IT IS A CLASSIC PROBLEM to give a homotopy theoretic criterion for an imbedding of the
$n$-sphere $S^n$ into a higher dimension $m$-sphere to be “equivalent” to the standard imbedding.
To make the problem more precise, one usually chooses to work in one of three categories: differential, piecewise-linear or topological. Then, the concept of a (locally-flat) submanifold
of $S^m$ and of isomorphism (i.e. diffeomorphism, piecewise-linear homeomorphism or
homeomorphism) is well-defined and the problem may be stated as follows. Let $M$ be a
submanifold of $S^m$, isomorphic to $S^n$; is there an isomorphism $h$ of $S^m$ such that
$h(M) = S^n \subset S^m$?

Many results are known. In the differential category, if $2m > 3(n + 1)$, $h$ always
exists, while if $2m \leq 3(n + 1)$, it may not [1, 18 and 16]. In the piecewise-linear and topo-
logical categories, $h$ always exists if $m - n \geq 3$ [11, 12, 8 and 15]. Finally, in the topological
category, if $m - n = 2$ and $n \geq 3$, $h$ exists if and only if $S^m - M$ is homotopy equivalent to
$S^1$ [8]. It is the main aim of this paper to examine the case $m - n = 2$ in the piecewise-linear
and differential categories and show that this criterion is the correct one here also; it is
necessary to exclude a few low values of $n$ and, in the piecewise-linear situation, impose a
condition of semi-local flatness.

The proofs will use the concept of spherical modifications [4, 5] and will follow almost
identical lines. To avoid repetition, therefore, we will work, simultaneously, in the differen-
tial and piecewise-linear categories. Unless stated otherwise, our manifolds, submanifolds,
mappings, imbeddings and isotopies will be understood to be differential or piecewise-linear,
consistently. All our statements will be treated, accordingly, as referring to the differential
or piecewise-linear category. All manifolds will be orientable.

Denote by $D^k$ the unit $k$-disk or a $k$-simplex in the differential or PL category, respec-
tively, and $S^{k-1} = \partial D^k$. Let $X$ be a manifold and $M$ a submanifold of $X$. We say $M$ is
collared if there is an imbedding $h : M \times D^k \rightarrow X$, where $k = \text{dim } X - \text{dim } M$, such that
$h(x, 0) = x$, for $x \in M$. We say $h$ is a collar of $M$. 

† The author is a National Science Foundation Postdoctoral Fellow.
The main theorem of this paper will be:

**Theorem (1).** Let \( n \geq 4 \); suppose \( M \) is a homology \( n \)-sphere imbedded as a closed collared submanifold of \( S^{n+2} \). Then, if \( S^{n+2} - M \) is homotopy equivalent to \( S^1 \), \( M \) bounds a contractible submanifold of \( S^{n+2} \).

§ 3.

We will need to use the notion of transverse regularity in our proof. It is, therefore, necessary to devote some attention to the piecewise-linear situation.

Let \( A \) be an unbounded PL-manifold, \( V \) a collared PL-submanifold. Let \( M \) be a compact PL-manifold and \( f: M \to A \) a PL-map such that \( f(M) \cap \partial V = \emptyset \). We say \( f \) is transverse regular on \( V \) if \( f^{-1}(V) = \emptyset \), a collared PL-submanifold of \( M \), with \( \partial N \subset \partial M \) and codim \( N = \text{codim} V = k \), and there are collars \( h_1, h_2 \) of \( N, h_2 \) of \( V \) such that:

1. \( h_1|\partial N \times \Delta^k \) is a collar of \( \partial N \) in \( \partial M \),
2. \( f h_1(N \times \Delta^k) \subset h_2(\Delta^k) \) and
3. \( h_2^{-1} f h_1: N \times \Delta^k \to V \times \Delta^k \) is level-preserving, i.e. \( \exists \) mapping \( f_1: N \times \Delta^k \to V \) so that \( h_2^{-1} f h_1(x, t) = (f_1(x, t), t) \).

**Lemma (1).** Let \( A, V, M \) as above and \( f: M \to A \) a PL-map which is transverse regular on \( V \) in a regular neighborhood \( B \) of \( \partial M \) and such that \( f(M) \cap \partial V = \emptyset \). Then \( f \) can be approximated by a PL-map \( f' : M \to A \) which is transverse-regular on \( V \) and such that \( f'|B = f|B \).

**Proof.** First consider the case \( A = \mathbb{R}^k, V = 0 \). A PL-map \( f: M \to \mathbb{R}^k \) is transverse regular at \( 0 \) if and only if there exist admissible triangulations \( K \) of \( M \) and \( L \) of \( \mathbb{R}^k \) such that \( 0 \) is an interior point of a \( k \)-simplex of \( L \) and \( f \) is simplicial with respect to \( K \) and \( L \). That the existence of such \( K \) and \( L \) imply transverse regularity we leave to the reader, or see [14]. Suppose \( f \) is transverse regular and \( h_1: N \times \Delta^k \to M, h_2: \Delta^k \to \mathbb{R}^k \) are collars as described above. Choose admissible triangulations \( K_1 \) of \( M, L_1 \) of \( \mathbb{R}^k \) such that \( f \) is simplicial with respect to \( K_1 \) and \( L_1 \) and the \( h_i \) are simplicial with respect to \( K_1 \) or \( L_1 \) and admissible triangulations of \( N \times \Delta^k \) or \( \Delta^k \). Let \( \Delta_1^k \) be a rectilinear \( k \)-simplex such that \( 0 \in \text{int} \Delta_1^k \subset \Delta_1^k \subset \text{int} \Delta^k \); let \( N_1 \) be an admissible triangulation of \( N \). Now define \( K \) and \( L \) as follows. On \( M - h_1(N \times \Delta_1^k) \) let \( K = K_1 \); on \( \mathbb{R}^k - h_2(\Delta^k) \), let \( L = L_1 \). On \( h_1(N \times \Delta_1^k) \), we carry over the product complex \( N_1 \times \Delta_1^k \); on \( h_2(\Delta_1^k) \), we carry over \( \Delta_1^k \). Note that \( f \) is simplicial with respect to these partial triangulations; it is an easy exercise to extend these triangulations over \( M \) and \( \mathbb{R}^k \) so that \( f \) is still simplicial. Note that we can choose \( L \) as fine as desired.

Now consider triangulations \( K_1 \) of \( B \) and \( L \) of \( \mathbb{R}^k \) such that \( 0 \) is an interior point of a \( k \)-simplex of \( L \) and \( f|B \) is simplicial with respect to \( K_1 \) and \( L \). We can extend \( K_1 \) to a triangulation \( K \) of \( M \). A theorem of [13] provides a simplicial map \( f': K \to L \) such that \( f'|K_1 = f|K_1 \) and, by choosing \( L \) fine enough, \( f' \) may be made to approximate \( f \) as close as desired.
Now consider the general case. Let $g_2$ be a collar of $V$ in $A$ such that $g_2(\partial V \times \Delta^k) \cap f(M) = \emptyset$. Now $f^{-1}(V) \subset \text{int} f^{-1}(\Delta^k \times V)$; let $E$ be a regular neighborhood of $f^{-1}(V)$ in $f^{-1}(\Delta^k \times V)$. Consider $g_2^{-1}f' = g' : E \to \Delta^k \times V$ and let $p : \Delta^k \times V \to \Delta^k$, $p' : \Delta^k \times V \to V$ be the projections. Now $pg' : E \to \Delta^k$ is transverse regular at 0 on a regular neighborhood $B'$ of $\partial E$, where $E \cap B \subset B'$; let $\phi : E \to \Delta^k$ be a transverse regular approximation to $pg'$, which coincides with $pg'$ on $B'$. Thus we have in $E$ a collar $g_1$ of $N = \phi^{-1}(0)$, a PL-submanifold of $M$, and a collar $\psi$ of 0 in $\Delta^k$ such that $\phi g_1(t, x) = \psi(t)$ for $t \in \Delta^k$. Now define $g : E \to \Delta^k \times V$ by $g(x) = (\phi(x), p'g'(x))$. Clearly $g = g'$ on $B'$ and $pgg_1 : E \to \Delta^k$ is transverse regular at 0 on a regular neighborhood $B' \subset B$; let $\phi : B' \to \Delta^k$ be a transverse regular approximation to $pg'$, which coincides with $pg'$ on $B'$. Thus we have in $E$ a collar $g_1$ of $N = \phi^{-1}(0)$, a PL-submanifold of $M$, and a collar $\psi$ of 0 in $\Delta^k$ such that $\phi g_1(t, x) = \psi(t)$ for $t \in \Delta^k$. Now define $f' : M \to A$ by $f'|E = g_2g$, $f'|M - E = f|M - E$. To exhibit the transverse regularity of $f'$, we define collars $h_1, h_2$ of $N$, $V$ by $h_1 = g_1$ and $h_2(t, x) = g_2(\psi(t), x)$. Thus we have

We remark that the differential version of Lemma (1) is proved in [9].

§4.

**Lemma (2).** Let $M$ be a closed collared $n$-submanifold of $S^{n+2}$. Then, if $H_1(M) = 0$, $M$ bounds a collared submanifold of $S^{n+2}$.

**Proof.** Let $g$ be a collar of $M$ and $X = S^{n+2} - g(D^2 \times M)$. Let $f : \partial X \to S^1$ be defined by projection on the “fiber”. Note that $f$ is transverse regular at every point of $S^1$. The only obstruction to extending $f$ over $X$ is in $H^2(X, \partial X; \mathbb{Z}) = H_1(M, \partial X; \mathbb{Z}) = 0$. If we choose $p \in S^1$, it follows from Lemma (1) or [9] that we may choose an extension $f'$ which is transverse regular at $p$. Now $f'^{-1}(p)$ is a collared submanifold of $S^{n+2}$ with boundary $f^{-1}(p)$. We can easily alter this to obtain the desired submanifold.

Let $V$ be a compact manifold and $\lambda : V \to I$ a mapping satisfying $\lambda^{-1}(0) = \partial V$. Define $W \subset V \times R$ as the set of points $(x, t)$ satisfying $|t| \leq \lambda(x)$; then $W$ is a submanifold of $V \times R$ (with a “corner” at $\partial V \times 0$, in the differential case). Note that $\partial W = V_0 \cup V_1$, where $V_1$ consists of the points $(x, (2t - 1)\lambda(x))$ and $V_0 \cap V_1 = \partial V_0 = \partial V_1 = \partial V \times 0$. Let $\varphi_1 : V \to V_1$ be the isomorphism defined by $\varphi_1(x) = (x, (2t - 1)\lambda(x))$.

If $V$ is a collared submanifold of an unbounded manifold $A$, of codimension one, then there is an imbedding $i : W \to A$ satisfying $i(x, 0) = x$ for $x \in V$. Let $Y = A - i(W)$; then $Y$ is homotopy equivalent to $A - V$ and $\partial Y = i(\partial W)$. Define $i_t = i \varphi_t : V \to Y$.

Suppose $f : S^k \times D^{n+1-k} \to int V$ is an imbedding. We define $\theta_f(V, f) = W \cup D^{k+1} \times D^{n+1-k}$, where the “handle” is attached by the imbedding $\varphi_f$ (in the differential case, the corners are rounded at $S^k \times S^{n-k}$); $\theta_f(V, f)$ is a manifold (with a corner at $\partial V \times 0$, in the differential case). Note that $\partial \theta_f(V, f) = V_s \cup V'$, where $s = t \pm 1$, $V'$ is isomorphic to $\chi(V, f)$, in the notation of [4] (extended, in the natural way, to the PL case), and $\partial V' = \partial V_s = \partial V \times 0$.

For the remainder of this paper, we assume $A = S^{n+2}$, $V$, a collared submanifold of $S^{n+2}$, has dimension $n + 1$ and is $(k - 1)$-connected where $k \geq 1$, and $\partial V$ is homology $(k - 2)$-connected. By Alexander duality and the van Kampen and Hurewicz theorems, it follows that $Y$ is $(k - 1)$-connected and, if $k \geq 2$ and $H_{k-1}(\partial V) = 0$, $\pi_k(Y) \approx \pi_k(V)$.
LEMA (3). Suppose (i) $n \geq 2k + 1$ or (ii) $n = 2k$ or $2k - 1$ and $n \geq 4$. Then, if $\alpha \in \pi_k(V)$ and $i_\ast(\alpha) = 0$, $i$ can be extended to an imbedding $i' : \partial_i(V, f) \to S^{n+k}$ where $f : S^k \times D^{n+1-k} \to int V$ is an imbedding representing $\alpha$.

Remark. Clearly $i'(V')$ will be collared in $A$ and $i'(\partial V') = \partial V$.

Proof. We begin by constructing an imbedding $g' : D^{k+1} \to Y$ such that:

$$g'(D^{k+1}) \cap \partial Y = g'(D^{k+1}) \cap V = g'(S^k),$$

and the intersection is normal in the differential case, and $g'|S^k = i_\ast f'$, where $f' : S^k \to int V$ represents $\alpha$.

In the PL situation we can apply the results of [3] to first construct $f'$ and then extend $i_\ast f'$ over $D^{k+1}$. In the differential situation we can, in the same way, use the results of [1] but, unfortunately, this does not cover the case $n = 5$ under hypothesis (ii). Instead we use the following argument.

Let $\alpha' \in \pi_k(\partial Y)$ correspond to $\alpha$ under the inclusion $V_i \subset Y$ (identifying $V_i$ with $V$). Since $i_\ast(\alpha) = 0$, $\alpha'$ is the boundary of an element $\beta' \in \pi_{k+1}(Y, \partial Y)$. Now $Y$ and $\partial Y$ are $1$-connected because $k \geq 2$ in hypothesis (ii). Since $(S^{n+2}, (W))$ is $k$-connected, it follows by excision that $(Y, \partial Y)$ is $k$-connected. We can, therefore, apply Lemma (1) of [17] to obtain an imbedding $g'' : D^{k+1} \to Y$, representing $\beta'$, such that $g''(D^{k+1})$ meets $\partial Y$ normally along $g''(S^k)$—we assume the corners at $\partial V_i$ are straightened. But we also need that $g''(S^k) \subset V_i$, representing $\alpha$. Now $V_i$, is $1$-connected and $(V_i, \partial V_i)$ is homology $(k - 1)$-connected; thus it follows by excision that $(\partial Y, V_i)$ is $(k - 1)$-connected. We can then apply Lemma (2) of [17] to isotopically deform $g''(S^k)$ into $V_i$ to represent $\alpha$. An application of the isotopy extension theorem to $g''$ yields the desired $g'$.

We now would like to extend $g'$ to an imbedding $g : D^{k+1} \times D^{n+1-k} \to Y$ such that $g(D^{k+1} \times D^{n+1-k}) \cap \partial Y = g(S^k \times D^{n+1-k}) \subset int V_i$ (and, in the differential case, the intersection is normal). A tubular neighborhood of $g'(D^{k+1})$ in $Y$ will satisfy these requirements in the differential case. In the PL case, choose a regular neighborhood $X$ of $g'(D^{k+1})$ in $Y$ such that $X \cap \partial Y = \partial X \cap \partial Y = \partial (int V_i)$ and $X \cap \partial Y$ is a regular neighborhood of $g'(S^k)$ in $\partial X$. It follows from [12] that $(\partial X, g'(S^k))$ is isomorphic to $(\partial D^{n+2}, \partial D^{k+1})$. Therefore, by [10], $(X, X \cap \partial Y)$ is isomorphic to $(D^{k+1} \times D^{n+1-k}, S^k \times D^{n+1-k})$. We may now define $f$ by $\varphi_1 f = g|S^k \times D^{n+1-k}$.

We will say $i'(V')$ is obtained by killing $\alpha$.

LEMA (4). Suppose (i) $i_\ast : \pi_k(V) \to \pi_k(Y)$ is a monomorphism for $t = 0, 1$ and (ii) $\pi_k(int V) = \pi_k(A - \partial V)$ is zero. Then $\pi_k(V) = 0$.

Proof. Let $\alpha \in \pi_k(V)$; by (ii) there is a mapping $f : D^{k+1} \to A - \partial V$ such that $f(S^k) \subset int V$ and $f|S^k$ represents $\alpha$.

We may assume $f$ is transverse regular on $V$; in fact, define $f$ in a neighborhood of $S^k$ first, using a collar of $V$, and then extend over $D^{k+1}$ and apply Lemma (1) or [9]. Thus $f^{-1}(V)$ is a, not necessarily connected, $k$-submanifold of $D^{k+1}$. We will show how to remove a component of $f^{-1}(V) \cap int D^{k+1}$, whenever one exists, leaving $f|S^k$ fixed. By a sequence of such modifications of $f$, we will have $f^{-1}(V) = S^k$: by transverse regularity of $f$, this implies $i_\ast(\alpha) = 0$, for some $t$. By (i), we will have $\alpha = 0$. 
Suppose \( f^{-1}(V) \cap \text{int } D^{k+1} \neq \emptyset \); choose an innermost component \( M \), i.e. such that there exists a connected submanifold \( W \) of \( \text{int } D^{k+1} \) such that \( \partial W = M \) and \( W \cap f^{-1}(V) = M \). Consider \( \partial M : M \rightarrow V \); we first show that \( f|\partial M \) extends over \( W \). In fact, the only obstruction to such an extension is the primary obstruction \( \beta \in H^{k+1}(W, M; \pi_k(V)) \approx \pi_k(V) \), since \( V \) is \((k-1)\)-connected. Now \( i_* \beta \in \pi_k(Y) \) is the primary obstruction to extending \( i_* f|\partial M : M \rightarrow Y \), since \( Y \) is \((k-1)\)-connected. Since \( W \cap f^{-1}(V) = M \) and \( f \) is transverse regular on \( V \), it is clear that \( i_* f|\partial M \) does extend over \( W \), for some \( t \). Thus \( i_* \beta = 0 \); by (i), this implies \( \beta = 0 \).

Let \( h_1, h_2 \) be collars of \( M, V \) respectively, satisfying the conditions of transverse regularity (these also exist in the differential case), (2) and (3). Assume \( h_1^{-1}(W) = [-1, 0] \times M \). Let \( g : \frac{1}{2} \times W \rightarrow \frac{1}{2} \times V \) be an extension of \( h_2^{-1} f h_1 : \frac{1}{2} \times M \rightarrow \frac{1}{2} \times V \). Define \( f_0 : W \rightarrow A \), where \( W_1 = W \cup h_1(D^1 \times M) \subset D^{k+1} \), as follows:

\[
\begin{align*}
    f_0 h_1(t, x) &= fh_1(t, x) & x \in M, \frac{1}{2} \leq t \leq 1 \\
    f_0 h_1(t, x) &= fh_1(\frac{1}{2}, x) & x \in M, 0 \leq t \leq \frac{1}{2} \\
    f_0(x) &= h_2 g(\frac{1}{2}, x) & x \in W.
\end{align*}
\]

Note that \( f_0(W_1) \cap V = \emptyset \) and \( f_0 = f \) on a neighborhood of \( \partial W_1 \) (\( f_0 \) will not be differentiable). It is clear that we may now define a map \( f' : D^{k+1} \rightarrow A \) such that \( f'|D^{k+1} - W_1 = \overline{f h(D^1 - \partial M)} - \overline{W_1} \) and \( f'|W_1 \) approximates \( f_0 \) closely. Obviously \( f' \) is transverse regular on \( V \) and \( f'^{-1}(V) = f(V) - M \).

\[\text{§5.}\]

Suppose \( n \geq 2k + 1 \); we will show how to replace \( V \) by a \( k \)-connected collared submanifold of \( S^{n+2} \) whose boundary coincides with \( \partial V \), under the assumption \( \pi_k(S^{n+2} - \partial V) \approx \pi_k(S^1) \).

First we treat the case \( k = 1 \). Let \( \{a_1, \ldots, a_r\} \) be a set of generators of \( \pi_1(V) \) and \( f_i : D^2 \rightarrow S^{n+2} - \partial V \), \( i = 1, \ldots, r \), be maps, transverse regular on \( V \), such that \( f_i(S^1) \subset \text{int } V \) represents \( a_i \). The \( f_i \) exist, as in Lemma (4), since \( a_i \) is null-homotopic in \( S^{n+2} - \partial V \). We define \( N(\{a_i\}, \{f_i\}) \) to be:

\[\sum_{i=1}^{r} \text{order } \pi_0(f_i(D^2) \cap V),\]

and \( N(V) \) to be the minimum of the \( N(\{a_i\}, \{f_i\}) \) for all choices of \( \{a_i\}, \{f_i\} \). Note the following facts.

(a) \( N(V) = 0 \) if and only if \( V \) is 1-connected.

(b) If an innermost component of \( f_i(D^2) \cap V \) is null-homotopic in \( V \), \( N(\{a_i\}, \{f_i\}) > N(V) \).

To prove (b), we use the construction in the proof of Lemma (4) to replace \( f_i \) by \( f'_i \) such that \( f'_i(D^2) \cap V \) has one less component than \( f_i(D^2) \cap V \).

We now show how to replace \( V \) by a new manifold \( V' \) satisfying \( N(V') < N(V) \), if \( N(V) > 0 \). By (a), a finite sequence of such alterations will kill \( \pi_1(V) \).
Choose \( \{x_i\}, \{f_i\} \) so that \( N(\{x_i\}, \{f_i\}) = N(V) \). Let \( \alpha \in \pi_1(V) \) be represented by an innermost component of \( f_i(D^2) \cap V \); then \( \alpha \in \text{Ker } i_* \), for some \( t \), as is pointed out in the proof of Lemma (4). We now apply Lemma (3) to kill \( \alpha \) to obtain our new manifold \( V' \). According to Lemma (2) of \([5]\), \( \pi_1(V') \) is a quotient of \( \pi_1(V) \) by a subgroup containing \( \alpha \). Let \( \alpha' \in \pi_1(V') \) correspond to \( \alpha \). If we assume that the "handle" used in the construction of \( V' \) meets none of the \( f_i(D^2) \)—including \( i = 1 \)—as we may by general position, then a slight deformation of the \( f_i \) will yield \( f'_i \), for which \( N(\{x_i\}, \{f_i\}) \) is defined and equal to \( N(\{x_i\}, \{f_i\}) = N(V) \). But an innermost component of \( f'_i(D^2) \cap V' \) is null-homotopic in \( V' \). Therefore, by (b), \( N(V') < N(V) \).

Suppose \( k \geq 2 \) and \( i_* \) is not a monomorphism. Let \( \alpha \) be a non-zero element of \( \text{Ker } i_* \). Let \( V' \) be obtained, according to Lemma (3), by killing \( \alpha \). Then \( V' \) is \((k - 1)\)-connected and \( \pi_1(V') \) is a proper quotient of \( \pi_1(V) \). Since \( \pi_1(V) \) is a finitely-generated abelian group, this procedure may be iterated only a finite number of times, after which \( i_0, i_1, \ldots \) will all be monomorphisms. Then, by Lemma (4), \( V \) is \( k \)-connected.

The above arguments, following an application of Lemma (2), have proved:

**THEOREM (2).** Let \( n \geq 2k + 1 \) and \( M \) be a closed collared \( n \)-dimensional submanifold of \( S^{n+2} \), such that \( H_i(M) = 0 \) for \( i \leq \max\{1, k - 2\} \). Then, if \( \pi_1(S^{n+2} - M) \approx \pi_k(S^1) \) for \( i \leq k \), \( M \) bounds a \( k \)-connected collared submanifold of \( S^{n+2} \).

Remark. The converse of this theorem is easy to prove, using the appropriate covering of \( S^{n+2} - M \).

§6.

To complete the proof of Theorem (1), we must show how to kill \( \pi_k(V) \), when \( n = 2k \) or \( 2k - 1 \). We first treat the case \( n = 2k \). Recall \( V \) is \((k - 1)\)-connected, and we assume \( k \geq 2 \).

**LEMMA (5).** Suppose \( V' \) is obtained, as in Lemma (3), by killing \( \alpha \in \pi_k(V) \). If \( \alpha \) is non-zero and of finite order, the torsion subgroup of \( \pi_k(V') \) is strictly smaller than that of \( \pi_k(V) \).

Remark. If \( k \) is even, this is proved, more generally, in \([4, \S 5]\).

Proof. It follows from \([4, \text{Lemma (5.6)}]\) (clearly valid in the PL case), that \( \pi_k(V)/\langle \alpha \rangle \approx \pi_k(V')/\langle \alpha' \rangle \), where \( \langle \alpha \rangle \) is the subgroup generated by \( \alpha \) and \( \alpha' \in \pi_k(V') \). To prove Lemma (5), it suffices, by an argument in \([4, \text{p. 519}]\) to show \( \alpha' \) has infinite order. Suppose \( \alpha \) is of order \( p \neq 0 \), \( \alpha' \) is of order \( p' \).

Let \( \theta(V,f) \) have a triangulation, which extends to one of \( S^{n+2} \) under \( i' \), and has, as subcomplexes, \( V \times 0, V' \) and disks \( D, D' \) defined by \( D = D^{k+1} \times y \cup C \) and \( D' = y' \times D^{k+1-k} \), where \( C \) is "cylinder" between \( f(S^k) \) and \( f(p(S^k)) \) in \( W \) and \( y, y' \) are interior points of \( D^{k+1-k}, D^{k+1} \), respectively. Let \( z, z' \) be cycles representing \( \alpha, \alpha' \), respectively carried by \( \partial D \subset V \times 0 \) and \( \partial D' \subset V' \), respectively. Then \( pz = \partial c, p'z' = \partial c' \), where \( c, c' \) are chains carried by \( V \times 0, V' \), respectively. Also \( z = \partial c_1, z' = \partial c_1' \), where \( c_1, c'_1 \) are chains carried by \( D, D' \), respectively. Note that the intersection numbers \( c \cdot c_1 = c_1 \cdot c' = c \cdot c' = 0 \), since \( V \times 0 \cap D' = D \cap V' = V \times 0 \cap V' = \emptyset \), and \( c_1 \cdot c'_1 = \pm 1 \), since the linking number of
$S^k \times y$ and $y' \times D^{n+1-k}$ in $D^{k+1} \times D^{n+1-k}$ is $\pm 1$. Therefore the intersection number $(c - pc_1)\cdot(c' - p'c'_1) = \pm pp'$. But the intersection number of two cycles in $S^{n+2}$ must be zero; since $p \neq 0$, $p' = 0$.

**Lemma (6).** Let $T \subset \pi_k(V)$ be the torsion subgroup. If $i_{*}|T$ is a monomorphism, $\ker i_{*}$ is generated by primitive elements (see [4, p. 516] for definition).

**Proof.** Let $x \in \ker i_{*}$; we shall show $x$ is a multiple of a primitive element of $\ker i_{*}$. Suppose $\alpha = px'$, where $x'$ is primitive; if $\beta = i_{*}(x')$, then $pp\beta = 0$. Since $i_{*}|T$ is a monomorphism and $\pi_k(Y) \approx \pi_k(V)$, $i_{*}|T$ is an isomorphism onto the torsion subgroup of $\pi_k(Y)$. Therefore, $\beta = i_{*}(y)$, where $p\gamma = 0$. Let $x' = x - \gamma$; clearly $i_{*}(x') = 0$ and $x'$ is primitive. Since $px' = px = x$, this completes the proof.

Suppose $i_{*}|T$ is not a monomorphism, for some $t$. We will describe an alteration of $V$ which results in a new $(k - 1)$-connected submanifold $V'$ of $S^{n+2}$, with $\partial V' = \partial V$, and satisfying:

(i) The torsion subgroup $T'$ of $\pi_k(V')$ is strictly smaller than that of $\pi_k(V)$, and

(ii) $\ker i_{*}$ (on $V'$) contains no primitive elements.

By (i), after a finite number of such alterations, we shall have $i_{*}|T'$ is a monomorphism, for $t = 0, 1$. But, by (ii) and Lemma (6), this means $\pi_k(V') = 0$. We have only to describe the required alteration to complete the proof of Theorem (1), when $n$ is even.

Since $i_{*}|T$ is not a monomorphism, we may, by Lemma (3), kill a non-zero element of $\ker i_{*}|T$. By Lemma (5), this results in a manifold with strictly smaller torsion subgroup of $\pi_k$. Next we examine the primitive elements of $\ker i_{*}$; if there is one, we may kill it and, by [4, p. 516], this reduces the rank of $\pi_k$ by one but does not alter the torsion. Thus we may kill all the primitive elements of $\ker i_{*}$; the resulting manifold $V'$ clearly satisfies (i) and (ii).

Now suppose $n = 2k - 1$; then $V$ is $(k - 1)$-connected and $\pi_k(V)$ is free abelian. Let $x' \in \ker i_{*}$; then $x' = px$, where $x$ is primitive. Since $\pi_k(Y) \approx \pi_k(V)$, $x \in \ker i_{*}$. Let $V'$ be obtained by killing $x$, by Lemma (3); we shall determine $\pi_k(V')$. Note that Lemma (3) tells us that $x$ is represented by an imbedded collared sphere; in particular, the self-intersection number $x \cdot x = 0$. Since $x$ is primitive and the intersection pairing of $V$ is non-singular, there exists $\beta \in \pi_k(V)$ such that $x \cdot \beta = 1$. Now, by an argument of [4, p. 527] and [5, p. 54], $\pi_k(V') \approx \pi_k(V)/(x, \beta)$, where $(x, \beta)$ is the subgroup generated by $x$ and $\beta$.

We see that, whenever $\ker i_{*} \neq 0$, we can reduce the rank of $\pi_k(V')$. Eventually $i_{*}$ will be a monomorphism for $t = 0, 1$; by Lemma (4), $\pi_k(V) = 0$. This completes the proof of Theorem (1).

§7.

We conclude by proving the promised unknotting theorem.

**Theorem (3).** Let $M$ be a homotopy $n$-sphere imbedded in $S^{n+2}$ such that $S^{n+2} - M$ is homotopy equivalent to $S^1$. Then, if $n \geq 5$ there is an isomorphism $h$ of $S^{n+2}$ onto itself such that $h(M)$ is the standard $S^n \subset S^{n+2}$. If $n = 4$, the conclusion follows if we assume that $M$ is already isomorphic to $S^n$. 


Proof. By Theorem (1), $M$ bounds a contractible submanifold $V$ of $S^{n+2}$. By [2, Theorem (3.1)], we may assume $V$ has a compatible differential structure; if $n = 4$, it follows from an unpublished result of Cerf, that $M$ is diffeomorphic to $S^4$. By a result of [7], $V$ is now diffeomorphic to $D^{n+1}$, since $\partial V$ is simply-connected and, if $n = 4$, is diffeomorphic to $S^4$. Therefore $V$ was already isomorphic to $D^{n+1}$. The theorem now follows from [6, Theorem (B)] and [11, p. 354].

REFERENCES


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