{Q}[t,t^{-1}]$ of Laurent polynomials with rational coefficients by Γ, and its field of fractions $\mathbb{Q}(t)$ by $Q\Gamma$. The involution $f(t) \to f(t^{-1})$ of Γ or $Q\Gamma$ will be denoted by J. The (multiplicative) infinite cyclic group is written as C_∞, and we assume that a generator t is fixed once for all. Let (M,φ) be a closed 3-manifold over C_∞; that is, M is a closed 3-manifold and φ is a homomorphism $H_1(M) \to C_\infty$. Suppose that M has the rational homology of $S^1 \times S^2$. Since $\Omega_3(K(C_\infty,1)) = 0$, we have $(M,\varphi) = \partial(W,\psi)$ for some compact 4-manifold (W,ψ) over C_∞.

REMARK. We do not assume that φ is onto, and it may be that $\varphi = 0$. In that case we always take W so that $H_1(M;\mathbb{Q}) \to H_1(W;\mathbb{Q})$ is injective and $\psi = 0$. Note that the injectivity is automatic if $\varphi \neq 0$. Here $\varphi = 0$ means that $\varphi(x) = 1$ for all x; in general we write $\mathrm{Hom}(A,B)$ additively even when B is multiplicative.

We define twisted homology and a twisted intersection pairing just as in [1]: if $\tilde{W}$ is the infinite cyclic covering of W determined by ψ, then $C_*(\tilde{W};\mathbb{Q})$ is a complex of Γ-modules, and we set

$$C_*^t(W;Q\Gamma) = C_*(\tilde{W};\mathbb{Q}) \otimes_\Gamma Q\Gamma .$$

The homology of this complex is written $H_*^t(W;Q\Gamma)$. There is a pairing $H_2^t(W;Q\Gamma) \times H_2^t(W;Q\Gamma) \to Q\Gamma$, Hermitian with respect to J, given at the chain level by

$$\langle x \otimes f, y \otimes g\rangle = fg^J \sum_{i=-\infty}^{\infty} (x \cdot t^i y)t^i \ , \ x,y \in C_2(\tilde{W};\mathbb{Q}),\ f,g \in Q\Gamma .$$

Here $x \cdot t^i y$ is the ordinary intersection number. The pairing is non-singular and so represents an element $t(W) = t_\psi(W)$ of the Witt group $W(Q\Gamma;J)$. For $\varphi \neq 0$ this is because the Milnor exact sequence for the infinite cyclic covering of (M,φ) [19] shows that $H_1^t(M;Q\Gamma) = 0$ (even if φ is not onto). If $\varphi = 0$, we have $H_1^t(M;Q\Gamma) = H_1(M;\mathbb{Q}) \otimes_{\mathbb{Q}} Q\Gamma$ and $H_1^t(W;Q\Gamma) = H_1(W;\mathbb{Q}) \otimes_{\mathbb{Q}} Q\Gamma$, so that $H_1^t(M;Q\Gamma) \to H_1^t(W;Q\Gamma)$ is injective. (See the remark above.) The ordinary intersection form on $H_2(W;\mathbb{Q})$ is also non-singular; let $t_0(W)$ be its image in $W(Q\Gamma;J)$. Define

$$\alpha(M,\varphi) = t(W) - t_0(W) \in W(Q\Gamma;J) .$$

The proof that this is well-defined is just like that for the Casson-Gordon invariants (for which see [1]).

REMARK. If $\varphi = 0$ then $\psi = 0$ so $t_\psi(W) = t_0(W)$. Hence $\alpha(M,0) = 0$. Our reason for being careful about the "trivial" case is that we have to deal with 4-manifolds over C_∞ of the form $(W_1,\psi_1) \cup (W_2,\psi_2)$ where one of ψ_1, ψ_2 may be zero.

Now let K be a knot in S^3. The manifold M_K obtained by 0-framed surgery along K comes with a preferred isomorphism $\varphi_K : H_1(M_K) \to C_\infty$ determined by the orientations of S^3 and K. We write $\alpha(K)$ or α_K for $\alpha(M_K,\varphi_K)$. It is not hard to see that $\alpha_K = 0$ if K is slice; together with Theorem 1 below this shows that α induces a homorphism

$$\alpha : \mathscr{C}^{3,1} \longrightarrow W(Q\Gamma;J)$$

where $\mathscr{C}^{3,1}$ is the knot cobordism group.

We now indicate why this invariant is equivalent to the algebraic cobordism class. Let $(M_K,\varphi_K) = \partial(W,\psi)$. Under the boundary homomorphism

$$\partial : W(Q\Gamma;J) \longrightarrow W(Q\Gamma/\Gamma;J)$$

of the localization exact sequence for $W(Q\Gamma;J)$, $t_0(W)$ dies and $t_\psi(W)$ is sent to minus the Witt class of the Blanchfield pairing on $H_1(\tilde{M}_K;\mathbb{Q})$, where $\tilde{M}_K$ is the infinite cyclic covering of M_K. According to Trotter [25] the isomorphism class of the Blanchfield pairing determines the rational S-equivalence class of a Seifert matrix V for K. It follows that $\partial\alpha_K$ determines the (rational) Witt class of V, which is to say, the algebraic cobordism class of K. (Recall that the homorphism from the integral algebraic cobordism group $W_S(\mathbb{Z})$ to the corresponding rational group $W_S(\mathbb{Q})$ is injective [12]). In the other direction we have:

PROPOSITION 1. *If* V *is a Seifert matrix for* K *then* $\alpha(K)$ *is represented by the matrix* $(1-t)V+(1-t^{-1})V^T$.

Here V^T is the transpose of V. Actually it can be shown that there is an isomorphism $W(Q\Gamma;J) \cong W_S(\mathbb{Q}) \oplus W(\mathbb{Q})$ under which $\alpha(K)$ corresponds to $([V],0)$; an account of this will be found in Appendix A.

Before giving the proof we describe an additivity property that we shall need frequently. Recall that if W_1 and W_2 are 4-manifolds with $\partial W_1 \cong -\partial W_2$ and W is the closed 4-manifold $W_1 \cup_\partial W_2$ then the signature of W is given by

$$\operatorname{sign}(W) = \operatorname{sign}(W_1) + \operatorname{sign}(W_2)$$

(Novikov additivity). However, if W_1 and W_2 are glued along only part of their boundaries, this may fail. This situation was studied by Wall [28]. Suppose that $\partial W_1 \cong M_1 \cup M_0$ and $\partial W_2 \cong M_2 \cup -M_0$, where for $i=1,2$, M_i and M_0 are 3-manifolds meeting only in their common boundary, and let $W = W_1 \cup_{M_0} W_2$. Let $F = \partial M_0 = \partial M_1 = \partial M_2$, and let $A_i = \ker(H_1(F;\mathbb{Q}) \to H_1(M_i;\mathbb{Q}))$ for $i=0,1,2$. Wall showed that the failure of additivity is measured by the signature of a bilinear form on

$$\frac{A_i \cap (A_j + A_k)}{(A_i \cap A_j)+(A_i \cap A_k)} \;, \quad \{i,j,k\} = \{1,2,3\} .$$

In fact, this holds on the level of the Witt classes of the intersection forms, and for twisted homology as well. We shall need only the special case in which at least two of A_0, A_1, A_2 are equal, when additivity does hold. We shall refer to this as *Wall additivity*.

We remark that Wall's result can be derived from Novikov additivity (or rather, the easy generalization to the case of gluing along some whole boundary components) by decomposing $W_1 \cup W_2$ into three pieces as indicated in Fig. 2. The "correction term" is the intersection form of the Θ-shaped piece.

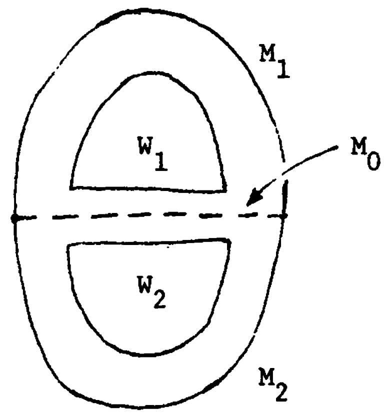

Figure 2

PROOF OF PROPOSITION 1. Let F be a spanning surface for K giving the Seifert matrix V. Let $\hat{F} \subset D^4$ be obtained by pushing $\operatorname{int} F$ into $\operatorname{int} D^4$. Set $W_1 = D^4 - (\hat{F} \times \operatorname{int} D^2)$ and let $\psi_1 : H_1(W_1) \longrightarrow C_\infty$ be given by linking number with $\hat{F}$. We shall show by a cut-and-paste construction of the infinite cyclic cover $\tilde{W}_1$ that $t(W_1)$ is represented by $(1-t)V + (1-t^{-1})V^T$; c.f. [8] Section 5, [27] Section 5. Let $F \times [-1,1]$ be a bicollar of F in S^3. Cutting W_1 open along the trace of the push yields D^4, and the faces exposed by the cut are $F \times [-1,-1/2]$ and $F \times [1/2,1]$. Thus $\tilde{W}_1$ is obtained by taking copies $t^i D^4$ of D^4 for $i \in \mathbb{Z}$ and identifying $t^i(F \times [-1,-1/2])$ with $t^{i+1}(F \times [1/2,1])$. It follows that $H_2(\tilde{W}_1;\mathbb{Q}) \cong H_1(F;\mathbb{Q}) \otimes_{\mathbb{Q}} \Gamma$. If x is a cycle on F, let $C_\pm x$ be the cone on $x \times \pm 1$ in D^4, and let

$$Sx = (C_-x) - t(C_+x) ,$$

a 2-cycle in $\tilde{W}_1$. This represents the element of $H_2(\tilde{W}_1;\mathbb{Q})$ corresponding to $[x] \otimes 1$. If Θ is the Seifert form on $H_1(F;\mathbb{Q})$ it follows easily that

$$\langle Sx,Sy \rangle = (1-t)\Theta([x],[y]) + (1-t^{-1})\Theta([y],[x]) .$$

This gives the result claimed. Note also that the ordinary intersection form on $H_2(W_1;\mathbb{Q})$ is identically zero.

Now let H be a solid handlebody with $\partial H = F \cup E^2$, where E^2 is a disc, and let $W = W_1 \cup_{F \times \partial D^2} H \times \partial D^2$. Then ψ_1 extends to $\psi : H_1(W) \longrightarrow C_\infty$, and $\partial(W,\psi) = (M_K, \varphi_K)$. We have $H_2^t(H \times \partial D^2; \mathbb{Q}\Gamma) = 0$, and the intersection form on $H_2(H \times \partial D^2; \mathbb{Q})$ is identically zero. Finally, Wall additivity applies to $W = W_1 \quad H \times \partial D^2$ in both ordinary and twisted homology (in this case, all three kernels are the same) to complete the proof. ///

Our next aim is to determine the algebraic cobordism class of a satellite knot. Although this follows from the results on the Blanchfield pairing in [10] and [17], we include a proof because it seems particularly simple from the point of view introduced above, and because it serves as a model for the proof of our theorem on the Casson-Gordon invariants. Before stating the result, we must discuss induced homomorphisms of $W(Q\Gamma;J)$.

Suppose $f:(\Gamma,J) \longrightarrow (F,J')$ is a homorphism of rings-with-involution, where F is a field. Let V be a finite dimensional vector space over $Q\Gamma$ with a non-singular Hermitian pairing $\varphi: V \times V \to Q\Gamma$. For any Γ-lattice L in V let

$$L^{\#} = \{x \in V \mid \varphi(x,y) \in \Gamma \text{ for all } y \in L\} .$$

We can choose L so that $L \leq L^{\#}$. Make F into a Γ-module via f. Then we have an induced Hermitian pairing φ_f on the F-vector space $L \otimes_{\Gamma} F$, namely

$$\varphi_f(x \otimes \alpha,\ y \otimes \beta) = \alpha\beta^{J'} f\varphi(x,y),\ x,y \in L,\ \alpha,\beta \in F .$$

In general, φ_f may be singular. However, one can show that the set of elements of $W(Q\Gamma;J)$ represented by (V,φ) for which L can be so chosen as to make φ_f non-singular forms a subgroup $\mathrm{Def}(f_*)$, say, and that the assignment

$$[V,\varphi] \longrightarrow [L \otimes_{\Gamma} F, \varphi_f]$$

is a well-defined homomorphism $f_*: \mathrm{Def}(f_*) \longrightarrow W(F;J')$. If $\alpha \in \mathrm{Def}(f_*)$ is represented by a matrix A over Γ then $f_*(\alpha)$ is represented by $f(A)$, provided this is non-singular. Clearly $\mathrm{Def}(f_*) = W(Q\Gamma;J)$ if f is injective; the same is true if $f(t) = 1$. (See Appendix A, where $\mathrm{Def}(f_*)$ is determined.)

If $f(t) = x$ and $\alpha \in \mathrm{Def}(f_*)$ we shall also write $\alpha[x]$ instead of $f_*(\alpha)$. In particular, we shall sometimes write an element α of $W(Q\Gamma;J)$ as $\alpha[t]$. For $\alpha \in W(Q\Gamma;J)$ and $\zeta \in S^1$, $\alpha[\zeta] \in W$ ($\mathbb{C}$, conjugation) is defined for all but finitely many ζ. We define $\sigma_\zeta(\alpha)$ to be the signature of $\alpha[\zeta]$ whenever possible, and elsewhere to be the average of the one-sided limits. (These signatures were introduced in a slightly different context by Casson and Gordon [1].) This gives a step function $\sigma_\bullet(\alpha): S^1 \to \mathbb{Z}$, all of whose discontinuities occur at points where $\alpha[\zeta]$ is not defined. In view of Proposition 1, for a knot K, $\sigma_\zeta(\alpha_K)$ is equal to the Tristram-Levine signature of K at ζ, except perhaps at finitely many points of S^1. We abbreviate $\sigma_\zeta(\alpha_K)$ to $\sigma_K(\zeta)$.

THEOREM 1. *Let S be a satellite of C with orbit K and winding number w. Then*

$$\alpha_S[t] = \alpha_K[t] + \alpha_C[t^w] .$$

We shall need the following lemma.

LEMMA 1. *Let (M,φ) be a closed 3-manifold over C_∞, and suppose that M has the rational homology of $S^1 \times S^2$. Then*

$$\alpha(M,w\varphi)[t] = \alpha(M,\varphi)[t^w]$$

<u>for any integer</u> w. <u>In particular,</u>

$$\alpha(M,\varphi)[1] = 0 \ .$$

<u>PROOF OF THEOREM 1, ASSUMING LEMMA 1</u>. Let (W_K,ψ_K) and (W_C,ψ_C) be compact 4-manifolds over C_∞ such that

$$\partial(W_K,\psi_K) = (M_K,\varphi_K)$$

and $$\partial(W_C,\psi_C) = (M_C,\varphi_C) \ .$$

Let $U_C \subset M_C$ be the surgery solid torus, and let $U_K \subset M_K$ be a small tubular neighborhood of the axis of K used to form S. We can construct

$$(W_S,\psi_S) = (W_K,\psi_K) \bigcup_{U_K \equiv U_C} (W_C,w\psi_C) \tag{1}$$

so that $\partial(W_S,\psi_S) = (M_S,\varphi_S)$. Wall additivity applies to (1) in both ordinary and twisted homology, since the kernels corresponding to the two pieces of ∂W_C are the same. Therefore

$$\begin{aligned}\alpha(M_S,\varphi_S) &= \alpha(M_K,\varphi_K) + \alpha(M_C,w\varphi_C)\\ &= \alpha(M_K,\varphi_K) + \alpha(M_C,\varphi_C)[t^w]\end{aligned}$$

by Lemma 1. This is the assertion of the theorem. ⁞⁞⁞

<u>PROOF OF LEMMA 1</u>. Let $(M,\varphi) = \partial(W,\psi)$. First suppose that $w \neq 0$. Let $\tilde{W}_\psi$, $\tilde{W}_{w\psi}$ be the infinite cyclic coverings of W determined by ψ, $w\psi$ respectively. Since $t_0(W)[t^w] = t_0(W)$, we need to show that

$$t_{w\psi}(W)[t] = t_\psi(W)[t^w] \ .$$

But this is easy, since $\tilde{W}_{w\psi}$ consists of $|w|$ copies of $\tilde{W}_\psi$; t permutes these copies cyclically, with t^w acting on each copy like t on $\tilde{W}_\psi$.

It remains to prove that $\alpha(M,\varphi)[1] = 0$. We may assume that φ is onto, since if $\varphi = 0$ there is nothing to prove, and otherwise we can use the previous case to replace φ by an epimorphism. Since $Q\Gamma$ is torsion-free over the PID Γ,

$$H_*^t(W;Q\Gamma) = H_*^t(W;\Gamma) \otimes_\Gamma Q\Gamma \ .$$

The intersection pairing on $H_*^t(W;Q\Gamma)$ comes from a pairing $H_2^t(W;\Gamma) \times H_2^t(W;\Gamma) \longrightarrow \Gamma$ by tensoring with $Q\Gamma$. This pairing induces one on

$$L = H_2^t(W;\Gamma)/\Gamma\text{-torsion},$$

and L is a Γ-lattice in $H_2^t(W;Q\Gamma)$. Now $H_*^t(W;\Gamma)$ is just the ordinary rational homology of the infinite cyclic covering $\tilde{W}$, regarded as a Γ-module. By doing surgery on W we may assume that $\pi_1(W) \cong C_\infty$. Then $\tilde{W}$ is simply connected. Also $H_1(M) \longrightarrow H_1(W)$ is onto, so $H_3(W;\mathbb{Q}) = 0$. From the exact sequence for the covering $\tilde{W} \to W$ [19] we therefore have

$$0 \to H_2^t(W;\Gamma) \xrightarrow{1-t} H_2^t(W;\Gamma) \longrightarrow H_2(W;\mathbb{Q}) \longrightarrow 0$$

exact. Thus $H_2(W;\mathbb{Q}) \cong H_2^t(W;\Gamma) \otimes_\Gamma \mathbb{Q}$; note that the intersection form on $H_2(W;\mathbb{Q})$ comes from that on $H_2^t(W;\Gamma)$ by tensoring with $\mathbb{Q}$. Also $H_2^t(W;\Gamma)$ has no $(1-t)$-torsion, so $H_2^t(W;\Gamma) \otimes_\Gamma \mathbb{Q} \cong L \otimes_\Gamma \mathbb{Q}$. Therefore $t_\psi(W)[1] = t_o(W)$, proving that $\alpha(M,\varphi)[1] = 0$. ⦙⦙⦙

REMARK. We could have defined $\alpha(M,\varphi)$ without the assumption that M has the rational homology of $S^1 \times S^2$, since this was only used to ensure non-singularity of the intersection forms and any Hermitian form over a field gives rise to a non-singular form on the quotient by its radical. However, the case $w=0$ of Lemma 1 would no longer hold. For instance, if M is the manifold obtained by 0-surgery on both components of the Whitehead link, and if φ sends the meridians of the components to t and 1 respectively, then $\alpha(M,\varphi)$ is the rank 1 form $\langle 1 \rangle$.

There is a related result that we shall need. In [6], Section 13 an invariant $\sigma(M,\varphi) \in \mathbb{Q}$ is associated to any closed 3-manifold (M,φ) over C_m, the finite cyclic group of m'th roots of unity.

LEMMA 2. Let K be a knot, let m be a power of a prime, and let $g: C_\infty \to C_m$ be a homomorphism. Let $\zeta = g(t)$. Then $\alpha_K[\zeta] \in W(\mathbb{C};$ conjugation) is defined and

$$\sigma_K(\zeta) = \sigma(M_K, g\varphi_K) \ .$$

PROOF. That $\alpha_K[\zeta]$ is defined follows from Proposition 1 and the fact that, if Δ is the Alexander polynomial of K, $\Delta(\zeta) \neq 0$ since ζ is a prime-power root of unity. (See [24], Lemma 2.5.) The second assertion follows from the identification of $\sigma(M,\varphi)$ with an eigenspace signature ([1], pp. 5-6), Lemma 3.1 of [2] and Proposition 1. ⦙⦙⦙

We conclude this section with a remark on surgery presentations of a knot K. In [22], Rolfsen shows how such a description gives rise to a presentation matrix $A(t)$ for the Alexander invariant of K. This matrix satisfies $A(t)^T = A(t^{-1})$, and $A(1)$ is a diagonal matrix with diagonal entries ± 1. It is evident from the definition that $A(t)$ represents the intersection form on $H_2^t(W;Q\Gamma)$ for a certain 4-manifold (W,ψ) over C_∞ with $\partial(W,\psi) = (M_K,\varphi_K)$; W is obtained by attaching 2-handles to B^4 as specified by the surgery description, and removing a neighborhood of an unknotted 2-disc spanning K. The intersection form on $H_2(W;\mathbb{Q})$ is represented by $A(1)$, and so

$$\alpha_K = [A(t)] - [A(1)]$$

where [..] denotes Witt class. (That the Tristram-Levine signatures of K can be computed from $A(t)$ was observed in [14], Section 12.)

3. THE CASSON-GORDON INVARIANTS...

In this section we set out our notation for these invariants and prove a technical lemma. If (M,φ) is a closed 3-manifold over $C_m \times C_\infty$, there is an invariant $\tau(M,\varphi) \in W(\mathbb{C}(t),J) \otimes_{\mathbb{Z}} \mathbb{Q}$ defined as in [6], Section 13. (The involution J of $\mathbb{C}(t)$ is given by $f(t)^J = \bar{f}(t^{-1})$.) Let K be a knot in S^3. Let $L = L_{K,n}$ be the n-fold branched cyclic covering of K, and let $M = M_{K,n}$ be obtained from L by O-surgery along the lift $\tilde{K}$ of K. Thus $M_{K,1}$ is the manifold M_K of the last section, and $M_{K,n}$ is an n-fold cyclic covering of $M_{K,1}$. We identify $H_1(M)$ with $H_1(L) \oplus C_\infty$, where the generator t of the C_∞ summand is represented by a meridian of $\tilde{K}$. Let $Ch_n(K) = \mathrm{Hom}(H_1(L), \mathbb{C}^*)$ be the group of characters of $H_1(L)$. We shall always assume that n is a power of a prime, so that L is a rational homology sphere, and any $\chi \in Ch_n(K)$ takes values in C_m for some m. Define $\chi^+ : H_1(M) \longrightarrow C_m \times C_\infty$ by

$$\chi^+(x,y) = (\chi(x),y) \ , \ x \in H_1(L),\ y \in C_\infty \ ,$$

and set

$$\tau(K,\chi) = \tau(M,\chi^+) \ .$$

(In [6] it is assumed that m is the order of χ, but it is easy to see that the choice of m is immaterial.)

Linking number gives a non-singular symmetric pairing $Lk: H_1(L) \times H_1(L) \longrightarrow \mathbb{Q}/\mathbb{Z}$, which yields another such pairing on $Ch_n(K)$, also denoted by Lk. We shall always think of $Ch_n(K)$ as carrying this form, and $-Ch_n(K)$ will denote $Ch_n(K)$ with the form $-Lk$. The theorem of Casson and Gordon ([1],[6]) is that if K is slice then (for any prime power n) $Ch_n(K)$ has a metaboliser $\mathcal{M}$ such that $\tau(K,\chi) = 0$ for all $\chi \in \mathcal{M}$ of prime-power order. (A <u>metaboliser</u> is a subgroup which is equal to its orthogonal complement.) The case $n = 2$ has received most attention; in Section 5 we shall have need of odd primes. We remark that the above makes sense for $n = 1$; in this case there is only one character, 0, and $\tau(K,0)$ is the image $\alpha_K^{\mathbb{C}}$ of α_K in $W(\mathbb{C}(t),J) \otimes \mathbb{Q}$. (If 0_n is the zero of $Ch_n(K)$, $\tau(K,0_n)$ may be non-zero for some $n > 1$ as well; it is determined by the algebraic cobordism class of K. See Appendix B.)

If K is a composite knot $K_1 \# K_2$, $Ch_n(K)$ may be identified with the orthogonal direct sum $Ch_n(K_1) \oplus Ch_n(K_2)$, and then

$$\tau(K,\chi_1 \oplus \chi_2) = \tau(K_1,\chi_1) + \tau(K_2,\chi_2) \ .$$

This is proved in [5], Proposition 3.2 for the case $n = 2$; it is a special case of Corollary 1 below.

Induced homomorphisms on $W(\mathbb{C}(t),J)$ are defined just as for $\mathbb{Q}(t)$ in Section 2.

LEMMA 3. *Let* $K \subset S^3$ *be a knot. Let* $\chi \in Ch_n(K)$ *take values in* C_m, *and suppose that* m *and* n *are both powers of primes. Let* $x \in C_m \times C_\infty \subset \mathbb{C}(t)$. *If* x *has finite order suppose further that* $n = 1$. *Then* $\tau(K,\chi)[x]$ *is defined and*

$$\tau(K,\chi)[x] = \tau(M_{K,n}, f\chi^+)$$

where $f: C_m \times C_\infty \longrightarrow C_m \times C_\infty$ *is defined by* $f(y) = y$ *for* $y \in C_m$ *and* $f(t) = x$.

PROOF. By $\tau(K,\chi)[x]$ we mean the image of $\tau(K,\chi)$ under the homomorphism $W(\mathbb{C}(t),J) \otimes \mathbb{Q} \longrightarrow W(\mathbb{C}(t),J) \otimes \mathbb{Q}$ induced by

$$\hat{f}: \mathbb{C}[t,t^{-1}] \longrightarrow \mathbb{C}(t);$$

$$\hat{f}(\alpha) = \alpha\ ,\ \alpha \in \mathbb{C},$$

$$\hat{f}(t) = x\ .$$

If x has infinite order, $\hat{f}$ is injective, and the proof is similar to the case $w \neq 0$ of Lemma 1. We leave this case to the reader. Suppose then that x has finite order, i.e. $x \in C_m$. By assumption $n = 1$, and so $\chi^+: H_1(M_K) \longrightarrow C_m \times C_\infty$ is given by $\chi^+(z) = (1, \varphi(z))$ where $\varphi = \varphi_K: H_1(M_K) \longrightarrow C_\infty$ is the canonical isomorphism of Section 2. Define $g: C_\infty \longrightarrow C_m$ by $g(t) = x$. Choose a compact 4-manifold (W,ψ) over C_m such that $\partial(W,\psi) = r(M_K, g\varphi)$ for some $r > 0$. Let $j: C_m \longrightarrow C_m \times C_\infty$ be the inclusion. Then $\partial(W, j\psi) = r(M, f\chi^+)$ over $C_m \times C_\infty$. Now the $C_m \times C_\infty$ covering of W determined by $j\psi$ is a trivial infinite cyclic covering of the C_m covering determined by ψ. It follows that $\tau(M, f\chi^+)$ lies in the image of $W(\mathbb{C}$, conjugation$) \otimes \mathbb{Q}$ and has signature $\sigma(M_K, g\varphi)$. On the other hand, $\tau(K,\chi) = \alpha_K^{\mathbb{C}}$. By Lemma 2, $\alpha_K[x] \in W(\mathbb{C}$, conjugation) is defined and has signature $\sigma(M_K, g\varphi)$. The result follows (remembering that signature gives an isomorphism $W(\mathbb{C}$, conjugation$) \cong \mathbb{Z}$). ∎

4. ....OF SATELLITE KNOTS

First we identify the character groups of a satellite.

LEMMA 4. *Let* S *be a satellite of* C *with orbit* K *and winding number* w. *Let* n *be a power of a prime, and set* $h = \mathrm{h.c.f.}(n,w)$ *and* $k = n/h$. *Then*

$$Ch_n(S) \cong Ch_n(K) \oplus h(Ch_k(C))\ ,$$

with the linking form on $Ch_n(S)$ *being the orthogonal sum of the forms on the summands.*

PROOF. We prove the corresponding statement for the dual groups $H_1(L_{-,n})$.

Let A be the axis of K used to construct S. In $L_{K,n}$, A is covered by h curves $\tilde{A}_1,\ldots,\tilde{A}_h$. Let $U_1,\ldots,U_h$ be disjoint tubular neighborhoods of the $\tilde{A}_i$. Let $L^u_{c,k}$ be the unbranched k-fold cyclic covering of S^3 less an open tubular neighborhood of C. Let $X = L_{K,n} - \operatorname{int}(U_1 \cup \cdots \cup U_h)$. We can construct $L_{S,n}$ by gluing a copy of $L^u_{c,k}$ to X along each ∂U_i via an appropriate gluing map. A Mayer-Vietoris argument gives

$$H_1(L_{S,n}) \cong H_1(L_{K,n}) \oplus h(H_1(L_{c,k})) .$$

It remains to determine the linking form. First let x belong to the i'th copy of $H_1(L_{c,k})$. Then x can be represented by a cycle ξ which lies in $L^u_{c,k}$ and represents a torsion element of $H_1(L^u_{c,k}) \cong H_1(L_{c,k}) \oplus \mathbb{Z}$. Let D be a 2-chain in $L^u_{c,k}$ with $\partial D = r\xi$, $r > 0$. For any $y \in H_1(L_{S,n})$ represented by a cycle η,

$$Lk(x,y) \equiv \frac{1}{r}(D\cdot\eta) \mod \mathbb{Z} .$$

It follows that each $H_1(L_{c,k})$ is an orthogonal summand and inherits the correct form from $H_1(L_{S,n})$.

Finally, if $x \in H_1(L_{K,n})$, represent x by a cycle ξ missing $U_1 \cup \cdots \cup U_n$, and let D be a 2-chain in $L_{K,n}$ with $\partial D = r\xi$, $r > 0$, and transverse to the $\tilde{A}_i$. We get a 2-chain D' in $L_{S,n}$ with $\partial D' = r\xi$ by replacing each component of $D \cap U_i$ with a 2-chain in a copy of $L_{c,k}$. It follows that for $y \in H_1(L_{K,n})$ we get the same value of $Lk(x,y)$ by working in $L_{K,n}$ or $L_{S,n}$.

THEOREM 2. *Let S be a satellite of C with orbit K, axis A and winding number w. Let n be a power of a prime,* $h = \text{h.c.f.}(n,w)$, $k = n/h$. *Let $x_i \in H_1(L_{K,n})$ be represented by the* i'th *lift of A,* $i = 1,\ldots,h$. *Identify $Ch_n(S)$ with $Ch_n(K) \oplus h(Ch_k(C))$ as in Lemma 4. Let $\chi_S = (\chi_K,\chi_1,\ldots,\chi_h) \in Ch_n(S)$ be of prime-power order. Then*

$$\tau(S,\chi_S)[t] = \tau(K,\chi_K)[t] + \sum_{i=1}^{h} \tau(C,\chi_i)[\chi_K(x_i)t^{w/h}] .$$

NOTES. (1) The terms under the summation sign are defined by Lemma 3, since either $w \neq 0$ or $\chi_i \in Ch_1(C)$.

(2) It is understood that the i'th lift of A corresponds to the i'th copy of $Ch_k(C)$.

The two extreme cases of this theorem embodied in the following corollaries are probably of most interest.

COROLLARY 1. *In the situation of Theorem 2, suppose that n is coprime to* w. Then $Ch_n(S) \equiv Ch_n(K) \oplus Ch_n(C)$ *and for* $\chi_S = (\chi_K, \chi_c) \in Ch_n(S)$ *of prime-power order we have*

$$\tau(S,\chi_S) = \tau(K,\chi_K) + \tau(C,\chi_c)[t^w] .$$

If $w=1$ this is always the case, and the Casson-Gordon invariants are the same as those of $K \# C$. In general, if n is coprime to w, the invariants associated to Ch_n cannot distinguish between elements of $\mathcal{S}_w(K,C)$.

PROOF. In the situation of Theorem 2 we always have $x_1 + \cdots + x_h = 0$. In the present case, $h=1$ and so $x_1 = 0$. ⁞⁞

COROLLARY 2. *In the situation of Theorem 2, suppose that n divides w. Then $Ch_n(S) \equiv Ch_n(K)$ and for $\chi \in Ch_n(S)$ of prime-power order we have*

$$\tau(S,\chi)[t] = \tau(K,\chi)[t] + \sum_{i=1}^{n} \alpha_C[\chi(x_i)t^{w/n}] .$$

PROOF. Here $Ch_k(C) = Ch_1(C) = \{0\}$ and $\tau(C,0)$ is the image of α_C in $W(\mathbb{C}(t),J) \otimes \mathbb{Q}$. ⁞⁞

PROOF OF THEOREM 2. Let m be the order of χ_S, and regard $\chi_K, \chi_1, \ldots, \chi_h$ as taking values in C_m. Take compact 4-manifolds (W_K,ψ_K), $(W_1,\psi_1), \ldots, (W_h,\psi_h)$ over $C_m \times C_\infty$ such that

$$\partial(W_K,\psi_K) = r(M_{K,n}, \chi_K^+)$$
$$\partial(W_i,\psi_i) = r(M_{C,k}, \chi_i^+) \quad , \quad i = 1,\ldots,h ,$$

for some $r > 0$. Note that $\chi_K^+(x_i) = (\chi_K(x_i), t^{w/h})$ for $i = 1,\ldots,h$. Recall that $M_{C,k}$ is obtained from $L_{C,k}$ by 0-surgery on the lift of C. Let $U \subset M_{C,k}$ be the surgery solid torus, and let $V_i \subset M_{K,n}$ be a tubular neighborhood of the i'th lift of A, with $V_1,\ldots,V_h$ disjoint. For $i = 1,\ldots,h$ and $j = 1,\ldots,r$, let U_{ij} be the copy of U in the j'th boundary component of W_i, and let V_{ij} be the copy of V_i in the j'th boundary component of W_K. We can construct

$$W_S = W_K \cup \bigcup_{i=1}^{h} W_i \tag{2}$$

where each U_{ij} is glued to V_{ij}, so that $\partial W_S = rM_{S,n}$. Define $f_i : C_m \times C_\infty \longrightarrow C_m \times C_\infty$ by $f_i(y) = y$ for $y \in C_m$ and $f_i(t) = (\chi_K(x_i), t^{w/h})$. Then $\psi_K | H_1(V_{ij})$ and $f_i\psi_i | H_1(U_{ij})$ agree under the identification, so we can combine ψ_K and the $f_i\psi_i$ to give $\psi_S : H_1(W_S) \longrightarrow C_m \times C_\infty$, and then

$$\partial(W_S,\psi_S) = r(M_S,\chi_S^+) .$$

Wall additivity applies to (2) in both ordinary and twisted homology, since the kernels corresponding to the pieces of the ∂W_i are the same. Therefore

$$\tau(M_{S,n}, \chi_S^+) = \tau(M_{K,n}, \chi_K^+) + \sum_{i=1}^{h} \tau(M_{C,k}, f_i\chi_i^+) .$$

But $\tau(M_{C,k}, f_i\chi_i^+) = \tau(C,\chi_i)[\chi_K(x_i)t^{w/h}]$ by Lemma 3, completing the proof. ⁞⁞

5. SATELLITES WHICH ARE INDEPENDENT IN $\mathscr{C}^{3,1}$.

In this section we shall prove:

THEOREM 3. *Let* $w \in \mathbb{Z}$ *be given. There exist knots* K *and* C *such that* $\mathscr{S}_w(K,C)$ *contains* $r \geq 2$ *knots representing linearly independent elements of* $\mathscr{C}^{3,1}$ *provided that at least* $2^{r-1} - 1$ *distinct primes divide* w.

Note in particular that for $w \neq \pm 1$ the condition holds with $r = 2$. Further, if $w = 0$ then r can be any integer. On the other hand, we know that the algebraic cobordism class and all the Casson-Gordon invariants are consistent with a positive answer to the following:

QUESTION. Is every member of $\mathscr{S}_1(K,C)$ cobordant to $K \# C$?

Combining Theorems 1 and 3 we have the following result of Jiang [7].

COROLLARY 3. *The cobordism group of algebraically slice knots contains a free abelian group of infinite rank.*

We are going to use Corollary 2. To get any mileage from this we need axes for a knot K whose lifts represent different elements of $H_1(L_{K,n})$ for some factor n of the winding number. This motivates the following definitions. Let A be an axis for K of winding number w, and let n divide w. Let $L = L_{K,n}$, and let $x_1,\dots,x_n \in H_1(L)$ be represented by the lifts of A. We say that A is n-*trivial* if $x_i = 0$ for $i = 1,\dots,n$, and that A is n-*generating* if $x_1,\dots,x_n$ generate $H_1(L)$. Note that, for any factor n' of n, if A is n-trivial then it is n'-trivial and also (since $H_1(L_{K,n}) \longrightarrow H_1(L_{K,n'})$ is onto) if A is n-generating then it is n'-generating. Given K, w and n it is easy to find an n-trivial axis for K of winding number w. In order for K to have an n-generating axis it is necessary for $H_1(L)$ to be cyclic as a $\mathbb{Z}[t,t^{-1}]$-module. It is not hard to see that this is also sufficient, but we shall not make use of this, as we now give specific examples of n-generating axes. If K is a torus knot of type (p,q) then K has two obvious "standard" axes A_p and A_q of winding numbers p and q respectively. (The satellite $\mathscr{S}(K,C;A_q)$ is the (p,q)-cable of C.)

LEMMA 5. *Let* K *be a torus knot of type* (p,q), *where* $p,q > 1$, *and let* n *be a factor of* q.

(i) $H_1(L_{K,n}) \cong (n-1)(\mathbb{Z}/p)$.

(ii) *The standard axis* A_q *is* n-*generating*.

PROOF. Let $L = L_{K,n}$, and let $\tilde{A}_1,\dots,\tilde{A}_n$ be the lifts of A_q to L, representing $x_1,\dots,x_n \in H_1(L)$. Let $\tilde{A}'$ be the single lift of A_p. Let U be a small tubular neighborhood of K, and let $L^u \subset L$ be the n-fold cyclic covering of $S^3 - \mathrm{int}(U)$. Let $y_1,\dots,y_n, y' \in H_1(L^u)$ be represented by $\tilde{A}_1,\dots,\tilde{A}_n, \tilde{A}'$, respectively. The decomposition of $S^3 - \mathrm{int}(U)$ into two solid tori with cores A_p and A_q lifts to a decomposition of L^u into $(n+1)$ solid tori. The Mayer-Vietoris sequence yields

$$H_1(L^u) = \langle y_1,\ldots,y_n,y' \mid py_i = (q/n)y',\ i=1,\ldots,n\rangle$$
$$\cong \mathbb{Z} \oplus (n-1)(\mathbb{Z}/p)$$

since p and q/n are coprime. Hence $H_1(L) \cong (n-1)(\mathbb{Z}/p)$. Since y_i maps to x_i and y' to 0 in $H_1(L)$, $x_1,\ldots,x_n$ generate $H_1(L)$. ⁞⁞

PROPOSITION 2. Let K be a torus knot of type (p,q), where $p,q>1$. Let $q=q'q''$ be a factorization of q into coprime integers, and let w be a multiple of q. Then there is an axis for K of winding number w which is q'-generating and q''-trivial.

PROOF. Let $L' = L_{K,q'}$ and $L'' = L_{K,q''}$. Let A''' be the standard axis for K of winding number q; it is both q'-generating and q''-generating by Lemma 5(ii). Modify A''' by winding it locally around K as in Fig. 3 to give an axis A'' of winding number q'. Because the modification

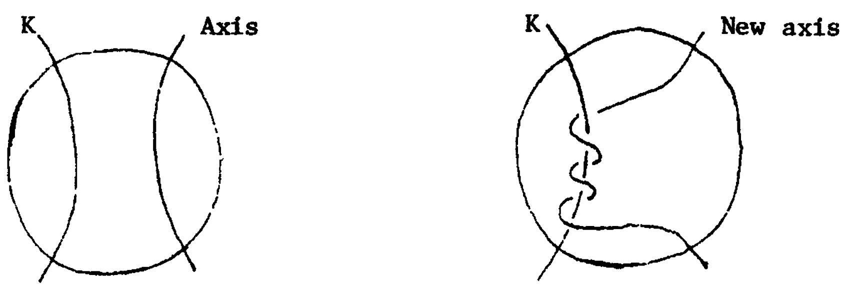

Figure 3

lifts to L' to give an isotopy of each lift of A''', A'' is also q'-generating. On the other hand, A'' is covered by a single, null-homologous curve in L''. Now let A' be a $(1,q'')$-cable about A''; A' is an axis of winding number q. Since each lift of A' to L' is homologous to q'' times the corresponding lift of A'', A' is still q'-generating (by Lemma 5(i)). Each lift of A' to L'' is homologous to the single curve over A'', and hence to zero; i.e. A' is q''-trivial. Finally use the modification of Fig. 3 again to give an axis A of winding number w. This time the modification lifts to both L' and L'', so A has the desired properties. ⁞⁞

The whole process is illustrated in Fig. 4 for the case $p=5$, $q=6$, $q'=3$, $q''=2$ and $w=12$.

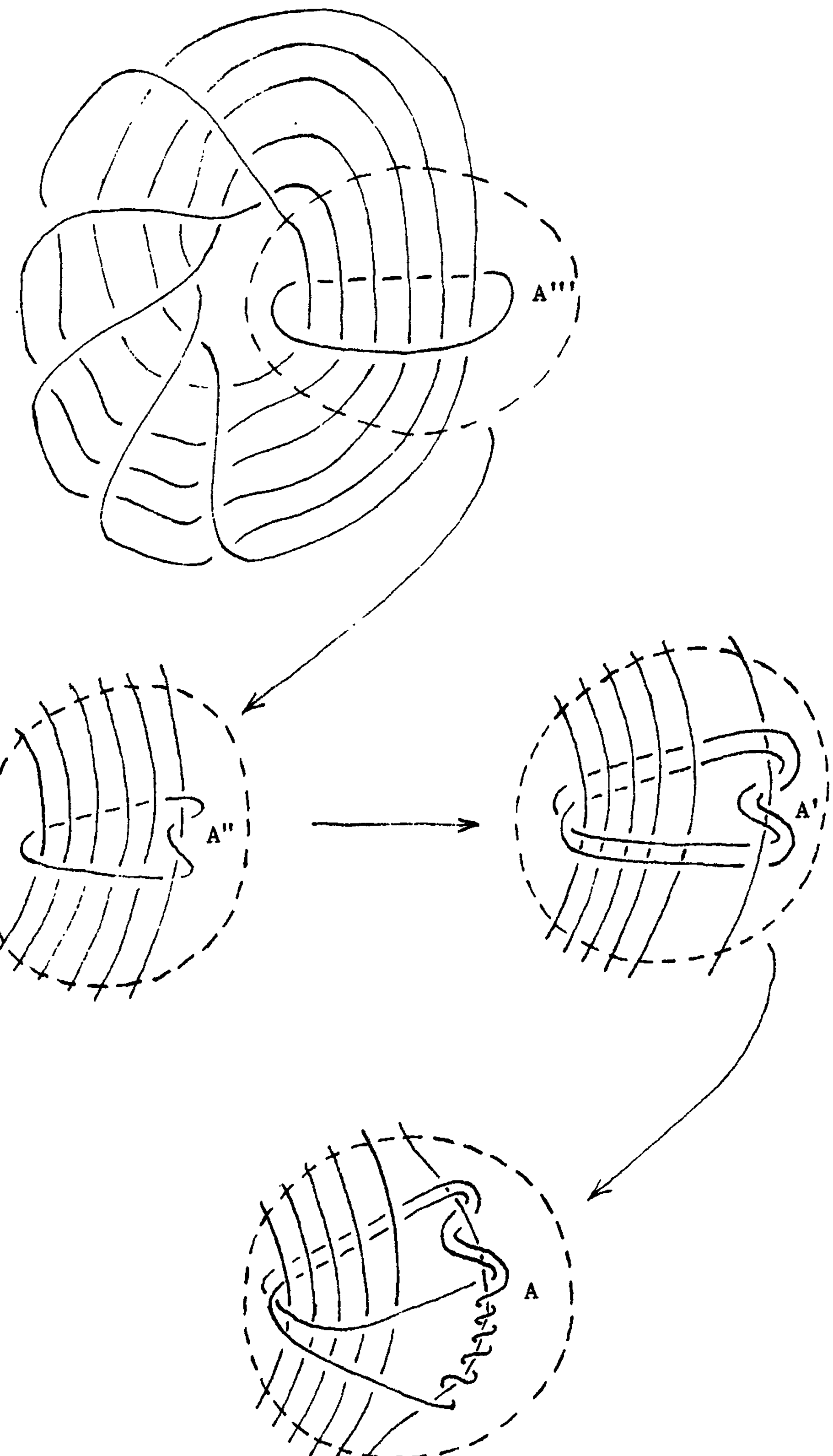

Figure 4

In order to avoid calculating $\tau(K,\chi)$ when applying Corollary 2, we want that term to be swamped by the contributions from C. The next lemma enables us to arrange this.

LEMMA 6. *Given an integer* N *and a neighborhood* U *of* 1 *in* S^1, *there is a knot* C *such that* $\sigma_C(\zeta) \geq N$ for $\zeta \notin U$.

PROOF. There are several ways of constructing such knots. For instance, the torus knot T_n of type $(-2,3^n)$ has $\sigma_{T_n}(e^{2\pi ix}) \geq 2$ for $1/2 \cdot 3^n < x < 1 - 1/2 \cdot 3^n$. (See [15], Proposition 1.) Thus we can take C to be the connected sum of M copies of T_n, $M \geq N/2$, where n is so large that $e^{2\pi ix} \in U$ for $|x| \leq 1/2 \cdot 3^n$. In fact a single copy of T_n will do if n is large enough; $e^{2\pi ix} \in U$ for $|x| \leq N/4 \cdot 3^{n-1} + 1/3^{n-1}$ will certainly suffice. ⦀

We also need a simple piece of linear algebra.

LEMMA 7. *Let* F *be a field and* V *a vector subspace of* F^ν. *Suppose that any element of* V *has fewer than* μ *non-zero coordinates, for some fixed* μ. *Then* $\dim V < \mu$.

PROOF. Let $\pi_i : V \to F$ be the restriction of the i'th coordinate function, $i=1,\dots,\nu$. Then $\pi_1,\dots,\pi_\nu$ generate the dual V^*, so we can pick out a basis $\pi_{i_1},\dots,\pi_{i_d}$ for V^*, where $d = \dim V$. There is an element v of V such that $\pi_{i_j}(v) = 1$ for $j=1,\dots,d$, so $d < \mu$. ⦀

PROOF OF THEOREM 3. First observe that if K is a torus knot of type (p,q) $(p,q > 1)$ then, for any knot C and any multiple w of q, each satellite $S \in \mathscr{S}_w(K,C)$ maps to an element of infinite order in $W_S(\mathbb{Z})$. This is because $\sigma_S(\zeta) = \sigma_K(\zeta)$ if $\zeta^w = 1$ (by [15] Theorem 2, or Theorem 1 above), while $\sigma_K(e^{2\pi i/q}) \neq 0$ (by [15], Proposition 1).

Now let w and r be as in the theorem. Choose distinct primes q_I dividing w, one for each non-empty subset I of $\{1,\dots,r-1\}$, and let p be any prime distinct from all the q_I. Let $q = \prod_I q_I$, and let K be the torus knot of type (p,q). Set

$$N = \max|\sigma_1 \tau(K,\chi)|$$

where χ ranges over $\bigcup_I Ch_{q_I}(K)$. For the definition of $\sigma_\zeta \tau$, where $\tau \in W(\mathbb{C}(t),J) \otimes \mathbb{Q}$, see [1] or [6], Section 13.) By Lemma 6, we can take a knot C so that

$$\sigma_C(\zeta) > 4N \quad \text{whenever} \quad \zeta^p = 1 \ , \ \zeta \neq 1 \ .$$

It remains to choose r axes for K. For $i=1,\dots,r$ we have a factorization $q = q_i' q_i''$ where $q_i' = \prod_{I: i \in I} q_I$ and $q_i'' = \prod_{I: i \notin I} q_I$. By Proposition 2 there is an axis A_i for K of winding number w which is q_i'-generating and q_i''-trivial. Let $S_i = \mathscr{S}(K,C;A_i)$. We claim that $S_1,\dots,S_r$ represent linearly independent elements of $\mathscr{C}^{3,1}$.

Suppose they do not. Any non-trivial relation can be written in the form

$$(3) \qquad \sum_{k=1}^{\ell} S_{i_k} \sim \sum_{k=1}^{\ell'} S_{j_k}$$

where "$\sim$" means "is cobordant to", $\ell > 0$, $1 \leq i_k, j_k \leq r$ and $i_k \neq j_{k'}$ for any k, k'. Since all the S_i map to the same element of infinite order in $W_S(\mathbb{Z})$, we have $\ell' = \ell$. Let $I = \{i_k | k=1,\ldots\ell\}$; I is non-empty and we may assume (by switching the sides of (3) if necessary) that $r \notin I$. Set $n = q_I$. Note that A_i is n-generating if $i \in I$, and n-trivial if not.

We shall use the τ-invariants associated to Ch_n. Let

$$\mathscr{W} = \sum_{k=1}^{\ell} Ch_n(S_{i_k}) \oplus \sum_{k=1}^{\ell} (-Ch_n(S_{j_k})) .$$

Since $n|w$, Lemma 4 gives

$$Ch_n(S_{i_k}) = Ch_n(S_{j_k}) = Ch_n(K) ,$$

which is isomorphic to $(n-1)(\mathbb{Z}/p)$ by Lemma 5(i). In particular, $\mathscr{W}$ has prime-power order, so the relation (3) implies that $\mathscr{W}$ has a metaboliser $\mathscr{M}$ such that

$$(4) \qquad \sum_{k=1}^{\ell} \tau(S_{i_k}, \chi_{i_k}) = \sum_{k=1}^{\ell} \tau(S_{j_k}, \chi_{j_k})$$

whenever $(\chi_{i_k}) \oplus (\chi_{j_k}) \in \mathscr{M}$. For any S_i and $\chi \in Ch_n(K)$, Corollary 2 gives

$$\sigma_1 \tau(S_i, \chi) = \sigma_1 \tau(K, \chi) + \sum_{s=1}^{n} \sigma_C(\chi x_s^{(i)})$$

where $x_1^{(i)}, \ldots, x_n^{(i)} \in H_1(L_{K,n})$ are represented by the lifts of A_i. If $i \notin I$ this simplifies to

$$\sigma_1 \tau(S_i, \chi) = \sigma_1 \tau(K, \chi)$$

(A_i being n-trivial). Therefore (4) gives

$$(5) \qquad \sum_{k=1}^{\ell} \sum_{s=1}^{n} \sigma_C(\chi_{i_k} x_s^{(i_k)}) = \sum_{k=1}^{\ell} (\sigma_1 \tau(K, \chi_{j_k}) - \sigma_1 \tau(K, \chi_{i_k})) \leq 2\ell N$$

for $(\chi_{i_k}) \oplus (\chi_{j_k}) \in \mathscr{M}$. Now for any $\chi \in Ch_n(K)$ and $x \in H_1(L_{K,n})$ we have $\chi(x)^p = 1$, so either $\sigma_C(\chi(x)) > 4N$ or $\chi(x) = 1$. Thus (5) implies that there are fewer than $\frac{1}{2}\ell$ values of k for which some $\chi_{i_k} x_s^{(i_k)} \neq 1$. Since A_{i_k} is n-generating

$$\chi_{i_k} x_s^{(i_k)} = 1 \quad \text{for} \quad s=1,\ldots,n \implies \chi_{i_k} = 0 .$$

Thus we have shown that if $(\chi_{i_k}) \oplus (\chi_{j_k}) \in \mathscr{M}$ then χ_{i_k} is non-zero for fewer than $\frac{1}{2}\ell$ values of k.

Write $\mathcal{W}$ as $\mathcal{W}_1 \oplus \mathcal{W}_2$, where $\mathcal{W}_1 = \Sigma\, \mathrm{Ch}_n(S_{i_k})$ and $\mathcal{W}_2 = \Sigma(-\mathrm{Ch}_n(S_{j_k}))$. Let $\mathcal{M}_1$ be the projection of $\mathcal{M}$ into $\mathcal{W}_1$ and let $\mathcal{M}_2 = \mathcal{M} \cap \mathcal{W}_2$, so that $\mathcal{M}_1 \cong \mathcal{M}/\mathcal{M}_2$. Identify $\mathrm{Ch}_n(K)$ with $(n-1)(\mathbb{Z}/p)$, and hence identify $\mathcal{W}_1$ with $\ell(n-1)(\mathbb{Z}/p)$. Under this identification, each element of $\mathcal{M}_1$ has fewer than $\frac{1}{2}\ell(n-1)$ non-zero coordinates, so by Lemma 7

$$\dim_{\mathbb{Z}/p} \mathcal{M}_1 < \tfrac{1}{2}\ell(n-1) = \tfrac{1}{2} \dim_{\mathbb{Z}/p} \mathcal{W}_1$$

or

$$|\mathcal{M}_1| < |\mathcal{W}_1|^{\frac{1}{2}}$$

Hence

$$|\mathcal{M}_2| > |\mathcal{W}_2|^{\frac{1}{2}} .$$

But $\mathcal{M}_2$ is a self-annihilating subgroup of $\mathcal{W}_2$, so this is impossible. This contradiction establishes the independence of $S_1,\ldots,S_r$ in $\mathcal{C}^{3,1}$. ▒

6. HOMOLOGY HANDLES.

In [9], Kawauchi defined a group $\Omega(S^1 \times S^2)$ which fits into a commutative diagram

$$\begin{tikzcd} \mathcal{C}^{3,1} \arrow[rr, "e"] \arrow[dr] & & \Omega(S^1 \times S^2) \arrow[dl] \\ & W_S(\mathbb{Z}) & \end{tikzcd}$$

and which may be described as follows. A <u>homology handle</u> is a 3-manifold with the integral homology of $S^1 \times S^2$, and a <u>special homology handle</u> is a homology handle M together with an isomorphism $\varphi: H_1(M) \to C_\infty$. (In comparing this with Kawauchi's definition, remember that all our manifolds are oriented.) Two such objects (M_1,φ_1), (M_2,φ_2) are $\tilde{H}$-<u>cobordant</u> if there is a compact 4-manifold (W,ψ) over C_∞ such that

$$\partial(W,\psi) = (M_1,\varphi_1) \cup (-M_2,\varphi_2)$$

and

$$H_*^t(W;Q\Gamma) = 0 .$$

The elements of $\Omega(S^1 \times S^2)$ are the $\tilde{H}$-cobordism classes of special homology handles. The group operation will be described later; the zero is represented by $S^1 \times S^2$ (with either φ), and the inverse of $[M,\varphi]$ is $[-M,\varphi]$. The homomorphism $e: \mathcal{C}^{3,1} \longrightarrow \Omega(S^1 \times S^2)$ is given by $[K] \longrightarrow [M_K,\varphi_K]$.

Now if (M_1,φ_1) and (M_2,φ_2) are $\tilde{H}$-cobordant, it is clear that

$$\alpha(M_1,\varphi_1) - \alpha(M_2,\varphi_2) = \alpha(M_1,\varphi_1)[1] - \alpha(M_2,\varphi_2)[1] ,$$

which is zero by Lemma 1. Therefore we have a commutative diagram

$$\begin{tikzcd} \mathscr{C}^{3,1} \arrow[rr, "e"] \arrow[dr, "\alpha"'] & & \Omega(S^1 \times S^2) \arrow[dl, "\alpha"] \\ & W(Q\Gamma, J) & \end{tikzcd}$$

It can be shown (see below) that $\alpha : \Omega(S^1 \times S^2) \longrightarrow W(Q\Gamma, J)$ is a homomorphism. It can also be shown, exactly as in Section 2, that it is equivalent to the homomorphism $\Omega(S^1 \times S^2) \longrightarrow W_S(\mathbb{Z})$ defined by Kawauchi.

We now define operations on $\Omega(S^1 \times S^2)$ analogous to forming satellite knots. Let (M_1, φ_1) and (M_2, φ_2) be special homology handles, and let w be an integer. Choose embeddings

$$j_i : S^1 \times D^2 \longrightarrow M_i \quad , \quad i=1,2,$$

such that j_1 is orientation preserving, j_2 is orientation reversing and

$$\varphi_1 j_{1*}[S^1 \times 0] = t^w \ ,$$

$$\varphi_2 j_{2*}[S^1 \times 0] = t \ .$$

Define a special homology handle (M, φ) by

$$M = (M_1 - j_1(S^1 \times \operatorname{int} D^2)) \bigcup_{\substack{j_1(x) \equiv j_2(x) \\ x \in S^1 \times \partial D^2}} (M_2 - j_2(S^1 \times \operatorname{int} D^2)) \ ,$$

$$\varphi | H_1(M_1) = \varphi_1 \ , \ \varphi | H_1(M_2) = w\varphi_2 \ .$$

We write

$$(M, \varphi) = (M_1, \varphi_1) \ O_w \ (M_2, \varphi_2) \ .$$

The case $w=1$ was considered by Kawauchi, who called it circle union. This construction is not well-defined, since in general (M, φ) depends on the choice of j_1 and j_2. However, we have:

PROPOSITION 3. (Kawauchi [9] in case $w=1$). Fix $w \neq 0$. Then the $\tilde{H}$-cobordism class of $(M_1, \varphi_1)\ O_w\ (M_2, \varphi_2)$ depends only on those of (M_1, φ_1) and (M_2, φ_2).

PROOF. (cf [9], Lemma 1.6) For $i = 1,2$ let (W_i, ψ_i) be an $\tilde{H}$-cobordism from (M_i, φ_i) to (M_i', φ_i'), and let

$$j_i : S^1 \times D^2 \longrightarrow M_i \quad , \quad j_i' : S^1 \times D^2 \longrightarrow M_i'$$

be embeddings as in the definition of O_w. Let

$$(M, \varphi) = (M_1, \varphi_1) \ O_w \ (M_2, \varphi_2) \ ,$$

$$(M', \varphi') = (M_1', \varphi_1') \ O_w \ (M_2', \varphi_2') \ ,$$

constructed using j_i, j_i' respectively. We must show that (M,φ) is $\tilde{H}$-cobordant to (M',φ'). Let W be obtained from the disjoint union $W_1 \sqcup W_2$ by making the identifications

$$j_1(x) \equiv j_2(x)\ ,\ j_1'(x) \equiv j_2'(x)\ ,\ x \in S^1 \times D^2$$

Define $\psi: H_1(W) \longrightarrow C_\infty$ by

$$\psi|H_1(W_1) = \psi_1\ ,\ \psi|H_1(W_2) = w\psi_2\ .$$

Then

$$\partial(W,\psi) = (M,\varphi) \cup (-M',\varphi')\ .$$

That $H_*^t(W,Q\Gamma) = 0$ follows from the Mayer-Vietoris sequence. (Note that, expanding the notation for twisted homology to indicate the twisting homorphism, $H_*^t((W_2,w\psi_2);Q\Gamma) = 0$ because the infinite cyclic covering of W_2 determined by $w\psi_2$ consists of $|w|$ copies of the one determined by ψ_2. This fails for $w=0$; $H_*^t((W_2,0);Q\Gamma) = H_*(W_2;\mathbb{Q}) \otimes_{\mathbb{Q}} Q\Gamma$.) ⦙⦙

Thus we have, for each $w \neq 0$, a well-defined binary operation $\circ_w$ in $\Omega(S^1 \times S^2)$. The addition in $\Omega(S^1 \times S^2)$ is $\circ_1$.

The next result if immediate from the definitions.

PROPOSITION 4. *Let* S *be a satellite of the knot* C *with orbit* K *and winding number* w. *Then*

$$(M_S,\varphi_S) = (M_K,\varphi_K)\ \circ_w\ (M_C,\varphi_C)\ .\quad ⦙⦙$$

Observe that the proof of Theorem 1 actually shows that if $(M,\varphi) = (M_1,\varphi_1)\ \circ_w\ (M_2,\varphi_2)$ then

$$\alpha(M,\varphi)[t] = \alpha(M_1,\varphi_1)[t] + \alpha(M_2,\varphi_2)[t^w]$$

(even for $w=0$). The case $w=1$ justifies our earlier claim that α induces a homomorphism $\Omega(S^1 \times S^2) \longrightarrow W(Q\Gamma;J)$.

From Propositions 3 and 4 we see that for any knots K and C and any nonzero integer w, $\mathcal{S}_w(K,C)$ maps to a single element of $\Omega(S^1 \times S^2)$. Together with Theorem 3 this gives:

THEOREM 4. *The kernel of the homorphism* $e: \mathcal{C}^{3,1} \longrightarrow \Omega(S^1 \times S^2)$ *contains a free abelian group of infinite rank.* ⦙⦙

Of course, whether or not this is really stronger than Corollary 3 depends on whether $\Omega(S^1 \times S^2) \longrightarrow W_S(\mathbb{Z})$ has kernel or not, which is an open question.

APPENDIX A. The Witt group of Hermitian forms over a function field.

A0. PREAMBLE. Let k be a field of characteristic different from 2, provided with an involution $x \mapsto \bar{x}$ (which may be trivial). Let the involution J of the rational function field $k(t)$ be given by $f(t)^J = \bar{f}(t^{-1})$. We study the Witt group $W(k(t),J)$ of Hermitian forms over $k(t)$, and prove a version of Milnor's exact sequence for the Witt group of symmetric forms [20]. If the involution of k is trivial then Milnor's proof can be carried over virtually unchanged, because the fixed field of $k(t)$ is $k(t+t^{-1})$. In the general case this does not work; our approach was suggested by Trotter's proof that the Blanchfield pairing of a knot determines its Seifert form. As special cases we obtain the isomorphism $W(\mathbb{Q}(t),J) \cong W_S(\mathbb{Q}) \oplus W(\mathbb{Q})$ mentioned in Section 2, and a computation of the group $W(\mathbb{C}(t),J)$ used by Casson and Gordon.

At the heart of Trotter's proof is his "trace" function $\mathbb{Q}(t)/Q[t,t^{-1}] \longrightarrow \mathbb{Q}$ ([25],[26]). We use a slightly different function which has a nice geometric interpretation, given in Section A4. We also give a new proof of a result of Matumoto [18].

A1. GENERALITIES

We recall the definitions and elementary results on Witt groups that we need. General references for this are [21] and [3]. In what follows, Γ is a PID with involution J, $\operatorname{char}\Gamma \neq 2$, $Q\Gamma$ is the field of fractions and $\Gamma^{\bullet}$ is the group of units. Also $(k,-)$ is a field-with-involution, $\operatorname{char} k \neq 2$, and $k[t,t^{-1}]$ is given the involution $f(t)^J = \bar{f}(t^{-1})$. The case of trivial involution is allowed.

Let N be a Γ-module with an involution, also called J, such that $(\gamma n)^J = \gamma^J n^J$ for $\gamma \in \Gamma$, $n \in N$. If M is a Γ-module and $\varphi: M\times M \to N$ is a sesquilinear pairing (linear in the first variable, conjugate linear in the second) we denote by φ^* the pairing $\varphi^*(x,y) = \varphi(y,x)^J$. Let $u \in \Gamma$ have form $uu^J = 1$. If $\varphi = u\varphi^*$ then φ is said to be <u>u-Hermitian</u>. We have the adjoint homomorphism

$$\operatorname{ad}\varphi: M \longrightarrow \overline{\operatorname{Hom}}(M,N) \quad ;$$
$$\operatorname{ad}\varphi(x)(y) = \varphi(x,y) \quad ,$$

where $\overline{\operatorname{Hom}}$ denotes the module of conjugate-linear maps. We say that φ is <u>non-singular</u> if $\operatorname{ad}\varphi$ and $\operatorname{ad}\varphi^*$ are isomorphisms.

We need four kinds of Witt groups; we now list the objects from which they are formed.

(a) $W_u(\Gamma,J)$ for $u \in \Gamma$ of norm 1. The objects are <u>u-Hermitian</u> <u>spaces</u> (V,φ); i.e. V is a finitely-generated free Γ-module and $\varphi: V\times V \to \Gamma$ is u-Hermitian and non-singular.

(b) $W(Q\Gamma/\Gamma,J)$. <u>Torsion forms</u> (M,φ); i.e. M is a finitely-generated torsion Γ-module and $\varphi: M\times M \longrightarrow Q\Gamma/\Gamma$ is Hermitian and non-singular.

(c) $W_-(C_\infty;k,-)$. <u>Skew-isometric structures</u> (V,φ,t); i.e. (V,φ) is a skew-Hermitian space over k and t is an isometry of (V,φ). Here a metaboliser is required to be t-invariant.

(d) $W_S(k,-)$. <u>Seifert forms</u> $(V,\mathcal{S})$, i.e. V is a finite dimensional k-vector space and $\mathcal{S}: V\times V \longrightarrow k$ is sesquilinear and non-singular, with $\mathcal{S}-\mathcal{S}^*$ also non-singular.

REMARK. The last group was defined by Levine [12] for the case of trivial involution, with two differences. Namely, he did not require that $\mathscr{P}$ be non-singular, but did require $\mathscr{P}+\mathscr{P}^*$ to be so. As to the first, it is shown in [13] that every Witt class has a non-singular representative. It will follow from Section A5 below that the second change does not affect the Witt group either.

If $v \in \Gamma$ and $\varphi: V \times V \longrightarrow \Gamma$ is u-Hermitian then $v\varphi$ is (uv/v^J)-Hermitian and we get an isomorphism $v: W_u(\Gamma,J) \longrightarrow W_{uv/v^J}(\Gamma,J)$. By Hilbert's Theorem 90 this gives:

PROPOSITION A1.

$$W_u(k,-) \cong \begin{cases} 0 & \underline{\text{if}} \ - \ \underline{\text{is trivial and}} \ u=-1 ; \\ W(k,-) & \underline{\text{otherwise}} . \end{cases}$$

Since $W(k,-)$ is generated by rank 1 forms, the same is true of $W_u(k,-)$. For $\gamma \in \Gamma^{\bullet}$ with $\gamma = u\gamma^J$ we denote the corresponding rank 1 form, or its class in $W_u(\Gamma,J)$, by $\langle\gamma\rangle$.

The following remarks apply to both (b) and (c). In case (b), let (M,φ) be a torsion form over Γ. In case (c), let (V,φ,t) be a skew-isometric structure over k. Set $\Gamma = k[t,t^{-1}]$, and think of (V,t) as a finitely generated torsion Γ-module M. If we restrict M to be -torsion, where is a symmetric ($= {}^J$) prime ideal of Γ, we obtain Witt groups $W(Q\Gamma/\Gamma,J)$ and $W_-(C_\infty;k,-)$. For any prime ideal of Γ let M denote the -torsion part of M. Then M is the orthogonal sum of M for $= {}^J$ and $M \oplus M_J$ for $\neq {}^J$, and the latter summands are metabolic. This gives canonical isomorphisms

(A1) $\quad W(Q\Gamma/\Gamma,J) \cong \ _{=}{}_J \ W(Q\Gamma/\Gamma,J) \quad ,$

(A2) $\quad W_-(C_\infty;k,-) = \ _{=}{}_J \ W_-(C_\infty;k,-) \quad .$

We denote by $W^O(Q\Gamma/\Gamma,J)$, $W_-^O(C_\infty;k,-)$ the sum of those terms on the right-hand side of (A1), (A2) (respectively) for which $\neq (t-1)$.

One can further show that any element of $W(Q\Gamma/\Gamma,J)$ or $W_-(C_\infty;k,-)$ can be represented by a form for which $M = 0$. In case (c) it follows that $W_-(C_\infty;k,-)_{(t-1)}$ can be identified with $W_-(k,-)$. In particular, if the involution of k is trivial, $W_-(C_\infty;k) = W_-^O(C_\infty;k)$. In case (b) it follows that the summands of (A1) are (non-canonically) isomorphic to groups of type (a). For let π be a generator of . Then $\pi^J = u\pi$ for some $u \in \Gamma^{\bullet}$. Since φ takes values in $(\frac{1}{\pi}\Gamma)/\Gamma$, we have a pairing

$$\pi^J\varphi: M \times M \longrightarrow \Gamma/ \quad .$$

Regarding M as a $\Gamma/\ $ - vector space, this is a non-singular, $\hat{u}$-Hermitian form, where $\hat{}$ denotes reduction modulo . It can be shown that $(M,\varphi) \longrightarrow (M,\pi^J\varphi)$ (for $M = 0$) induces an isomorphism.

$$\pi^J_*: W(Q\Gamma/\Gamma,J) \longrightarrow W_{\hat{u}}(\Gamma/\ ,J) .$$

Finally, we recall the "localization" exact sequence and deal more fully with the subject of induced homorphisms treated in Section 2. The sequence is

$$0 \longrightarrow W(\Gamma,J) \xrightarrow{i_*} W(Q\Gamma,J) \xrightarrow{\partial} W(Q\Gamma/\Gamma,J) .$$

The first homomorphism is induced by the inclusion $\Gamma \longrightarrow Q\Gamma$. (Homomorphisms induced by injections cause no problems, of course.) The definition of the second runs as follows. Let (V,φ) be a Hermitian space over $Q\Gamma$. If L is a Γ-lattice in V with $L \leq L^{\#}$ (definition as in Section 2) then $\partial[V,\varphi]$ is represented by the torsion form

$$\varphi': L^{\#}/L \times L^{\#}/L \longrightarrow Q\Gamma/\Gamma \ ;$$

$$\varphi'([x],[y]) \equiv \varphi(x,y) \mod \Gamma \ .$$

We denote the composition of ∂ with the projection to $W(Q\Gamma/\Gamma,J)$ by ∂ , and if π is a generator of we write ∂_π for $\pi^J_*\partial : W(Q\Gamma,J) \longrightarrow W_{\hat{u}}(\Gamma/\ ,J)$. Now $W(Q\Gamma,J)$ is generated by $\langle\gamma\rangle$ for $\gamma \in Q\Gamma^{\bullet}$, $\gamma = \gamma^J$. We may assume that $\gamma \in \Gamma$ and that either γ is coprime to π or $\gamma = \pi\delta$ with δ coprime to π. Computation shows that

$$\partial_\pi\langle\gamma\rangle = 0 \qquad \text{for } \gamma \text{ coprime to } \pi \ ;$$

$$\partial_\pi\langle\pi\delta\rangle = \langle\hat{\delta}\rangle \qquad \text{for } \delta \text{ coprime to } \pi \ .$$

(cf. [21], Chapter IV, (1.2).)

If $L \leq L^{\#}$ we obtain a Hermitian pairing φ on the $\Gamma/\ $ vector space $L = L \otimes_\Gamma \Gamma/\ $ by setting

$$\varphi\ (x \otimes \alpha, y \otimes \beta) = \alpha\beta^J\varphi(x,y)^{\wedge} \ , \quad x,y \in L,\ \alpha,\ \beta \in \Gamma/\ \ .$$

<u>PROPOSITION A2</u>. <u>We can choose</u> L <u>so that</u> φ <u>is non-singular if and only if</u> $\partial\ [V,\varphi] = 0$.

PROOF. Identifying $L^{\#}$ with $\overline{\mathrm{Hom}}_\Gamma(L,\Gamma)$ we have an exact sequence

$$0 \rightarrow L \xrightarrow{\mathrm{ad}\,\varphi} \overline{\mathrm{Hom}}_\Gamma(L,\Gamma) \longrightarrow L^{\#}/L \longrightarrow 0 \ .$$

Tensoring with $\Gamma/\ $ gives an exact sequence

$$\mathrm{Tor}_\Gamma(L^{\#}/L,\Gamma/\ ) \longrightarrow L \xrightarrow{\mathrm{ad}\,\varphi} \overline{\mathrm{Hom}}_{\Gamma/}\ (L\ ,\Gamma/\ ) \longrightarrow (L^{\#}/L) \otimes_\Gamma \Gamma/\ \longrightarrow 0.$$

Thus φ is non-singular iff $L^{\#}/L$ has no -torsion. If this is the case, certainly $\partial\ [V,\varphi] = 0$. For the converse, let L_1 be any lattice with $L_1 \leq L_1^{\#}$. Set $M = L_1/L_1^{\#}$, and write $M = M \ \oplus M$ where M is the part of M with torsion coprime to .

This is an orthogonal sum. If $\partial\,[V,\varphi] = 0$, $M.$ has a metabolizer N. Let $p{:}L_1^{\#} \to M$ be the quotient map. Then $L = p^{-1}(N)$ is a lattice with $L_1 \leq L \leq L_1^{\#}$.

$$L^{\#} = p^{-1}(N^{\perp}) = p^{-1}(N \oplus M \quad) ,$$

so $L \leq L^{\#}$ and $L^{\#}/L \cong M$ has no -torsion. ▥

It is now not hard to show that one can define a homomorphism $\ker\partial \longrightarrow W(\Gamma/\ ,J)$ by $[V,\varphi] \longrightarrow [L\ ,\ \varphi\ ]$, where L is chosen so that φ is non-singular. If $f{:}\,(\Gamma,J) \longrightarrow (\Gamma',J')$ is a homomorphism of PID's-with-involution and $= \ker f$, one gets an induced homomorphism

$$f_*{:}\ \ker\partial \longrightarrow W(\Gamma/\ ,J) \longrightarrow W(\Gamma',J') .$$

Thus $\ker\partial$ is the subgroup called $\mathrm{Def}(f_*)$ in Section 2. If $\Gamma = k[t,t^{-1}]$ we use the notation $\tau[x] = f_*(\tau)$ introduced in Section 2.

A2. THE LOCALIZATION SEQUENCE FOR A FUNCTION FIELD

For the rest of this appendix, $(k,-)$ is a field-with-involution, $\mathrm{char}(k) \neq 2$, $\Gamma = k[t,t^{-1}]$, $Q\Gamma$ is the quotient field $k(t)$ and J is the involution $f(t)^J = \bar{f}(t^{-1})$ of Γ or $Q\Gamma$.

<u>LEMMA A1</u>. <u>The sequence</u>

$$0 \longrightarrow W(k,-) \xrightarrow{i_*} W(Q\Gamma,J) \xrightarrow{\partial} W(Q\Gamma/\Gamma,J)$$

<u>is exact, where</u> i_* <u>is induced by the inclusion</u> $k \longrightarrow Q\Gamma$.

<u>PROOF</u>. This amounts to showing that the map $W(k,-) \longrightarrow W(\Gamma,J)$ induced by inclusion is an isomorphism. It has a left inverse π given by $\pi(\tau) = \tau[1]$, so it is enough to show that π is injective. First we show that $\ker(\pi)$ is generated by forms of rank 2. Let (L,φ) be a Hermitian space over Γ, and suppose that $\pi[L,\varphi] = 0$. This means that (L,φ) becomes metabolic upon tensoring with k, so there exists a non-zero x in L such that $\varphi(x,x) = f$ and $f(1) = 0$. Without loss of generality we may assume that x is primitive, so there exists y in L with $\varphi(x,y) = 1$. Let W be the submodule of L spanned by x and y. Suppose that $z \in W \cap W^{\perp}$, and let $z = ax + by$, $a,b \in \Gamma$. Then

$$0 = \varphi(z,x) = a\varphi(x,x) + b$$

and

$$0 = \varphi(z,y) = a + b\varphi(y,y) .$$

It follows from the first equation that $b(1) = 0$, and then from the second that $a(1) = 0$. Hence $z = (1-t)z'$ with $z' \in W \cap W^{\perp}$. Since this process can be repeated indefinitely, z must be zero, and we have $W \cap W^{\perp} = 0$. Therefore $L = W^{\perp} \oplus W^{\perp\perp}$, and $\varphi|W^{\perp\perp}$ is a rank 2 form representing an element of $\ker(\pi)$. By induction, φ is a sum of such forms.

Now consider a rank 2 form (L,φ) representing an element of $\ker(\pi)$. Let A be a matrix for φ. Then $\det A(1) = -\alpha\bar{\alpha}$ for some $\alpha \in k^{\bullet}$. Since φ is non-singular over Γ, $\det A \in k^{\bullet}$, so $\det A = -\alpha\bar{\alpha}$. Changing basis, we may assume that $\det A = -1$. Let the corresponding basis of L be x,y. Set $f = \varphi(x,x)$, $g = \varphi(x,y)$ and $h = \varphi(y,y)$. If $h = 0$, φ is metabolic. If not, let $z = hx + (1-g)y \neq 0$. We have

$$\begin{aligned}\varphi(z,z) &= h^2 f + h(1-g)^J g + h(1-g)g^J + (1-g)(1-g)^J h \\ &= h^2 f + h(1-gg^J) \\ &= 0 \quad \text{since} \quad fh - gg^J = \det A = -1 .\end{aligned}$$

Thus φ is metabolic in any case, and so $\ker(\pi) = 0$ as claimed. ⫴

The formula (A1) and the discussion following it show that the computation of $W(Q\Gamma/\Gamma,J)$ reduces to the computation of the Hermitian Witt groups of finite extensions of k. These are known if k is a finite extension of the rationals (Landherr [11]), or $\mathbb{R}$ or $\mathbb{C}$. Below we determine the image of ∂ and show that the sequence splits, which determines $W(Q\Gamma,J) = W(k(t),J)$ in these cases.

A3. A TRACE FUNCTION, TORSION FORMS AND SKEW ISOMETRIC STRUCTURES

We are going to define a k-linear function $\chi: Q\Gamma/\Gamma \to k$. Let Γ^* be the Γ-module of all Laurent power series $\sum_{i=-\infty}^{\infty} a_i t^i$, $a_i \in k$, and extend J to Γ^*. There are two fields

$$\Gamma_+ = \{ \sum_{i=m}^{\infty} a_i t^i \mid m \in \mathbb{Z} , a_i \in k\} ,$$

and

$$\Gamma_- = \{ \sum_{i=-\infty}^{n} b_i t^i \mid n \in \mathbb{Z} , b_i \in k\}$$

inside Γ^*. These give rise to two Γ-linear embeddings $i_+, i_-: Q\Gamma \to \Gamma^*$. Since $\Gamma_+ \cap \Gamma_- = \Gamma$, $i_+ - i_-$ induces a Γ-linear embedding $j: Q\Gamma/\Gamma \to \Gamma^*$. Let $\text{const}: \Gamma^* \to k$ be given by $\text{const}(\Sigma a_i t^i) = a_0$, and let χ be the k-linear map $(\text{const})\, j$. The properties of χ that we need are:

<u>PROPOSITION A3.</u>

(i) $\chi(x^J) = -\overline{\chi(x)}$ <u>for</u> $x \in Q\Gamma/\Gamma$;

(ii) $\chi_*: \text{Hom}_\Gamma(M,Q\Gamma/\Gamma) \longrightarrow \text{Hom}_k(M,k)$ <u>is an isomorphism for every torsion Γ-module</u> M.

Part (ii) says that χ is a universal element for the functor $\text{Hom}_k(-,k)$ of torsion Γ-modules.

<u>PROOF</u> (i) This follows from the observation that

$$i_+(f^J) = i_-(f)^J \quad \text{for} \quad f \in Q\Gamma .$$

(ii) First we show that χ_* is injective. Note that if N is a Γ-submodule of Γ^* and $\text{const}(N) = 0$ then $N = 0$. Let $\varphi \in \ker \chi_*$. Then $\text{const}(j\varphi M) = 0$, so $j\varphi M = 0$ and since j is injective $\varphi M = 0$, i.e. $\varphi = 0$. If M is finitely generated then $\text{Hom}_\Gamma(M,Q\Gamma/\Gamma)$ and $\text{Hom}_k(M,k)$ have the same finite dimension over k, so χ_* is an isomorphism. The general case follows because M is the union of its finitely generated submodules. |||

Let M be a finitely generated torsion Γ-module. We claim that there is a bijection between Hermitian forms $\varphi: M\times M \longrightarrow Q\Gamma/\Gamma$ and skew-Hermitian forms $\psi: M\times M \to k$ with the property that $\psi(tx,y) = \psi(x,t^{-1}y)$ for $x,y\in M$, given by $\varphi \to \chi\varphi$. To see this, let $\bar{M}$ denote M with the conjugate action of Γ. Regard forms of the first type as elements of $\mathrm{Hom}_\Gamma(M\otimes_\Gamma \bar{M}, Q\Gamma/\Gamma)$ such that

$$\begin{array}{ccc} M\otimes_\Gamma \bar{M} & \xrightarrow{\varphi} & Q\Gamma/\Gamma \\ \downarrow{\scriptstyle\sigma} & & \uparrow{\scriptstyle J} \\ M\otimes_\Gamma \bar{M} & \xrightarrow{\varphi} & Q\Gamma/\Gamma \end{array}$$

commutes, where σ switches the factors. Similarly, forms of the second type are elements of $\mathrm{Hom}_k(M\otimes_\Gamma \bar{M},k)$ such that

$$\begin{array}{ccc} M\otimes_\Gamma \bar{M} & \xrightarrow{\psi} & k \\ \downarrow{\scriptstyle\sigma} & & \uparrow{\scriptstyle\alpha} \\ M\otimes_\Gamma \bar{M} & \xrightarrow{\psi} & k \end{array}$$

commutes, where $\alpha(x) = -\bar{x}$. The claim follows on using Proposition A3. Moreover, φ is non-singular if and only if $\chi\varphi$ is, as one sees by regarding them as elements of $\mathrm{Hom}_\Gamma(M,\mathrm{Hom}_\Gamma(\bar{M},Q\Gamma/\Gamma))$ and $\mathrm{Hom}_\Gamma(M,\mathrm{Hom}_k(\bar{M},k))$ respectively and using the universal property of χ. Therefore if (M,φ) is a torsion form over Γ then $(M,\chi\varphi)$ is a skew-isometric structure over k. Further, (M,φ) is metabolic if and only if $(M,\chi\varphi)$ is. (Use the universal property again for the "if" part.) Thus we have:

<u>LEMMA A2</u>. <u>The trace function</u> χ <u>induces on isomorphism</u>

$$\chi_*: W(Q\Gamma/\Gamma,J) \longrightarrow W_-(C_\infty;k,-) \ . \quad |||$$

This isomorphism respects the splittings (A1) and (A2); in particular it takes $W^0(Q\Gamma/\Gamma,J)$ onto $W^0_-(C_\infty;k,-)$. Recall the homomorphism $\partial: W(Q\Gamma,J) \longrightarrow W(Q\Gamma/\Gamma,J)$. Let $\mathrm{forget}: W_-(C_\infty;k,-) \longrightarrow W_-(k,-)$ be the homomorphism which forgets the action of t.

<u>LEMMA A3</u>. $\chi_*(\mathrm{Im}\ \partial) \leq \mathrm{Ker}(\mathrm{forget})$.

<u>PROOF</u>. Consider an arbitrary generator $\langle\gamma\rangle$ of $W(Q\Gamma,J)$, where $\gamma\in Q\Gamma^\bullet$ and $\gamma = \gamma^J$. We may assume that $\gamma\in\Gamma$. Then $\partial\langle\gamma\rangle$ is represented by a form φ on a cyclic Γ-module M of order γ, where for a generator x we have

$$\varphi(x,x) \equiv 1/\gamma \mod \Gamma \ .$$

Let $\gamma = \sum_{i=-n}^{n} a_i t^i$ where $a_{-i} = \bar{a}_i$ and $a_n \neq 0$. Then as a k-vector space M

has a basis $t^i x$, $-n \le i < n$, and

$$\chi\varphi(t^i x, t^j x) = \chi\left(\frac{t^{i-j}}{\gamma}\right) .$$

Now $i_+(1/\gamma) = \sum_{i=n}^{\infty} b_i^+ t^i$, $i_-(1/\gamma) = \sum_{i=-\infty}^{-n} b_i^- t^i$ for some $b_i^+, b_i^- \in k$, and so $j(1/\gamma) = \sum_{i=-\infty}^{\infty} b_i t^i$ with $b_i = 0$ for $-n < i < n$. It follows that

$$\chi\varphi(t^i x, t^j x) = 0 \quad \text{for} \quad |i-j| < n .$$

In particular, $x, tx, \ldots, t^{n-1}x$ span a metaboliser for $\chi\varphi$ (considered just as a skew-Hermitian form over k). |||

Consider the splitting

$$\text{(A3)} \qquad W(Q\Gamma/\Gamma, J) \cong W^0(Q\Gamma/\Gamma, J) \oplus W(Q\Gamma/\Gamma, J)_{(t-1)} .$$

Since the restriction of forget to $W_-(C_\infty; k, -)_{(t-1)}$ is an isomorphism it follows that $(\mathrm{Im}\,\partial) \cap W(Q\Gamma/\Gamma, J)_{(t-1)} = 0$, so if we let $\partial^0 : W(Q\Gamma, J) \longrightarrow W^0(Q\Gamma/\Gamma, J)$ be the composite of ∂ and projection on the first factor in (A3), we have proved:

<u>LEMMA A4</u>. <u>The sequence</u>

$$0 \longrightarrow W(k,-) \longrightarrow W(Q\Gamma, J) \xrightarrow{\partial^0} W^0(Q\Gamma/\Gamma, J)$$

<u>is exact</u>. |||

A4. A KNOT-THEORETIC INTERLUDE

In this section the base field k will be the rationals. Let $K \subset S^3$ be a knot. The rational Blanchfield pairing β of K represents an element of $W(Q\Gamma/\Gamma, J)$, while the skew-symmetric Milnor pairing μ defined in [19], represents an element of $W_-(C_\infty; \mathbb{Q})$. It can be seen by computing matrix representatives in terms of a Seifert matrix for K that $\chi\beta = -\mu$ (cf. Section A6; this is also true for Trotter's trace function). We give here a direct geometric proof of this; in fact this suggested our definition of χ in the first place.

We first recall the definitions of β and μ. Let $M = M_K$, the result of 0-surgery along K, and let $\tilde{M}$ be the infinite cyclic covering of M. Let $H = H_1(\tilde{M}; \mathbb{Q})$, a finitely generated torsion Γ-module. Let x be a (rational) 1-cycle in $\tilde{M}$. There exist $f \in \Gamma$, $f \neq 0$, and a 2-chain C such that $\partial C = fx$. Then one defines

$$\beta([x],[y]) = \frac{1}{f} \sum_{i=-\infty}^{\infty} (t^i C \cdot y) t^{-i}$$

where $\cdot$ denotes ordinary intersection number. The Milnor pairing arises from an isomorphism $\partial : H_2^\infty(\tilde{M}; \mathbb{Q}) \longrightarrow H_1(\tilde{M}; \mathbb{Q})$ where H_*^∞ is homology based on infinite chains. The desired pairing μ is the composite

$$H \times H \xrightarrow{\partial^{-1} \times \mathrm{id}} H_2^\infty(\tilde{M}; \mathbb{Q}) \times H \longrightarrow \mathbb{Q}$$

where the final arrow is the ordinary intersection pairing. To see μ geometrically we need the definition of ∂. Let ε_+ (respectively ε_-) be the end of $\tilde{M}$ such that for $x \in \tilde{M}$, $t^i x \to \varepsilon_+$ (respectively ε_-) as $i \to +\infty$ (respectively $-\infty$). The chain complex $C_*^\infty(\tilde{M};\mathbb{Q})$ has subcomplexes $C_*(\tilde{M},\varepsilon_+;\mathbb{Q})$, $C_*(\tilde{M},\varepsilon_-;\mathbb{Q})$ consisting of those chains whose support lies outside some neighborhood of ε_-, ε_+ respectively. There is an exact sequence

$$0 \to C_*(\tilde{M};\mathbb{Q}) \longrightarrow C_*(\tilde{M},\varepsilon_+;\mathbb{Q}) \oplus C_*(\tilde{M},\varepsilon_-;\mathbb{Q}) \longrightarrow C_*^\infty(\tilde{M};\mathbb{Q}) \to 0 .$$

One shows that $H_*(\tilde{M},\varepsilon_\pm;\mathbb{Q}) = 0$, and defines ∂ to be the connecting homomorphism in the long exact homology sequence. Thus given (finite) 1-cycles x and y there are 2-chains $C_+ \in C_2(\tilde{M},\varepsilon_+; \ )$, $C_- \in C_2(\tilde{M},\varepsilon_-; \ )$ with $\partial C_+ = x = \partial C_-$ and

$$\mu([x],[y]) = (C_- - C_+) \cdot y .$$

<u>REMARKS</u>. (1) There is lots of scope in this area for conflicting sign conventions. The one we use means that for a fibered knot μ is the same as the intersection pairing on the fiber.

(2) Milnor [19] used the dual cohomology pairing.

<u>THEOREM A1</u>. <u>If</u> β,μ <u>are the Blanchfield and Milnor pairings of the knot</u> K, <u>then</u> $\mu = -\chi\beta$. <u>In particular</u>, $\chi_*\partial\alpha_K = [\mu]$ <u>in</u> $W_-(C_\infty;\mathbb{Q})$.

<u>PROOF</u>. Note that if $\Delta \in C_i(\tilde{M};\mathbb{Q})$ and $\gamma \in \Gamma_+$ there is a chain $\gamma\Delta \in C_i(\tilde{M},\varepsilon_+;\mathbb{Q})$ and $\partial(\gamma\Delta) = \gamma\partial\Delta$. Similarly if $\gamma \in \Gamma_-$.

Let x,y be (finite) 1-cycles; and choose f and C as in the definition of β. Then we have

$$i_\pm(1/f)C \in C_2(\tilde{M},\varepsilon_\pm;\mathbb{Q}) ,$$

$$\partial(i_\pm(1/f)C) = i_\pm(1/f)fx = x .$$

Hence

$$\mu([x],[y]) = \{(i_-(1/f) - i_+(1/f))C\} \cdot y$$

$$= - \sum_{i=-\infty}^{\infty} a_i(t^i C \cdot y)$$

where $i_+(1/f) - i_-(1/f) = \sum_{i=-\infty}^{\infty} a_i t^i$. On the other hand,

$$\chi\beta([x],[y]) = \chi\{\frac{1}{f} \sum_{i=-\infty}^{\infty} (t^i C \cdot y)t^{-i}\}$$

$$= \text{const}\{(\sum_{i=-\infty}^{\infty} a_i t^i)(\sum_{i=-\infty}^{\infty} (t^i C \cdot y)t^{-i})\}$$

$$= \sum_{i=-\infty}^{\infty} a_i(t^i C \cdot y) .$$

That is, $\chi\beta([x],[y] = -\mu([x],[y])$. ⁞⁞⁞

A5. SKEW-ISOMETRIC STRUCTURES AND SEIFERT FORMS

Let V be a finite dimensional vector space over k. There is a 1-1 correspondence between skew-isometric structures (V,φ,t) such that $1-t$ is an automorphism of V, and Seifert forms $(V,\mathscr{S})$, given by the formulae

$$\varphi = \mathscr{S}-\mathscr{S}^*: V\times V \to k$$

$$t = (\mathrm{ad}\mathscr{S})^{-1}\mathrm{ad}\mathscr{S}^*: V\to V$$

and

$$\mathscr{S}(x,y) = \varphi((1-t)^{-1}x,y) \ , \ x,y\in V \ . \tag{A4}$$

Moreover a subspace W of V is a metaboliser for $\mathscr{S}$ iff it is a t-invariant metaboliser for φ. (We leave it to the reader to supply the easy proofs of these assertions, which are straightforward generalisations from the case of a trivial involution.) Thus we have:

LEMMA A5. There is an isomorphism

$$\lambda: W_-^0(C_\infty;k,-) \longrightarrow W_S(k,-)$$

given by the formula (A4). ⁞⁞⁞

Note that if (V,φ,t) and $(V,\mathscr{S})$ correspond as above then

$$(1+t) = (\mathrm{ad}\mathscr{S})^{-1}\mathrm{ad}(\mathscr{S}+\mathscr{S}^*) \ .$$

Thus $\mathscr{S}+\mathscr{S}^*$ is non-singular if and only if $(1+t)$ is an automorphism of V. If the involution on k is trivial then $W_-(C_\infty;k)_{(1+t)} \cong W_-(k) = 0$, so we can always assume that $1+t$ is an automorphism. This justifies the assertion made in Section A1 that it does not affect $W_S(k)$ if we insist that $\mathscr{S}+\mathscr{S}^*$ is non-singular.

A6. DETERMINATION OF $W(Q\Gamma,J)$.

THEOREM A2. (cf. [20] Theorem 5.3). The sequence

$$0\to W(k,-) \longrightarrow W(Q\Gamma,J) \xrightarrow{\partial^0} W^0(Q\Gamma/\Gamma,J) \to 0$$

is split exact.

PROOF. In view of Lemmas A4, A2 and A5, it is enough to produce a homomorphism $\nu: W_S(k,-) \longrightarrow W(Q\Gamma,J)$ such that $\lambda\chi_*\partial^0\nu$ is the identity of $W_S(k,-)$, where λ is as in Lemma A5. Let $\sigma\in W_S(k,-)$ be represented by a matrix S, and define $\nu(\sigma)$ to be the element of $W(Q\Gamma,J)$ represented by

$$S_t = (1-t)S + (1-t^{-1})S^* \ .$$

Note that $\det(S_t) = (1-t)^n\det(S-t^{-1}S^*)$ if S is $n\times n$, and since $\det(S-S^*)\neq 0$, S_t is non-singular. It is straightforward to check that ν is a well-defined homomorphism. The proof that $\lambda\chi_*\partial^0\nu = \mathrm{id}$ is a simple matrix calculation. First, $\partial\nu(\sigma) = \partial[S_t]$ is represented by a form φ on the

Γ-module M presented by S_t, and φ is given by

$$\varphi(\tilde{x},\tilde{y}) \equiv xS_t^{-1}y^* \mod \Gamma$$

where $\tilde{x},\tilde{y}$ are the images in M of the row vectors x,y. Since $S_t = (1-t)(S-t^{-1}S^*)$ and $\det(S-t^{-1}S^*)$ is coprime to $1-t$, $\partial^O[S_t]$ is represented by the restriction φ^O of φ to $M^O = (1-t)M$. A presentation matrix for M^O is $S-t^{-1}S^*$, and relative to this presentation φ^O is given by

$$\begin{aligned}\varphi^O(\tilde{x},\tilde{y}) &\equiv (1-t)(1-t^{-1})xS_t^{-1}y^* \\ &\equiv (1-t^{-1})x(S-t^{-1}S^*)^{-1}y^* \mod \Gamma .\end{aligned}$$

Making a change of basis we see that M^O also has a presentation matrix $tI-S^*S^{-1}$, and the corresponding representation of φ^O is

$$\begin{aligned}\varphi^O(\tilde{x},\tilde{y}) &\equiv (1-t^{-1})(t^{-1}xS)(S-t^{-1}S^*)^{-1}(t^{-1}yS)^* \\ &\equiv (1-t)x(S^*S^{-1}-tI)^{-1}S^*y^* \mod \Gamma\end{aligned}$$

Thus as a vector space over k, M^O has dimension equal to the size of S, and the automorphism t has matrix S^*S^{-1}. In other words, if S represents the Seifert form $\mathscr{S}$, $t = (\mathrm{ad}\mathscr{S})^{-1}(\mathrm{ad}\mathscr{S}^*)$. (The order is reversed since matrices act on the right of row vectors.) If ξ and η are row vectors over k we have

$$\begin{aligned}\chi\varphi^O(\xi,\eta) &= \chi\{(1-t)\xi(S^*S^{-1}-tI)^{-1}S^*\eta^*\} \\ &= \mathrm{const}\{(1-t)\xi(\sum_{i=-\infty}^{\infty}(S^*S^{-1})^{-(i+1)}t^i)S^*\eta^*\} \\ &= \xi(S-S^*)\eta^* .\end{aligned}$$

Comparing this with Section A5 we see that $\chi_*\partial^O[S_t] = \lambda^{-1}[S]$, as claimed. ▒

We have isomorphisms

$$\begin{aligned}W(Q\Gamma,J) &\cong W(k,-) \oplus W^O(Q\Gamma/\Gamma,J) \\ &\cong W(k,-) \oplus W_-^O(C_\infty;k,-) \\ &\cong W(k,-) \oplus W_S(k,-)\end{aligned}$$

ADDENDUM TO THEOREM. Let $K \subset S^3$ be a knot with Seifert form Θ, Blanchfield pairing β and (skew-symmetric) Milnor pairing μ. Under the above isomorphisms with $k = \mathbb{Q}$, α_K corresponds to $(0,[-\beta])$, $(0,[\mu])$ and $(0,[\Theta])$ respectively.

PROOF. By Theorem A1, it is enough to show that $\nu[\Theta] = \alpha_K$. But this is Proposition 1. ▒

A7. THE GROUP $W(\mathbb{C}(t),J)$.

In this section we take $(k,-) = (\mathbb{C}$, conjugation$)$, so that $\Gamma = \mathbb{C}[t,t^{-1}]$ and $Q\Gamma = \mathbb{C}(t)$. By a balanced function we mean a function $f:S^1 \to \mathbb{Z}$ with a finite number of discontinuities such that (in an obvious notation) $f(\xi) = 1/2(f(\xi+) + f(\xi-))$ for all $\xi \in S^1$. Recall that for $\tau \in W(Q\Gamma,J)$ and $\xi \in S^1$, $\sigma_\xi \tau$ is defined to be $\operatorname{sign}(\tau[\xi])$ whenever $\tau[\xi]$ exists, and for the remaining ξ it is defined to make $\sigma_\bullet \tau : S^1 \to \mathbb{Z}$ a balanced function (see [1]). Thus we have a homomorphism $\sigma_\bullet$ from $W(Q\Gamma,J)$ to the group of balanced functions.

We can also associate signatures to an element υ of $W(Q\Gamma/\Gamma,J)$, in two equivalent ways. The symmetric prime ideals of Γ are $(t-\xi)$ for $\xi \in S^1$, so we have

$$W(Q\Gamma/\Gamma,J) \cong \bigcup_{\xi \in S^1} W(Q\Gamma/\Gamma,J)_{(t-\xi)}$$

and isomorphisms

$$\begin{aligned} W(Q\Gamma/\Gamma,J)_{(t-\xi)} &\xrightarrow{(t-\xi)^J_*} W_{(-t^{-1}\bar{\xi})}\hat{}(\Gamma/(t-\xi),J) \\ &\xrightarrow{\cong} W_{-\bar{\xi}^2}(\mathbb{C}, \text{conjugation}) \\ &\xrightarrow{i\xi} W(\mathbb{C}, \text{conjugation}) \ . \end{aligned}$$

Denote the image of the ξ'th component of υ in $W(\mathbb{C}$, conjugation) by υ_ξ. Then we have the signatures $\operatorname{sign}(\upsilon_\xi)$.

Secondly, we have

$$W_-(C_\infty;\mathbb{C}, \text{conjugation}) \cong \bigoplus_{\xi \in S^1} W_-(C_\infty;\mathbb{C}, \text{conjugation})_{(t-\xi)}$$

and each summand is isomorphic to $W_-(\mathbb{C}$, conjugation) by forgetting the action of t. Denote the ξ'th component of $\chi_*\upsilon$ by $(\chi_*\upsilon)_\xi$. We claim that $(\chi_*\upsilon)_\xi = -i\upsilon_\xi$. It is enough to check this when υ is represented by the form φ on a cyclic Γ-module M of order $t-\xi$, $\xi \in S^1$, given by

$$\varphi(x,x) \equiv it/(t-\xi) \mod \Gamma$$

where x generates M. For $\zeta \neq \xi$ we have $\upsilon_\zeta = 0 = (\chi_*\upsilon)_\zeta$, while

$$\begin{aligned} \upsilon_\xi &= \langle 1 \rangle \\ (\chi_*\upsilon)_\xi &= \langle \chi(\tfrac{it}{t-\xi}) \rangle = \langle -i \rangle \ . \end{aligned}$$

Thus we can define

$$\sigma_\zeta \upsilon = \operatorname{sign}(\upsilon_\zeta) = \operatorname{sign}(i(\chi_*\upsilon)_\zeta) \ .$$

<u>REMARK</u>. In [18] Matumoto studies two families of signatures associated to a Seifert matrix S over $\mathbb{C}$. These are essentially the same as the signatures of $\nu[S]$ and $\partial\nu[S]$ defined above. Thus our next result, which says that the signatures of $\partial\tau$ are the jumps in $\sigma_\bullet\tau$, $\tau \in W(Q\Gamma,J)$, is just that part of

Matumoto's theorem which does not consider the value at a discontinuity. Our proof is a trivial computation.

THEOREM A3. (Matumoto [18]) *For* $\tau \in W(\mathbb{C}(t),J)$ *and* $\zeta \in S^1$ *we have*

$$\sigma_{\zeta+}\tau - \sigma_{\zeta-}\tau = 2\sigma_\zeta(\partial\tau) .$$

PROOF. It suffices to consider the cases

$$\tau = \langle\gamma\rangle , \ \gamma \in \Gamma \ \text{ coprime to } \ (t-\zeta)$$

and $$\tau = \langle(t-\zeta)\delta\rangle , \ \delta \in \Gamma \ \text{ coprime to } \ (t-\zeta) .$$

In the first, $(\partial\tau)_\zeta = 0$ so $\tau[\zeta]$ is defined and both sides of the asserted equality are zero. In the second, for $\xi \in S^1$ close to ζ we have

$$\tau[\xi] = \langle(\xi-\zeta)\delta(\xi)\rangle = \langle(\xi-\zeta)\delta(\xi)/|\xi-\zeta|\rangle ,$$

whence

$$\sigma_{\zeta\pm}\tau = \operatorname{sign}(\pm i\zeta\delta(\zeta)) .$$

On the other hand

$$(\partial\tau)_\zeta = \langle i\zeta\hat{\delta}\rangle = \langle i\zeta\delta(\zeta)\rangle . \ ⦀$$

Combining this with Theorem A2 we have:

COROLLARY A1. *The map* $\sigma_\bullet$ *is an isomorphism from* $W(\mathbb{C}(t),J)$ *to the group of balanced functions.* ⦀

One can deal similarly with the cases $k = \mathbb{R}$ or k an algebraic number field. In the first case, $W(\mathbb{R}(t),J)$ is isomorphic to the group of balanced functions f for which $f(\zeta) = f(\bar{\zeta})$. In the second, $W(k(t),J)$/torsion is determined by the functions $\sigma_\bullet$ associated to the involution-preserving embeddings of k in $\mathbb{C}$.

APPENDIX B. RELATIONS BETWEEN CASSON-GORDON INVARIANTS.

Let K be a knot, and let n and N be powers of the same prime with $n < N$. Let $p: L_{K,N} \to L_{K,n}$ be the covering projection. Each $\chi \in Ch_n(K)$ gives rise to $\chi p_* \in Ch_N(K)$. We show how $\tau(K,\chi)$ determines $\tau(K,\chi p_*)$. The main purpose of this is to shed some light on the multiplicative behavior of $\sigma(K,\chi)$ for certain K noted in [2]. In fact, we show that $\tau(K,\chi)$ has the same behavior, and identify the properties of the knots responsible. Throughout, $\Gamma = \mathbb{C}[t,t^{-1}]$, and for $\gamma \in \Gamma$ we write $\gamma|x$ instead of $\gamma(x)$.

THEOREM B1. *In the above situation, let* $\nu = N/n$. *Then we have*

$$\sigma_\zeta\tau(K,\chi p_*) = \sum_{\xi:\xi^\nu=\zeta} \sigma_\xi\tau(K,\chi) - \sum_{\omega:\omega^N=1} \sigma_K(\omega) + \nu \sum_{\eta:\eta^n=1} \sigma_K(\eta) .$$

REMARK. This determines $\tau(K,\chi p_*)$ by Corollary A1.

COROLLARY B1. *In the situation of the theorem, suppose further that* K *is algebraically slice and that* $\tau(K,\chi)$ *is in the image of* $W(\mathbb{C}$, conjugation) $\otimes \mathbb{Q}$. *Then*

$$\tau(K,\chi p_*) = \nu\tau(K,\chi) \quad . \quad |||$$

REMARK. By Theorem (3.5) of [5], the last hypothesis is satisfied when $n=2$ and K has genus 1. This is the case in [2].

COROLLARY B2. *Let* K *be a knot and* ν *a prime power. Let* O_ν *denote the zero of* $Ch_\nu(K)$. *Then*

$$\sigma_\zeta\tau(K,O_\nu) = \sum_{\xi:\xi^\nu=\zeta}\sigma_K(\xi) - \sum_{\omega:\omega^\nu=1}\sigma_K(\omega) \quad .$$

PROOF. In the theorem take $n=1$, $N=\nu$, and recall that $\tau(K,O_1)=\alpha_K^{\mathbb{C}}$. |||

REMARK. Of course, one is only interested in $\tau(K,\chi)$ for K algebraically slice. However, if K is a sum of two non-algebraically-slice knots then this result shows that some care must be taken. Note however that $\sigma_1\tau(K,O_\nu)$ is always zero.

PROOF OF THEOREM B1. Let χ take values in C_m. Let p denote also the projection $M_{K,N}\to M_{K,n}$; we have

$$\chi^+p_* = (\chi p_*)^+ : H_1(M_{K,N}) \to C_m \times C_\infty \quad .$$

Choose (W^4,ψ) such that

$$\partial(W,\psi) = r(M_{K,n},\chi^+) \quad , \quad r>0 \quad .$$

There is a ν-fold cyclic covering $q:W_\nu\to W$ such that

$$\partial(W_\nu,\psi q_*) = r(M_{K,N},\chi^+p_*) \quad .$$

From the approach to knot signatures via branched covering spaces (see [4] or [15]) it follows that

$$\frac{1}{r}(\mathrm{sign}(W_\nu) - \mathrm{sign}(W)) = \sum_{\omega:\omega^N=1}\sigma_K(\omega) - \nu\sum_{\eta:\eta^n=1}\sigma_K(\eta)$$

and hence that the desired result is equivalent to

$$\sigma_\zeta t_{\psi q_*}(W_\nu) = \sum_{\xi:\xi^\nu=\zeta}\sigma_\xi t_\psi(W) \quad . \tag{B1}$$

In what follows, ω and η will be variables ranging over the ν'th roots of unity. Let $e,e_\omega:\Gamma\to Q\Gamma$ be the injections given by

$$e(\gamma) = \gamma|t^\nu \ , \ e_\omega(\gamma) = \gamma|\omega t \quad .$$

We show that

$$e_*t_{\psi q_*}(W_\nu) = \sum_\omega e_{\omega *}t_\psi(W) \quad ,$$

from which (B1) follows upon taking σ_ξ for some νth root ξ of ζ.

Let $L = H_2^t(W;\Gamma)$/torsion and $L_* = H_2^t(W_\nu;\Gamma)$/torsion, which are Γ-lattices in $H_2^t(W;Q\Gamma)$ and $H_2^t(W_\nu;Q\Gamma)$ respectively, with $L \leq L^\#$, $L_* \leq L_*^\#$. Now the $C_m \times C_\infty$ covering $\tilde{W}$ of W determined by ψ is also the covering of W_ν determined by ψq_*, so as $\mathbb{C}$-vector spaces

$$H_2^t(W;\Gamma) = H_2^t(W_\nu;\Gamma) = H_2(\tilde{W};\mathbb{C}) \ .$$

For $x \in H_2^t(W;\Gamma)$, let x_* be the corresponding element of $H_2^t(W_\nu;\Gamma)$. The Γ-module structures are related by

$$t \cdot x_* = (t^\nu x)_* \ .$$

Then L and L^* are related in the same way. Further, the intersection forms φ, φ_* on L, L_* are connected by the formulae

$$\text{(B2)} \qquad \varphi(x,y) = \sum_{i=-\infty}^{\infty} a_i t^i \ , \ \varphi_*(x_*,y_*) = \sum_{i=-\infty}^{\infty} a_{\nu i} t^i$$

for $x,y \in L$. Let $Q\Gamma$ with the Γ-module structures induced by e, e_ω be denoted by $Q\Gamma_e$ $Q\Gamma_\omega$. Then $e_* t_{\psi q*}(W_\nu)$ is represented by $(L_* \otimes Q\Gamma_e, \varphi_e)$ where

$$\varphi_e(x_* \otimes \gamma, y_* \otimes \delta) = \gamma\delta^J(\varphi_*(x_*,y_*)|t^\nu) \ ,$$

and $e_{\omega *} t_\psi(W)$ by $(L \otimes Q\Gamma_\omega, \varphi_\omega)$ where

$$\varphi_\omega(x \otimes \gamma, y \otimes \delta) = \gamma\delta^J(\varphi(x,y)|\omega t) \ .$$

There is an isometry T of $(L_* \otimes Q\Gamma_e, \varphi_e)$ defined by

$$T(x_* \otimes \gamma) = (tx)_* \otimes \gamma$$

and T^ν is multiplication by t^ν. Therefore $L_* \otimes Q\Gamma_e$ splits as an orthogonal direct sum $\bigoplus_\omega E_\omega$ where E_ω is the ωt-eigenspace of T. The proof is completed by showing that

$$(E_\omega, \varphi_e|E_\omega) \cong (L \otimes Q\Gamma_\omega, \varphi_\omega) \ .$$

Define homomorphisms $\alpha_\omega : L_* \otimes Q\Gamma_e \longrightarrow L \otimes Q\Gamma_\omega$ and $\beta_\omega : L \otimes Q\Gamma_\omega \longrightarrow L_* \otimes Q\Gamma_e$ by

$$\alpha_\omega(x_* \otimes \gamma) = \frac{1}{\sqrt{\nu}}(x \otimes \gamma) \ ,$$

$$\beta_\omega(x \otimes \gamma) = \frac{1}{\sqrt{\nu}} \sum_{i \bmod \nu} (\omega t)^{-i} T^i(x_* \otimes \gamma) \ .$$

We leave it to the reader to verify that these are well-defined and satisfy

$$\beta_\omega(L \otimes Q\Gamma_\omega) \leq E_\omega \ ,$$

$$\alpha_\eta\beta_\omega = \begin{cases} \text{id} & \text{if } \eta = \omega \ , \\ 0 & \text{if } \eta \neq \omega \ , \end{cases}$$

$$\sum_\omega \beta_\omega\alpha_\omega = \text{id} \ .$$

Thus β_ω maps $L \otimes Q\Gamma_\omega$ isomorphically onto E_ω. Finally, we have

$$\varphi_e(\beta_\omega(x \otimes \gamma) , \beta_\omega(y \otimes \delta))$$

$$= \frac{\gamma\delta^J}{\nu} \sum_{i,j \bmod \nu} (\omega t)^{j-i} \varphi_e(x_* \otimes 1, T^{j-i}(y_* \otimes 1))$$

$$= \gamma\delta^J \sum_{k \bmod \nu} (\omega t)^k (\varphi_*(x_*, (t^k y)_*) | t^\nu) .$$

From (B2),

$$\varphi(x,y) = \sum_{k \bmod \nu} t^k (\varphi_*(x_*, (t^k y)_*) | t^\nu) ,$$

so

$$\varphi_e(\beta_\omega(x \otimes \gamma) , \beta_\omega(y \otimes \delta)) = \gamma\delta^J (\varphi(x,y) | \omega t)$$

$$= \varphi_\omega(x \otimes \gamma, y \otimes \delta) . \quad ⦀$$

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