LXVI.

LISTING'S *TOPOLOGIE*.

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SOME of you may have been puzzled by the advertised title of this Address. But certainly not more puzzled than I was while seeking a title for it.

I intend to speak (necessarily from a very elementary point of view) of those space-relations which are independent of *measure*, though not always of *number*, and of which perhaps the very best instance is afforded by the various convolutions of a knot on an endless string or wire. For, once we have tied a knot, of whatever complexity, on a string and have joined the free ends of the string together, we have an arrangement which, however its apparent form may be altered (as by teazing out, tightening, twisting, or flyping of individual parts), retains, until the string is again cut, certain perfectly definite and characteristic properties altogether independent of the relative lengths of its various convolutions.

Another excellent example is supplied by Crum Brown's chemical *Graphic Formulæ*. These, of course, do not pretend to represent the actual positions of the constituents of a compound molecule, but merely their relative connection.

From this point of view all figures, however distorted by projection &c., are considered to be unchanged. We deal with grouping (as in a *quincunx*), with motion by starts (as in the chess-knight's move), with the necessary relation among numbers of intersections, of areas, and of bounding lines in a plane figure; or among the numbers of corners, edges, faces, and volumes of a complex solid figure, &c.

For this branch of science there is at present no definitely recognized title except that suggested by Listing, which I have therefore been obliged to adopt. *Geometrie der Lage* has now come, like the *Géométrie de Position* of Carnot, to mean something very different from our present subject; and the *Geometria sitûs* of Leibnitz, though intended (as Listing shows) to have specially designated it, turned out, in its inventor's hands, to be almost unconnected with it. The subject is one of very great importance, and has been recognized as such by many of the greatest investigators, including Gauss and others; but each, before and after Listing's time, has made his separate contributions to it without any attempt at establishing a connected account of it as an independent branch of science.

It is time that a distinctive and unobjectionable name were found for it; and once that is secured, there will soon be a crop of *Treatises.* What is wanted is an erudite, not necessarily a very original, mathematician. The materials already to hand are very numerous. But it is not easy (in English at all events) to find a name for it without coining some altogether new word from Latin or Greek roots. *Topology* has a perfectly definite meaning of its own, altogether unconnected with our subject. *Position*, with our mathematicians at least, has come to imply measure. *Situation* is not as yet so definitely associated with measure; for we can speak of a situation to right or left of an object without inquiring *how far off.* So that till a better term is devised, we may call our subject, in our own language, the *Science* (not the *Geometry*, for that implies measure) *of Situation.*

Listing, to whom we owe the first rapid and elementary, though highly suggestive, sketch of this science, as well as a developed investigation of one important branch of it, was in many respects a remarkable man. It is to be hoped that much may be recovered from his posthumous papers; for there can be little doubt that in consequence of his disinclination to publish (a disinclination so strong that his best-known discovery was actually published for him by another), what we know of his work is a mere fragment of the results of his long and active life.

In what follows I shall not confine my illustrations to those given by Listing, though I shall use them freely; but I shall also introduce such as have more prominently forced themselves on my own mind in connection mainly with pure physical subjects. It is nearly a quarter of a century since I ceased to be a Professor of Mathematics: and the branches of that great science which I have since cultivated are especially those which have immediate bearing on Physics. But the subject before us is so extensive that, even with this restriction, there would be ample material, in my own reading, for a whole series of strictly elementary lectures.

I ought not to omit to say, before proceeding to our business, that it is by no means creditable to British science to find that Listing's papers on this subject—the *Vorstudien zur Topologie* (*Göttinger Studien*, 1847), and *Der Census räumlicher Complexe* (Göttingen *Abhandlungen*, 1861)—have not yet been rescued from their most undeserved obscurity, and published in an English dress, especially when so much that is comparatively worthless, or at least not so worthy, has already secured these honours. I was altogether ignorant of the existence of the *Vorstudien* till it was pointed out to me by Clerk-Maxwell, after I had sent him one of my earlier papers on *Knots*; and I had to seek, in the Cambridge University Library, what was perhaps the only then accessible copy.

direction in which a stone falls, or in which a plummet hangs, and its reverse. Even below-decks, when the vessel is lying over under a steady breeze, and we "have our sea-legs on," we instinctively keep our bodies vertical, without any thought of setting ourselves perpendicular to the cabin-floor. And this definition holds in every region of space where the earth's attraction is the paramount force. In an imaginary cavity at the earth's centre the terms would cease to have any meaning.

East, in the sense of "*Orient*," is the quarter in which the sun rises; and *this* distinction is correct all over the earth except at the poles, where it has no meaning. But if we were to define *South* as the region in which the sun is seen at midday, our definition would be always wrong if we were placed beyond the tropic of Capricorn, and at particular seasons even if we were merely beyond that of Cancer. Still there is a certain *consensus* of opinion which enables all to understand what is meant by South without the need of any formal definition.

But the distinction between *Right* and *Left* is still more difficult to define. We must employ some such artifice as "A man's right side is that which is turned eastwards, when he lies on his face with his head to the north." For, in the lapse of ages of development, one may perhaps be right in saying, with Molière's physician, "*Nous avons changé tout cela*"; and men's hearts may have migrated by degrees to the other side of their bodies, as does one of the eyes of a growing flounder. Or some hitherto unsuspected superiority of left-handed men may lead to their sole survival; and then the definition of the right hand, as that which the majority of men habitually employ most often, would be false.

I will not speak further of these things, which I have introduced merely to show how difficult it sometimes is to formulate precisely in words what every one in his senses knows perfectly well; and thus to prepare you to expect difficulties of a higher order, even in the very elements of matters not much more recondite.

(2) But there is a very simple method of turning a man's right hand into his left, and *vice versâ*, and of shifting his heart to the right-hand side, without waiting for the (problematical) results of untold ages of development or evolution. We have only to look at him with the assistance of a plane mirror or looking-glass, and these extraordinary transformations are instantly effected. Behind the looking-glass the world and every object in it are *perverted* (*verkehrt*, as Listing calls it). Seen through an astronomical telescope, everything is *inverted* merely (*umgekehrt*). Particular cases of this distinction, which is of very considerable importance, were of course known to the old geometers. For two halves of a circle are congruent; one semicircle has only to be made to rotate through two right angles *in its own plane* to be superposable on the other. But how about the halves of an isosceles triangle formed by the bisector of the angle between the equal sides? They are equal in every respect except congruency; one has to be *turned over* before it can be exactly superposed on the other.

Listing gives many examples of this distinction, of which the following is the simplest:—

Inversion:—(English) V and (Greek) Λ.
Perversion:—(English) R and (Russian) Я.
Inversion and perversion:—(English) L and (Greek) Γ.

He also gives an elaborate discussion of the different relative situations of two dice whose edges are parallel, taking account of the *points* on the various sides.

When we *flype* a glove (as in taking it off when very wet, or as we skin a hare), we perform an operation which (not describable in English by any shorter phrase than "*turning outside in*") changes its character from a right-hand glove to a left. A pair of trousers or a so-called *reversible* water-proof coat is, after this operation has been performed, still a pair of trousers or a coat, but the legs or arms are interchanged; unless the garments, like those of "Paddius à Corko," are buttoned behind[1].

(3) The germ of the whole of this part of the subject lies in the two ways in which we can choose the three rectangular axes of x, y, z; and is intimately connected with the kinematical theory of rotation of a solid.

Thus we can make the body rotate through two right angles about one axis, so that each of the other two is inverted. Such an operation does *not* change their relative situation.

But to invert one only, or all three, of the axes requires that the body should (as it were) be *pulled through itself*, a process perfectly conceivable from the kinematical, but not from the physical, point of view. By *this* process the relative situation of the axes is changed.

When we think of the rotation about the axis of x which shall bring Oy where Oz was, we see that it must be of opposite character in these two cases. And it is a mere matter of convention which of the two systems we shall choose as our normal or positive one.

That which seems of late to have become the more usual is that in which a quadrantal rotation about x (which may be any one of the three) shall change Oy into the former Oz (*i.e.* in the cyclical order x, y, z), when it is applied in the sense in which the earth turns about the *northern* end of its polar axis. Thus we may represent the three axes, in cyclical order, by a northward, an upward, and an eastward line. So that we change any one into its cyclical successor by seizing the positive end of the third, and, as it were, *unscrewing* through a quadrant[2].

The hands of a watch, looked at from the side on which the face is situated, thus move round in the *negative* direction; but if we could see *through* the watch, they would appear to move round in the *positive* direction. This universally employed construction arises probably from watch-dials having been originally made to behave as much as possible like sun-dials, on which the hours follow the apparent daily course of the sun, *i.e.* the *opposite* direction to that of the earth's rotation about its axis.

(4) This leads us into another very important elementary branch of our subject,

[1] When a Treatise comes to be written (in English) on this science, great care will have to be taken in *exactly* defining the senses in which such words as inversion, reversion, perversion, &c. are to be employed. There is much danger of confusion unless authoritative definitions be given once for all, and *not too late.*

[2] These relations, and many which follow, were illustrated by *models*, not by diagrams; and the reader

one in which Listing (it is to be feared) introduced complication rather than simplification, by his endeavours to extricate the botanists from the frightful chaos in which they had involved themselves by their irreconcilable descriptions of vegetable spirals. [He devotes a good many pages to showing how great was this confusion.]

When we compare the tendrils of a hop with those of a vine, we see that while they both grow upwards, as in coiling themselves round a vertical pole, the end of the hop tendril goes round *with the sun* (*secundum solem*), that of the vine tendril *against the sun* (*contra solem*).

Thus the vine tendril forms an ordinary or (as we call it) right-handed screw, the hop tendril a left-handed screw.

Now, if a point move in a circle in the plane of yz in the positive direction, and if the circle itself move in the direction of x positive, the resultant path of the point will be a vine-, or right-handed screw. But if the circle's motion as a whole, *or* the motion of the point in the circle, be reversed, we have a left-handed screw; while if *both* be reversed, it remains right-handed. Every one knows the combination of the rotatory and translatory motions involved in the use of an ordinary corkscrew; but there are comparatively few who know that a screw is *the same at either end*—that it has, in fact, what is called *dipolar symmetry*.

With a view to assist the botanists, Listing introduced a fancied resemblance between the threads of the two kinds of (double-threaded) screws and the Greek letters λ and δ, for the latter of which he also proposed the long ∫ used as a sign of integration; thus λλλλ and δδδδ, or ∫∫∫∫.

The first, which is our vine- or right-handed screw, he calls from his point of view (which is taken *in* the axis of the screw) *laeotrop*, the other *dexiotrop*. He also proposes to describe them as *lambda-* or *delta-Windungen*. But it is clear that all this "makes confusion worse confounded." Every one knows an ordinary screw. *It* is right-handed or positive. Hence he can name, at a glance, any vegetable or other helix.

(5) A symmetrical solid of revolution, an ellipsoid for instance (whether prolate or oblate), has, *if at rest*, dipolar symmetry. But if it rotate about its axis, we can at once distinguish one end of the axis from the other, and there is *dipolar asymmetry*.

This distinction is dynamical as well as kinematical, as every one knows who is conversant with gyroscopes or gyrostats.

A flat spiral spring, such as a watch- or clock-spring, or the gong of an American clock, if the inner coils be pulled out to one side, becomes a right-handed screw; if to the other, a left-handed screw. In either case it retains the dipolar symmetry which it had at first, while plane.

But when we pass an electric current round a circle of wire, we at once give it dipolar asymmetry. The current appears, from the one side, to be going round in the positive direction; from the other, in the negative. This is, in fact, the *point* of Ampère's explanation of magnetism.

A straight wire heated at one end has dipolar asymmetry, not only because of the different temperatures of its ends, but because of the differences of their electric potential (due to the "Thomson effect").

The same is generally true of every vector (or directed) quantity, such as a velocity, a force, a flux, an axis of rotation, &c.

(6) An excellent example of our science is furnished by the *Quincunx*, which is the basis of the subject of *Phyllotaxis* in botany, as well as of the arrangement of scales on a fish.

A quincunx (from the scientific point of view) is merely the system of points of intersection of two series of equidistant parallel lines in the same plane. By a simple shear parallel to one of the two series of lines, combined (if necessary) with mere uniform extensions or contractions along either or both series, any one quincunx can be changed into any other. Hence the problems connected with the elements of the subject are very simple; for it follows from the above statements that any quincunx can be reduced to square order. The botanist who studies the arrangement of buds or leaf-stalks on a stem, or of the scales on a fir-cone, seeks the *fundamental spiral*, as he calls it, that on which all the buds or scales lie. And he then fully characterizes each particular arrangement by specifying whether this spiral is a right- or left-handed screw, and what is its *divergence*. The divergence is the angle (taken as never greater than π) of rotation about the axis of the fundamental spiral from one bud or scale to the next.

(7) It is clear that if the stem or cone (supposed cylindrical) were inked and rolled on a sheet of paper, a quincunx (Plate III. fig. 1) would be traced, consisting of continuously repeated (but, of course, *perverted*) impressions of the whole surface. Hence if A, A_1, be successive prints of the *same* scale, B a scale which can be reached from A by a right-handed spiral, AB, of m steps, or by a left-handed spiral, A_1B, of n steps, these two spirals being so chosen that *all* the scales lie on n spirals parallel to AB and also on m spirals parallel to A_1B, we shall find a scale of the fundamental spiral by seeking the scale *nearest* to AA_1 within the space ABA_1.

Here continued fractions perforce come in. Let μ/ν be the last convergent to m/n. Then, if it be greater than m/n, count μ leaves or scales from A along AB, and thence ν leaves or scales parallel to BA_1, and we arrive at the required leaf or scale. If the last convergent be less than m/n, count ν leaves along A_1B, and thence μ parallel to BA. If the leaf, a, so found in either case, be nearer to A than to A_1, the fundamental spiral (as printed, *i.e. perverted*) is right-handed; and *vice versâ*. Thus the first criterion is settled.

To find the divergence, take the case of μ/ν greater than m/n; and a, so found, nearer to A than to A_1. Draw ac perpendicular to AA_1, and let the spirals through a, parallel to BA and BA_1 respectively, cut AA_1 in d and e. Then the divergence is $2\pi Ac/AA_1$. This is obviously greater than $2\pi Ad/AA_1$ (*i.e.* $2\pi\nu/n$), and less than $2\pi Ae/AA_1$ (*i.e.* $2\pi\mu/m$); and can be altered by shearing the diagram parallel to AA_1, or (what comes to the same thing) *twisting* the stem or cone. To find its exact value, draw through B a line perpendicular to AA_1 (*i.e.* parallel to the axis of the stem or cone), and let C the first leaf ...

seen to be $2\pi(\mu s+\nu r)/(ms+nr)$; and we have the complete description of the object, so far as our science goes.

In the figure, which is taken from an ordinary cone of *Pinus pinaster*, we have $m=5$, $n=8$; whence $\mu=2$, $\nu=3$. Also $r=3$, $s=2$; and the fundamental spiral (*perverted*) is therefore right-handed, with divergence $2\pi 13/34$.

Should m and n have a common divisor p, it is easily seen that the leaves are arranged in *whorls*; and, instead of one fundamental spiral, there is a group of p such spirals, forming a multiple-threaded screw. Each is to be treated by a process similar to that above.

(8) The last statement hints at a subject treated by Listing, which he calls *paradromic winding*. Some of his results are very curious and instructive.

Take a long narrow tape or strip of paper. Give it any number, m, of half-twists, then bend it round and paste its ends together.

If m be zero, or any other even number, the two-sided surface thus formed has two edges, which are paradromic. If the strip be now slit up midway between the edges, it will be split into two. These have each $m/2$ full twists, like the original, and (except when there is no twist, when of course the two can be separated) are $m/2$ times *linked* together.

But if m be odd, there is *but one surface* and *one edge*; so that we may draw a line on the paper from any point of the original front of the strip to any point of the back, *without crossing the edge*. Hence, when the strip is slit up midway, it remains one, but with m *full* twists, and (if $m>1$) it is *knotted*. It becomes, in fact, as its single edge was before slitting, a paradromic knot, a double clear coil with m crossings.

[This simple result of Listing's was the sole basis of an elaborate pamphlet which a few years ago had an extensive sale in Vienna; its object being to show how to perform (without the usual conjuror's or spiritualist's deception) the celebrated trick of tying a knot on an endless cord.]

The study of the one-sided autotomic surface which is generated by increasing indefinitely the breadth of the paper band, in cases where m is odd, is highly interesting and instructive. But we must get on.

(9) I may merely mention, in passing, as instances of our subject, the whole question of the *Integral Curvature* of a closed plane curve; with allied questions such as "In an assigned walk through the streets of Edinburgh, how often has one rotated *relatively to* some prominent object, such as St Giles' (supposed within the path) or Arthur's Seat (supposed external to it)?" We may vary the question by supposing that he walks so as always to turn his face to a particular object, and then inquire how often he has turned about his own axis. But here we tread on Jellinger Symonds' ground, the *non*-rotation of the moon about her axis!

But the subject of the *area* of an autotomic plane curve is interesting. It is one of Listing's examples. De Morgan, W. Thomson, and others in this country have also developed it as a supposed new subject. But its main principles (as Muir has shown in *Phil. Mag.* June, 1873) were given by Meister 113 years ago. It is now so well known that I need not dilate upon it.

(10) A curious problem, which my colleague Chrystal recently mentioned to me, appears to be capable of adaptation as a good example of our subject. It was to this effect:—

Draw the circle of least area which includes four given points in one plane.

In this form it is a question of ordinary geometry. But we may modify it as follows:—

Given three points in a plane; divide the whole surface into regions such that wherever in any one of those regions a fourth point be chosen, the rule for constructing the least circle surrounding the four shall be the same.

There are two distinct cases (with a transition case which may be referred to either), according as the given points A, B, C (suppose) form an acute- or an obtuse-angled triangle.

(α) When ABC is acute-angled (fig. 2). Draw from the ends of each side perpendiculars towards the quarter where the triangle lies, and produce each of them indefinitely from the point in which it again intersects the circumscribing circle.

The circle ABC is itself the required one, so long as D (the fourth point) lies within it.

If D lie *between* perpendiculars drawn (as above) from the ends of a side, as AB, then ABD is the required circle.

If it lie in any other region, the required circle has D for one extremity of a diameter, and the most distant of A, B, C for the other.

(β) When there is an obtuse angle, at C say (fig. 3). Make the same construction as before, but, in addition, describe the circle whose diameter is AB. All is as before, except that AB is the circle required, if D lie within it; and that if D lie within the middle portion of the larger of the two lunes formed the required circle is ABD.

[In figs. 2, 3, 4, which refer to these two cases in order, and to the intermediate case in which the triangle is right-angled at C, each region is denoted by three or by two letters. When there are three, the meaning is that the required circle passes through the corresponding points; when there are but two, these are the ends of a diameter. The separate regions are, throughout, bounded by full lines; the dotted lines merely indicate constructions.]

(11) A very celebrated question, directly connected with our subject, is to make a Knight (at chess) move to each square on the board once only till it returns to its original position. From the time of Euler onwards numerous solutions have been given. To these I need not refer further.

A much simpler question is the motion of a Rook, and to this the lately popular American "15-puzzle" is easily reduced. For *any* closed path of a rook contains an even number of squares, *since it must pass from white to black alternately.* [This furnishes a good instance of the extreme simplicity which often characterises *even* number of interchanges of pieces will give the required result, the puzzle can be solved; if not, the arrangement is irreducible.

(12) A few weeks ago, in a railway-train, I saw the following problem proposed:—Place four sovereigns and four shillings in close *alternate* order in a line. Required, in four moves, each of *two* contiguous pieces (without altering the relative position of the two), to form a *continuous* line of four sovereigns followed by four shillings. Let sovereigns be represented by the letter B, shillings by A.

One solution is as follows:—

Before starting:—	.	.	A	B	A	B	A	B	A	B
1st move	B	A	A	B	A	B	A	.	.	B
2nd „	B	A	A	B	.	.	A	A	B	B
3rd „	B	.	.	B	A	A	A	A	B	B
4th „	B	B	B	B	A	A	A	A	.	.

If we suppose the pieces to be originally arranged in circular order, with two contiguous blank spaces, the law of this process is obvious. Operate always with the *penultimate* and *antepenultimate*, the gap being looked on as the end for the time being. With this hint it is easy to generalize, so as to get the nature of the solution of the corresponding problem in any particular case, whatever be the number of coins. It is also interesting to vary the problem by making it a condition that the two coins to be moved at any instant shall first be made to change places.

(13) Another illustration, commented on by Listing, but since developed from a different point of view in a quite unexpected direction, was originated by a very simple question propounded by Clausen in the *Astronomische Nachrichten* (No. 494). In its general form it is merely the question, "What is the smallest number of pen-strokes with which a given figure, consisting of lines only, can be traced?" No line is to be gone over twice, and every time the pen has to be lifted counts one.

The obvious solution is:—Count the number of points in the figure at each of which an *odd* number of lines meet. There must always be an even number of such (zero included). Half of this number is the number of necessary separate strokes (except in the zero case, when the number of course *must* be unity). Thus the boundaries of the squares of a chess-board can be traced at 14 separate pen-strokes; the usual figure for Euclid I. 47 at 4 pen-strokes; and fig. 5 at one.

(14) But, if $2n$ points in a plane be joined by $3n$ lines, no two of which intersect, (*i.e.* so that *every* point is a terminal of 3 different lines), the figure requires n separate pen-strokes. It has been shown that in this case (unless the points be divided into two groups, between which there is but *one* connecting line, fig. 7) the $3n$ lines may be divided into 3 groups of n each, such that one of each group ends at each of the $2n$ points. See fig. 6, in which the lines are distinguished as α, β, or γ. Also note that $\alpha\beta\alpha\beta$ &c., and $\alpha\gamma\alpha\gamma$ &c. form entire cycles passing through all the trivia, while $\beta\gamma\beta\gamma$ &c. breaks up into detached subcycles.

Thus, if a Labyrinth or Maze be made, such that every intersection of roads is

Trivium, it may always be arranged so that the several roads meeting at each intersection may be one a grass-path, one gravel, and the other pavement. To make sure of getting out of such a Labyrinth (if it be possible), we must select two kinds of road to be taken alternately at each successive trivium. Thus we may elect to take grass, gravel, grass, gravel, &c., in which case we *must* either come to the exit point or (without reaching it) return to our starting-point, to try a new combination. For it is obvious that, if we follow our rule, we cannot possibly pass through the same trivium twice before returning to our starting-point.

(15) This leads to a very simple solution of the problem of *Map-colouring with four colours*, originally proposed by Guthrie, and since treated by Cayley, Kempe, and others.

The boundaries of the counties in a map generally meet in threes. But if four, or more, meet at certain points, let a small county be inserted surrounding each such point; and there will then be trivia of boundaries only. These various boundaries may, by our last result, be divided (usually in many different ways) into three categories, α, β, γ suppose, such that each trivium is formed by the meeting of one from each category. Now take four colours, A, B, C, D, and apply them, according to rule, as follows; so that

α separates A and B or C and D,
β „ A and C „ B and D,
γ „ A and D „ B and C,

and the thing is done. For the small counties, which were introduced for the sake of the construction, may now be made to contract without limit till the boundaries become as they were at first.

The connection between these two theorems gives an excellent illustration of the principle involved in the reduction of a biquadratic equation to a cubic.

Kempe has pointed out that four colours do not in general suffice for a map drawn upon a multiply-connected surface, such as that of a *tore* or anchor-ring. This you can easily prove for yourselves by establishing *one* simple instance. (This is an example of a case of Listing's *Census*.)

(16) From the very nature of our science, the systems of trivia, as we described them in § 14, may be regarded as mere distorted *plane projections of polyhedra which have trihedral summits only.* There are two obvious classes of exceptions, which will be at once understood from the simple figures 7 and 8. Their characteristic is that parts of the figure containing closed circuits (*i.e. faces* of the polyhedron) are connected to the rest by *one* or by *two* lines (*edges*) only. The lines are always $3n$ in number, and, excluding only the first class of exceptions, can be marked in 3 groups α, β, γ, one of each group ending at each point (*trihedral angle*).

Now in *every one* of the great variety of cases which I have tried (where the figure was, like fig. 6, a projection of a *true* polyhedron) I have found that a *complete* circuit of edges, alternately of two of these groups (such as $\alpha\beta\alpha\beta$ &c.) can each of the angles. That is, in another form, every such polyhedron may be projected in a figure of the type shown in fig. 9, where the *dotted* lines are supposed to lie below the full lines. But, in the words of the extraordinary mathematician Kirkman, whom I consulted on the subject, "the theorem......has this provoking interest, that it mocks alike at doubt and proof[1]." Probably the proof of this curious proposition has (§ 11) hitherto escaped detection from its sheer simplicity. Habitual stargazers are apt to miss the beauties of the more humble terrestrial objects.

(17) Kirkman himself was the first to show, so long ago as 1858, that a "clear circle of edges" of a unique type passes through all the summits of a pentagonal dodecahedron. Then Hamilton pounced on the result and made it the foundation of his *Icosian Game*, and also of a new calculus of a very singular kind. See figures 9, 10, 11, which are all equivalent projections of a pentagonal dodecahedron.

At every trivium you must go either to right or to left. Denote these operations by r and l respectively. In the pentagonal dodecahedron, start where you will, either r^5 or l^5 brings you back to whence you started. Thus, in this case, r and l are to be regarded as operational symbols—each (in a sense) a *fifth* root of $+1$. In this notation Kirkman's Theorem is formulated by the expression

$$rlrlrrrlllrlrlrrrlll = 1\,;$$

or, as we may write it more compactly,

$$[(rl)^2 r^3 l^3]^2 = 1, \quad \text{or} \quad [(lr)^3 r^2 l^2]^2 = 1.$$

It may be put in a great many apparently different, but really equivalent, forms; for, so long as the *order* of the operations is unchanged, we may begin the cycle where we please. Also we may, of course, interchange r and l throughout, in consequence of the symmetry of the figure.

It is curious to study, in such a case as this, where it can easily be done, the essential nature of the various kinds of necessarily abortive attempts to get out of such a labyrinth. Thus if we go according to such routes as $(rl)^2 lrl^3$, or $r^3 lr^3$ (sequences which do not occur in the general cycle), the next step, whatever it be, brings us to a point already passed through. We thus obtain other relations between the symbols r and l. We can make special partial circuits of this kind, including any number of operations from 7 up to 19.

All of these remarks will be obvious from any one of the three (equivalent) diagrams 9, 10, or 11.

(18) As I have already said, the subject of knots affords one of the most typical applications of our science. I had been working at it for some time, in consequence of Thomson's admirable idea of Vortex-atoms, before Clerk-Maxwell referred me to Listing's Essay; and I had made out for myself, though by methods entirely different from those of Listing, all but one of his published results. Listing's remarks on this fascinating branch of the subject are, unfortunately, very brief; and it is here especially, I hope, that we shall learn much from his posthumous papers. In the *Vorstudien* he

[1] 'Reprint of Math. Papers from the Ed. Times,' 1881, p. 113.

looks upon knots simply from the point of view of screwing or winding; and he designates the angles at a crossing of two laps of the cord by the use of his λ and δ notation (§ 4). Fig. 12 will show the nature of such crossings. Figs. 13, 14, and 15 show what he calls *reducible* and *reduced* knots. In a reducible knot the angles in some compartments at least are not all λ or all δ (the *converse* is not necessarily true). In a *reduced* knot, each compartment is all λ or all δ.

(19) My first object was to *classify* the simpler forms of knots, so as to find to what degree of complexity of knotting we should have to go to obtain a special form of knotted vortex for each of the known elements. Hence it was necessary to devise a mode of notation, by means of which any knot could be so fully described that it might, from the description alone, be distinguished from all others, and (if requisite) constructed in cord or wire.

This I obtained, in a manner equally simple and sufficient, from the theorem which follows, one which (to judge from sculptured stones, engraved arabesques, &c.) must have been at least *practically* known for very many centuries.

Any closed plane curve, which has double points only, may be looked upon as the projection of a knot *in which each portion of the cord passes alternately under and over the successive laps it meets.* [The same is easily seen to hold for any number of self-intersecting, and mutually intersecting, closed plane curves, in which cases we have in general both *linking* and *locking* in addition to knotting.]

The proof is excessively simple (§ 11). If both ends of one continuous line lie *on the same side* of a second line, there must be an even number of crossings.

(20) To apply it, go continuously round the projection of a knot (fig. 16), putting *A*, *B*, *C*, &c. at the *first*, *third*, *fifth*, &c. crossing you pass, until you have put letters to all. Then go round again, writing down the name of *each* crossing in the order in which you reach it. The list will consist of each letter employed, taken twice over. *A*, *B*, *C*, &c. will occupy, in order, the first, third, fifth, &c. places; but *the way in which these letters occur in the even places fully characterizes the drawing of the projected knot.* It may therefore be described by the order of the letters in the even places alone; and it does not seem possible that *any* briefer description could be given.

To prove that this description is complete, so far as the projection is concerned, all that is required is to show that from it we can at once construct the diagram. Thus let it be, as in fig. 16, *E F B A C D*. Then the full statement is

$$A\,E\,B\,F\,C\,B\,D\,A\,E\,C\,F\,D/A \text{ \&c.}$$

(21) To draw from such a statement, choose in it two apparitions of the same letter, *between which no other letter appears twice.* Thus *A E C F D/A* (at the end of the statement) forms such a group. It must form a loop of the curve. Draw such a loop, putting *A* at the point where the ends cross, and the other letters in order (either way) round the loop. Proceed to fill in the rest of the cycle in the same way. The figures thus obtained may present very different appearances; but they are all projections of the *same* definite knot. The only further information we require for

This can be at once supplied by a + or − sign attached to each letter where it occurs in the statement of the order in the even places.

(22) Furnished with this process, we find that it becomes a mere question of skilled labour to draw all the possible knots having any assigned number of crossings. The requisite labour increases with extreme rapidity as the number of crossings is increased. For we must take every possible arrangement of the letters in the even places, and try whether it is compatible with the properties of a self-intersecting plane curve. Simple rules for rejecting useless or impracticable combinations are easily formed. But then we have again to go through the list of survivors, and reject all but one of each of the numerous groups of different distortions of one and the same species of knot.

I have not been able to find time to carry out this process further than the knots with *seven* crossings. But it is very remarkable that, so far as I have gone, the number of knots of each class belongs to the series of powers of 2. Thus:

Number of crossings 3, 4, 5, 6, 7,
Number of distinct forms... 1, 1, 2, 4, 8.

It is greatly to be desired that some one, with the requisite leisure, should try to extend this list, if possible up to 11, as the next prime number. The labour, great as it would be, would not bear comparison with that of the calculation of π to 600 places, and it would certainly be much more useful. [But see Nos. XL, XLI, which are of later date than this Address. 1899.]

Besides, it is probable that modern methods of analysis may enable us (by a single "happy thought" as it were) to avoid the larger part of the labour. It is in matters like this that we have the true "raison d'être" of mathematicians.

(23) There is one very curious point about knots which, so far as I know, has as yet no analogue elsewhere. In general the perversion of a knot (*i.e.* its image in a plane mirror) is non-congruent with the knot itself. Thus, as in fact Listing points out, it is impossible to change even the simple form (fig. 14) into its image (fig. 15). But I have shown that there is at least one form, for every *even* number of crossings, which is congruent with its own perversion. The unique form with four crossings gave me the first hint of this curious fact. Take one of the larger laps of fig. 17, and turn it *over* the rest of the knot, fig. 18 (which is the perversion) will be produced.

We see its nature better from the following process (one of an infinite number) for forming *Amphicheiral* knots. Knot a cord as in fig. 19, the number of complete figures of "eight" being at pleasure. Turn the figure upside down, and it is seen to be merely its own image. Hence, when the ends are joined, it forms a knot which is congruent with its own perversion.

(24) The general treatment of links is, unless the separate cords be also knotted, much simpler than that of knots—*i.e.* the measurement of *belinkedness* is far easier than that of *beknottedness.*

I believe the explanation of this curious result to lie mainly in the fact that it is possible to interweave three or more continuous cords, *so that they cannot be separated, and yet no one shall be knotted, nor any two linked together.*

This is obvious at once from the simplest possible case, shown in fig. 20. Here the three rings are not linked but *locked* together.

Now mere linkings and mere lockings are very easy to study. But the various loops of a knot may be linked *or* locked with one another. Thus the full study of a knot requires in general the consideration of linking and locking also.

(25) But it is time to close, in spite of the special interest of this part of the subject. And I have left myself barely time to mention the very interesting portion of the *Topologie* which Listing worked out in detail. You will find a brief synopsis of a part of it prefixed to Clerk-Maxwell's *Electricity and Magnetism*, and Cayley has contributed an elementary statement of its contents to the *Messenger of Mathematics* for 1873; but there can be no doubt that so important a paper as the *Census räumlicher Complexe* ought to be translated into English.

To give an exceedingly simple notion of its contents I may merely say that Listing explains and generalizes the so-called *Theorem of Euler about Polyhedra* (which all of us, whose reading dates some twenty years back or more, remember in Snowball's or Hymers' *Trigonometry*), viz. that "if S be the number of solid angles of a polyhedron, F the number of its faces, and E the number of its edges, then

$$S + F = E + 2."$$

The mysterious 2 in this formula is shown by Listing to be the number of *spaces* involved; *i.e.* the content of the polyhedron, and the *Amplexum*, the rest of infinite space.

And he establishes a perfectly general relation of the form

$$V - S + L - P = 0,$$

where V is the number of spaces, S of surfaces, L of lines, and P of points in any complex; these numbers having previously been *purged* in accordance with the amount of *Cyclosis* in the arrangement studied. But to make even the elements of this intelligible I should require to devote at least one whole lecture to them.

Meanwhile I hope I have succeeded in showing to you how very important is our subject, loose and intangible as it may have at first appeared to you; and in proving, if only by special examples, that there are profound difficulties (of a kind different altogether from those usually attacked) which are to be met with even on the very threshold of the Science of Situation.

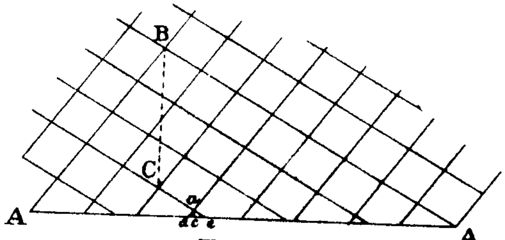


Fig. 1.

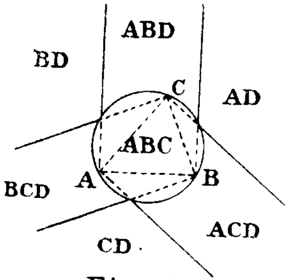


Fig. 2.

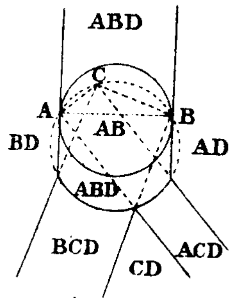


Fig. 3.

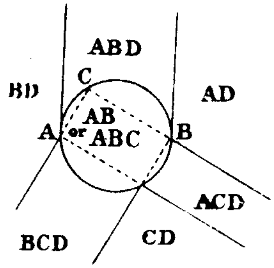


Fig. 4.

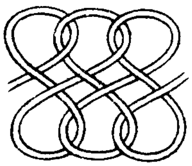


Fig. 19.

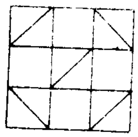


Fig. 5.

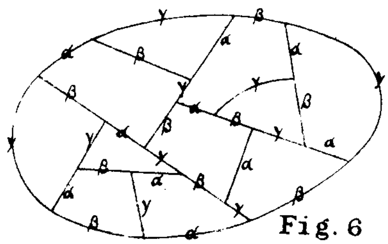


Fig. 6

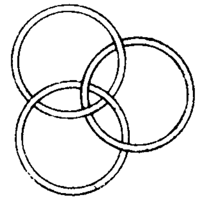
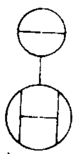


Fig. 20.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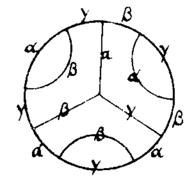
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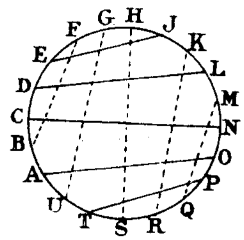


Fig. 9

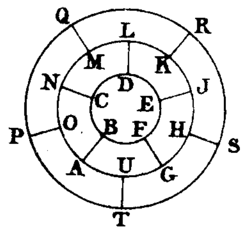


Fig. 11.

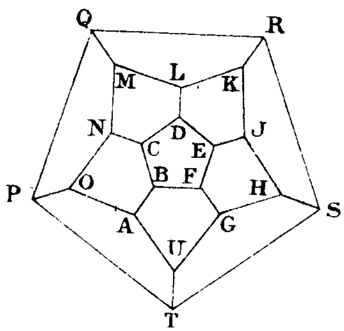


Fig. 10.

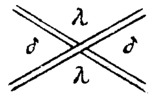
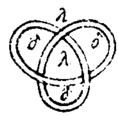


Fig. 12.


Fig. 13.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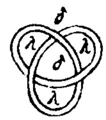
Fig. 15.

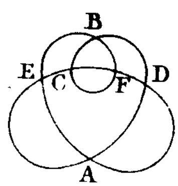

Fig. 16.

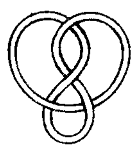
Fig. 17.

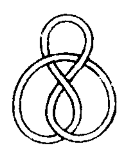
Fig. 18.