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Geometrically defined group structures in 3-dimensional surgery.

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CHAPTER 1

Introduction

Given an n-dimensional Poincaré complex X there is a bijection between the topological manifold structure set $\mathcal{S}^{TOP}(X)$ and the $n+1$'st relative homotopy group of a map of simplicial Ω-spectra $\sigma_* : X_+ \wedge \mathbb{L}_0 \to \mathbb{L}_0(\pi_1 X)$, provided that the former is non-empty. See [**Ran78**]. This group structure arises by algebraic methods, but surgery theory as in [**Wal70**] and [**Bro72**] is geometric. Therefore it is interesting to look for a geometric definition of a group structure of $\mathcal{S}^{TOP}(X)$. The group structure should fit into the exact sequence of surgery. This is accomplished when X is a closed 3-dimensional manifold. In this case the methods used also gives a group structure for the smooth manifold structure set $\mathcal{S}^{O}(X)$.

Chapter 2 reviews surgery theory and describes the necessary changes in dimension 3. In chapter 3 we define the group structure for the smooth manifold structure set of a closed 3-dimensional manifold M and show that it is well defined modulo a technical result. Chapter 4 concerns the exact sequence of surgery and in chapter 5 we see that the topological case follows as a corollary of the smooth case. Chapter 6 reproves a result of Rong and Wang [**RW92**], a result which is a central ingredient in the construction of the operation. Chapter 7 finishes the technical part of the well definedness proof. Stable uniqueness of Heegaard splittings is a well known result, see [**Sav99**] and [**Sin33**], but appendix A offers a nice Morse-theoretic proof.

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Tore August Kro

CHAPTER 2

Preliminaries

2.1. The exact sequence of surgery theory

How can closed smooth manifolds simply homotopy equivalent to a given closed n-manifold M be described?

We can study this question in the differentiable, the piecewise linear or the topological category. Let CAT be O, PL or TOP according to which category we choose to work in.

Two simple homotopy equivalences $f_i : N_i \to M$, $i = 1, 2$ should be considered to be equal if there is diffeomorphism $g : N_1 \to N_2$ such that $f_1 \circ g$ is homotopic to f_2. In view of the s-cobordism theorem this is the same as asserting that there exists an s-cobordism between N_1 and N_2 and an extension of the maps to the cobordism. At least when $n \geq 5$. Therefore we use the following definition:

DEFINITION 2.1. Define $\mathcal{S}(M)$ to be equivalence classes of simple homotopy equivalences $f : N \to M$ from a compact manifold N to M. Two objects $f_i : N_i \to M$, $i = 1, 2$ are equivalent if there exists a cobordism V between N_1 and N_2 and a simple homotopy equivalence $f : (V, N_1, N_2) \to (M \times I, M \times \{0\}, M \times \{1\})$ extending the maps f_i.

If we need to specify the category we write $\mathcal{S}^{CAT}(M)$.

The question we are asking is how to compute the these sets $\mathcal{S}(M)$? To answer this we start with an object that is easier to compute.

DEFINITION 2.2. A normal map is a triple (f, ν, F) where $f : N \to M$ is a degree 1 map, ν a vector bundle over M and F a stable trivialization of $\tau_N \oplus f^*\nu$.

To ease the notation we will suppress the bundle data and just speak of the normal map $f : N \to M$.

DEFINITION 2.3. A normal cobordism consists of a compact manifold V with boundary the disjoint union of N_1 and N_2, a degree 1 map $f : (V, N_1, N_2) \to (M \times I, M \times \{0\}, M \times \{1\})$, a bundle ν over M and a stable trivialization G of $\tau_V \oplus f^* \operatorname{pr}^* \nu$.

Here pr is the projection $M \times I \to M$. When restricting a normal cobordism to the boundary we get two normal maps in the following manner: Identify $\tau_V|N_i$ with $\varepsilon^1 \oplus \tau_{N_i}$ by using the inward normal along N_1 and the outward normal along N_2. Restricting f to N_i we get a degree 1 map $f_i : N_i \to M$. G restricts to a stable trivialization of $\varepsilon^1 \oplus \tau_{N_i} \oplus f_i^*\nu$. Now let F_i be a stable trivialization of $\tau_{N_i} \oplus f_i^*\nu$ such that $id_{\varepsilon^1} \oplus F_i$ is homotopic to the restriction of G. We say that $V \to M \times I$ is a normal cobordism between $f_1 : N_1 \to M$ and $f_2 : N_2 \to M$.

Definition 2.4. Define $\mathcal{N}(M)$ to be the equivalence classes of normal maps $N \to M$. Two normal maps (f_i, ν_i, F_i) are considered to be equivalent if there is a normal cobordism between (f_1, ν_1, F_1) and (f_2, ν_1, F_2') and a stable isomorphism $H : \nu_1 \to \nu_2$ such that F_2 is homotopic to the composition

$$\tau_{N_2} \oplus f_2^*\nu_2 \overset{id \oplus f_2^* H}{\to} \tau_{N_2} \oplus f_2^*\nu_1 \overset{F_2'}{\to} \varepsilon$$

Notice the difference between being normally cobordant and being equivalent in $\mathcal{N}(M)$, in the latter we also allow more flexibility with respect to the bundle ν over M.

Again we write $\mathcal{N}^{CAT}(M)$ whenever we need to indicate which category we use.

There also exists relative versions. If W is a compact manifold with boundary then we may define $\mathcal{N}(W \text{ rel } \partial W)$ to be the equivalence classes of normal maps $g : V \to W$ such that g restricted to ∂V is some fixed simple homotopy equivalence $f : \partial V \to \partial W$. Two such normal maps $g_i : V_i \to W$ are equivalent if there exists a normal cobordism $h : P \to W \times I$ between the two normal maps with bundle data as above and such that $\partial P = V_1 \cup \partial V_i \times I \cup V_2$ and the restriction of g to $\partial V_i \times I$ is $f \times id$.

One reason for introducing the sets $\mathcal{N}(M)$ is that they can be computed. The following result is well known, see for example [**MM79**].

Theorem 2.5. *There is a bijection $\mathcal{N}^{CAT}(M) \approx [M, G/CAT]$. In the relative case there is a bijection $\mathcal{N}^{CAT}(W \text{ rel } \partial W) \approx [W/\partial W, G/CAT]$.*

Another reason for introducing $\mathcal{N}(M)$ is that it may be compared to $\mathcal{S}(M)$. That is we can define a map $\eta(M) : \mathcal{S}(M) \to \mathcal{N}(M)$ as follows: Let $f : N \to M$ be a simple homotopy equivalence. Let ν' be a normal bundle for N. Then there is a trivialization F' of $\tau_N \oplus \nu'$. Since f is a homotopy equivalence pullback of bundles by f induces a isomorphism $f^* : [M, BCAT] \to [N, BCAT]$. Choose ν such that $f^*\nu = \nu'$. Using F' we get a stable trivialization F of $\tau_N \oplus f^*\nu$. The same construction works when replacing N with a s-cobordism V and f by a map $V \to M \times I$. Thus the map $\eta(M)$ is well defined.

Now there are two natural questions which arise:

> Which elements of $\mathcal{N}(M)$ lie in the image of $\mathcal{S}(M)$?
> How can elements of $\mathcal{S}(M)$ mapping to the same element of $\mathcal{N}(M)$ be classified?

Wall defines the Abelian groups $L_n(\mathbb{Z}[\pi_1 M], w_1)$ to solve these questions. The groups depend on the fundamental group of M and the first Stiefel-Whitney class and are periodic in the sense that $L_{n+4}(\mathbb{Z}[\pi_1 M], w_1)$ is isomorphic to $L_n(\mathbb{Z}[\pi_1 M], w_1)$. One important result is that when the dimension n is greater than 5 the sequence

$$\cdots \to \mathcal{N}(M \times I \text{ rel } \partial M \times I) \to L_{n+1}(\mathbb{Z}[\pi_1 M], w_1) \\ \to \mathcal{S}(M) \to \mathcal{N}(M) \to L_n(\mathbb{Z}[\pi_1 M], w_1)$$

is exact. See Wall [**Wal70**].

Denote the map $\mathcal{N}(M) \to L_n(\mathbb{Z}[\pi_1 M], w_1)$ by $\theta(M)$. This answers the first question. The inverse image of the identity under $\theta(M)$ is equal to the image of $\mathcal{S}(M)$ under $\eta(M)$. The fact that $\operatorname{im} \eta(M)$ is contained in $\theta(M)^{-1}(0)$ follows from the definitions. See [**Wal70**]. To prove that a normal map $N \to M$ with trivial surgery obstruction is normally cobordant to a simple homotopy equivalence we proceed as follows. By elementary surgery we may assume that the map is k-connected where k is the largest integer less than or equal to $\frac{1}{2}n$. Now let $K_k(N)$ be the kernel of $H_k(N) \to H_k(M)$ where we use the twisted coefficients $\mathbb{Z}[\pi_1 M]$. Wall shows that we may kill this kernel by surgery if and only if the surgery obstruction vanishes.

There is a group action $\gamma(M)$ of $L_{n+1}(\mathbb{Z}[\pi_1 M], w_1)$ on $\mathcal{S}(M)$. This action answers the second question. Two elements of $\mathcal{S}(M)$ maps to the same element in $\mathcal{N}(M)$ if and only if they lie in the same orbit. Given a simple homotopy equivalence $N \to M$ and an element θ of $L_{n+1}(\mathbb{Z}[\pi_1 M], w_1)$ we are actually able to construct a normal cobordism $V \to M \times I$ with surgery obstruction θ going from $N \to M$ to another simple homotopy equivalence $N' \to M$ which we define to be the image of $N \to M$ under the action of θ. Well definedness follows since a normal cobordism with trivial surgery obstruction may be replaced by an s-cobordism.

2.2. Group structures on $\mathcal{N}^{TOP}(M)$ and $\mathcal{N}^{CAT}(M \times I \text{ rel } \partial M \times I)$

In the TOP-category we can give $\mathcal{N}^{TOP}(M)$ a group structure. Use the homotopy equivalence $\mathbb{Z} \times G/TOP \simeq \Omega^4(\mathbb{Z} \times G/TOP)$ to define a H-space structure on G/TOP. Then the set $[M, G/TOP]$ has a group structure. With respect to this group structure the map $\theta(M)$ is a homomorphism. See **[KS77]** Essay V theorem C.4.

Consider also the sets $\mathcal{N}^{CAT}(M \times I \text{ rel } \partial M \times I)$. They are equivalent to $[\Sigma M, G/CAT]$. ΣM has a coH-space structure which gives rise to a group structure. In $\mathcal{N}^{CAT}(M \times I \text{ rel } \partial M \times I)$ this corresponds to taking the disjoint union of two normal maps $f_i : V_i \to M \times I$, $i = 1, 2$ and gluing $f_1^{-1}(M \times \{0\}) \to M \times \{0\}$ to $f_2^{-1}(M \times \{1\}) \to M \times \{1\}$. The condition on the boundary ensures well definedness. The result is another normal map $V \to M \times I$. Additivity of surgery obstructions with disjoint support assures that the map $\mathcal{N}^{CAT}(M \times I \text{ rel } \partial M \times I) \to L_4(\mathbb{Z}[\pi_1 M], w_1)$ is a homomorphism with this group structure.

We have now defined two group structures on $\mathcal{N}^{CAT}(M \times I \text{ rel } \partial M \times I) \approx [\Sigma M, G/TOP]$. The first comes from the H-space structure on G/TOP, while the second comes from the coH-space structure of ΣM. The two group structures coincide according to theorem III.5.21 in **[Whi78]**.

2.3. What happens in dimension 3?

There are problems with the exact sequence of surgery theory in dimension 3. What follows is necessary modifications of $\mathcal{S}(M)$. See **[KT]**.

The first problem involves lifting normal maps with trivial surgery obstruction to simple homotopy equivalences. We can do surgery on 1-cycles in $N \to M$, but this changes the fundamental group of N. The kernel $K_1(N)$ is killed, but this does not imply that the fundamental groups of N and M are isomorphic by f_*. This weakens the conclusion. We are only able to get simple homology equivalences.

The second problem involves the action of $L_4(\mathbb{Z}[\pi_1 M], w_1)$ on a simple homology equivalence. For every element θ of $L_4(\mathbb{Z}[\pi_1 M], w_1)$ and simple homology equivalence $N \to M$ with coefficients in $\mathbb{Z}[\pi_1 M]$ it is still possible to construct a normal cobordism with surgery obstruction θ from $N \to M$ to another simple homology equivalence $N' \to M$. But the possible failure of 4-dimensional surgery means that this action is only well defined up to normal cobordism with trivial surgery obstruction, not s-cobordism as was the case for the higher dimensions.

Therefore we modify the definition of $\mathcal{S}(M)$ for dimension 3 to be:

DEFINITION 2.6. For a closed 3-manifold M let $\mathcal{S}(M)$ be the set of simple homology equivalences $N \to M$ with coefficients in $\mathbb{Z}[\pi_1 M]$ where $N_1 \to M$ and $N_2 \to M$ are considered to be equal if there exists a normal cobordism $V \to M \times I$ with trivial surgery obstruction between the two simple homology equivalences.

The relation given above is obviously symmetric and transitive, but reflexivity is a problem. The reason is the following: A simple homology equivalence $N \to M$ does not necessarily induce an isomorphism of fundamental groups. The kernel of $\pi_1 N \to \pi_1 M$ might be a non-trivial perfect group. If so it is not obvious that the cross product $N \times I \to M \times I$ with bundle data is a normal cobordism with trivial surgery obstruction since the map is not 1-connected. Remember from chapter 5 in **[Wal70]** that we calculate the surgery obstruction of a normal cobordism $V \to M \times I$ by first doing surgery below the middle dimension until the map is 1-connected, then taking the class of $K_2(V)$ in $L_4(\mathbb{Z}[\pi_1 M], w_1)$. A small proof is needed here:

LEMMA 2.7. *There exists a normal cobordism with trivial surgery obstruction between two copies of any simple homology equivalence $f : N \to M$.*

This result also holds if M has boundary.

PROOF. Start with $N \times I \to M \times I$. Represent the kernel of $\pi_1(N \times I) \to \pi_1(M \times I)$ by disjoint embeddings $\alpha_i : S^1 \times D^3 \to N \times I$ preferred by the bundle data. Do surgery on all the α_i's and denote the result by $V \to M \times I$. If we let $V_0 = N \times I \setminus \bigcup_i \operatorname{int} \alpha_i(S^1 \times D^3)$ then $V = V_0 \cup (\bigcup_i D^2 \times S^2)$. Let $\beta_i : D^2 \times S^2$ be the embedding representing the copy of $D^2 \times S^2$ corresponding to α_i. For each i there is a direct summand of $K_2(V)$ isomorphic to $\mathbb{Z}[\pi_1 M] \oplus \mathbb{Z}[\pi_1 M]$. Let e_i be the element of $K_2(V)$ represented by the composition

$$S^2 \to \{0\} \times S^2 \to D^2 \times S^2 \xrightarrow{\beta} V$$

This map $S^2 \to V$ is preferred by the bundle data. And the self intersection number of e_i is 0 because e_i is represented by an embedding. Since α_i represent an element of the kernel of $\pi_1(N \times I) \to \pi_1(M \times I)$ there exists a 2-chain in $N \times I$ with boundary $\alpha_i | S^1 \times \{1\}$. We may assume that this 2-chain lies in V_0. Gluing $\beta_i(D^2 \times \{1\})$ to this chain we get an element f_i'' of $H_2(V)$. Since $H_2(V_0) \to H_2(N \times I)$ is surjective and $H_2(N \times I) \to H_2(M \times I)$ an isomorphism there exists an element of $H_2(V_0)$ with the same image in $H_2(M \times I)$ as f_i''. Subtracting this element form f_i'' we get an element f_i' of $K_2(V)$ such that $\{e_i, f_i'\}$ is a preferred base of $K_2(V)$ and that $\mu(e_i) = 0$, $\lambda(e_i, e_j) = 0$ and $\lambda(e_i, f_j') = \delta_{ij}$. Thus the submodule H of $K_2(V)$ generated by the e_i's is a subkernel. And lemma 5.3 in **[Wal70]** then shows that $K_2(V)$ is a kernel. Hence the surgery obstruction of $V \to M \times I$ is trivial. $\square$

Using definition 2.6 we see that the sequence

$$\cdots \to \mathcal{N}(M \times I \text{ rel } \partial M \times I) \to L_4(\mathbb{Z}[\pi_1 M], w_1) \\ \to \mathcal{S}(M) \to \mathcal{N}(M) \to L_3(\mathbb{Z}[\pi_1 M], w_1)$$

still is exact.

REMARK 2.8. When defining the map $\eta(M) : \mathcal{S}(M) \to \mathcal{N}(M)$ we used that $f^* : [M, BCAT] \to [N, BCAT]$ is an isomorphism when $f : N \to M$ is a homotopy equivalence. This is true also for homology equivalences $f : N \to M$ since the $BCAT$'s are loop spaces.

In dimension 3 the categories of smooth, piecewise linear and topological manifolds are all equivalent. But there are differences in dimension 4. The smooth and the piecewise linear categories are still the same, but there exist topological 4-manifolds without any smooth structure. Hence $\mathcal{S}^O(M)$ and $\mathcal{S}^{PL}(M)$ are equal, while $\mathcal{S}^{TOP}(M)$ has the same objects, but more equivalences. The map $\mathcal{S}^O(M) \to \mathcal{S}^{TOP}(M)$ defined by forgetting the smooth structure is surjective.

$\mathcal{N}^{CAT}(M)$ is the same for all categories when M is a 3-manifold. The reason is that in this range the spaces G/O, G/PL and G/TOP are equivalent.

CHAPTER 3

An Abelian group-structure on $\mathcal{S}^O(M)$

In this chapter we will define a group structure on the smooth structure set $\mathcal{S}^O(M)$ for a 3-manifold.

PROPOSITION 3.1. *Let $M = H_1 \cup H_2$ be any Heegaard splitting and choose i to be 1 or 2. Every class in $\mathcal{S}^O(M)$ contains a representative $f : N \to M$ such that $f| : f^{-1}(H_i) \to H_i$ is a diffeomorphism.*

PROOF. This follows directly from theorem 6.8. □

CONSTRUCTION 3.2. Let M be a closed and connected 3-manifold, $f_i : N_i \to M$, $i = 1, 2$ two homology equivalences and $H_1 \cup H_2$ a Heegaard splitting of M. Assume that $f_i| : f_i^{-1}(H_i) \to H_i$ is a diffeomorphism for $i = 1, 2$. Define $N_1 + N_2$ to be the union of $f_1^{-1}(H_2)$ and $f_2^{-1}(H_1)$ identified along the boundary. That is $x \in f_1^{-1}(H_2)$ is equivalent to $y \in f_2^{-1}(H_1)$ if $f_1(x) = f_2(y) \in H_1 \cap H_2$. There also is a map $f_1 + f_2 : N_1 + N_2 \to M$ defined by

$$(f_1 + f_2)(x) = \begin{cases} f_1(x) & \text{if } x \in f_1^{-1}(H_2) \\ f_2(x) & \text{if } x \in f_2^{-1}(H_1) \end{cases}$$

See figure 3.1.

We can use construction as an addition in the $\mathcal{S}^O(M)$. Given two elements in $\mathcal{S}^O(M)$ we choose a Heegaard splitting $H_1 \cup H_2$ of M and pick representatives $f_1 : N_1 \to M$ and $f_2 : N_2 \to M$ for each of the two elements such that $f_i| : f_i^{-1}(H_i) \to H_i$ is a diffeomorphism for $i = 1, 2$. Now use the construction above to form $f_1 + f_2 : N_1 + N_2 \to M$. The main result is:

THEOREM 3.3. *With the operation described above the smooth structure set $\mathcal{S}^O(M)$ is an Abelian group.*

The proof will fill the rest of this chapter. We begin by showing that $N_1 + N_2 \to M$ is a simple homology equivalence.

LEMMA 3.4. *$f_1 + f_2 : N_1 + N_2 \to M$ is a simple homology equivalence.*

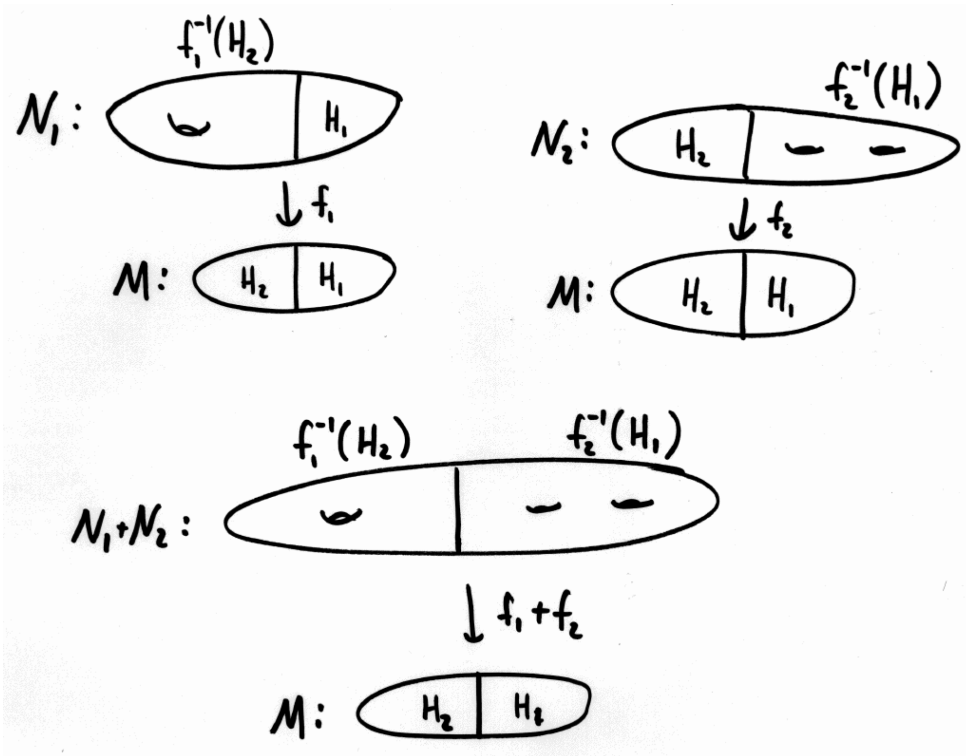

FIGURE 3.1. The operation $+$.

PROOF. Consider manifolds N with a map f into M and a splitting $N = K_1 \cup K_2$ such that $K_i = f^{-1}(H_i)$ and $f : K_1 \cap K_2 \to H_1 \cap H_2$ is a homeomorphism. Write F for both $K_1 \cap K_2$ and $H_1 \cap H_2$. We have Mayer-Vietoris sequences and maps between them

$$\begin{array}{ccccccccc}
\longrightarrow & H_q^t(F) & \longrightarrow & H_q^t(K_1) \oplus H_q^t(K_2) & \longrightarrow & H_q^t(N) & \longrightarrow & H_{q-1}^t(F) & \longrightarrow \\
 & \downarrow \cong & & \downarrow & & \downarrow & & \downarrow \cong & \\
\longrightarrow & H_q^t(F) & \longrightarrow & H_q^t(H_1) \oplus H_q^t(H_1) & \longrightarrow & H_q^t(M) & \longrightarrow & H_q^t(F) & \longrightarrow
\end{array}$$

Here the coefficients are $\mathbb{Z}[\pi_1 M]$ over M and the respective pull backs over the other spaces. So if two of the three maps $N \to M$, $K_1 \to H_1$ and $K_2 \to H_2$ are homology equivalences with twisted coefficients in $\mathbb{Z}[\pi_1 M]$ then the last map also is a homology equivalence with the same coefficients.

It follows that $N_1 + N_2 \to M$ is a homology equivalence by first considering the two cases $N = N_1$ and $N = N_2$ and then the case $N = N_1 + N_2$.

It remains to show that this homology equivalence is simple. There is a short exact sequence of the algebraic mapping cones:

$$0 \to C(f|F) \to C(f|K_1) \oplus C(f|K_2) \to C(f) \to 0$$

These algebraic mapping cones are acyclic and have preferred bases. Using that $\tau(C(f|F)) = 0$, $\tau(C(f|K_1) \oplus C(f|K_2)) = \tau(C(f|K_1)) + \tau(C(f|K_2))$ and theorem 3.1 in [**Mil66**] we see that

$$\tau(C(f)) = \tau(C(f|K_1)) + \tau(C(f|K_2))$$

Thus if two of these three torsions vanishes then the last one must also be 0. It follows that the homology equivalence $N_1 + N_2 \to M$ is simple because both $N_1 \to M$ and $N_2 \to M$ are. □

3.1. The operation is well defined

The operation is defined on some of the objects in the equivalence classes of $\mathcal{S}^O(M)$. In this section it will be shown that this is a well defined operation. There are essentially two choices involved in the construction of $N_1 + N_2 \to M$. These are:

i) The choice of representations $N \to M$.
ii) The choice of Heegaard splitting of the manifold M.

So it must be shown that up to the equivalence relation defining $\mathcal{S}^O(M)$ these choices are irrelevant.

We start by fixing a Heegaard splitting $H_1 \cup H_2$ of M. Given two classes of $\mathcal{S}^O(M)$ we may, by proposition 3.1, choose $f_1 : N_1 \to M$ and $f_2 : N_2 \to M$ from the first and the second respectively such that $f_i^{-1}(H_i)$ is mapped diffeomorphically to H_i by f_i. Our first aim is to show that the class of $N_1 + N_2 \to M$ is independent of this choice. By symmetry we can fix one of the simple homology equivalences, say $N_2 \to M$, and just consider two different choices of $N_1 \to M$. The following result, theorem 7.17, will be shown in a later chapter:

Assume that $f_1 : N_1 \to M$, $f_1' : N_1' \to M$ and $f_2 : N_2 \to M$ are simple homology equivalences such that the restriction of the maps to $f_1^{-1}(H_1)$, $f_1'^{-1}(H_1)$ and $f_2^{-1}(H_2)$ are diffeomorphism and that there exists a normal cobordism between f_1 and f_1' with trivial surgery obstruction then there exists a normal cobordism between $f_1 + f_2$ and $f_1' + f_2$ with trivial surgery obstruction.

This shows that the operation $+$ is independent of the choice of representatives $N_i \to M$.

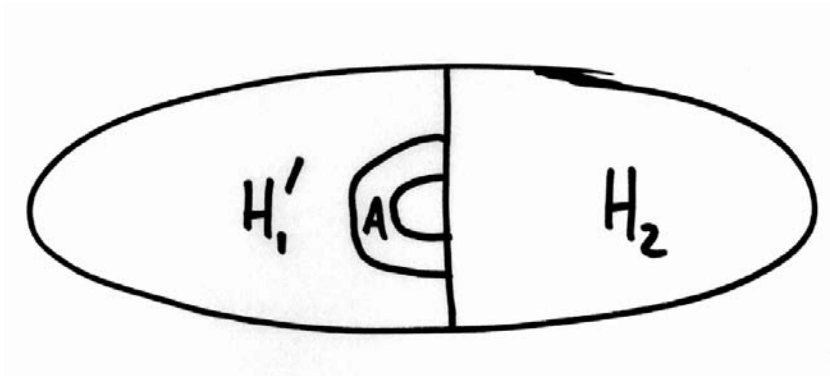

FIGURE 3.2. Two Heegaard splittings of M differing by a handle A.

3.1.1. The choice of Heegaard splitting. Now that the first step of the program is completed we move on to study the effect of choosing different Heegaard splittings.

Since homotopy is a stronger relation than normal bordism with trivial surgery obstruction it is clear that we are allowed to deform the maps $N_i \to M$ whenever needed.

LEMMA 3.5. *If $H_1' \cup H_2'$ is a stabilization of the Heegaard splitting $H_1 \cup H_2$ then the two different sums $(N_1 + N_2)'$ and $N_1 + N_2$ obtained by using the two Heegaard splittings are diffeomorphic over M.*

Being diffeomorphic over M means that the following diagram commute

$$\begin{array}{ccc} N_1 + N_2 & \longrightarrow & M \\ \downarrow \cong & \nearrow & \\ (N_1 + N_2)' & & \end{array}$$

PROOF. It suffices to prove this result for the case where H_1 is H_1' union a trivial handle A which lies close to ∂H_2. The case with several handles follows by induction. Think of M as the union $H_1' \cup A \cup H_2$. See figure 3.2.

Let F be the boundary of H_1. There exists an embedding of $F \times [-1, 1]$ into M such that $F \times [0, 1]$ and $F \times [-1, 0]$ are collars of the boundary in H_1 and H_2 respectively and the handle A is contained in $F \times [0, 1]$.

Now assume that f_i restricted to $f_i^{-1}(F \times [-1, 1])$ is an embedding for $i = 1, 2$. If this is not the case the maps f_i may be deformed by a homotopy fixing $f_i^{-1}(H_i)$ so that it becomes true.

Now write $N_i = f_i^{-1}(H_2) \cup f_i^{-1}(A) \cup f_i^{-1}(H_1')$ for $i = 1, 2$. It is clear that

$$N_1 + N_2 = f_1^{-1}(H_2) \cup f_1^{-1}(A) \cup f_2^{-1}(H_1')$$

and

$$(N_1 + N_2)' = f_1^{-1}(H_2) \cup f_2^{-1}(A) \cup f_2^{-1}(H_1')$$

But $f_1^{-1}(A)$ and $f_2^{-1}(A)$ are both diffeomorphic to A. The result follows. □

LEMMA 3.6. *Let $G : M \times I \to M$ be an isotopy between the Heegaard splittings $H_1 \cup H_2$ and $H_1' \cup H_2'$. Then $N_1 + N_2$ and $(N_1 + N_2)'$, formed using the Heegaard splittings $H_1 \cup H_2$ and $H_1' \cup H_2'$ respectively, are equivalent.*

PROOF. We can form $N_1 + N_2 \to M$ using some maps $f_i : N_i \to M$ which have the usual property that $f_i^{-1}(H_i)$ is mapped diffeomorphically to H_i by f_i. To form $(N_1 + N_2)'$ we can then choose the maps $G_1 f_i : N_i \to M$. Then it is clear that $N_1 + N_2 = (N_1 + N_2)'$ and $G_1 f_1 + G_1 f_2$ is equal to $G_1(f_1 + f_2)$, hence a homotopy is given by $G_t(f_1 + f_2)$ where $t \in I$. □

COROLLARY 3.7. *The class of $N_1 + N_2 \to M$ in $\mathcal{S}^O(M)$ is independent of the choice of Heegaard splitting.*

PROOF. This follows from the preceding lemmas and the stable equivalence of Heegaard splittings. See theorem A.1. □

3.2. Verification of the axioms for an Abelian group

We will finnish the proof of theorem 3.3 by verifying that the axioms for an Abelian group is sattisfied for $\mathcal{S}^O(M)$ with the operation $+$.

3.2.1. Identity. The class of the homology equivalence $id : M \to M$ is clearly an identity element for the sum operation.

3.2.2. Associativity. There is an alternative way to view the operation. Let $M = H_1 \cup H_2$ be a Heegaard splitting. Move the two pieces slightly apart in order to write M as a union of codimension 0 submanifolds H_1, $F \times I$ and H_2 such that $H_1 \cap F \times I = \partial H_1 = F \times \{0\}$ and $H_2 \cap F \times I = \partial H_2 = F \times \{1\}$.

Now let $f_1 : N_1 \to M$ and $f_2 : N_2 \to M$ be two homology equivalences. It can be assumed that $f_i^{-1}(H_j) \to H_j$ is a diffeomorphism for all i and j. Let K_i be the inverse image of $F \times I$ under f_i. The interesting part is $K_i \to F \times I$. See figure 3.3. It is clear that $N_1 + N_2$

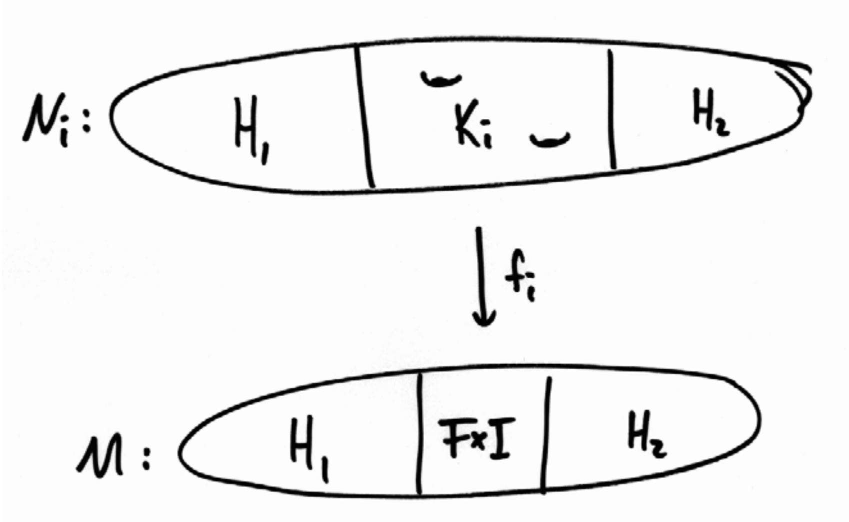

FIGURE 3.3. An alternative description of the operation.

can be thought of as the union $H_1 \cup K_1 \cup K_2 \cup H_2$. Here we glue at the appropriate places. This manifold maps to $H_1 \cup F \times I \cup F \times I \cup H_2$ which is diffeomorphic to M.

If $f_3 : N_3 \to M$ is a simple homology equivalence such that the restriction of f_3 to both $f_3^{-1}(H_1)$ and $f_3^{-1}(H_2)$ are diffeomorphisms onto their image then it is clear from this construction that both $(N_1 + N_2) + N_3 \to M$ and $N_1 + (N_2 + N_3) \to M$ can be identified with the simple homology equivalence

$$H_1 \cup K_1 \cup K_2 \cup K_3 \cup H_2 \to H_1 \cup F \times I \cup F \times I \cup F \times I \cup H_2$$

This implies associativity.

3.2.3. Inverse. To prove the existence of an inverse use the description of the operation given above. Let $f' : K \to F \times I$ be the composition of $f|K : K \to F \times I$ with the map $(x, t) \mapsto (x, 1 - t)$. Define N' to be the disjoint union of H_1, K and H_2 where x in the boundary of H_1 or H_2 is identified with $y \in \partial K$ if $f'(y) = x$. We can thus extend f' to a map $N' \to M$. Informally we have removed K from N and flipped it upside down.

We now want to construct a normal cobordism $V \to M \times I$ with trivial surgery obstruction between $N + N' \to M$ and $M \to M$. Notice that ∂K is two copies of F. Let V be the union of $M \times I$ and $K \times I$ where $F \times I \times \{1\}$ in $M \times I$ is identified with one of the copies of $F \times I$ in $\partial K \times I$. If we do the same construction but replace K by $F \times I$ we get a manifold whose smoothing is diffeomorphic to $M \times I$.

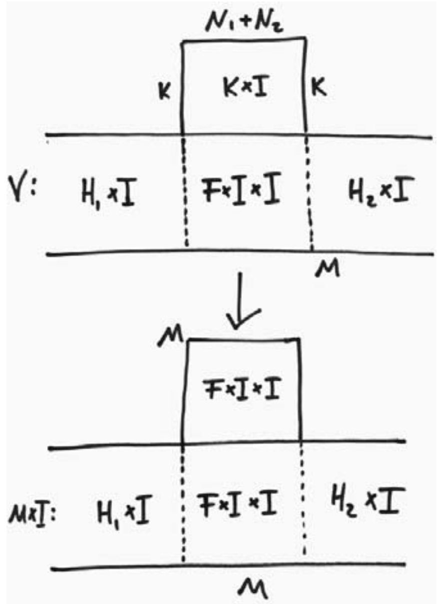

FIGURE 3.4. The normal cobordism V.

This shows that we can construct a map $V \to M \times I$. See figure 3.4 Simple calculations using the Mayer-Vietoris sequence shows that this is a simple homology equivalence. Thus bundle data can be constructed as in the definition of $\eta(M)$ in chapter 2. See Remark 2.8.

The boundary of V has two components. One of the components is M the other is the sum of N and N'. Thus $N' \to M$ is an inverse for $N \to M$.

3.2.4. Commutativity. Commutativity of the operation follows from the operations independence of the choice of Heegaard splitting. If $N_1 \to M$ and $N_2 \to M$ are two homology equivalences and $M = H_1 \cup H_2$ a Heegaard splitting then $N_2 + N_1 \to M$ is formed by removing H_2 from N_1 and H_1 from N_2, but this is the same as using the opposite Heegaard splitting $M = H_2 \cup H_1$ to form $N_1 + N_2 \to M$. Hence commutativity follows.

CHAPTER 4

The mappings of the surgery sequence

4.1. The action of $L_4(M)$ on $\mathcal{S}^O(M)$

In the surgery sequence we are given an action of $L_4(M)$ on $\mathcal{S}^O(M)$. Now that we have a group structure on $\mathcal{S}^O(M)$ it is natural to ask if this action gives rise to a group homomorphism.

PROPOSITION 4.1. *Let $f : N \to M$ represent an element of $\mathcal{S}^O(M)$ then $\theta \in L_4(M)$ acts according to the formula:*

$$\theta(f : N \to M) = \theta(id : M \to M) + (f : N \to M)$$

PROOF. Let $M = H_1 \cup H_2$ be a Heegaard splitting. We can assume that $f^{-1}(H_1)$ is mapped diffeomorphically to H_1 by f. When doing the plumbing on $f : N \to M$ as described in Wall's theorem 5.8 [**Wal70**] we can ensure that all the action takes place inside $f^{-1}(H_1) \to H_1$. In this case the formula above is obvious. $\square$

COROLLARY 4.2. *The action of $L_4(M)$ on $\mathcal{S}^O(M)$ gives a group homomorphism defined by $\theta \mapsto \theta(id : M \to M)$.*

PROOF. Let $\theta_1, \theta_2 \in L_4(M)$. Then

$$(\theta_1 + \theta_2)(id) = \theta_1(\theta_2(id)) = \theta_1(id) + \theta_2(id)$$

The first equality follows since we have a group action, the second follows from the formula above. $\square$

4.2. The map $\mathcal{S}^O(M) \to \mathcal{N}(M)$

Every simple homology equivalence $N \to M$ can be given bundle data in a canonical way. See chapter 2. We thus have a map from $\mathcal{S}^O(M)$ into $\mathcal{N}(M)$. $\mathcal{N}(M)$ has an Abelian group structure and we want to investigate if the map is a homomorphism. This is done by giving a description of the group operation of $\mathcal{N}(M)$ analougus to the definition of the group operation of $\mathcal{S}^O(M)$.

Since M is a 3-manifold $\mathcal{N}(M)$ is independent of CAT. We concentrate on the TOP case:

$$\mathcal{N}(M) \approx [M, G/CAT] \approx [M, G/TOP]$$

Now we can give a group structure by a H-space structure $\mu : G/TOP\times G/TOP \to G/TOP$.

Let $M = H_1 \cup H_2$ be a Heegaard splitting.

PROPOSITION 4.3. *Wedge gives an addition* $[H_1/\partial, G/O]\times[H_2/\partial, G/O] \to [M, G/O]$ *which corresponds to gluing relative normal maps* $f_i : K_i \to H_i$ *for* $i = 1, 2$ *together to a map* $K_1 \cup K_2 \to M$.

PROOF. We trace the proof of the bijection $[X/\partial, G/O] \xrightarrow{\approx} \mathcal{N}(X \text{ rel } \partial)$. See [**MM79**] theorem 2.23 for the proof.

Let F be the common boundary of H_1 and H_2. Given maps $\bar{g}_i : H_i/\partial \to G/O$ we form $\bar{g} : M \to G/O$ by composing the wedge $\bar{g}_1 \vee \bar{g}_1$ with the quotient map $M \to M/F = H_1/\partial \vee H_2/\partial$. This map $\bar{g}$ is equivalent to specifying a bundle $\lambda \xrightarrow{\pi} M$ and a fiber homotopy equivalence $t : \lambda \to \varepsilon^l$. And t restricts to an isomorphism over F since $\bar{g}$ factorizes through M/F. Moreover the maps $\bar{g}_i$ corresponds to the restriction of λ and t to H_i. To produce normal maps we make t transversal to the 0-section of ε^l. This can be done over each H_i separately and we can keep t fixed over F since it already is transversal here. Taking the inverse image under t of the 0-sections H_1, H_2 and M contained in the trivial bundle we get manifolds K_1, K_2 and N respectively. And the projection $\pi : \lambda \to M$ restricts to degree 1 maps $f_1 : K_1 \to H_1$, $f_2 : K_2 \to H_2$ and $f : N \to M$. Since the restriction of t to F is an isomorphism the boundaries of the K_i's are mapped diffeomorphically to F by the f_i's and N is the union of K_1 and K_2 along the boundary.

Bundle data are constructed as follows. Let $\lambda^\perp$ be an inverse bundle for λ and let ν be a normal bundle of M. Then $\pi^*(\lambda^\perp \oplus \nu)$ is a normal bundle for the total space of λ. Fix a trivialization $F' : \tau_\lambda \oplus \pi^*\lambda^\perp \oplus \pi^*\nu \cong \varepsilon^k$. Since N is the inverse image under t of the 0-section of ε^l the normal bundle of N in the total space of λ has a canonical trivialization $\nu_N \cong \varepsilon^l$. Now restrict F' to N and identify $\tau_\lambda|N \cong \tau_N \oplus \nu_N \cong \tau_N \oplus \varepsilon^l$ to get at stable trivialization F of $\tau_N \oplus f^*(\lambda^\perp \oplus \nu)$.

Restriction to K_1 and K_2 gives bundle data for the maps f_1 and f_2.

Now the normal maps $f_i : K_i \to H_i$ corresponds to $\bar{g}_i : H_i/\partial \to G/O$ and $f : N \to M$ corresponds to the map $\bar{g} : M \to G/O$. ☐

LEMMA 4.4. *The map* $\mathcal{N}(H_1 \text{ rel } \partial) \to \mathcal{N}(M)$ *given by gluing the relative normal map* $f : K \to H_1$ *to* $id : H_2 \to H_2$ *is surjective.*

PROOF. Theorem 6.8 says that any normal map $f : N \to M$ is homotopic to one such that $f^{-1}(H_2) \to H_2$ is a diffeomorphism. Removing the interior of $f^{-1}(H_2)$ and H_2 from N and M respectively we get an element of $\mathcal{N}(H_1 \text{ rel } \partial)$ mapping to $f : N \to M$. □

By symmetry this result is also true for H_2. Therefore if $f_i : N_i \to M$ are normal maps for $i = 1, 2$ we may lift f_i to an element $\bar{f}_i : K_i \to H_i$ of $\mathcal{N}(H_i \text{ rel } \partial)$. Let $f_1 + f_2 : N_1 + N_2 \to M$ denote the result when we glue the two liftings together.

PROPOSITION 4.5. *$f_1 + f_2$ is the sum of f_1 and f_2 with respect to the group structure on $\mathcal{N}(M)$ given by the H-space structure on G/TOP.*

PROOF. The relative normal maps $\bar{f}_i : K_i \to H_i$ give maps $\bar{g}_i : H_i/\partial H_i \to G/TOP$ while $f_1 + f_2$ determines the map $\bar{g}_1 \vee \bar{g}_2 : H_1/\partial \vee H_2/\partial \to G/TOP$.

Look at the commutative diagram:

$$\begin{array}{ccccccc}
H_1/\partial \vee H_2/\partial & \longrightarrow & H_1/\partial \times H_2/\partial & \xrightarrow{\bar{g}_1 \times \bar{g}_2} & G/TOP \times G/TOP & \xrightarrow{\mu} & G/TOP \\
\uparrow & & \uparrow & & = \uparrow & & = \uparrow \\
M & \xrightarrow{\Delta} & M \times M & \xrightarrow{g_1 \times g_2} & G/TOP \times G/TOP & \xrightarrow{\mu} & G/TOP
\end{array}$$

Here g_i is the composition $M \to H_i/\partial \xrightarrow{\bar{g}_i} G/TOP$ which corresponds to $f_i : N_i \to M$. The bottom part is the definition of sum in a H-space. The composition at the top is equal to $\bar{g}_1 \vee \bar{g}_2$. □

COROLLARY 4.6. *The map $\mathcal{S}^O(M) \to \mathcal{N}(M)$ is a homomorphism.*

PROOF. This follows by the similarity of the descriptions of the addition in the two groups. □

CHAPTER 5

The topological case

Theorem 3.3 and the corollaries 4.2 and 4.6 show that in the smooth category we have an exact sequence of Abelian groups:

$$\cdots \to \mathcal{N}^O(M \times I \text{ rel } \partial M \times I) \to L_4(\mathbb{Z}[\pi_1 M], w_1) \\ \to \mathcal{S}^O(M) \to \mathcal{N}(M) \to L_3(\mathbb{Z}[\pi_1 M], w_1)$$

In this chapter we show that the result also holds in the topological case.

THEOREM 5.1. *Let M be a closed 3-manifold. The topological structure set $\mathcal{S}^{TOP}(M)$ is an Abelian group, the action of $L_4(\mathbb{Z}[\pi_1 M], w_1)$ on $\mathcal{S}^{TOP}(M)$ gives a homomorphism defined by $\theta \mapsto \theta(id)$ and the map $\mathcal{S}^{TOP} \to \mathcal{N}(M)$ is a homomorphism.*

PROOF. We will show this result by comparing the topological and the smooth case by the map which forgets the smooth structure.

$$\begin{array}{ccccccc} \mathcal{N}^O(M \times I \text{ rel } \partial) & \longrightarrow & L_4(\mathbb{Z}[\pi_1 M], w_1) & \longrightarrow & \mathcal{S}^O(M) & \longrightarrow & \mathcal{N}(M) \\ \downarrow & & \downarrow = & & \downarrow & & \downarrow = \\ \mathcal{N}^{TOP}(M \times I \text{ rel } \partial) & \longrightarrow & L_4(\mathbb{Z}[\pi_1 M], w_1) & \longrightarrow & \mathcal{S}^{TOP}(M) & \longrightarrow & \mathcal{N}(M) \end{array}$$

We have already noticed that the map $\mathcal{S}^O(M) \to \mathcal{S}^{TOP}(M)$ is surjective. The image of $\mathcal{N}^O(M \times I \text{ rel } \partial)$ in $L_4(\mathbb{Z}[\pi_1 M], w_1)$ is contained in the image of $\mathcal{N}^{TOP}(M \times I \text{ rel } \partial)$. Let C be the quotient group. Chasing the diagram above we see that the sequence

$$0 \to C \to \mathcal{S}^O(M) \to \mathcal{S}^{TOP}(M) \to 0$$

is exact. The map $C \to \mathcal{S}^O(M)$ is a homomorphism because $L_4(\mathbb{Z}[\pi_1 M], w_1) \to \mathcal{S}^O(M)$ is. Hence $\mathcal{S}^{TOP}(M)$ can be identified with the quotient group $\mathcal{S}^O(M)/C$. Since $\mathcal{S}^O(M) \to \mathcal{N}(M)$ is a homomorphism the map $\mathcal{S}^{TOP}(M) \to \mathcal{N}(M)$ must be a homomorphism too. $\square$

CHAPTER 6

Techniques from differential topology

Corollary 7 in [**RW92**] states that for a degree 1 map $f : N \to M$ between closed oriented 3-manifolds and any handlebody H of genus g in N, f is homotopic to f_1 such that f_1 maps $f_1^{-1}(H)$ onto H homeomorphically.

In what follows techniques from differential topology will be used to reprove and improve this result. Particulary the condition that the manifolds must be oriented is removed.

The following transversality theorem is proved in [**Boa65**]:

THEOREM 6.1. *If $f : N \to M$ is transversal to a submanifold A in M when restricted to a closed subset L of N then there is a map f_0 homotopic to f relative to L such that f_0 is transversal to A on all of N.*

It will be used on several occasions.

The first two results describe how maps between manifolds may be deformed to get a nice inverse image of a point.

LEMMA 6.2. *Let $f : N \to M$ be a map between compact n-manifolds of dimension $n \geq 3$ taking ∂N to ∂M and having $q \in \operatorname{int} M$ as a regular value. Assume that $\alpha : I \to N$ is an embedded path between two different points in the inverse image of q such that f has opposite orientations at $\alpha(0)$ and $\alpha(1)$ with respect to transporting the orientation along α. If $f\alpha$ is nullhomotopic in M then for any neighborhood U of $\alpha(I)$ there is a homotopy of f to f_1 which is fixed outside U such that $f_1^{-1}(q) = f^{-1}(q) \setminus \{\alpha(0), \alpha(1)\}$.*

The first part of the proof in the same as lemma 5.1.7 in [**Hir76**].

PROOF. The aim is to construct a homotopy f_t from $f = f_0$ to a map f_1 as described above. This homotopy is a map $N \times I \to M$ and will be constructed piece by piece. See figure 6.1. It is already given on $N \times 0$.

First deform α inside U such that it only meets the inverse image of q in the end points. Let $\bar{\alpha} : (I, \partial I) \to (N \times I, N \times 0)$ be α pushed into $N \times I$ that is $\bar{\alpha}$ is an embedding such that $\operatorname{pr}_1 \bar{\alpha} = \alpha$, the image of

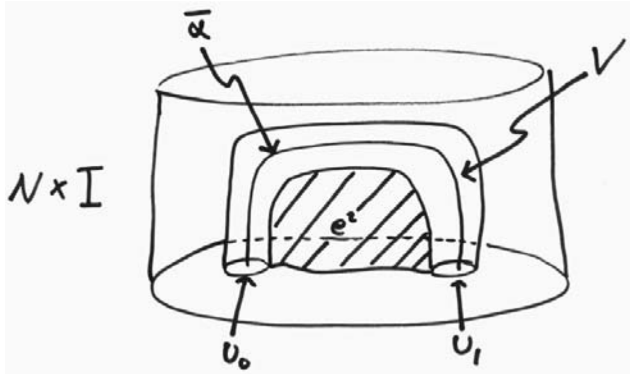

FIGURE 6.1. Constructing the homotopy piece by piece.

$\bar{\alpha}$ meets $N \times 0$ only in the end points and the tangents of $\bar{\alpha}$ are normal to $N \times 0$ at the intersection points. A closed tubular neighborhood, $\bar{V}$ of $\bar{\alpha}$ is the next piece on which the homotopy will be extended to.

Choose a chart $\phi : \mathbb{R}^n \to M$ sending 0 to q. And let U_0 and U_1 be the components of $f^{-1}(\phi(\mathbb{R}^n))$ containing $\alpha(0)$ and $\alpha(1)$. Since q is a regular value we may assume that the restrictions of f to U_1 and U_2 are diffeomorphisms onto $\phi(\mathbb{R}^n)$. Identify the union of U_0 and U_1 as a tubular neighborhood of $\partial\bar{\alpha}(I)$ in $N \times 0$. By theorem 4.6.4 in [**Hir76**] there is an extension to V a open tubular neighborhood of $\bar{\alpha}(I)$ in $N \times I$. All of these neighborhoods may be assumed to be so small that $V \subset U \times I$.

Choose a trivialization $t : V \to I \times \mathbb{R}^n$ of the tubular neighborhood V. Restricting to the fibers U_0 and U_1 and composing with the inverse of f we get linear maps

$$\mathbb{R}^n \stackrel{f|U_i}{\leftarrow} U_i \stackrel{t|U_i}{\rightarrow} \{i\} \times \mathbb{R}^n = \mathbb{R}^n$$

for $i = 0, 1$. Let A_i be the invertible matrices representing these maps. The sign of the determinant is the same for both A_0 and A_1 since f has opposite orientation at $\alpha(0)$ and $\alpha(1)$ with respect to transporting the orientation along α.

So there is a path $A : I \to GL(n)$ between them. This can be used to extend the homotopy over $\bar{V}$, the closed tubular neighborhood inside V of radius 1. Just define $\bar{V} \to M$ by

$$\bar{V} \stackrel{\bar{t}}{\cong} I \times D^n \stackrel{A_*}{\to} \mathbb{R}^n \stackrel{\phi}{\to} M$$

where $A_* : I \times D^n \to \mathbb{R}^n$ is given by matrix multiplication, $A_*(t, v) = A(t)v$. This defines $f_t(p)$ for $(p, t) \in N \times 0 \cup \bar{V}$.

The next piece to extend over is the closed 2-cell $e^2 \subset N \times I$ which is the part of $\alpha(I) \times I$ underneath $\operatorname{int} \bar{V}$. f_t is already given on the boundary of e^2. On $e^2 \cap \bar{V}$ it is a path in $\phi(\mathbb{R}^n \setminus 0)$ and on $e^2 \cap N \times 0$ it is the restriction of $f\alpha$ to a subinterval of I. Considering $f\alpha$ as a map $S^1 \to M$ we see that it is homotopic to the map $S^1 = \partial e^2 \xrightarrow{f_t} M$, thus there is a nullhomotopy $\beta : D^2 \to M$ of $f_t(\partial e^2)$. And when $n \geq 3$ this nullhomotopy may be pushed away from q. Use this modified nullhomotopy to extend f_t over e^2.

Now $f_t(p)$ is defined for (p, t) in $N \times 0 \cup \bar{V} \cup e^2$. Clearly there exists a retraction $r : N \times I \to N \times 0 \cup \bar{V} \cup e^2$ which is the projection to the first factor outside $U \times I$ and the image of $U \times I \setminus \operatorname{int} \bar{V}$ does not meet $\operatorname{int} \bar{V}$. Use this to construct the homotopy. That is we redefine f_t to be $f_t \circ r$. □

By repeated use of this lemma the following global result about degree 1 maps is achieved.

THEOREM 6.3. *Let $f : N \to M$ be a degree 1 map between compact n-manifolds, $n \geq 3$, taking ∂N to ∂M and inducing a π_0 bijection. For any $q \in \operatorname{int} M$ there is a homotopy of f to f_1 rel ∂ such that q is a regular value for f_1 and $f_1^{-1}(q)$ is a single point.*

PROOF. It is enough to consider the case where M is path connected.

We start by looking at the orientable case. Deform f such that q becomes a regular value. Now $f^{-1}(q)$ will consist of a finite number of points. Since f is degree 1 this number will be odd, that is equal to $2k - 1$ for some positive integer k. Near k of these points f will be orientation preserving and the function will reverse orientation near the last $k - 1$ points in $f^{-1}(q)$.

If $k > 1$ then we may cancel a pair of these points as follows: Any degree 1 map is a surjection on fundamental groups so we may choose a path $\alpha : I \to N$ between points in $f^{-1}(q)$ of different orientations such that $f\alpha$ is a nullhomotopic loop in M. Now lemma 6.2 applied to α reduces k by 1. This finishes the proof in the orientable case.

If the manifolds are nonorientable then f induces a degree 1 map $\bar{f} : \bar{N} \to \bar{M}$ of double orientable covering spaces. Make q regular by deforming f. Let $\bar{q}$ be one of q's liftings. Let $2k - 1$ be the number of points in $f^{-1}(q)$. This is the same as the number of points in $\bar{f}^{-1}(\bar{q})$. If $k > 1$ choose $\bar{\alpha}$ to be a path in $\bar{N}$ between points in $\bar{f}^{-1}(\bar{q})$ of different orientation such that $\bar{f}\bar{\alpha}$ is nullhomotopic in $\bar{M}$. Then by applying

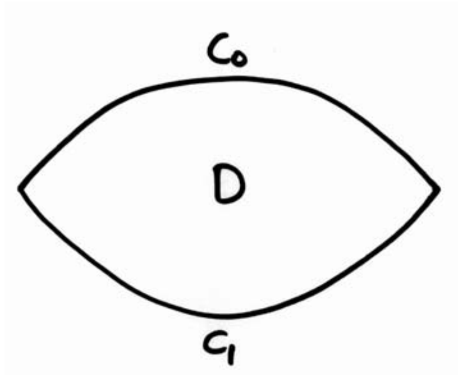

FIGURE 6.2. The disk with two corners.

lemma 6.2 to $p\bar{\alpha} : I \to N$ k is reduced by 1. Here $p : \bar{N} \to N$ is the covering space. $\square$

The next step is to consider the inverse image of a path. If a map behaves well the inverse image should also be a path. And in order to prove results in this direction a handle exchange theorem is needed, but first a definition:

DEFINITION 6.4. Let $\beta : I \to M$ be a smooth path intersecting a submanifold Y only at the end points such that the tangent to β at an endpoint is normal to Y. Then β is called strongly homotopic to zero in (M, Y) if there is a smooth map γ from the 2-disk D with two corners, see figure 6.2, to M such that γ restricted to C_0 is β, the inverse image $\gamma^{-1}(Y)$ is C_1 and a normal vector of C_1 in D maps under $\gamma_* : \tau_D \to \tau_M$ to a normal vector of Y in M. Here C_0 and C_1 are two different pieces of ∂D having the corners as end points.

LEMMA 6.5. *Every map $\alpha : I \to Y$ into a smooth manifold of dimension $k \geq 2$ such that $\alpha(0) \neq \alpha(1)$ is homotopic rel end points to an embedding. If $k = 1$ then α is homotopic rel end points to an immersion.*

PROOF. Transversality gives the result for $k \geq 3$.

For $k = 2$ transversality only gives an immersion, but crossings may be pushed along one branch and thus eliminated. See figure 6.3.

For $k = 1$ we lift to the universal covering space of Y. Throw away the components not containing the lifted path $\tilde{\alpha}$. What is left is just the real line $\mathbb{R}$. And any path $\tilde{\alpha} : I \to \mathbb{R}$ is homotopic rel endpoints to a linear map $I \to \mathbb{R}$. This gives an immersion $I \to Y$. $\square$

Rong and Wang [**RW92**] state this theorem:

FIGURE 6.3. Eliminating selfintersections of a path.

THEOREM 6.6. *Suppose that a proper map $f : N \to M$ between compact connected n-manifolds is transverse to a proper k-submanifold $Y \subset M$, where $n \geq 3$ and $n > k > 0$. If there is an smooth embedded path β in N intersecting $f^{-1}(Y)$ only in $\beta(0)$ and $\beta(1)$ such that the tangents to β at the end points are normal to $f^{-1}(Y)$ and $f\beta$ is strongly homotopic to zero in (M, Y), then f is homotopic by a homotopy fixed outside a tubular neighborhood of the path β to a map f_0 which is transverse to Y such that $f_0^{-1}(Y)$ is diffeomorphic to $f^{-1}(Y)$ with the points $\beta(0)$ and $\beta(1)$ removed and a 1-handle attached instead.*

The analogous result for $k = 0$ is lemma 6.2.

PROOF. We will relate the general situation to an "universal" example. The proof will start by describing the example.

Let $c : \mathbb{R} \to \mathbb{R}^2$ be an embedded path such that

$c(s) = (s + 1, -1)$ for $s \leq \delta - 1$,
$c(s) = (1 - s, 1)$ for $s \geq 1 - \delta$ and
the first coordinate of c has maximum value 1 for $s = 0$ and this is the only point where the tangent is vertical.

Here $\delta > 0$ is some small real number.

Let N_1 be a vector field along c of unit vectors orthogonal to c. Let $\mathbb{R}^2$ be included in $\mathbb{R}^n$ as the first two coordinates. Let $N_2, \ldots, N_{n-1}$ be the unit vector fields along c in $\mathbb{R}^n$ such that N_i points in the direction of the $i+1$'th unit vector e_{i+1} in $\mathbb{R}^n$. We may think of the N_i's as maps $\mathbb{R} \to \mathbb{R}^n$. Choose $\varepsilon > 0$ so small that $g : \mathbb{R} \times D_\varepsilon^{n-1}$ of c is given by

$$g(s, x_1, \ldots, x_{n-1}) = c(s) + \sum_i x_i N_i(s)$$

defines a tubular neighborhood of c such that $g([\delta - 1, 1 - \delta] \times D_\varepsilon^{n-1})$ is contained in the open half plane $(0, \infty) \times \mathbb{R}$. Here D_ε^{n-1} is the disk of points $(x_1, \ldots, x_{n-1})$ in $\mathbb{R}^{n-1}$ of length less than or equal to ε.

Let $\rho : \mathbb{R} \to \mathbb{R}$ be a function such that

$\rho(t) = 0$ for $t \leq \frac{1}{4}$,

$\rho(t) = -2$ for $t \geq \frac{3}{4}$,
$\rho'(t) = 0$ for $t \in \left(-\infty, \frac{1}{4}\right] \cup \left[\frac{3}{4}, \infty\right)$ and $\rho'(t) < 0$ for $t \in \left(\frac{1}{4}, \frac{3}{4}\right)$
and
the equation $\rho'(t) = a$ has no more than two solutions when a is a real number less than 0.

And let $\mu : \mathbb{R} \to [0,1]$ be a bump function such that $\mu(s) = 1$ for $|s| \leq 3$ and $\mu(s) = 0$ for $|s| \geq 4$.

Define the homotopy f_t of g by

$$f_t(s,x) = g(s,x) + \mu(s)\rho(t\frac{\varepsilon - \|x\|}{\varepsilon})e_1$$

Here $t \in I$, $x \in D_\varepsilon^{n-1}$ and e_1 is the vector $(1,0,0,\dots,0) \in \mathbb{R}^n$. Observe that this homotopy is fixed when $|s| \geq 4$ or $\|x\| \geq \frac{3\varepsilon}{4}$.

Let P be the k-plane in $\mathbb{R}^n$ spanned by $\{e_2, e_3, e_4, \dots, e_{k+1}\}$. We are interested in the inverse image of P under f_0 and f_1.

f_0 is clearly transversal to P and $f_0^{-1}(P)$ is seen to be $\{-1,1\} \times D_\varepsilon^k$, where D_ε^k is the intersection of D_ε^{n-1} with $\mathbb{R}^k \subset \mathbb{R}^{n-1}$.

Now we verify that f_1 is transversal to P. Since $\mathrm{pr}_{i+1} f_1(s, x_1, x_2, \dots, x_{n-1}) = x_i$ for $i = 2, \dots, n-1$ it is clear that e_{i+1} lies in the image of the derivative of f_1 the only difficulty is to show that e_1 also lies in the image at a point of the inverse image of P. Let $\mathrm{pr}_1 : \mathbb{R}^n \to \mathbb{R}$ be the projection onto the first coordinate. Consider

$$\mathrm{pr}_1 f_1(s, x_1, x_2, \dots, x_{n-1}) = \mathrm{pr}_1 c(s) + x_1 \mathrm{pr}_1 N_1(s) + \mu(s)\rho(\frac{\varepsilon - \|x\|}{\varepsilon})$$

We see that $(s, x_1, \dots, x_{n-1})$ is contained in $f_1^{-1}(P)$ only if the expression above is 0. Since $\mathrm{pr}_1 g(s, x_1, \dots, x_{n-1}) = \mathrm{pr}_1 c(s) + x_1 \mathrm{pr}_1 N_1(s) \geq 0$ if and only if $s \in [-1,1]$ and $\mu(s)\rho(\frac{\varepsilon - \|x\|}{\varepsilon}) \leq 0$ it follows that s must lie in $[-1,1]$ for the expression to be 0. Hence $\mu(s)$ is 1 in the cases we consider.

First consider the case where $s \in (\delta - 1, 1 - \delta)$. We fix some unit vector $v = (v_1, \dots, v_{n-1})$ in $\mathbb{R}^{n-1}$ and define $d : [0, \varepsilon] \to \mathbb{R}$ by

$$d(r) = \mathrm{pr}_1 f_1(s, rv)$$

Computing this we get that

$$d(r) = \mathrm{pr}_1 c(s) + rv_1 \mathrm{pr}_1 N_1(s) + \rho(\frac{\varepsilon - r}{\varepsilon})$$

We see that $d(0) < 0$ and $d(\varepsilon) > 0$. The derivative of d is

$$d'(r) = v_1 \mathrm{pr}_1 N_1(s) - \frac{1}{\varepsilon}\rho'(\frac{\varepsilon - r}{\varepsilon})$$

Recall that $\rho'(t) = 0$ for $t \in \left(-\infty, \frac{1}{4}\right] \cup \left[\frac{3}{4}, \infty\right)$ and $\rho'(t) < 0$ for $t \in \left(\frac{1}{4}, \frac{3}{4}\right)$. This implies that if $v_1 \operatorname{pr}_1 N_1(s) \geq 0$ then d is an increasing function which has a single transverse intersection with $0 \in \mathbb{R}$, and if $v_1 \operatorname{pr}_1 N_1(s) < 0$ then $d'(0) < 0$ while $d'(\varepsilon) > 0$ and $d'(r) = 0$ has not more than two solutions, by the last condition on ρ, so again d is an increasing function which has a single transverse intersection with $0 \in \mathbb{R}$.

This shows that the intersection of f_1 with P is transverse for $(s, x) \in (\delta - 1, 1 - \delta) \times D_\varepsilon^{n-1}$ and in each fiber $\{s_0\} \times D_\varepsilon^{n-1}$ the inverse image $f_1^{-1}(P)$ is a $k-1$-sphere contained in $\{s_0\} \times D_\varepsilon^k$.

For $s \in [-1, \delta - 1] \cup [1 - \delta, 1]$ we start by fixing some $x \in D_\varepsilon^{n-1}$ and we look at the function $d : [-1, \delta - 1] \cup [1 - \delta, 1] \to \mathbb{R}$ defined by

$$d(s) = \operatorname{pr}_1 f_1(s, x)$$

Computing we get that

$$d(s) = \operatorname{pr}_1 c(s) + \rho(\frac{\varepsilon - \|x\|}{\varepsilon})$$

We find that the derivative of d is nonzero for all s since $\frac{d \operatorname{pr}_1 c}{ds}(s) = 0$ only for $s = 0$ which does not lie in the domain of d.

Moreover we see that for $s_0 \in [-1, \delta - 1] \cup [1 - \delta, 1]$ the part of $f_1^{-1}(P)$ which lies in the fiber $\{s_0\} \times D_\varepsilon^{n-1}$ is a $k-1$-sphere contained in $\{s_0\} \times D_\varepsilon^k$ except for $s_0 = \pm 1$. In this case it is an annulus in $\{s_0\} \times D_\varepsilon^k$.

Thus $f_1^{-1}(P)$ is diffeomorphic to $f_0^{-1}(P)$ with the points $f_0^{-1}(c(-1))$ and $f_0^{-1}(c(1))$ removed and a 1-handle attached instead.

The condition that $f\beta$ is strongly homotopic to zero in (M, Y) ensures that it is possible to find a map $\phi : \mathbb{R}^n \to M$ such that

> ϕ is transversal to Y and $\phi^{-1}(Y) = P$,
> the restriction of ϕ to $(-\infty, \delta] \times \mathbb{R}^{n-1}$ is an embedding if $k > 1$ and an immersion if $k = 1$,
> the restriction of f to an appropriate tubular neighborhood T of $\beta(I)$ factorizes through ϕ and the lifting $\tilde{f} : T \to \mathbb{R}$ is given by
>
> $$\tilde{f}(s, x) = g(2s - 1, x)$$
>
> where $(s, x) \in I \times D_\varepsilon^{n-1} \cong T$ and g is the function defined above.

Once we have such a map ϕ and lifting $\tilde{f}$ we use the homotopy f_t of g to deform f by a homotopy which is ϕf_t inside T and fixed outside. If f was transversal to Y then the resulting map f' also is transversal to Y and $f'^{-1}(Y)$ is $f^{-1}(Y)$ with the points $\beta(0)$ and $\beta(1)$ removed and a 1-handle glued on instead.

Let $\gamma : D \to M$ be as in definition 6.4. Observing that any homotopy of $\gamma|C_1$ in Y gives rise to a deformation of the strong homotopy γ. Therefore we may assume by lemma 6.5 that $\gamma|C_1$ is an embedding if $k > 1$ and an immersion if $k = 1$.

Now construct the map ϕ on $(-\infty, \delta] \times \mathbb{R}^{n-1}$ by letting it be a tubular neighborhood of the embedded (immersed) path $\gamma|C_1$. It is clear that we may assume that

$\phi^{-1}(Y) = P$ and
$\phi^{-1}(\gamma(D)) = [0, \delta] \times [-1, 1] \times 0$

Extend the domain of ϕ to $g(\mathbb{R} \times D^{n-1}_\varepsilon) \cup (-\infty, \delta] \times \mathbb{R}^{n-1}$ by choosing an appropriate tubular neighborhood T of $\beta(I)$ which we identify with $g([-1, 1] \times D^{n-1}_\varepsilon)$ and define ϕ to be f over the image of g. We may now fill inn a 2-cell in the domain of ϕ and extend the map over this cell by the strong homotopy γ. Since $\gamma^{-1}(Y) = C_1$ we may assume that $\phi^{-1}(Y)$ still is P. Finally extend the domain of ϕ to be all of $\mathbb{R}^n$ by using a retraction from $\mathbb{R}^n$ to the subset where we have defined ϕ already. □

LEMMA 6.7. *Given a degree* 1 *map* $f : (N, \partial N) \to (M, \partial M)$ *between manifolds of dimension* $n \geq 3$*, two points* p_0*,* p_1 *in* ∂N *such that* f *is a local diffeomorphism at* p_0 *and* p_1 *and* $f^{-1}(f(p_i))$ *contains only* p_i*,* $i = 0, 1$*. Then for any embedded path* $\alpha : I \to M$ *between* $f(p_0)$ *and* $f(p_1)$ *intersecting* ∂M *only at these two points, the intersection being transversal, there exist homotopy of* f *to* f_1 *relative to a neighborhood of* ∂N *such that* f_1 *is a diffeomorphism in a neighborhood of* $f_1^{-1}(\alpha(I))$*.*

PROOF. In this proof observe that every homotopy can be assumed to be relative to a neighborhood of ∂N.

By theorem 6.1 it can be assumed that f meets $\alpha(I)$ transversally. Then $f^{-1}(\alpha(I))$ will be a submanifold of dimension 1. The assumptions concerning the p's implies that this submanifold consists of one arch and a finite number of circles. These circles will be eliminated by handle exchanging. Choose a tubular neighborhood of $\alpha(I)$. Pulling this back by f we get a tubular neighborhood of $f^{-1}(\alpha(I))$ assuming that the first neighborhood was sufficiently small. Let D be the subbundle of the tubular neighborhood of $f^{-1}(\alpha(I))$ consisting of disks of radius 1.

Restrict f to a map $f| : N \setminus \operatorname{int} D \to M \setminus \operatorname{int} f(D)$. This has degree 1 since $f : N \to M$ has degree 1, hence it induces a surjection $\pi_1(N \setminus \operatorname{int} D) \to \pi_1(M \setminus \operatorname{int} f(D))$. Let $\beta : I \to N$ be a path between different components of $f^{-1}(\alpha(I))$. Deforming β by a homotopy rel end points we may assume that there is some $\delta > 0$ such that the image of

β meets D in $\beta([0,\delta]\cup[1-\delta,1])$ and that

$$\beta(t)=\begin{cases}\frac{1}{\delta}tv_0\in D\subset N & \text{for } t\in[0,\delta]\\ \frac{1}{\delta}(1-t)v_1\in D\subset N & \text{for } t\in[1-\delta,1]\end{cases}$$

Here v_0 and v_1 are unit vectors in the fiber above $\beta(0)$ and $\beta(1)$ in the tubular neighborhood of $f^{-1}(\alpha(I))$. We are interested in finding an appropriate choice of β such that the restriction of $f\beta$ to $[\delta,1-\delta]$ can be deformed down into $\partial f(D)$ by a homotopy relative to the end points. If this is the case then $f\beta$ is clearly strongly homotopic to zero in $(M,\alpha(I))$ and using theorem 6.6 will reduce the number of components in $f^{-1}(\alpha(I))$ by 1. Such β always exists. To see this let γ be a path in $\partial f(D)$ from $f\beta(\delta)$ to $f\beta(1-\delta)$. Since the map $\pi_1(N\smallsetminus \operatorname{int} D)\to\pi_1(M\smallsetminus\operatorname{int} f(D))$ induced by $f|$ is surjective we may suppose that β have been chosen so that $f\beta$ followed by γ is nullhomotopic in $M\smallsetminus\operatorname{int} f(D)$. Inductively this shows that it may be assumed that $f^{-1}(\alpha(I))$ is connected.

So one can assume that:

f is transversal to $\alpha(I)$.

$f^{-1}(\alpha(I))$ is an arch.

$D\subset N$ and $E\subset M$ are closed tubular neighborhoods of $f^{-1}(\alpha(I))$ and $\alpha(I)$ respectively both diffeomorphic to $I\times D^{n-1}$.

With respect to these diffeomorphisms f restricted to D has the form $f(t,x)=(g(t),h_t(x))$ where

$g:(I,\operatorname{int} I,0,1)\to(I,\operatorname{int} I,0,1)$ is a diffeomorphism near ∂I and

$h_t:D^{n-1}\to D^{n-1}$ is linear.

The aim is to straighten f to a diffeomorphism near $f^{-1}(\alpha(I))$.

Let $g_s:I\to I$ be a homotopy from $g=g_0$ to a diffeomorphism g_1 relative to a neighborhood of ∂I. And let $\lambda_s:I\to I$ be a smooth family of bump functions, that is λ_s is monotonic and constant near ∂I, such that $\lambda_s(0)=0$ and $\lambda_s(1)=s$. Define $f_s(p)$ to be $f(p)$ outside D and let $f_s(t,x)=(g_{\lambda_s(1-\|x\|)}(t),h_t(x))$ for $(t,x)\in I\times D^{n-1}\cong D$. This is a straightening, that is $f_0=f$ and f_1 is a diffeomorphism near $f^{-1}(\alpha(I))$. □

This result is the improved version of Rong and Wang's corollary 7. See [**RW92**].

THEOREM 6.8. *Let $f:N\to M$ be a degree 1 map between compact 3-manifolds taking ∂N to ∂M. Then for any handlebody H of genus g in* $\operatorname{int} M$, *f is homotopic to f_1 relative to a neighborhood of the boundary* *such that* f_1 *maps* $f_1^{-1}(H)$ *onto* H *diffeomorphically, and if* $H' \subset H$ *is a handlebody such that* H *is obtained from* H' *by adding* 1*-handles and* f *already maps* $f^{-1}(H')$ *onto* H' *diffeomorphically, then the homotopy of* f *to* f_1 *may be assumed to be relative to a neighborhood of* H'*.*

This is proved by essentially the same method as in [**RW92**] except that I am working in the differentiable category while Rong and Wang uses PL techniques.

PROOF. If no H' is given then choose H' be a 3-cell of H such that H can be obtained from H' by adding 1-handles. f can easily be deformed such that $f| : f^{-1}(H') \to H'$ is a diffeomorphism. Use Theorem 6.3.

Let H'' be H' plus one of the handles. Represent this handle by a tubular neighborhood of an embedding $\alpha : I \to M \setminus \operatorname{int} H'$ which meets $\partial H''$ only in the two points $\alpha(0)$ and $\alpha(1)$, the intersection being transversal. Now use lemma 6.7 to deform f relative to a neighborhood of H' such that f maps a tubular neighborhood D of $f^{-1}(\alpha(I))$ diffeomorphically onto a tubular neighborhood E of $\alpha(I)$. By isotopy of tubular neighborhoods E may be assumed to be the 1-handle.

Continue inductively to obtain H. □

CHAPTER 7

Surgery

In this chapter we work in the smooth category.

7.1. Connected sum and normal maps

Let W_1 and W_2 be smooth path connected compact manifolds. We want to construct the connected sum of two normal maps $g_i : V_i \to W_i$ for $i = 1, 2$.

Suppose that q_i is a regular value for the map g_i contained in the interior of W_i and that $g_i^{-1}(q_i)$ is a single point p_i in V_i. Then we may construct the connected sum as follows. Let U_i be neighborhoods of q_i in W_i. We demand that the closure of U_i is contained in a open set in W_i which is diffeomorphic to $\mathbb{R}^n$ and that U_i is diffeomorphic to $\mathbb{R}^n$ by chosen charts $\phi_i : U_i \to \mathbb{R}^n$ which takes q_i to $0 \in \mathbb{R}^n$. Suppose that the restriction of g_i to $g_i^{-1}(U_i)$ is a diffeomorphism. If W_i is oriented we demand that ϕ_i is orientation preserving for $i = 1$ and orientation reversing for $i = 2$. For unorientable manifolds the choice of orientation does not matter since there exists a loop with non-trivial first Stiefel-Whitney class. Let $W_1 \sharp W_2$ be the disjoint union of $W_1 \setminus \{q_1\}$ and $W_2 \setminus \{q_2\}$ where we identify $x \in U_1 \setminus \{q_1\}$ with $y \in U_2 \setminus \{q_2\}$ if $\phi_1(x) = \frac{\phi_2(y)}{||\phi_2(y)||^2}$. We form $V_1 \sharp V_2$ similarly. Just use $\phi_i g_i : g^{-1}(U_i) \to \mathbb{R}^n$ instead of $\phi_i : U_i \to \mathbb{R}^n$.

There also is a map $g_1 \sharp g_2 : V_1 \sharp V_2 \to W_1 \sharp W_2$ defined by

$$g_1 \sharp g_2(p) = \begin{cases} g_1(p) & \text{for } p \in V_1 \setminus \{p_1\} \\ g_2(p) & \text{for } p \in V_2 \setminus \{p_2\} \end{cases}$$

We now want to extend bundle data so that this map becomes a normal map. The given bundle data are ν_i a bundle over W_i and F_i a stable trivialization of $\tau_{V_i} \oplus g_i^* \nu_i$. Assume that ν_1 and ν_2 both are l-dimensional. And that the stable trivializations are maps $F_i : \tau_{V_i} \oplus g_i^* \nu_i \to \varepsilon^{n+l}$.

Let V_i' be V_i where $g_i^{-1}\phi_i^{-1}(\operatorname{int} D^n)$ has been removed. Let $S_i \subset \partial V_i'$ be $g_i^{-1}\phi_i^{-1}(S^{n-1})$. Identify the tangent bundle of $V_1 \sharp V_2$ as the disjoint union of $\tau_{V_1'}$ and $\tau_{V_2'}$ where we identify $v \in \tau_{V_1'}|S_1$ with $\alpha(v) \in \tau_{V_2'}|S_2$. Here $\alpha : \tau_{V_1'}|S_1 \to \tau_{V_2'}|S_2$ is some clutching map over $g_2^{-1}\phi_2^{-1}\phi_1 g_1|S_1$.

Choose a clutching map $\beta' : g_1^*\nu_1|S_1 \to g_2^*\nu_2|S_2$ such that the following diagram commutes up to homotopy:

$$\begin{array}{ccc} \tau_{V_1'}|S_1 \oplus g_1^*\nu_1|S_1 & \xrightarrow{F_1|} & \varepsilon^{n+l} \\ \downarrow{\scriptstyle \alpha\oplus\beta'} & & \downarrow{\scriptstyle =} \\ \tau_{V_2'}|S_2 \oplus g_2^*\nu_2|S_2 & \xrightarrow{F_2|} & \varepsilon^{n+l} \end{array}$$

By changing for example F_2 by a homotopy we may suppose that the diagram actually commutes.

Since the restriction of g_i to S_i is a diffeomorphism and that $g_i^{-1}(g_i(S_i)) = S_i$ for both $i = 1, 2$ there exists a clutching map $\beta : \nu_1|g_1(S_1) \to \nu_2|g_2(S_2)$ which pulls back to β'. Use β to define the bundle ν over $W_1\sharp W_2$.

Now define $F : \tau_{V_1\sharp V_2} \oplus (g_1\sharp g_2)^*\nu \to \varepsilon^{n+l}$ to be $F_i|V_i'$ over V_i' for $i = 1, 2$. Since the diagram above commutes (after F_2 has been changed) this map is well defined.

Proposition 7.1. *Connected sum defines map $\mathcal{N}^O(W_1 \text{ rel } \partial) \times \mathcal{N}^O(W_2 \text{ rel } \partial) \to \mathcal{N}^O(W_1\sharp W_2 \text{ rel } \partial)$. At least when $n \geq 3$.*

Here q_i, U_i and ϕ_i are fixed.

Proof. In order to be able to carry out the construction of connected sum there was a list of requirements for the normal maps $g_i : V_i \to W_i$. The most essential of these requirements was that q_i was a regular value for g_i and that $g_i^{-1}(q_i)$ was a single point.

If this holds then we may deform g_i by a homotopy such that the restriction of g_i to $g_i^{-1}(U_i)$ is a diffeomorphism. This follows by uniqueness of tubular neighborhoods.

Theorem 6.3 ensures that every normal cobordism class contains a representative $g_i : V_i \to W_i$ such that q_i is regular and $g_i^{-1}(q_i)$ is a single point p_i. We must show that the construction of connected sum does not depend on the normal map we choose to represent the class. So let $h_i : P_i \to W_i \times I$ be normal cobordisms between $g_i : V_i \to W_i$ and another choice $g_i' : V_i' \to W_i$. We may assume that the P_i's are both path connected. And by lemma 6.7 we may suppose that h_i is transversal to $\{q_i\} \times I$ and the restriction of h_i to $h_i^{-1}(\{q_i\} \times I)$ is a diffeomorphism onto its image. Since h_i is transversal to $\{q_i\} \times I$ the restriction of h_i to a tubular neighborhood of $h_i^{-1}(\{q_i\} \times I)$ is a diffeomorphism onto its image. And by uniqueness of tubular neighborhoods up to isotopy we may even suppose that this tubular neighborhood is $h_i^{-1}(U_i \times I)$.

Now we repeat the construction of connected sum, but replacing q_i by $\{q_i\} \times I$, U_i by $U_i \times I$ and ϕ_i by $\phi_i \times \mathrm{pr} : U_i \times I \to \mathbb{R}^n \times I$.

To be more specific we start by forming $(W_1 \sharp W_2) \times I$ from $W_1 \times I$ and $W_2 \times I$ by taking the disjoin union of $W_1 \times I \setminus \{q_1\} \times I$ and $W_2 \times I \setminus \{q_2\} \times I$ where we identify $(x, t) \in U_1 \times I \setminus \{q_1\} \times I$ with $(y, t) \in U_2 \times I \setminus \{q_2\} \times I$ if $\phi_1(x) = \frac{\phi_2(y)}{\|\phi_2(y)\|^2}$. As before we form $P_1 \sharp' P_2$ similarly. Take the disjoint union of $P_1 \setminus h_1^{-1}(\{q_1\} \times I)$ and $P_2 \setminus h_2^{-1}(\{q_2\} \times I)$ and identify $x' \in h_1^{-1}(U_1 \times I \setminus \{q_1\} \times I)$ with $y' \in h_2^{-1}(U_2 \times I \setminus \{q_2\} \times I)$ if $h_1(x') = (x, t)$, $h_2(y') = (y, t)$ and $\phi_1(x) = \frac{\phi_2(y)}{\|\phi_2(y)\|^2}$. And we also have a map $h_1 \sharp' h_2 : P_1 \sharp' P_2 \to (W_1 \sharp W_2) \times I$. The proof that this is a normal map goes as above. □

We can describe the argument above as follows: Let $S^{n-1} \subset W_1 \sharp W_2$ be the $n-1$ sphere which separates the manifold into W_1 and W_2 with open disks removed. We write $W_1 \sharp W_2$ as the union $W_1' \cup_{S^{n-1}} W_2'$. We have shown that the map $\mathcal{N}^O(W_i' \text{ rel } \partial) \to \mathcal{N}^O(W_i \text{ rel } \partial)$ given by gluing a disk into a normal map $V_i' \to W_i'$ is a bijection. So when forming the connected sum of two normal maps $V_1 \to W_1$ and $V_2 \to W_2$ we first lift them to $\mathcal{N}^O(W_i' \text{ rel } \partial)$. Then we glue these two maps together along $S^{n-1} \subset \partial W_i$ to obtain a normal map over $W_1 \sharp W_2$. Using the bijection $\mathcal{N}^O(W \text{ rel } \partial) \approx [W/\partial, G/O]$ the following diagram describes the operation:

$$\begin{array}{ccc} [W_1'/\partial, G/O] \times [W_2'/\partial, G/O] & & \\ \downarrow \approx & \searrow^{\vee} & \\ [W_1/\partial, G/O] \times [W_2/\partial, G/O] & \xrightarrow{\sharp} & [(W_1 \sharp W_2)/\partial, G/O] \end{array}$$

Remember that gluing corresponds to wedge. See 4.3.

Proposition 7.2. *If $\pi_{n-1}(G/O)$ is trivial, where n is the dimension of the manifolds W_1 and W_2, then the map defined in proposition 7.1 is surjective.*

This result should give the reader associations to the Mayer-Vietoris sequence.

Proof. According to the remarks above it is enough to show that the following sequence is exact:

$$[W_1'/\partial, G/O] \times [W_2'/\partial, G/O] \xrightarrow{\vee} [(W_1 \sharp W_2)/\partial, G/O] \xrightarrow{j^*} [S^{n-1}, G/O]$$

where the map j^* is induced by the inclusion $j : S^{n-1} \to W_1 \sharp W_2$ of the sphere separating the connected sum into the two pieces W_1' and W_2'.

It is clear that the composition of the two maps are zero.

If we have some map $h : W_1 \sharp W_2 \to G/O$ such that $jh : S^{n-1} \to G/O$ is nullhomotopic, then we may find, by the homotopy extension property, a homotopy of h such that jh actually equals the constant map. And in this case it is clear that h lies in the image of $[W_1'/\partial, G/O] \times [W_2'/\partial, G/O]$. $\square$

PROPOSITION 7.3. *The following diagram commutes when $n \geq 4$ is even.*

$$\begin{array}{ccc} \mathcal{N}^O(W_1 \ rel \ \partial) \times \mathcal{N}^O(W_2 \ rel \ \partial) & \xrightarrow{\sharp} & \mathcal{N}^O(W_1 \sharp W_2 \ rel \ \partial) \\ \downarrow & & \downarrow \\ L_n(\mathbb{Z}[\pi_1 W_1], w_1) \times L_n(\mathbb{Z}[\pi_1 W_2], w_1) & \xrightarrow{\oplus} & L_n(\mathbb{Z}[\pi_1 W_1 * \pi_1 W_2], w_1) \end{array}$$

This is a formula for computing the surgery obstruction of a connected sum. A similar result probably holds for n odd, but since the definition of the L-groups are different in this case another proof is needed. For our purposes the even case is sufficient.

PROOF. We must show that the surgery obstruction of a connected sum is the direct sum of the surgery obstructions.

For n even the surgery obstruction of a normal map $V \to W$ is calculated by first doing surgery such that $V \to W$ is $\frac{n}{2}$-connected. Let $k = \frac{n}{2}$. Every element in the kernel $K_k(V)$ of the induced homomorphism $H_k(V) \to H_k(W)$ can now be represented by immersed spheres $S^k \to V$ such that the composition $S^k \to V \to W$ is nullhomotopic. The bundle data determines which regular homotopy class of immersions we should choose. On $K_k(V)$ there are defined two forms: λ, the intersection form and μ, the self intersection form. The elements of the groups $L_n(\mathbb{Z}[\pi_1 W], w_1)$ are equivalence classes of the type (G, λ, μ) and the operation is direct sum. The surgery obstruction of $V \to W$ is the class of $(K_k(V), \lambda, \mu)$.

So we must show that for a connected sum of two k-connected normal maps $g_1 : V_1 \to W_1$ and $g_2 : V_2 \to W_2$ the class of $(K_k(V_1 \sharp V_2), \lambda, \mu)$ is the same as $(K_k(V_1), \lambda_1, \mu_1) \oplus (K_k(V_2), \lambda_2, \mu_2)$. This follows easily form the observation that every immersion $\alpha : S^k \to V_i$ may be deformed by a regular homotopy away from the neighborhood $g_i^{-1}(U_i)$ of p_i where the connected sum is formed. We must also make sure that there exists a nullhomotopy of $g_i \alpha$ in $W_1 \sharp W_2$. This might fail for 2-manifolds. Since $n \geq 4$ we may deform a nullhomotopy of $g_i \alpha$ away from U_i in W_i.

There is a small complication here as $K_k(V_1 \sharp V_2)$, $K_k(V_1)$ and $K_k(V_2)$ are modules over different rings, but we can consider $K_k(V_1)$ as a module over $\mathbb{Z}[\pi_1 W_1 * \pi_1 W_2]$ by letting the elements of $\pi_1 W_2$ act trivially. Similarly $K_k(V_2)$ is also a module over $\mathbb{Z}[\pi_1 W_1 * \pi_1 W_2]$. □

REMARK 7.4. The formula holds for 2-manifolds if W_1 or W_2 is a sphere.

PROOF. The condition that at least one of W_1 and W_2 is a sphere ensures the existence of nullhomotopies of the $g_i\alpha$'s. □

LEMMA 7.5. *If $g : V \to W$ is a normal map between compact 4-manifolds and the second Stiefel-Whitney class, $w_2(\tau_W)$, is 0 then the signature of V is divisible by 16.*

PROOF. Let ν_W be the normal bundle of W and let ν be the bundle over W which pulls back to a normal bundle ν_V for V. We wish to prove that $w_2(\tau_V) = 0$. Since ν_W and ν both are stably fiber homotopy equivalent to the Spivak normal bundle there exists a bundle ξ^k classified by a map $W \to BO$ which factorizes through $G/O \to BO$ and such that $\nu_W \oplus \varepsilon^k = \nu \oplus \xi^k$. We now have that both $w_1(\xi^k)$ and $w_2(\xi^k)$ are zero. This implies that $w_1(\nu_W) = w_1(\nu)$ and $w_2(\nu_W) = w_2(\nu)$. We have the following formula for $w_2(\tau_V)$:

$$w_2(\tau_V) = (w_1(\nu_V))^2 + w_2(\nu_V)$$

The same formula holds of W. Since we know that $g^*\nu$ is a normal bundle for V we have:

$$w_2(\tau_V) = (g^*w_1(\nu))^2 + g^*w_2(\nu) = (g^*w_1(\nu_W))^2 + g^*w_2(\nu_W) = g^*w_2(\tau_W)$$

Thus $w_2(\tau_V) = 0$ and we may apply Rohlin's theorem to V. See [**MK58**]. This says that the first Pontrjagin number of V, $p_1[V]$, is divisible by 48. By the signature theorem we have that the signature of V is $\sigma(V) = \frac{1}{3}p_1[V]$, hence divisible by 16. See [**MS74**] for the formulas above and the signature theorem. □

LEMMA 7.6. *There exists a normal map $V \to S^4$ where the signature of V is* 16.

See also [**MM79**] remark 2.16.

PROOF. By theorem 2 of [**MK58**] there exists a almost parallelizable manifold V with signature 16. Almost parallelizable means that there exists a disk $D^4 \subset V$ such that the tangent bundle of V is trivial over $V \setminus \operatorname{int} D^4$.

Let ν_V be the normal bundle of V for some embedding of V in $\mathbb{R}^{4+k}$. Since V is almost parallelizable we may write ν_V as the union of the trivial bundle over $V \setminus \operatorname{int} D^4$ and the trivial bundle over D^4 where we identify according to some clutching map $\beta : S^3 \to O(k)$. Construct a degree 1 map $g : V \to S^4$ by sending $V \setminus \operatorname{int} D^4$ to the northern hemisphere D_+ of S^4 and sending $D^4 \subset V$ diffeomorphically to the southern hemisphere D_-. Now let ν be the bundle over S^4 obtained from identifying the trivial k-dimensional bundles over D_+ and D_- using the clutching map β. Now there exists a bundle map $\nu_V \to \nu$ covering g. So we have a normal map. □

LEMMA 7.7. *$\mathcal{N}^O(S^4)$ and $\mathcal{N}^O(S^2 \times S^2)$ has the same image in the group $L_4(\mathbb{Z}, 0)$.*

PROOF. Let $\mathcal{N}^O(S^4) \to \mathcal{N}^O(S^2 \times S^2)$ be the map defined by taking connected sum with the identity map $S^2 \times S^2 \to S^2 \times S^2$. By proposition 7.3 the following diagram commutes:

$$\begin{array}{ccc} \mathcal{N}^O(S^4) & \longrightarrow & \mathcal{N}^O(S^2 \times S^2) \\ \downarrow & & \downarrow \\ L_4(\mathbb{Z}, 0) & \xrightarrow{=} & L_4(\mathbb{Z}, 0) \end{array}$$

Hence the image of $\mathcal{N}^O(S^4)$ in $L_4(\mathbb{Z}, 0)$ is contained in the image of $\mathcal{N}^O(S^2 \times S^2)$.

We know that $L_4(\mathbb{Z}, 0)$ is isomorphic to $\mathbb{Z}$ and the surgery obstruction is $\frac{1}{8}$ times the difference in signatures. Let $g : V \to S^2 \times S^2$ be a normal map. The tangent bundle of $S^2 \times S^2$ is trivial hence $w_2(\tau_{S^2 \times S^2}) = 0$ and it follows from lemma 7.5 that the signature of V is divisible by 16. But by the lemma 7.6 there exists a normal map $V' \to S^4$ such that the signature of V' equals the signature of V. This shows that the image of $\mathcal{N}^O(S^2 \times S^2)$ is contained in the image of $\mathcal{N}^O(S^4)$. □

PROPOSITION 7.8. *Let W be a 4-dimensional smooth manifold possibly with boundary. Then the image of $\mathcal{N}^O(W \text{ rel } \partial)$ in $L_4(\mathbb{Z}[\pi_1 W], w_1)$ is equal to the image of $\mathcal{N}^O(W \sharp S^2 \times S^2 \text{ rel } \partial)$.*

PROOF. By proposition 7.2 the map

$$\mathcal{N}^O(W \text{ rel } \partial) \times \mathcal{N}^O(S^2 \times S^2) \to \mathcal{N}^O(W \sharp S^2 \times S^2 \text{ rel } \partial)$$

is surjective. So we may write any normal map $g' : V' \to W \sharp S^2 \times S^2$ as the connected sum of a normal map $g : V \to W$ and a normal map $h : T \to S^2 \times S^2$. Now choose some map $h' : P \to S^4$ having -1 times the surgery obstruction of h. Denote by $\bar{h}' : \bar{P} \to S^4$ the normal map obtained from h' by switching the orientation. h' and $\bar{h}'$ has opposite surgery obstructions. It is clear that the connected sum

$$(g\sharp\bar{h}')\sharp(h\sharp h') : (V\sharp\bar{P})\sharp(T\sharp P) \to (W\sharp S^4)\sharp(S^2 \times S^2\sharp S^4) = W\sharp S^2 \times S^2$$

has the same surgery obstruction as g'. Hence we have written g' as the connected sum of a normal map in $\mathcal{N}^O(W \text{ rel } \partial)$ and a normal map in $\mathcal{N}^O(S^2 \times S^2)$ with trivial surgery obstruction. The result now follows by the commutativity of the diagram:

$$\begin{array}{ccc} \mathcal{N}^O(W \text{ rel } \partial) \times \mathcal{N}^O(S^2 \times S^2) & \xrightarrow{\sharp} & \mathcal{N}^O(W\sharp S^2 \times S^2 \text{ rel } \partial) \\ \downarrow & & \downarrow \\ L_n(\mathbb{Z}[\pi_1 W], w_1) \times L_n(\mathbb{Z}, 0) & \xrightarrow{\oplus} & L_n(\mathbb{Z}[\pi_1 W], w_1) \end{array}$$

□

7.2. Submanifolds and normal maps

Let $D \subset W$ be a submanifold intersecting the boundary of W transversally.

PROPOSITION 7.9. *There is a map $\mathcal{N}^O(W \text{ rel } \partial) \to \mathcal{N}^O(D \text{ rel } \partial)$ defined by sending a normal map $g : V \to W$ which is transversal to D to the restriction $g| : g^{-1}(D) \to D$.*

PROOF. Start by checking that the restriction $g| : g^{-1}(D) \to D$ has degree 1. We show that the degree of the map $g|$ depends only on the homotopy class of g in the following sense. Assume that g_1 is homotopic to g rel ∂W and that g_1 is transversal to D. Then there is a map $G : V \times I \to W \times I$ such that $\operatorname{pr} G_0 = g$ and $\operatorname{pr} G_1 = g_1$ where pr denotes the projection $W \times I \to W$. We may assume that G is transversal to $D \times I \subset W \times I$. We now get a cobordism $G^{-1}(D \times I)$ between $g^{-1}(D)$ and $g_1^{-1}(D)$ and an extension of the restricted maps $g|$ and $g_1|$ to a map $G| : G^{-1}(D \times I) \to D \times I$. This shows that $g|$ and $g_1|$ has the same degree.

Let $q \in D$ be a point in the interior of W. Theorem 6.3 says that we may find a map $g_1 : V \to W$ homotopic to g rel ∂W such that q is a regular value for g_1 and $g_1^{-1}(q)$ consists of a single point. One implication of the fact that q is a regular value is that g_1 is already transversal to D near q, so by a small deformation of g_1 we may assume that g_1 is transversal to D while preserving the property that q is a regular value and $g_1^{-1}(q)$ is a single point. Restricting g_1 to $g_1^{-1}(D)$ we see that the degree must be ± 1 since there exists a point $q \in D$ such that $g_1|^{-1}(q)$ is a single point. And by choosing the right orientation the degree is 1.

We must now show how to define bundle data for the restricted map, then we go on to show that the map $\mathcal{N}^O(W \text{ rel } \partial) \to \mathcal{N}^O(D \text{ rel } \partial)$ is well defined.

Let $g : V \to W$ be a normal map. The following bundle data are given: ν a bundle over W and $F : \tau_V \oplus g^*\nu \to \varepsilon$ a stable trivialization. Define η over D to be the direct sum of $\nu|D$ and the normal bundle $\nu_D(W)$ of D in W. Since g is transversal to D the inverse image $T = g^{-1}(D)$ is a submanifold of V and g induces an isomorphism $\alpha : \nu_T(V) \to g|^*\nu_D(W)$. Define $F_T : \tau_T \oplus g|^*\eta \to \varepsilon$ to be the composition

$$\tau_T \oplus g|^*\eta \cong \tau_T \oplus g|^*\nu_D(W) \oplus g|^*(\nu|D) \overset{id\oplus\alpha^{-1}\oplus id}{\to}$$
$$\tau_T \oplus \nu_T(V) \oplus g|^*(\nu|D) \cong (\tau_V \oplus g^*\nu)|T \overset{F|T}{\to} \varepsilon$$

This constructs bundle data for $g| : T \to D$.

Now assume that $h : P \to W \times I$ is a normal cobordism between two normal maps $g : V \to W$ and $g' : V' \to W$, both being transversal to D. By a small perturbation relative to $V \cup V' \in \partial P$ we may also assume that $h : P \to W \times I$ is transversal to $D \times I$. The construction above shows that $h| : h^{-1}(D \times I) \to D \times I$ is a normal cobordism between $g| : T \to D$ and $g'| : T' \to D$. Hence the map $\mathcal{N}^O(W \text{ rel } \partial) \to \mathcal{N}^O(D \text{ rel } \partial)$ is well defined. □

PROPOSITION 7.10. *Any normal cobordism $h : F \to D \times I$ of $g| : T \to D$ may be extended to a normal cobordism of $g : V \to W$.*

PROOF. Let $\nu_D(W)$ be the normal bundle of D in W. And let $\mathrm{pr}^*\,\nu_D(W)$ be the bundle over $D \times I$ obtained by pulling $\nu_D(W)$ back by the projection $\mathrm{pr} : D \times I \to D$. We want to thicken $h : F \to D \times I$. This is done as follows: Consider the map

$$h^*\,\mathrm{pr}^*\,\nu_D(W) \to \mathrm{pr}^*\,\nu_D(W)$$

over h. Let $\bar{P}$ and $\bar{Q}$ be the disksubbundles of vectors of length ≤ 1 in $h^*\,\mathrm{pr}^*\,\nu_D(W)$ and $\mathrm{pr}^*\,\nu_D(W)$ respectively. The map $\bar{h} : \bar{P} \to \bar{Q}$ is the thickening we are looking for.

Let Q be the union of $W \times I$ and $\bar{Q}$ where we identify points in a tubular neighborhood of D in $W \times 1$ with points over $D \times 0$ in $\bar{Q} \subset \mathrm{pr}^*\,\nu_D(W)$. See figure 7.1.

Similarly let P be the union of $V \times I$ and $\bar{P}$ where we make similar identifications.

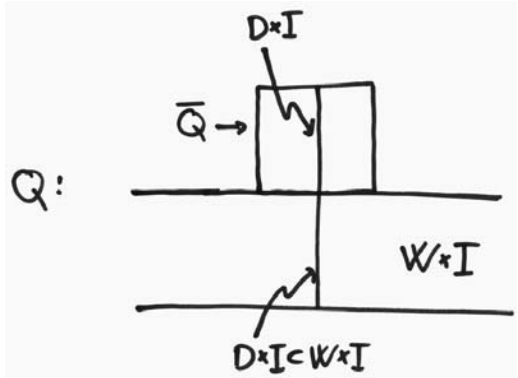

FIGURE 7.1. The manifold Q.

It is clear that there is a map $g' : P \to Q$. Composing with a homeomorphism $Q \to W \times I$ we get an extension of the cobordism.

Next we discuss the bundle data. We have given ν a bundle over W, $F : \tau_V \oplus g^*\nu \to \varepsilon$ a stable trivialization, the bundle η over D which is the direct sum of $\nu|D$ and $\nu_D(W)$, and a stable trivialization $G_F : \tau_F \oplus h^* \operatorname{pr}^* \eta \to \varepsilon$. Q is the union of $W \times I$ and the cross product of tubular neighborhood of D and I. So we have a projection map $Q \to W$ by mapping to the first factor. Let ν_Q be ν pulled back over this map. We want a stable trivialization G of $\tau_P \oplus {g'}^* \nu_Q$. Over $V \times I \to W \times I$ we let G correspond to F pulled back over the projection. To extend over $\bar{P}$ it is enough to extend over the submanifold $F \subset \bar{P}$. Since $\eta \cong \nu_D(W) \oplus \nu|D$ we see that G_F gives a stable trivialization of $\tau_F \oplus h^* \operatorname{pr}^* \nu_D(W) \oplus h^* \operatorname{pr}^* \nu|D$. But $\tau_F \oplus h^* \operatorname{pr}^* \nu_D(W)$ is the restriction of $\tau_{\bar{P}}$ to F. Hence we can extend bundle data. $\square$

We have the diagram

$$\begin{array}{ccc} \mathcal{N}^O(W \text{ rel } \partial) & \longrightarrow & \mathcal{N}^O(D \text{ rel } \partial) \\ \downarrow & & \downarrow \\ L_n(\mathbb{Z}[\pi_1 W], w_1(W)) & & L_k(\mathbb{Z}[\pi_1 D], w_1(D)) \end{array}$$

Now we could ask if there is a homomorphism $L_n(\mathbb{Z}[\pi_1 W], w_1(W)) \to L_k(\mathbb{Z}[\pi_1 D], w_1(D))$ which makes the diagram commute. In general the answer to this question is no as the following examples show.

We will construct two normal maps over $S^2 \times S^2$ both having trivial surgery obstruction such that the restriction to $S^2 \times 0$ has trivial surgery obstruction for the first map and nontrivial surgery obstruction for the second normal map.

The first example is just the identity map $S^2 \times S^2 \to S^2 \times S^2$.

The next example depends on the construction of a normal map $f : T^2 \to S^2$ with nontrivial surgery obstruction. Let T^2 be the torus and $f : T^2 \to S^2$ the degree 1 map which sends $S^1 \vee S^1 \subset T^2$ to a basepoint in S^2 and is injective outside $S^1 \vee S^1$. Let η be the trivial r-dimensional bundle over S^2 and let $F_T : \tau_{T^2} \oplus f^*\eta \to \varepsilon^{2+r}$ be defined as follows: $T^2 = S^1 \times S^1$ therefore $\tau_{T^2} = \tau_{S^1} \times \tau_{S^1}$ and τ_{S^1} has a canonical trivialization $\tau_{S^1} \cong \varepsilon^1$. This induces a trivialization $\tau_{T^2} \cong \varepsilon^2$. And since η is trivial we have a canonical trivialization of $f^*\eta$. Putting these trivialization together we get a trivialization $F'_T : \tau_{T^2} \oplus f^*\eta \cong \varepsilon^{2+r}$. There is a inclusion i of S^1 in $SO(2+r)$. Define $F_T(v) = i(\mathrm{pr}_1(\pi(v)))i(\mathrm{pr}_2(\pi(v)))F'_T(v)$ for $v \in \tau_{T^2} \oplus f^*\eta$. Here $\mathrm{pr}_1 : T^2 \to S^1$ and $\mathrm{pr}_2 : T^2 \to S^1$ is the projection onto the first and second factor of $T^2 = S^1 \times S^1$ respectively and $\pi : \tau_{T^2} \oplus f^*\eta \to T^2$ is the projection map of the bundle. The Kervaire invariant,which is equal to the surgery obstruction, of the normal map $f : T^2 \to S^2$ with η and F_T as bundle data is nontrivial.

Take the cross product of $f : T^2 \to S^2$ with the identity map $S^2 \to S^2$ to obtain a normal map $h : T^2 \times S^2 \to S^2 \times S^2$. It is clear that the restriction of h to $T^2 \times 0 = g^{-1}(S^2 \times 0)$ has nontrivial surgery obstruction. We must now check that the normal map f itself has trivial surgery obstruction. In the simply connected 4-dimensional case the surgery obstruction for a normal map $V \to W$ is $\frac{1}{8}$ times the difference between the signature of V and W. We can compute the signature of $T^2 \times S^2$ and $S^2 \times S^2$ as follows: We see that $H^2(T^2 \times S^2; \mathbb{Q})$ and $H^2(S^2 \times S^2; \mathbb{Q})$ are free of dimension 2. The fundamental class of the first and second factor gives generators of the second homology group. Let b_1 and b_2 be the dual elements in cohomology. They form a basis and the matrix $(\langle b_i \smile b_j, \mu\rangle)$, where μ is the fundamental class, is

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

in both cases. Hence the signature is 0. It follows that the surgery obstruction of the normal map $h : T^2 \times S^2 \to S^2 \times S^2$ is trivial.

7.3. Stabilizing the relations of $\mathcal{S}^O(M)$

In this section M is a closed 3-manifold.

Definition 7.11. Let $\mathcal{S}'(M)$ have the same objects as $\mathcal{S}^O(M)$. Two simple homology equivalences $f_1 : N_1 \to M$ and $f_2 : N_2 \to M$ are considered equivalent in $\mathcal{S}'(M)$ if there exists a normal map $g : V \to W$ with trivial surgery obstruction such that

∂V is the disjoint union of N_1 and N_2
W is the connected sum of $M \times I$ and a finite number of copies of $S^2 \times S^2$, hence ∂W is the disjoint union of two copies of M.
the restriction of g to ∂V is the disjoin union of f_1 and f_2

There are more relations in $\mathcal{S}'(M)$ than in $\mathcal{S}^O(M)$. But there is a natural surjection of sets $\mathcal{S}^O(M) \to \mathcal{S}'(M)$. In this section we will prove that this surjection actually is a bijection. This is done by showing that $\mathcal{S}'(M)$ fits into a exact sequence similar to the exact sequence of surgery.

Analogous to the definition of $\mathcal{S}'(M)$ we can define $\mathcal{N}'(M)$ to be normal maps $N \to M$ where two normal maps $f_1 : N_1 \to M$ and $f_2 : N_2 \to M$ are considered to be equivalent if there exists a normal map $g : V \to W$ such that

∂V is the disjoint union of N_1 and N_2.
W is the connected sum of $M \times I$ and a finite number of copies of $S^2 \times S^2$, hence ∂W is the disjoint union of two copies of M.
the restriction of g to ∂V is the disjoin union of f_1 and f_2.

Again there is a natural surjection $\mathcal{N}(M) \to \mathcal{N}'(M)$.

LEMMA 7.12. *The natural surjection $\mathcal{N}(M) \to \mathcal{N}'(M)$ is a bijection.*

PROOF. Let $g : V \to W$ be a normal map with the properties mentioned above. This is an element of $\mathcal{N}^O(W \text{ rel } \partial)$ where we demand that normal maps equals the disjoint union of $f_1 : N_1 \to M$ and $f_2 : N_2 \to M$ at the boundary. Write W as $M \times I \sharp W'$ where W' is the connected sum of a finite number of copies $S^2 \times S^2$. Since the map

$$\mathcal{N}^O(M \times I \text{ rel } \partial) \times \mathcal{N}^O(W') \to \mathcal{N}^O(W \text{ rel } \partial)$$

is surjective by proposition 7.2 there exists a normal cobordism $g' : V' \to M \times I$ between f_1 and f_2. □

Analogous to $\mathcal{N}^O(M \times I \text{ rel } \partial)$ we consider the following: We may stabilize by taking the connected sum with the identity element of $\mathcal{N}^O(S^2 \times S^2)$. This gives a map

$$\mathcal{N}^O(W \text{ rel } \partial) \to \mathcal{N}^O(W \sharp S^2 \times S^2 \text{ rel } \partial)$$

Define $\mathcal{N}'(M \times I \text{ rel } \partial M \times I)$ to be the direct limit of the sequence

$$\mathcal{N}^O(M \times I \text{ rel } \partial) \to \mathcal{N}^O(M \times I \sharp S^2 \times S^2 \text{ rel } \partial) \to \\ \mathcal{N}^O(M \times I \sharp S^2 \times S^2 \sharp S^2 \times S^2 \text{ rel } \partial) \to \cdots$$

Using the definitions above we have:

PROPOSITION 7.13. *There is an exact sequence*

$$\mathcal{N}'(M \times I \textit{ rel } \partial M \times I) \to L_4(\mathbb{Z}[\pi_1 M], w_1) \\ \to \mathcal{S}'(M) \to \mathcal{N}'(M) \to L_3(\mathbb{Z}[\pi_1 M], w_1)$$

and a natural map from the exact sequence of surgery to the exact sequence above.

PROOF. We clearly have a natural map from the exact sequence of surgery to the stabilized sequence. See the diagram below:

$$\begin{array}{ccc}
\mathcal{N}^O(M \times I \text{ rel } \partial) & \longrightarrow & L_4(\mathbb{Z}[\pi_1 M], w_1) \\
\downarrow & & \downarrow = \\
\mathcal{N}'(M \times I \text{ rel } \partial) & \longrightarrow & L_4(\mathbb{Z}[\pi_1 M], w_1)
\end{array}$$

$$\begin{array}{ccccccc}
\longrightarrow & \mathcal{S}^O(M) & \longrightarrow & \mathcal{N}(M) & \longrightarrow & L_3(\mathbb{Z}[\pi_1 M], w_1) \\
& \downarrow & & \downarrow & & \downarrow = \\
\longrightarrow & \mathcal{S}'(M) & \longrightarrow & \mathcal{N}'(M) & \longrightarrow & L_3(\mathbb{Z}[\pi_1 M], w_1)
\end{array}$$

We use this diagram to check exactness of the bottom sequence.

Exactness at $\mathcal{N}'(M)$: The composition $\mathcal{S}'(M) \to \mathcal{N}'(M) \to L_3(\mathbb{Z}[\pi_1 M], w_1)$ is zero since the natural map $\mathcal{S}^O(M) \to \mathcal{S}'(M)$ is surjective. And the kernel of $\mathcal{N}'(M) \to L_3(\mathbb{Z}[\pi_1 M], w_1)$ is contained in the image of $\mathcal{S}'(M)$ since the kernel consists of normal maps $N \to M$ with trivial surgery obstruction and such maps are normally cobordant to simple homology equivalences. Hence $N \to M$ lies in the image of $\mathcal{S}'(M)$.

Exactness at $\mathcal{S}'(M)$: We let $\theta \in L_4(\mathbb{Z}[\pi_1 M], w_1)$ act on a simple homology equivalence $N \to M$ by constructing a normal map $g : V \to W$ with surgery obstruction θ, where W is the connected sum of $M \times I$ and a finite number of copies of $S^2 \times S^2$ and the restriction of g to the boundary of V is the disjoint union of $N \to M$ and another simple homology equivalence $N' \to M$. This can be done for any θ and $N \to M$. We let $N' \to M$ be the image of $N \times M$ under θ. This action is clearly well defined.

By construction of the action there exists a normal cobordism $V \to W$ with surgery obstruction θ between $N \to M$ and $N' \to M$, where W is the connected sum of $M \times I$ and a finite number of copies of $S^2 \times S^2$. This shows that $N \to M$ and $N' \to M$ represent the same element in $\mathcal{N}'(M)$.

Conversely suppose that $N \to M$ and $N' \to M$ are simple homology equivalences which maps th the same element of $\mathcal{N}'(M)$. Since $\mathcal{N}(M) \to \mathcal{N}'(M)$ is a bijection there exists a normal cobordism $V \to M \times I$ between the two simple homology equivalences. By the exactness of $L_4(\mathbb{Z}[\pi_1 M], w_1) \to \mathcal{S}^O(M) \to \mathcal{N}(M)$ there exists an element θ of $L_4(\mathbb{Z}[\pi_1 M], w_1)$ which sends $N \to M$ to $N' \to M$.

Exactness at $L_4(\mathbb{Z}[\pi_1 M], w_1)$*:* At this point it is important make the correct interpretation of the map $\mathcal{N}^O(M \times I \text{ rel } \partial) \to L_4(\mathbb{Z}[\pi_1 M], w_1)$. See [**KT**].

An element of $\mathcal{N}'(M \times I \text{ rel } \partial)$ is represented by a normal map $V \to W$ restricting to two copies of the same simple homology equivalence $N \to M$ at the boundary of V, where W is the connected sum of $M \times I$ and a finite number of copies of $S^2 \times S^2$. Let θ be the surgery obstruction of $V \to W$. Then it is clear that θ acts trivially on $N \to M$ by the existence of $V \to W$. This shows that the composition $\mathcal{N}'(M \times I \text{ rel } \partial) \to L_4(\mathbb{Z}[\pi_1 M], w_1) \to \mathcal{S}'(M)$ is zero.

Suppose that $\theta \in L_4(\mathbb{Z}[\pi_1 M], w_1)$ acts trivially on the simple homology equivalence $N \to M$ when considered as an element of $\mathcal{S}'(M)$. Then there exists a normal map $V \to W$ with surgery obstruction θ restricting to two copies of $N \to M$ at the boundary, where W is the connected sum of $M \times I$ and a finite number of copies of $S^2 \times S^2$. But this is an element of $\mathcal{N}'(M \times I \text{ rel } \partial)$. □

LEMMA 7.14. *The image of* $\mathcal{N}^O(M \times I \textit{ rel } \partial)$ *in* $L_4(\mathbb{Z}[\pi_1 M], w_1)$ *is identical to the image of* $\mathcal{N}'(M \times I \textit{ rel } \partial)$*.*

PROOF. This is a corollary of proposition 7.8. □

THEOREM 7.15. *The natural surjection* $\mathcal{S}^O(M) \to \mathcal{S}'(M)$ *is a bijection.*

PROOF. This follows from a diagram chase in the map between the two exact sequences above, using the fact that the image of $\mathcal{N}^O(M \times I \text{ rel } \partial)$ in $L_4(\mathbb{Z}[\pi_1 M], w_1)$ is identical to the image of $\mathcal{N}'(M \times I \text{ rel } \partial)$.

□

7.4. Well definedness of the operation

THEOREM 7.16. *Let H be a handle body contained in a closed 3-manifold M and let $g' : V' \to M \times I$ be a normal cobordism with trivial surgery obstruction between $f : N \to M$ and $f' : N' \to M$. Assume that the restriction of f and f' to $f^{-1}(H)$ and ${f'}^{-1}(H)$ respectively are diffeomorphism, then there exists a normal map $g : V \to W$ with trivial surgery obstruction such that the restriction of g to ∂V is the disjoint union of f and f', W is the connected sum of $M \times I$ and finite number of copies of $S^2 \times S^2$ and there exists an embedding of $H \times I$ in W which intersects $\partial W = M \amalg M$ in $H \amalg H$ such that the restriction of g to $g^{-1}(H \times I)$ is a diffeomorphism.*

PROOF. We will modify the normal cobordism $V' \to M \times I$.

Let S be a wedge of copies of S^1, one for each handle of H. There is an embedding of S in H such that S is a deformation retraction of H. We will call S the spine of H. And we think about H as a "tubular neighborhood" of S. S is a CW-complex with a single 0-cell and one 1-cell for each handle of H. Let q denote the 0-cell.

$q \times I$ is a nice 1-dimensional submanifold of $M \times I$. We may assume that g' is transversal to $q \times I$ and the restriction of g' maps ${g'}^{-1}(q \times I)$ diffeomorphically onto $q \times I$. Here we allow ourselves to choose another representative for g' which lies in the same normal cobordism class rel ∂. To see this we first do surgery on $g' : V' \to M \times I$ to make V' connected. Then we may by lemma 6.7 deform g' such that g' is transversal to $q \times I$ and the restriction of g' to ${g'}^{-1}(q \times I)$ is a diffeomorphism.

There exists some open tubular neighborhood U of $q \times I$ such that the restriction of g' to ${g'}^{-1}(U)$ is a diffeomorphism. We remove ${g'}^{-1}(U)$ and U from V' and $M \times I$ respectively to get a normal map $g'_1 : V'_1 \to (M \times I \setminus U)$. We may assume that the intersection of $S \times I$ and $M \times I \setminus U$, which we will denote by D, has one component for each handle of H and that each of these components are 2-disks which meets boundary of $M \times I \setminus U$ transversally. Let D_i^2 be the i'th component of D. Make g'_1 transversal to D. The restriction of g'_1 to ${g'_1}^{-1}(D)$ is the disjoint union of normal maps $f_i : T_i \to D_1^2$, where $T_i = {g'_1}^{-1}(D_i^2)$. Our main problem is that some of these normal maps may have nontrivial surgery obstruction. If this is the case we modify the normal map $g'_1 : V'_1 \to (M \times I \setminus U)$ as follows:

Remember that there exists a normal map $h : T^2 \times S^2 \to S^2 \times S^2$ with trivial surgery obstruction such that h is transversal to $S^2 \times 1$ and the restriction of h to $h^{-1}(S^2 \times 1)$ is a normal map with nontrivial surgery obstruction in $L_2(\mathbb{Z}, 0)$. Remember that this group is cyclic of order 2. By construction there exists a point $q' \in S^2 \times 1$ which is a regular value for h and the inverse image $h^{-1}(q')$ is a single point.

For each $f_i : T_i \to D_i^2$ with nontrivial surgery obstruction we pick a point q_i in D_i^2 which is regular for f_i and such that $f_i^{-1}(q_i)$ is a single point. There exists such a point q_i since f_i is a diffeomorphism near the boundary.

Let $g_0' : V_0' \to W_0$ be the normal map obtained from $g_1' : V_1' \to (M \times I \setminus U)$ by taking the connected sum with $h : T^2 \times S^2 \to S^2 \times S^2$ at the points q_i and q' for each $f_i : T_i \to D_i^2$ with nontrivial surgery obstruction. The effect on the restriction to the D_i^2's are as follows: D_i^2 is replaced by $D_i^2 \sharp S^2$ which also is a 2-disk. We call this disk ${D'}_i^2$. T_i is replaced by the connected sum of T_i and the torus T^2. We write T_i' for this space. And the surgery obstruction for the map $f_i' : T_i' \to {D'}_i^2$ is trivial since it is the connected sum of two nontrivial normal maps.

Since the restriction of g_0' to each component of ${g_0'}^{-1}(D)$ is a normal map into a 2-disk with trivial surgery obstruction there exists a normal cobordism from $g_0' : V_0' \to W_0$ to a normal map $g_0 : V_0 \to W_0$ where the restriction of g_0 to ${g_0}^{-1}(D)$ is a diffeomorphism. This uses proposition 7.10. We may glue the tubular neighborhood U back in to get a normal map $g : V \to W$. Here $V = V_0 \cup {g'}^{-1}(U)$ and $W = W_0 \cup U$. Since g' and h both have trivial surgery obstruction so does g. We see that there is an embedding of $H \times I$ in W such that the inverse image $g^{-1}(H \times I)$ is mapped diffeomorphically onto $H \times I$ by g. □

As announced in chapter 3 we now prove the following:

THEOREM 7.17. *Assume that $f_1 : N_1 \to M$, $f_1' : N_1' \to M$ and $f_2 : N_2 \to M$ are simple homology equivalences such that the restriction of the maps to $f_1^{-1}(H_1)$, ${f_1'}^{-1}(H_1)$ and $f_2^{-1}(H_2)$ are diffeomorphism and that there exists a normal cobordism between f_1 and f_1' with trivial surgery obstruction, then there exists a normal cobordism between $f_1 + f_2$ and $f_1' + f_2$ with trivial surgery obstruction.*

Here M is a closed 3-manifold and the Heegaard splitting $H_1 \cup H_2$ of M is fixed.

PROOF. By theorem 7.15 it is enough to show that $f_1 + f_2$ and $f_1' + f_2$ are equivalent in $\mathcal{S}'(M)$. We use coefficients in $\mathbb{Z}[\pi_1 M]$ throughout this proof.

By theorem 7.16 there exists a normal map $g : V \to W$ restricting to the disjoint union of f_1 and ${f'}_1$ at the boundary where W is the connected sum of $M \times I$ and a finite number of copies of $S^2 \times S^2$ and there exists an embedding of $H_1 \times I$ in W which agrees with the inclusion $H_1 \subset M$ near ∂W such that the inverse image of $H_1 \times I$ under g is mapped diffeomorphically to $H_1 \times I \subset W$ by g and that g has trivial surgery obstruction.

Let V_0 and W_0 be the closure of the complement of $H_1 \times I$ in V and W respectively. The restriction of g is a normal map $g_0 : V_0 \to W_0$. By doing surgery we may suppose that g_0 induces an isomorphism on fundamental groups. See lemma 2.7. We have a Mayer-Vietoris sequence of K groups:

$$\cdots \to K_2(V_0 \cap H_1 \times I) \to K_2(V_0) \oplus K_2(H_1 \times I) \to \\ K_2(V) \to K_1(V_0 \cap H_1 \times I) \to \cdots$$

But $K_*(V_0 \cap H_1 \times I)$ and $K_*(H_1 \times I)$ are trivial so the inclusion induces an isomorphism $K_2(V_0) \to K_2(V)$. And since $g_0 : V_0 \to W_0$ induces an isomorphism of fundamental groups we have that every element of $K_2(V_0)$ may be represented by a map $S^2 \to V_0$. Hence the surgery obstruction of g_0 maps to 0 under the map $L_4(\mathbb{Z}[\pi_1 W_0], w_1) \to L_4(\mathbb{Z}[\pi_1 W], w_1)$ induced by the inclusion.

The restriction of f_2 to $f_2^{-1}(H_1)$ is a simple homology equivalence $f_2^{-1}(H_1) \to H_1$ which restricts to a diffeomorphism at the boundary. By lemma 2.7 there exists a normal cobordism $g_2 : V_2 \to H_1 \times I$ which has trivial surgery obstruction and induces a isomorphism $\pi_1 V_2 \to \pi_1 H_1 \times I$. The boundary of V_2 consists of two copies of $f_2^{-1}(H_1)$ glued together along a copy of $\partial f_2^{-1}(H_1) \times I$ and the restriction of g_2 to ∂V_2 is equal to f_2 on the copies of $f_2^{-1}(H_2)$ and maps $\partial f_2^{-1}(H_1) \times I$ diffeomorphically onto $\partial H_1 \times I$.

Form the union $\tilde{V} = V_0 \cup_{\partial H_1 \times I} V_2$ by gluing along $\partial H_1 \times I$. This maps into W by a map $\tilde{g} : \tilde{V} \to W$. We must construct bundle data for the map $\tilde{g}$. Bundle data over $g_0 : V_0 \to W_0$ consist of a bundle $\nu|$ over W_0 and a stable trivialization F_{V_0} of $\tau_{V_0} \oplus g_0^* \nu|$. Over $g_2 : V_2 \to H_1 \times I$ we have $\nu_{H_1 \times I}$ over $H_1 \times I$ and a stable trivialization F_{V_2} of $\tau_{V_2} \oplus g_2^* \nu_{H_1 \times I}$. The tangent bundle of $\tilde{V}$ can be obtained form τ_{V_0} and τ_{V_2} by gluing along a clutching map $\alpha : \tau_{V_0} | \partial H_1 \times I \to \tau_{V_2} | \partial H_1 \times I$. We seek a clutching map $\beta : \nu | \partial H_1 \times I \to \nu_{H_1 \times I} | \partial H_1 \times I$ which we can use to construct a bundle $\tilde{\nu}$ over W from $\nu|$ and $\nu_{H_1 \times I}$. We also want to find a stable trivialization $F_{\tilde{V}}$ of $\tau_{\tilde{V}} \oplus \tilde{g}^* \tilde{\nu}$ which up to homotopy restricts to F_{V_0} and F_{V_2} over V_0 and V_1 respectively.

To construct this we restrict the stable trivializations to $\partial H_1 \times I$, then we get maps

$$\tau_{V_0}|\partial H_1 \times I \oplus g_0^*\nu|\partial H_1 \times I \to \varepsilon^k$$

and

$$\tau_{V_2}|\partial H_1 \times I \oplus g_2^*\nu_{H_1\times I}|\partial H_1 \times I \to \varepsilon^k$$

This shows that $g_0^*\nu|\partial H_1 \times I$ and $g_2^*\nu_{H_1\times I}|\partial H_1 \times I$ are stably isomorphic. Therefore it is possible to choose some clutching map $\beta' : g_0^*\nu|\partial H_1 \times I \to g_2^*\nu|\partial H_1 \times I$ making the following diagram commute up to homotopy.

$$\begin{array}{ccc}
\tau_{V_0}|\partial H_1 \times I \oplus g_0^*\nu|\partial H_1 \times I & \xrightarrow{F_{V_0}|} & \varepsilon^k \\
\downarrow{\scriptstyle \alpha\oplus\beta'} & & \downarrow{=} \\
\tau_{V_2}|\partial H_1 \times I \oplus g_2^*\nu_{H_1\times I}|\partial H_1 \times I & \xrightarrow{F_{V_2}|} & \varepsilon^k
\end{array}$$

But since g_0 and g_2 restricted to $\partial H_1 \times I$ are diffeomorphism onto their image we have a clutching map $\beta : \nu|\partial H_1 \times I \to \nu_{H_1\times I}|\partial H_1 \times I$ corresponding to β'. This is the map we seek and we may construct $\tilde{\nu}$ and the stable trivialization $F_{\tilde{V}}$ of $\tau_{\tilde{V}} \oplus \tilde{g}^*\tilde{\nu}$.

With these bundle data $\tilde{g}$ has trivial surgery obstruction. This shows that $(f_1 + f_2) : (N_1 + N_2) \to M$ and $({f'}_1 + f_2) : ({N'}_1 + N_2) \to M$ are equivalent in $\mathcal{S}'(M)$. $\square$

APPENDIX A

Stable uniqueness of Heegaard splittings

We are interested in classifying Heegaard splittings of a given closed, connected and smooth 3-manifold M. The aim is to prove the following theorem:

THEOREM A.1. *Any two Heegaard splittings $H_1 \cup H_2$ and $H_1' \cup H_2'$ of M have stabilizations $K_1 \cup K_2$ and $K_1' \cup K_2'$ which are ambient isotopic.*

A Heegaard splitting $K_1 \cup K_2$ is a stabilization of $H_1 \cup H_2$ if it is obtained from the latter by adding a finite number of trivial 1-handles. And $K_1 \cup K_2$ and $K_1' \cup K_2'$ are ambient isotopic if there exists an isotopy $F : I \times M \to M$ such that $F_0 = id$ and F_1 carries K_1 and K_2 to K_1' and K_2' respectively.

Here is an outline of the proof: First choose Morse functions f_0 and f_1 of M that corresponds to the two different Heegaard splittings. There are a homotopy $f : I \times M \to \mathbb{R}$ between f_0 and f_1 since the target is contractible. Using techniques from Hatcher and Wagoner's article [**HW73**] we deform this homotopy until the theorem can be read from f's graphic.

A Morse function $f : M \to \mathbb{R}$ is called ordered if $p > q$ implies $f(p) > f(q)$ whenever p and q are critical points.

LEMMA A.2. *For every Heegaard splitting $M = H_1 \cup H_2$ there exists an ordered Morse function $f : M \to \mathbb{R}$ such that*

all critical points have different values,
$H_1 = f^{-1}(-\infty, 0]$ *and* $H_2 = f^{-1}[0, \infty)$,
if $p \leq 1$ then $f(p) < 0$ and if $p \geq 2$ then $f(p) > 0$ for all critical points p.

We say that $f : M \to \mathbb{R}$ corresponds to the Heegaard splitting $H_1 \cup H_2$.

PROOF. The boundary $\partial H_1 = \partial H_2 = F$ has a tubular neighborhood $F \times (-1, 1) \subset M$ mapping $F \times (-1, 0]$ into H_1 and $F \times [0, 1)$ into H_2.

Any handlebody H possesses a Morse function $f : H \to (-\infty, 0]$ with critical points of index 0 and 1 only. We may assume that all critical points lie in the interior of H and that f maps the boundary ∂H to 0. Since there are no critical points near ∂H there exists a collar $\partial H \times (-\varepsilon, 0] \subset H$ such that f corresponds to the projection onto $(-\varepsilon, 0]$ in the collar. Here $0 < \varepsilon < 1$ Let f_1 , f_2 be such Morse functions for H_1 and H_2 respectively.

By uniqueness of collars there are isotopies taking the collars above to the collars determined by the tubular neighborhood of F. Therefore we may suppose that f_1 and $-f_2$ coincides with the projection onto $(-\varepsilon, 0]$ and $[0, \varepsilon)$ respectively in the collars determined by the tubular neighborhood of F.

Now define $f : M \to \mathbb{R}$ to be

$$f(x) = \begin{cases} f_1(x) & \text{for } x \in H_1 \\ -f_2(x) & \text{for } x \in H_2 \end{cases}$$

This gives the required Morse function. □

Now let f_0 and f_1 be Morse functions on M corresponding to two different Heegaard splittings $H_1 \cup H_2$ and $H_1' \cup H_2'$ respectively. There exists a homotopy $f : M \times I \to \mathbb{R}$ between f_0 and f_1. We may assume that $f_t = f_0$ for $t \in [0, \varepsilon)$ and $f_t = f_1$ for $t \in (1-\varepsilon, 1]$ for some small $\varepsilon > 0$. We will now start to deform the homotopy f. When doing this we may always assume that the deformations are fixed in a neighborhood of $M \times \{0, 1\}$.

The homotopy f may be thought of as a one-parameter family of functions $M \to \mathbb{R}$. It also gives rise to a map $F : I \times M \to I \times \mathbb{R}$ given by $F(t, x) = (t, f_t(x))$. Define Σ to be the set

$$\Sigma = \{(t, p) \in I \times M |\ p \text{ is a critical point for } f_t : M \to \mathbb{R}$$

This will be called the set of singularities. The graphic of f is $F(\Sigma) \subset I \times \mathbb{R}$. When drawing the graphic we will sometimes mark different parts of $F(\Sigma)$ with a number indicating the index of the critical point it corresponds to.

Following Chapter I.§2 of [**HW73**] we may assume after deforming the homotopy f that

- Σ is a 1-dimensional submanifold of $I \times M$
- Σ is the disjoint union of two strata Σ_0 and Σ_1
- Σ_0 is 1-dimensional and consists of the non-degenerate critical points of f_t
- Σ_1 is a finite set of points
- for every point (t_0, p) in Σ_1 there exists a one-parameter family of embeddings $\phi_t : \mathbb{R}^3 \to M$ defined for t in a neighborhood of

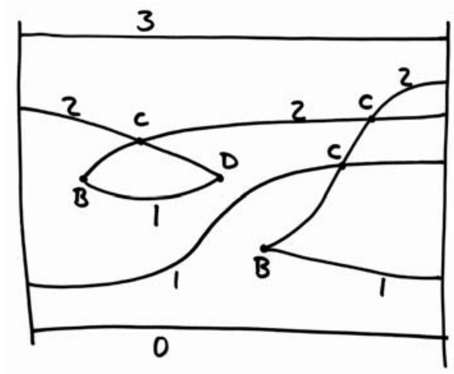

FIGURE A.1. An example of a graphic.

t_0 with $\phi_{t_0}(0) = p$ such that

$$f\phi_t(x_1, x_2, x_3) - f_{t_0}(p) = \pm x_1^2 \pm x_2^2 \pm (t - t_0)x_3 + x_3^3$$

- all points of Σ_1 has different parameter values
- F restricted to each of the strata Σ_0 and Σ_1 is an immersion
- if Σ_i^α and Σ_j^β are opens subsets of Σ_i and Σ_j then $F : \Sigma_i^\alpha \to I \times \mathbb{R}$ and $F : \Sigma_j^\beta \to I \times \mathbb{R}$ are in general position
- $F : \Sigma \to I \times \mathbb{R}$ has no triple points

From this list of properties we can say much about what the graphic of f looks like locally. A simple example of such a graphic is shown in figure A.1. Mostly it consists of lines corresponding to non-degenerate critical points of a some index. There are also point (marked C) where we have transverse intersections. These are called crossing points. At last there are the points (marked B and D) coming from the stratum Σ_1. They are called birth(B) and death(D) points.

Now choose $r_0 < r_1 < r_2 = 0 < r_3$. We may assume that if p is a critical point of f_i of index j then $r_{j-1} < f_i(p) < r_j$ for $i = 0, 1$. According to proposition 8.1 in chapter I of [**HW73**], or at least the proof, we may deform f such that

- the properties above still holds
- if p is a non-degenerate critical point for f_t of index j then $r_{j-1} < f_t(p) < r_j$
- if p is a birth or death point of index $j/(j-1)$ for f_t then $f_t(p) = r_j$

Now each f_t is an ordered function and if t is not the parameter for any birth or death points then f_t is a Morse function and $f_t^{-1}(-\infty, 0] \cup f_t^{-1}[0, \infty)$ is a Heegaard splitting of M.

We want to compare the Heegaard splittings corresponding to the parameter values t_0 and t_1 whenever f_t has no 2/1 birth or death points for $t \in [t_0, t_1]$. By the list of properties we now assume f satisfies this is the same as saying that 0 is a regular value for all f_t when t lies in the interval.

LEMMA A.3. *If f_t has no 2/1 birth or death points for t in the interval $[t_0, t_1]$ then the Heegaard splittings $f_{t_0}^{-1}(-\infty, 0] \cup f_{t_0}^{-1}[0, \infty)$ and $f_{t_1}^{-1}(-\infty, 0] \cup f_{t_1}^{-1}[0, \infty)$ are ambient isotopic.*

PROOF. Restrict f to $[t_0, t_1] \times M$ and look at $f^{-1}(0)$. Let p be the restriction of the projection $[t_0, t_1] \times M \to [t_0, t_1]$ to $f^{-1}(0)$. $f^{-1}(0)$ is a submanifold and p has no critical points since 0 is a regular value for f_t when $t \in [t_0, t_1]$. Theorem 3.4 in [**Mil65**] tells us that $f^{-1}(0)$ is diffeomorphic to $[t_0, t_1] \times f_{t_0}^{-1}(0)$ and that the following diagram commutes:

$$\begin{array}{ccc} [t_0, t_1] \times f_{t_0}^{-1}(0) & \xrightarrow{\cong} & f^{-1}(0) \\ & \searrow^{\text{pr}} & \downarrow p \\ & & [t_0, t_1] \end{array}$$

We thus have an isotopy taking $f_{t_0}^{-1}(0)$ to $f_{t_1}^{-1}(0)$. This can be extended to an ambient isotopy. See theorem 8.1.3 in [**Hir76**]. □

If the parameter passes a birth or death point this corresponds to stabilization of the Heegaard splitting.

LEMMA A.4. *If t_0 is the parameter value for a 2/1 birth point then for some $\varepsilon > 0$ the Heegaard splitting $f_{t_0+\varepsilon}^{-1}(-\infty, 0] \cup f_{t_0+\varepsilon}^{-1}[0, \infty)$ is a stabilization of $f_{t_0-\varepsilon}^{-1}(-\infty, 0] \cup f_{t_0-\varepsilon}^{-1}[0, \infty)$.*

PROOF. Let $(t_0, p) \in \Sigma_1$ be the 2/1 birth point. Outside a suitable small neighborhood of (t_0, p) the space $f^{-1}(0)$ is the trace of a isotopy from $f_{t_0-\varepsilon}^{-1}(0)$ to $f_{t_0+\varepsilon}^{-1}(0)$, but interesting changes occur inside this neighborhood. We choose coordinates near (t_0, p) such that

$$f\phi_t(x_1, x_2, x_3) = -x_1^2 + x_2^2 - (t - t_0)x_3 + x_3^3$$

We sketch the level surface $f\phi_t(x_1, x_2, x_3) = 0$ for $t < t_0$and $t > t_0$ and see that a handle is attached. See figures A.2 and A.3 □

To complete the proof of the theorem it only remains to show that we may deform the homotopy f such that all 2/1-birth points have lower parameter values than all 2/1-death points.

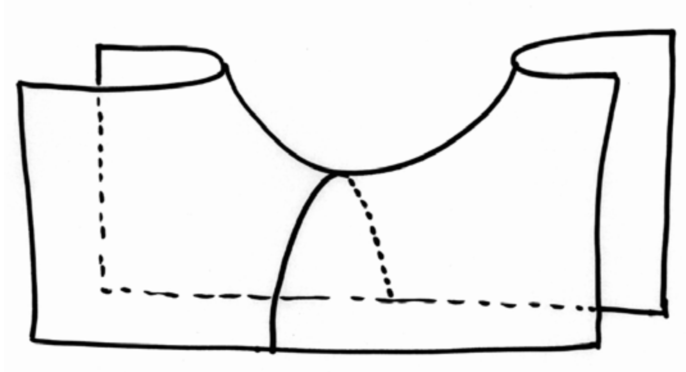

FIGURE A.2. The level surface for $t < t_0$.

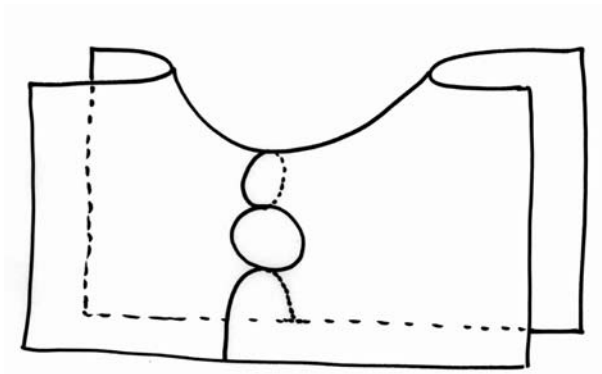

FIGURE A.3. The level surface for $t > t_0$.

LEMMA A.5. *The homotopy f can be deformed such that if (t_b, p_b) is a 2/1-birth point and (t_d, p_d) is a 2/1-death point then $t_b < t_d$.*

PROOF. Assume that there exists a 2/1-birth point (t_b, p_b) and a 2/1-death point (t_d, p_d) with $t_d < t_b$. We may assume that there are no other 2/1-birth or -death points between the two chosen. If there are we just pick another pair. Now by Chapter V.§2 in [**HW73**] it is possible because M is connected to eliminate these two points without introducing other birth or death points. See figure A.4. This process may be repeated until f satisfies the conclusion. □

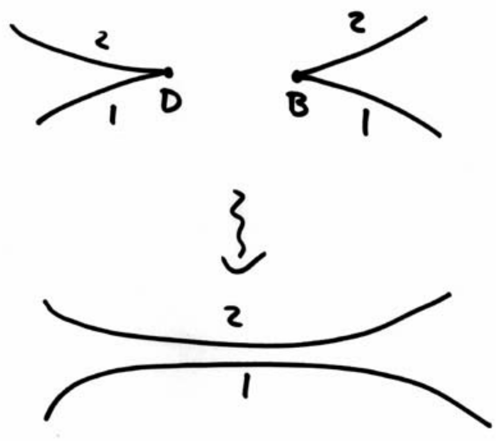

FIGURE A.4. Elimination of a birth point and a death point.

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