INERTIA GROUPS OF MANIFOLDS AND DIFFEOMORPHISMS
OF SPHERES.

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The inertia group of a closed smooth manifold $M$ consists of those
topological spheres which do not change the diffeomorphism class of $M$ by
connected sum. It is often non-zero; examples have been constructed by
Tamura [27] and Brown-Steer [10]. On the other hand, limitations on the
size of this group have been given by Wall [30], Browder [7], Kosinski [17]
and Novikov [24].

Another inertia group can be defined as those diffeomorphisms of a disk,
the identity on the boundary, which, when used to change a diffeomorphism
of $M$, don't change its isotopy class. It is technically more practical to replace
isotopy by concordance (see §1)—according to a result of Cerf [11], these
concepts coincide if $M$ is simply-connected and of large enough dimension.
In case $M$ is a topological sphere, this inertia group determines the group of
concordance classes of diffeomorphisms of $M$.

Our study will be based upon a general method of constructing elements
of inertia groups—using a generalization of a construction of Milnor [10].
A special case of this result has been previously obtained by Munkres [21].
In some cases this will enable us to completely determine inertia groups;
also, most existing examples of non-zero inertia groups—and many more—
will emerge.

Some of these results have been obtained independently by A. Kosinski
(unpublished) and R. de Sapio [33].

Two Inertia Groups.

1. All manifolds are smooth and oriented; diffeomorphisms and em-
beddings with codimension zero are orientation preserving. $\Gamma^n$ is the group
of diffeomorphism classes of smooth topological $n$-spheres under connected
sum (see [28]). If $\sigma \in \Gamma^n$, then $\Sigma_\sigma$ will be used to denote a representative
manifold. Two other interpretations of $\Gamma^n$ will be used. They are: (1) the

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group of concordance classes of diffeomorphisms of $S^{n-1}$ (two diffeomorphisms of $M$ are concordant if they extend to a diffeomorphism of $I \times M$—see [31], where the term quasi-diffeotopy is used), under composition. (2) the group of concordance classes rel $\partial D^{n-1}$ of diffeomorphisms of $D^{n-1}$ which are $1$ on $\partial D^{n-1}$ (a concordance rel $\partial D^{n-1}$ is one which is $1$ on $I \times \partial D^{n-1}$).

In either case, if $\sigma \in \Gamma^n$, let $h_\sigma$ be used to denote a representative diffeomorphism. The correspondence between the interpretations is given as follows. Given $h_\sigma$, a diffeomorphism of $S^{n-1}$, which can be taken to be $1$ on a hemisphere $D_0^{n-1}$, then $h_\sigma \big| D^{n-1}$ (the opposite hemisphere) is a corresponding diffeomorphism of $D^{n-1}$, and $\Sigma_\sigma$ can be defined as the union of two copies of $D^n$ with boundaries identified by $h_\sigma$. See [28], [31] for more details.

2. We will use $M^n$ to denote a closed manifold of dimension $n$. We consider two subgroups $I_0(M) \subset \Gamma^n$, $I_1(M) \subset \Gamma^{n+2}$ called the inertia groups of $M$. $I_0(M)$ consists of all $\sigma \in \Gamma^n$ such that the connected sum $M \# \Sigma_\sigma$ is diffeomorphic to $M$ (see [17]). $I_1(M)$ consists of all $\sigma \in \Gamma^{n+2}$ such that the diffeomorphism of $M$ which differs from $1$ only on an $n$-disk $D \subset M$, and there coincides with $h_\sigma$, is concordant to $1$. These groups are obviously of importance in the classification of diffeomorphism classes of manifolds homeomorphic to $M$ and concordance classes of diffeomorphisms of $M$.

We also define reduced inertia groups $\tilde{I}_0(M)$, $\tilde{I}_1(M)$. Let $bP^{n+1} \subset \Gamma^n$ be the subgroup of those $\sigma$ such that $\Sigma_\sigma$ bounds a parallelizable manifold (see [16]). Then we define:

$$\tilde{I}_0(M) = I_0(M) / I_0(M) \cap bP^{n+1}, \quad \tilde{I}_1(M) = I_1(M) / I_1(M) \cap bP^{n+2}$$

—subgroups of $\Gamma^n / bP^{n+1} = \hat{\Gamma}^n$, and $\hat{\Gamma}^{n+1}$, respectively.

3. We relate the two inertia groups by:

**Proposition 1.** $I_1(M) = I_0(M \times S^1)$.

Recall the mapping torus $M_h$ of a diffeomorphism $h$ of $M$. This is the manifold obtained from $M \times I$ by identifying $(x, 0)$ with $(h(x), 1)$, for every $x \in M$ (see [8]).

**Lemma 1.** If $\sigma \in \Gamma^{n+1}$ and $h$ is a diffeomorphism of $M$, let $h'$ be obtained from $h$ by changing it on an $n$-disk $D \subset M$ by $h_\sigma$. Then $M_{h'}$ is diffeomorphic to $M_h \# \Sigma_\sigma$.

See [8] for a proof.

**Lemma 2.** If $h$ is a diffeomorphism of $M$ and $n \geq 4$ then $h$ is concordant
to 1 if and only if \( M_h \) is diffeomorphic to \( M_1 = M \times S^1 \), by a diffeomorphism yielding a homotopy-commutative diagram:

\[
\begin{array}{c}
M_h \\
\downarrow p \\
S^1
\end{array} \quad \begin{array}{c}
\rightarrow \\
\downarrow p^1
\end{array} \quad \begin{array}{c}
M \times S^1
\end{array}
\]

where \( p, p^1 \) are the natural fibrations.

A more general fact is proved in [8] when \( M \) is 1-connected, \( n \geq 5 \). But the proof actually shows that if \( M_h \) is diffeomorphic to \( M \times S^1 \), then there exists an \( h \)-cobordism \( V \) of \( M \) with itself and a diffeomorphism \( g \) of \( V \) which is \( h \) on one end and 1 on the other. In the case \( n \geq 4 \), it is proved in [26] that \( V \) is invertible, i.e., there exists another \( h \)-cobordism \( W \) from \( M \) to \( M \) such that \( V \cup W \) identified along the end of \( V \) where \( g = 1 \) is diffeomorphic to \( I \times M \). If we extend \( g \) to a diffeomorphism of \( V \cup W \) which is 1 on \( W \), we get a concordance from \( h \) to 1.

Now Proposition 1 follows easily. If \( n \leq 5 \), both groups are zero, since \( \Gamma^{n+1} \equiv 0 \). When \( n \geq 4 \), it follows from Lemmas 1 and 2.

**Diffeomorphisms of Spheres.**

4. When \( M \) is a topological sphere, \( I_1(M) \) assumes added significance.

**Proposition 2.** If \( M \) is a topological sphere, \( I_1(M) \) contains at most two elements and \( \Gamma^{n+1}/I_1(M) \) is naturally isomorphic to \( \Gamma(M) \), the group of concordance classes of diffeomorphisms of \( M \).

Define a homomorphism \( \phi: \Gamma^{n+1} \rightarrow \Gamma(M) \) by changing 1 on a disk \( D \subset M \), as described in §2. The kernel is clearly \( I_1(M) \). Since the closure of the complement of \( D \) is a disk \( D_0 \) and any diffeomorphism of \( M \) is isotopic to one which is 1 on \( D_0 \), \( \phi \) is onto.

We introduce the group \( \Gamma(M \text{ rel } D_0) \) of concordance classes rel \( D_0 \) of diffeomorphisms of \( M \) which are 1 on \( D_0 \) ([31]), and the obvious homomorphism \( \psi: \Gamma(M \text{ rel } D_0) \rightarrow \Gamma(M) \). If \( n \geq 3 \), it is proved in [31] that \( \psi \) is onto and the kernel has order at most two. Moreover, a diffeomorphism of \( M \) represents the generator of \( \Gamma(M \text{ rel } D_0) \) if and only if it is concordant to 1 by a concordance which restricts to the non-trivial bundle map \( I \times D_0 \rightarrow I \times D_0 \) (bundles over \( I \)) which is 1 over \( \partial I \) i.e., the one corresponding to the non-trivial homotopy class \( (I, \partial I) \rightarrow (SO_n, e) \).

There is a natural isomorphism \( \Gamma^{n+1} \leftrightarrow \Gamma(M \text{ rel } D_0) \) obtained by asso-
ciating to any diffeomorphism of $M$, which is $1$ on $D_0$, its restriction to a
diffeomorphism of $D$. Clearly $\phi$ corresponds to $\psi$ under this isomorphism.

This completes the proof of Proposition 2.

5. $I_1(M)$ also is related to a question of “rotational symmetry” of $M$,
when it is a topological sphere. It follows from § 4 that $I_1(M) = \text{Kernel } \psi = 0$
if and only if the non-trivial isotopy from 1 to 1 on $D_0$ extends to a con-
cordance—and therefore an isotopy, when $n \geq 6$, according to [11]—from 1
to 1 on $M$. This can be restated.

**Proposition 3.** If $M$ is a topological sphere, then $I_1(M) = 0$ if and
only if a non-trivial orthogonal action of $S^1$ on any disk in $M$ extends to an
action of $S^1$ on $M$.¹

6. Define a function $\gamma: \Gamma^n \to \Gamma^{n+1}$ by:

$$\gamma(\sigma) = \text{generator of } I_1(\Sigma_{\sigma}).$$

**Proposition 4.** $\gamma$ is a homomorphism.

We use the following characterization of $\gamma(\sigma)$. Let $\{f_t\}$ be the non-
trivial linear isotopy from 1 to 1 on $D^n$. Then a diffeomorphism $h_{\gamma(\sigma)}$ of
$D^n$ represents $\gamma(\sigma)$ if and only if the isotopy $\{h_{\sigma} \circ f_t \circ h_{\sigma}^{-1}\}$ of $S^{n-1}$ extends
to a concordance from 1 to $h_{\gamma(\sigma)}$. This follows readily from § 4. Suppose
$\tau \in \Gamma^n$ also. We may assume $h_{\sigma} \big| D_+^{n-1} = 1$ and $h_{\tau} \big| D_-^{n-1} = 1$; then
$h_{\sigma+\tau} = h_{\sigma} \circ h_{\tau}$ agrees with $h_{\tau}$ on $D_+^{n-1}$ and $h_{\sigma}$ on $D_-^{n-1}$. Since $D_+^{n-1}$ and
$D_-^{n-1}$ are invariant under $f_t$, $\gamma(\sigma)$ may be chosen to be 1 on $D_+^{n} \subset D^n$
($D_+^{n}$ is the “half-moon” defined by a coordinate being non-negative); also
$h_{\gamma(\tau)} = 1$ on $D_-^{n}$. To construct $h_{\gamma(\sigma+\tau)}$ we need an isotopy from 1 on $D^n$
which extends $h_{\sigma+\tau} \circ f_t \circ h_{\sigma+\tau}^{-1}$ on $S^{n-1}$. But this can be done by piecing
together the isotopy from 1 to $h_{\gamma(\sigma)}$ on $D_-^{n}$, and from 1 to $h_{\gamma(\tau)}$ on $D_+^{n}$.
Then we see that $h_{\gamma(\sigma+\tau)} = h_{\gamma(\sigma)} \circ h_{\gamma(\tau)}$, which says $\gamma(\sigma + \tau) = \gamma(\sigma) + \gamma(\tau)$.

**Corollary.** $I_1(\Sigma_{\sigma}) = 0$ if $\sigma = 2\alpha'$ for some $\alpha' \in \Gamma^n$.

The homomorphism $\gamma$ can be shown to coincide with the special case of the $\Lambda_2$ of Munkres [20]:

$$\Lambda_2: H^{n-1}(X; \Gamma^n) \to H^{n+1}(X; \Gamma^{n+1}),$$

where $X$ is the non-trivial $(n-1)$ sphere bundle over $S^2$.²

¹ The actions referred to are not group actions i.e. do not satisfy the formula
$gh(x) = g(h(x))$.

² Using the Hirsch-Mazur isomorphism $\Gamma^n = \Pi_4(Pl/0)$, $\gamma$ corresponds to composition with the generator $\eta \in \Pi_{n+1}(S^n)$. 
Construction of Some Inertial Spheres.

7. Let \( n \) and \( k \) be positive integers. Choose elements
\[
\sigma \in \Gamma^{n+1}, \quad \tau \in \Gamma^{k+1}, \quad \alpha \in \pi_n(SO_k), \quad \beta \in \pi_k(SO_n).
\]
By a slight generalization of a construction of Milnor [19], we define an element \( \delta = \delta(\sigma, \alpha; \tau, \beta) \in \Gamma^{n+k+1} \).

Let \( h_\sigma, h_\tau \) be representative diffeomorphisms of \( S^n \) and \( D^k \)—we may assume \( h_\tau = 1 \) in a neighborhood \( N \) of \( S^{k-1} \) and \( h_\sigma = 1 \) on a hemisphere \( D \subset S^n \). Let \( f : (S^n, D) \to (SO_k, e) \) and \( g : (D^k, S^{k-1}) \to (SO_n, e) \) represent \( \alpha, \beta \) respectively—we may assume \( g \) maps all of \( N \) onto \( e \). Now define diffeomorphisms \( d_1, d_2 \) of \( S^n \times D^k \) by:
\[
\begin{align*}
d_1(x, y) &= (h_\sigma(x), f(x) \cdot y) \\
d_2(x, y) &= (g(y) \cdot x, h_\tau(y)),
\end{align*}
\]
using the (suspended) action of \( SO_n \) on \( S^n \) and the usual action of \( SO_k \) on \( D^k \).

**Lemma 3.**

(a) \( d_1 \mid D \times D^k = 1 \)

(b) \( d_2 \mid S^n \times N = 1 \)

(c) \( d_2(D \times D^k) = D \times D^k \)

(d) \( d_2 \) extends to a diffeomorphism of \( D^{n+1} \times D^k \) which is 1 on \( D^{n+1} \times N \).

One checks (a) and (b) immediately; (c) and (d) follow from the fact that the action of \( SO_n \) on \( S^n \) preserves \( D \) and extends to an action on \( D^{n+1} \).

Now define \( \delta = d_1^{-1}d_2^{-1}d_1d_2 \). It follows directly from Lemma 3 that \( \delta = 1 \) on a neighborhood of \( S^n \times S^{k-1} \cup D \times D^k \). Thus \( \delta = 1 \) outside of an interior disk \( D_0 \subset S^n \times D^k \). Let \( \delta \in \Gamma^{n+k+1} \) be the element represented by \( \delta \mid D_0 \); it clearly depends only upon \( \sigma, \tau, \alpha \) and \( \beta \).

When \( \sigma = \tau = 0 \), this agrees with Milnor’s construction. When \( \alpha = 0 \) and \( \tau = 0 \), for example, it is related to a construction of Novikov [25], the twist-spinning operation of Hsiang-Sanderson [13], and a pairing of Bredon [32].

8. The following theorem is basic.

**Theorem 1.** Let \( M \) be a closed, smooth \((n + k + 1)\)-manifold and suppose \( \Sigma_\sigma \) is embedded in \( M \) with normal bundle associated to \( \alpha \in \pi_n(SO_k) \). Then, for any \( \tau \in \Gamma^{k+1}, \beta \in \pi_k(SO_n) \), we have:
\[
\delta(\sigma, \alpha; \tau, \beta) \in I_0(M).
\]

An immediate consequence of Theorem 1 and Proposition 1 is:
THEOREM 2. Let $M$ be a closed, smooth $(n+k)$-manifold, and suppose $\Sigma_\sigma$ is embedded in $M$ with normal bundle associated to $x \in \pi_n(SO_{k-1})$. Then, if $S: \pi_n(SO_{k-1}) \to \pi_n(SO_k)$ is suspension, for any $\tau \in \Gamma^{k+1}$, $\beta \in \pi_k(SO_n)$, we have:

$$\delta(\sigma, S(x); \tau, \beta) \in I_1(M).$$

For example, in both theorems, $M$ can be taken to be the sphere-bundle over $\Sigma_\sigma$ associated with $S(x)$. See [33] for a similar result.

Let $T$ be a tubular neighborhood of $\Sigma_\sigma$ in $M$; then $T$ is diffeomorphic to the disk bundle over $\Sigma_\sigma$ associated with $x$. We will show that, if the connected sum $M \# \Sigma_\delta$ is formed along a disk interior to $T$, then it is diffeomorphic to $M$ by a diffeomorphism which reduces to 1 on $M - T$. Equivalently, we simply show that $T$ is diffeomorphic to $T \# \Sigma_\delta$ (along an interior disk) by a diffeomorphism which is 1 near $\partial T$.

Let $d_1$ be as in § 7; then $T$ can be described as the union of two copies of $D^{n+1} \times D^k$ identified along $S^n \times S^{k-1}$ by $d_1$. We denote this by $X(d_1)$. Theorem 1 will now follow from the two facts:

1. $X(d_1)$ is diffeomorphic to $X(d_2^{-1}d_1d_2)$ by a diffeomorphism which is 1 near the boundary—this makes sense since, by Lemma 1-(b), $d_1 = d_2^{-1}d_1d_2$ near $S^n \times S^{k-1}$.

2. $X(d_1d)$ is diffeomorphic to $X(d_1) \# \Sigma_\delta$ by a diffeomorphism which is 1 near the boundary ($d = 1$ near $S^n \times S^{k-1}$).

Since $d_1d = d_2^{-1}d_1d_2$, Theorem 1 follows.

Fact (2) is proven by an argument similar to that which proves Lemma 1.

Some Invariants of $\mathfrak{g}$.

9. We now investigate various techniques for proving non-triviality of $\delta(\sigma, x; \tau, \beta)$.

We will need the following alternative description of $\delta$. Let $X_1$ be the disk-bundle over $\Sigma_\sigma$ associated with $S(x)$ and $X_2$ the disk-bundle over $\Sigma_\tau$ associated with $S(\beta)$. We then form $X_\delta$ by the operation of "plumbing" $X_1$ and $X_2$: $X_\delta$ is just the union of $X_1$ and $X_2$ with an identification of the sub-bundle in $X_1$ over a disk in $\Sigma_\sigma$ with a similar sub-bundle in $X_2$—both sub-bundles admit obvious diffeomorphisms with $D^{n+1} \times D^{k+1}$. Now $\Sigma_\delta$ can be taken to be $\partial X_\delta$. See [19] for more details in the case $\sigma = \tau = 0$; the argument is precisely the same for general $\sigma, \tau$. 
In the case of \( n = k \) even, \( \sigma = \tau = 0 \) and \( \alpha = \beta \) a desuspension of the tangent bundle of \( S^{n+1} \), \( \Sigma_{n} \) is just the Kervaire sphere \([15]\). In fact, even if \( \sigma, \tau \) are unrestricted \( X_{\delta} \) is an \( n \)-connected parallelizable \((2n + 2)\)-manifold with Arf invariant 1. By \([16]\), \( \delta \) is the generator of \( bP^{2n+2} \), which is zero, if \( n = 2 \) or 6, \( Z_{2} \) if \( n \equiv 0 \mod 4 \), and 0 or \( Z_{2} \) otherwise (see \([9]\)).

As a consequence of Theorem 1 we, therefore, have:

**Example 1** (Brown-Steer \([10]\)). \( I_{0}(V_{n+1,2}) \supset bP^{2n+2} \) if \( n \) is even, where \( V_{n+1,2} \) is the Stiefel-manifold of 2-frames in \((n + 1)\)-space.

10. We now use the Eells-Kuiper invariant \([12]\) to study \( \delta(\sigma, \alpha; \tau, \beta) \). Suppose \( r, s \geq 1 \) are integers. We define:

\[
\mu_{r,s} = \frac{a_{r}a_{s}B_{r}B_{s}(2^{2r} - 1)(2^{2s} - 1)}{16a_{r+s}r^{8}(2^{2r+2s-1} - 1)} \mod 1
\]

where \( B_{r} \) is the \( r \)-th Bernoulli number and \( a_{r} = 1 \) or 2 as \( r \) is even or odd.

For example \( \mu_{2,1} = 1/112; \mu_{1,2} = \mu_{2,1} = 1/3968; \mu_{2,2} = 1/32,5122 \).

Let \( \Gamma_{n_{\text{spin}}} \) be the subgroup of \( \sigma \) such that \( \Sigma_{\sigma} \) bounds a spin-manifold.

It follows from \([3]\), \([3]\) that \( \Gamma_{n_{\text{spin}}} = \Gamma_{n} \) unless \( n \equiv 1 \) or 2 \( \mod 8 \), in which case it is a subgroup of index 2.

Suppose \( n = 4r - 1, k = 4s - 1 \). The \( \mu \) invariant of Eells-Kuiper \([12]\) defines a homomorphism:

\[
\mu : \Gamma_{n+k+1} \to \mathbb{Q}/\mathbb{Z}
\]

since \( \Gamma_{n+k+1} = \Gamma_{n+k+1_{\text{spin}}} \).

If \( \alpha \in \pi_{n}(SO_{k}) \), then the suspension of \( \alpha \) into the stable group \( \pi_{n}(SO) \equiv Z \) determines a unique non-negative integer, denoted \( |\alpha| \). If \( n \geq 2k + 1, |\alpha| = 0 \) \(([19, \text{Lemma 5}])\).

**Proposition 5.** If \( \delta = \delta(\sigma, \alpha; \tau, \beta) \), then:

\[
\mu(\delta) = \mu_{r,s} |\alpha| |\beta|.
\]

This is proved in \([12]\) for \( \sigma = \tau = 0 \), using the relation

\[
p_{r}(\alpha) = \pm a_{r}(2r - 1)! |\alpha|
\]

(see also \([19]\)), where \( p_{r}(\alpha) \) is the Pontragin class of \( \alpha \). The more general case is proved identically.

**Example 2.** Suppose \( s < 2r, n = 4r - 1, k = 4s - 1 \). If \( \lambda \in H_{n+1}(M^{n+k+1}; Z) \)

is represented by an imbedded sphere, then \( I_{0}(M) \) has order a multiple of
the denominator of the fractions: \( \epsilon_{r,s} = \frac{(p_r(M) \cdot \lambda)}{a_r(2r - 1)} \) where \( \epsilon_{r,s} = 2 \), if \( r = s = 1 \) or 2 or \( r = 3, s = 4 \), and 1 otherwise.

This follows from Theorem 1 and the fact that \( M \) contains a copy of a disk bundle associated to \( \alpha \in \pi_n(SO_k) \), where \( |\alpha| = p_r(M) \cdot \lambda \) and \( \beta \) can be chosen to satisfy \( |\beta| = \epsilon_{r,s} \) (see [6]). For any given \( \alpha \in \pi_n(SO_k) \), we can choose \( M \) as the sphere bundle over \( S^{n+1} \) associated with \( S(a) \) to satisfy the hypotheses of Example 2.

In the special case \( r = s = 1 \), if we choose \( |\alpha| = 2 \), which is possible, we find that \( I_o(M) = \Gamma^{n-1} \) — a result of Tamura [27]. More generally, \( |\alpha| \) can be chosen to be \( \epsilon_{s,r} \), if \( r < 2s \).

**COROLLARY.** If \( s < 2r < 4s \), there exists a \( k \)-sphere bundle \( M \) over \( S^{n+1} \) such that \( I_o(M) \) has order a multiple of the denominator of \( \epsilon_{r,s} \).

The next non-trivial example is a 7-sphere bundle \( M \) over \( S^8 \) with \( I_o(M) \) a subgroup of \( \Gamma^{15} \) of index \( \leq 2 \).

**Reduced Inertia Groups.**

11. In §§ 9, 10 we studied \( \delta \) by techniques which are particularly sensitive for distinguishing elements of \( bP^{n+1} \cdot 2 \). We now examine the reduced inertia groups (see § 2). It is possible to obtain some of these results using the Browder-Novikov theory [23], [24].

Recall the homomorphism:

\[ T: \Gamma^n \to \text{Cokernel } \{ J_n: \pi_n(SO) \to \pi_n(S) \} \]

defined by the Thom construction, where \( J_n \) is the Hopf-Whitehead homomorphism (see [16]). The kernel of \( T \) is precisely \( bP^{n+1} \); the associated monomorphism, \( \Gamma^n \to \text{Cok } J_n \), will also be denoted by \( T \). Recall that \( T \) is onto, unless \( n = 2, 6 \) or 14, when the image is a subgroup of index 2, or \( n \equiv 6 \mod 8 \), when it is a subgroup of index \( \leq 2 \) (see [9] and [18]).

We determine \( T(\delta) \), when \( \alpha \) or \( \beta \) is zero. If \( \sigma \in \Gamma^n \), denote the corresponding element of \( \Gamma^n \) by \( \dot{\sigma} \).

We use the bilinear anti-commutative composition pairing [29]:

\[ \pi_i(S) \times \pi_j(S) \to \pi_{i+j}(S), \quad (\xi, \eta) \to \nu \circ \xi. \]

If \( \nu \in \pi_{i+j}(S^j) \), \( \theta \in \pi_i(SO) \), then the composition \( \theta \circ \nu \in \pi_{i+j}(SO) \) is defined. Let \( E: \pi_{i+j}(S^j) \to \pi_{i+j+1}(S^{j+1}) \) be the suspension homomorphism — then \( E^\infty \) will denote suspension into the stable stem. The following formula holds [17]:

\[ J_{i+j}(\theta \circ \nu) = \pm J_j(\theta) \circ E^\infty (\nu). \]
This implies the existence of an induced bilinear composition pairing:

\[ \pi_{l+j}(S^j) \times \text{Cok } J_j \to \text{Cok } J_{l+j}. \]

**Proposition 6.** If \( \delta = \delta(\sigma, \alpha; \tau, \beta) \) and \( J: \pi_4(SO_j) \to \pi_{l+j}(S^j) \) is the (non-stable) Hopf-Whitehead homomorphism:

\[
T(\delta) = \pm EJ(\alpha) \circ T(\tau) \quad \text{if } \beta = 0,
\]

\[
\pm EJ(\beta) \circ T(\sigma) \quad \text{if } \alpha = 0.
\]

It follows from Proposition 6, the above formula, and consideration of suspension [17], that \( T(\delta) = 0 \) if \( n > k \) and \( \beta = 0 \), or \( k > n \) and \( \alpha = 0 \).

Proposition 6, and its proof, is closely related to [25, Lemma 6]. A similar fact has been proved by Milnor [21] and Bredon [32].

12. Since \( \delta(\sigma, \alpha; \tau, \beta) = -\delta(\tau, \beta; \sigma, \alpha) \), it suffices to consider only \( \beta = 0 \).

It follows from the description of \( \delta \) in \$9\$, that \( \Sigma_\delta \) arises from a spherical modification ([16]) on \( \partial X_2 \). In case \( \beta = 0 \), \( \partial X_2^\prime = S^n \times \Sigma_\tau \) and the modification is constructed from an imbedding:

\[ i: S^n \times D^{k+1} \to S^n \times \Sigma_\tau \]

defined by \( i(x, y) = (h_\sigma(x), f(x) \cdot y) \), where \( f \) represents \( S(\alpha) \in \pi_n(SO_{k+1}) \) and \( D^{k+1} \) is identified with a disk in \( \Sigma_\tau \). Then:

\[ \Sigma_\delta = S^n \times \Sigma_\tau \to i(S^n \times D^{k+1}) \cup D^{n+1} \times S^k \]

where the boundaries are identified by \( i/S^n \times S^k \).

\( \Sigma_\delta \) and \( S^n \times \Sigma_\tau \) are connected by the cobordism

\[ X = I \times S^n \times \Sigma_\tau \cup D^{n+1} \times D^{k+1} \]

where the pieces are attached by the imbedding \( S^n \times D^{k+1} \to 1 \times S^n \times \Sigma_\tau \) corresponding to \( i \).

Suppose \( S^n \subset R^N \), \( N \gg n \), has a normal frame \( F_0 \) obtained from the standard normal frame by a “twist” by a map representing

\[ -S^{N-n-k}(\alpha) \in \pi_n(SO_{N-n}). \]

Consider \( \Sigma_\tau \subset R^M \), \( M \gg k \), with a normal frame \( F_1 \). Then the product imbedding \( S^n \times \Sigma_\tau \subset R^N \times R^M \), with the product framing \( F_0 \times F_1 \), defines, by the Thom construction, a representative of \( \pm EJ(\alpha) \circ T(\tau) \) (see [14]). The theorem will be proved by extending this to a framed imbedding of \( X \) in \( R \times R^N \times R^M \).
An imbedding of $X$ is defined by merely extending the composite imbedding:

$$i: S^n \times D^{k+1} \to S^n \times \Sigma_r \subset R^N \times R^M$$

to an imbedding of

$$D^{n+1} \times D^{k+1} \subset \{0, \infty\} \times R^N \times R^M$$

which meets $0 \times R^N \times R^M$ transversely along $i(S^n \times D^{k+1})$.

Now suppose $i(S^n \times 0) = S^n \times a$, $a \in \Sigma_r$. The normal frame $F_2$ to $S^n \times a$ in $S^n \times \Sigma_r$ induced by the differential of $i$, is obtained from the standard normal frame by twisting with $S(a)$. To extend $i$ to an imbedding of $D^{n+1} \times D^{k+1}$, we may first extend $i \mid S^n \times 0$ to an imbedding $i^1$ of $D^{n+1} \times 0$ (transverse to $0 \times R^N \times R^M$ along $i(S^n \times 0)$) and then extend $F_2$ to a normal $(k+1)$-frame to $i^1(D^{n+1} \times 0)$ in $R \times R^N \times R^M$. Therefore an extension of $F_0 \times F_1$ to a normal framing of $X$ is equivalent to an extension of $F_0 \times F_1 \times F_2$ (a normal frame to $S^n \times a$ in $R^N \times R^M$) to a normal framing of $i^1(D^{n+1} \times 0)$ in $R \times R^N \times R^M$.

But $F_0 \times F_1 \mid S^n \times a = F_0 \times (F_1 \mid a)$, and $F_1 \mid a$ is trivial. Since $F_0$, $F_2$ are obtained from trivial frames by twisting by $-a$ and $a$, respectively, it follows that $F_0 \times F_1 \times F_2$ is homotopic to a trivial frame on $S^n \times a$, which will extend to a normal frame on an imbedding $i^1$ of $D^{n+1} \times 0$.

This completes the proof of Proposition 6.

13. Proposition 6, together with Theorems 1 and 2, have obvious consequences about the reduced inertia groups.

**Example 4.** (see [24, 13.3]) If $M$ contains an imbedded topological $(n+1)$-sphere with normal bundle associated to $a \in \pi_n(SO_k)$, then $T(\tilde{I}_0(M))$ contains, as a subgroup $EJ(a) \circ T(\tilde{r}^{k+1})$.

**Example 5.** If $M$ contains an imbedded topological $(n+1)$-sphere with normal bundle associated to $a \in \pi_n(SO_{k-1})$, then $T(\tilde{I}_1(M))$ contains, as a subgroup, $E^2J(a) \circ T(\tilde{r}^{k+1})$.

**Example 6.** If $M$ contains an imbedded $(n+1)$-sphere $\Sigma_a$ with trivial normal $k$-plane bundle, then $T(\tilde{I}_0(M))$ contains, as a subgroup

$$J\pi_k(SO_{n+1}) \circ T(\tilde{\sigma}),$$

and $T(\tilde{I}_1(M))$ contains, as a subgroup, $EJ\pi_{k+1}(SO_n) \circ T(\tilde{\sigma})$. 
Example 6 follows by noticing that $EJ\pi_i(SO_j) = J\pi_i(SO_{j+1})$, when $j > i$, and, when $j \leq i$, their compositions with an element of the form $T(\sigma)$, $\sigma \in \Gamma_j$, are zero, according to remarks on §11.

In Examples 4 and 5, a sample $M$ is the sphere bundle over $S^{n+1}$ associated with $S(\alpha)$. In Example 6, we can take for $M$ a manifold of the form $X_\sigma \times V$, where $V$ is any $k$-manifold.

If $\alpha$ is the non-zero element of $\pi_1(SO_k)$ ($k > 2$) then there exist $\tau \in \Gamma^{k+1}$ such that $EJ(\alpha) \circ T(\tau)$ is non-zero for $k = 7, 13, 15$ and $k \equiv 0 \mod 8$. This follows from [5] and [29]. Therefore, we have, as a consequence of Example 4 (see [24, Lemma 13.4]) for similar results:

**Corollary 1.** Suppose $M$ is a manifold of dimension $9, 15, 17$ or $8t + 2$ ($t \geq 1$) satisfying:

(a) $M$ is not a spin-manifold.
(b) $H_2(\pi_1(M); Z_2) = 0$, e.g., $\pi_1(M) = 0, Z$, or finite of odd order.

Then $\tilde{I}_0(M)$ is non-zero.

Condition (b) implies $H_2(M; Z_2)$ is entirely spherical. Then, (a) implies there is an imbedded 2-sphere with non-trivial normal bundle. A similar fact is proved in [21].

Similarly, we derive from Example 5:

**Corollary 2.** If $M$ satisfies (a), (b) of Corollary 1 and has dimension $8, 14, 16$ or $8t + 1$ ($t \geq 1$), then $\tilde{I}_1(M)$ is non-zero.

As an application of Example 6 we compute reduced inertia groups in some special cases (see also [33]).

**Corollary 3.** If $\sigma \in \Gamma^n$, $\tau \in \Gamma^k$, $n \geq k$, then

$$I_0(\Sigma_\sigma \times \Sigma_\tau) = J\pi_k(SO_n) \circ T(\sigma).$$

The inclusion $I_0(\Sigma_\sigma \times \Sigma_\tau) \supset J\pi_k(SO_n) \circ T(\sigma)$ follows from Example 6. For the reverse inclusion we examine the subset $P$ of $\text{Cok} J_{n+k}$ determined, from the Thom construction, by all possible normal framing of $\Sigma_\sigma \times \Sigma_\tau$ ([16]). By the additivity of this operation ([16, Lemma 4.4]), every element of $\tilde{I}_0(\Sigma_\sigma \times \Sigma_\tau)$ is the difference of two elements of $P$.

It follows by obstruction theory that any normal frame to $\Sigma_\sigma \times \Sigma_\tau$ is homotopic to a product framing $F_\sigma \times F_\tau$, on the complement of a point;
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where $F_\sigma, F_\tau$ are normal frames to $\Sigma_\sigma, \Sigma_\tau$, respectively. Thus any element of $P$ is represented by a composition $a_\sigma \circ a_\tau$, where $a_\sigma \in T(\sigma), a_\tau \in T(\tau)$. Now the difference of two such elements is a sum $J_k a_1 \circ a_\sigma + J_n a_2 \circ a_\tau$, where $a_1 \in \pi_n(SO), a_2 \in \pi_n(SO)$. Since $n \geq k$, $J_n a_2 \circ a_\tau \in \text{Image} J_{n+k}$. We only need to show that $(\text{Image} J_k) \circ a_\sigma \subset J_n a_2 \circ a_\tau \circ T(\sigma)$. If $n \geq k + 1$, this is clear. When $n = k$, the composition is zero, according to a remark in §11.

As an application of Corollary 3, we notice that there exist $\pi$-manifolds $M$ with non-zero reduced inertia group $I_0(M)$. This disproves a conjecture of Novikov [24].

**Corollary 4.** If $\sigma \in \Gamma^n$, then $I_1(\Sigma_\sigma) = J_{n+1}(SO) \circ T(\sigma)$.

This follows from Corollary 3 and Proposition 1.

**Diffeomorphisms of Spheres (continued).**

14. We now study the homomorphism $\gamma: \Gamma^n \to \Gamma^{n+1}$ defined in §6. It follows from Corollary 4 that $\gamma$ induces a commutative diagram

$$
0 \to b P^{n+1} \longrightarrow \Gamma^n \longrightarrow \text{Cok} J_n \quad \gamma \quad \bar{\gamma}
$$

where $\bar{\gamma}(\theta) = \eta \circ \theta$, $\eta$ the generator of $\pi_1(S)$. It follows from the non-zero compositions, mentioned in §13, that $\bar{\gamma} \neq 0$ when $n = 8, 14, 16$ or $n = 1 \mod 8, n > 1$. We point out a few more facts about $\gamma$.

**Proposition 7.** $\gamma(\Gamma^n_{\text{spin}}) \subset \Gamma^{n+1}_{\text{spin}}$ and the induced homomorphism $\Gamma^n/\Gamma^n_{\text{spin}} \to \Gamma^{n+1}/\Gamma^{n+1}_{\text{spin}}$ is an isomorphism for $n = 1 \mod 8$ and zero otherwise. Recall (10) that $\Gamma^n/\Gamma^n_{\text{spin}} \approx Z_2$ for $n = 1$ or $2 \mod 8$ and zero otherwise.

If $\Sigma_\sigma$ bounds a Spin-manifold $M$ and $(\Sigma_\sigma \times S^1) \nexists \Sigma_\gamma$ is diffeomorphic to $\Sigma_\sigma \times S^1$ we define a new manifold as follows. Consider the connected sum along the boundary of $I \times \Sigma_\sigma \times S^1$ and $I \times \Sigma_\gamma$. The boundary consists of three components $(0 \times \Sigma_\sigma \times S^1) \nexists (0 \times \Sigma_\gamma), 1 \times \Sigma_\gamma \times S^1$ and $1 \times \Sigma_\gamma$. To the first two components attach copies of $M \times S^1$. The resulting manifold $W$ has boundary $\Sigma_\gamma$.

That $W$ is a Spin-manifold follows from a Mayer-Vietoris argument, as in [7], which proves that:
\[ H^2(W) \to H^2(M \times S^1) \oplus H^2(M \times S^1) \] (coefficients in \( \mathbb{Z}_2 \))

is injective, while \( M \times S^1 \) is a Spin-manifold.

Finally, it follows from [2], [3] that, if \( n \equiv 1 \mod 8 \), \( \sigma \in \Gamma^n \) and \( \Sigma_{\sigma} \) does not bound a Spin-manifold then \( \Sigma_{\sigma} \times S^1 \) (\( S^1 \) has the non-trivial Spin structure) does not bound a Spin-manifold. It follows that \( \eta \circ T(\sigma) \) cannot be represented by an element of \( \Gamma^{n+1}_{\text{spin}} \); thus \( \gamma(\sigma) \notin \Gamma^{n+1}_{\text{spin}} \). This completes the proof of Proposition 7.

15. Of special interest is whether \( \gamma(\sigma) \) is zero, in view of Propositions 2, 3. This is determined by \( \gamma \), when \( n \) is odd. For \( n \) even, we must consider whether \( \gamma(\sigma) \) can be non-zero in \( bP^{n+2} \). This is answered in some cases by:

**Proposition 8.** Suppose \( \sigma \in \Gamma^n_{\text{spin}}, \ n = 4t - 2 \), and \( \gamma(\sigma) \in bP^{n+2} \). If \( t \leq 5 \), or \( t \) is odd, or, more generally, if:

\[(*) \quad \text{order}(\text{Image } J_{n+1}) = \text{denominator } \frac{B_t}{4t}, \]

then \( \gamma(\sigma) = 0 \).

That \((*)\) holds for \( t \) odd is a theorem of Adams [1]. It is conjectured to hold for all \( t \).

If \( \gamma = \gamma(\sigma) \in bP^{n+2} \), then \( \Sigma_{\gamma} \) bounds a parallelizable manifold \( V \). Suppose \( \gamma \neq 0 \); then, by Proposition 2, \( 2\gamma = 0 \), and it follows from [16] that one may assume:

\[ \text{index } V = 2^t(2^{2t-1} - 1) \quad \text{numerator } \frac{4B_t}{t}, \quad \text{for the given value of } t. \]

Since \( \sigma \in \Gamma^n_{\text{spin}} \), we can construct a Spin manifold \( W \), as in the proof of Proposition 7. If we adjoin the manifold \( V \) along \( \partial W \), we obtain a closed manifold \( X \). Clearly \( X \) is a Spin-manifold, because \( W \) and \( V \) are.

We now compute the \( J \)-genus of \( X \) [4]. Coefficients of cohomology are rational. First notice that all the decomposable Pontragin numbers of \( X \) are zero. In fact, we have the isomorphism:

\[ H^i(X, \Sigma_{\sigma} \times S^1) \cong H^i(M \times S^1, \Sigma_{\sigma} \times S^1) \oplus H^i(M \times S^1, \Sigma_{\sigma} \times S^1) \oplus H^i(V, \Sigma_{\gamma}). \]

Any Pontragin class \( p_i(X) \) pulls back to a class \( \alpha_i \in H^i(X, \Sigma_{\sigma} \times S^1) \)—since \( H^i(\Sigma_{\sigma} \times S^1) = 0 \). Under the above isomorphism \( \alpha_i \leftrightarrow \alpha_i + \alpha_i'' + \alpha_i''' \), where \( \alpha_i', \alpha_i'' \) are pull-backs of the Pontragin classes of \( M \times S^1 \) and \( V \). Thus a decomposable Pontragin number in \( H^{n+2}(X) \) pulls back to \( \alpha \in H^{n+2}(X, \Sigma_{\sigma} \times S^1), \alpha \leftrightarrow \alpha' + \alpha'' + \alpha''' \), where \( \alpha', \alpha'', \alpha''' \) are products of the
\( \alpha', \alpha'', \alpha''', \) respectively. But \( H^*(M \times S^1, \Sigma \times S^1) \approx H^*(M, \Sigma) \oplus H^*(S^1) \) and \( \alpha', \alpha'' \) are of the form \( \beta' \otimes 1, \beta'' \otimes 1 \). Thus, their products in \( H^{m+2} \) are all zero. Finally \( \alpha''' = 0 \), since \( V \) is parallelizable.

Now, it is easily seen that the index of \( X \) is equal to the index of \( V \), since the index of the pair \( (M \times S^1, \Sigma \times S^1) \) is zero and

\[
H^{2t}(\Sigma \times S^1) = H^{2t-1}(\Sigma \times S^1) = 0.
\]

Using in addition the index theorem and the vanishing of the decomposable Pontrajin classes of \( X \), we have the formula (\([12]\)).

\[
A(X) = \frac{-\text{index } V}{2^{2t+1}(2^{2t-1}-1)} = -\frac{1}{2} \text{ numerator } \frac{4B_t}{t}
\]

using the calculation of index \( V \). It is a consequence of a theorem of von Staudt [22] that the 2-primary part of numerator \( \frac{4B_t}{t} \) is 1, if \( t \) is even, and 2, if \( t \) is odd. But this violates the Atiyah-Hirzebruch Theorem [4], which asserts that \( A(X) \) must be integral and, when \( t \) is odd, divisible by 2.

16. In conclusion, we discuss \( \gamma \) for \( n \leq 18 \), using the computations in [29], and our preceding results. For \( n \leq 7 \) and \( n = 11, 12, 13, 15, \gamma = 0 \). For \( n = 8, 14 \) and 16, \( \gamma \) (and, therefore, \( \gamma \)) is a monomorphism. For \( n = 10 \) and 18, \( \gamma | \Gamma_{\text{spin}}^n = 0 \); \( \Gamma_{\text{spin}}^n \) is a subgroup of index 2 of \( \Gamma^n \), and I do not know whether \( \gamma = 0 \). For \( n = 9, \gamma(\Gamma^9) \approx \tilde{\gamma}(\tilde{\Gamma}^9) \approx \mathbb{Z}_2 \) and \( \text{Ker } \gamma = \Gamma_{\text{spin}}^9 \). For \( n = 17, \gamma(\Gamma^{17}) \approx \gamma(\tilde{\Gamma}^{17}) \approx \mathbb{Z}_2 + \mathbb{Z}_2 \) and \( \text{Ker } \gamma \subset \Gamma_{\text{spin}}^{17} \).

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