AN EXTENSION OF RESULTS OF BROWDER, NOVIKOV AND WALL ABOUT SURGERY ON COMPACT MANIFOLDS

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§ 0 Introduction

The s-cobordism Theorem provides a strong tool for the classification of manifolds. The basic idea of its application to the classification of manifolds is the following: Given two closed manifolds M and N one first has to decide whether they are bordant by a manifold W with $\partial W = M + N$. Then, in a second step one tries to transform W into an s-cobordism W'. If this is possible then, if $\dim M > 4$, M and N are diffeomorphic.

More precisely one asks whether in the bordism class of W rel. boundary there is contained an s-cobordism. As by the Morse Lemma every bordism is the union of a sequence of addition of handles [26] which means that W is transformed into W' by a sequence of surgeries it is natural to study surgical modifications: Embed $S^r \times D^{m-r}$ into W ($\dim W = m \geq 5$), cut it out and replace it by $D^{r+1} \times S^{m-r-1}$. This is very natural as for $r < (m-1)/_2$ it has the same effect for the homotopy groups of W of dim $\leq r$ as attaching a $(r+1)$- cell: the element represented by $S^r \times \{0\}$ is killed. On the other hand W is an s-cobordism if and only if the homotopy groups $\pi_*(W,M)$ and $\pi_*(W,N)$ vanish (by Poincaré duality it is enough to repuire this for $* \leq m/_2$) and the Whitehead torsion vanishes.

Thus one tries to kill $\pi_*(W,M)$ and $\pi_*(W,N)$ for $* \leq m/_2$ by a sequence of surgeries. At least for $* < (m-1)/_2$ the facts indicated above imply that one can do it if certain informations are at hand: Given $\alpha \in \pi_r(W,M)$ it must be possible to represent α by an embedding $S^r \times D^{m-r} \hookrightarrow W$ or equivalently that α is in the image of $\pi_r(W) \longrightarrow \pi_r(W,M)$ and that there exists a representative $\beta \in \pi_r(w)$ which has stable trivial normal bundle.

The fundamental idea of Browder [4] and Novikov [29] is to prepare these informations by the following data: Let $f : W \longrightarrow B$ be a map whose restriction to M and N is a simple homotopy equivalence and which is covered by a bundle map $\bar{f} : \nu(W) \longrightarrow \xi$ where ξ is some stable vector bundle over B.We call $(f,\bar{f})$ a normal map and the restrictions to M and N simple normal smoothings. Then we obtain a splitting $W \xrightarrow{f} B \xrightarrow{(f|M)^{-1}} M$ of the inclusion $M \longrightarrow W$ leading to an isomorphism between Ker $f_* : \pi_r(W) \longrightarrow \pi_r(B)$ and $\pi_r(W, M)$ and similar to $\pi_r(W,N)$. On the other hand elements in Ker f_* have trivial normal bundles. Thus one can kill Ker f_* for $r < (m-1)/_2$ by a sequence of surgeries leaving the problem of killing the kernel in dim $[m/_2]$ ([5], IV§1). In general this kernel can't be killed but there are obstructions which in the 1-connected case considered by Browder and Novikov are rather simple, given by the signature and Arf-invariant and in the non-simply connected case studied by Wall [41] more complicated invariants in his L-groups.

This approach seems to be perfect for the classification of manifolds especially in the reformulation of Sullivan in terms of the Spivak normal bundle [38]. But even if there are a lot of famous results obtained with this method the applications to the problem of deciding whether given manifolds M and N are diffeomorphic are rather limited. One reason is that the approach of Browder and Novikov requires some data which are not easy to obtain: First one has to decide whether M and N are homotopy equi valent, then one has to choose a homotopy equivalence f and a bundle map $\bar{f}$ into a Poincaré complex B (=M) and to extend it to a bordism W. Finally one has to compute the surgery obstruction of W in L_m. And if this is non-trivial one has to repeat this process by changing all data $(f,\bar{f})$ and W. O ne can only conclude that M and N are not diffeomorphic if the surgery obstruction is non-trivial for all choices.

If we look again at the informations collected in the approach of Browder and Novikov to guarant that one can kill $\pi_r(W,M)$ and $\pi_r(W,N)$ for $r < (m-1)/_2$ we see that it is not necessary to require for the map $f : W \longrightarrow B$ that the restrictions to M and N are homotopy-equivalences. A $[(m-1)/_2]$-equivalence is enough as this also implies Ker $\pi_r(W) \longrightarrow \pi_r(B)$ isomorphic to $\pi_r(W,M)$ for $r < (m-1)/_2$.

$\pi_r(f) = 0$ for $r \leq k+1$

A map $(f,\bar{f})\colon (M, \nu(M)) \longrightarrow (B, \xi)$ is called a <u>normal k-smoothing in (B, ξ)</u> if f is a $(k+1)$-equivalence. If $(f,\bar{f})\colon (W, \nu(W)) \longrightarrow (B, \xi)$ is a normal map such that the restriction to $\partial W = M + N$ are normal k-smoothings of M and N and if $k+1 \geq [(m-1)/_2]$ the remarks above imply that we can kill $\pi_r(W,M)$ and $\pi_r(W,N)$ for $r < (m-1)/_2$. Our aim is to find an obstruction for killing these homotopy groups for $r = [m/_2]$. As for $k < m$ the information of a normal k-smoothing is weaker than a normal smoothing one expects more difficult obstructions. But this is not the case if $k \geq (m-1)/_2$ with one slight modification. As we don't have the notion of a simple normal k-smoothing we have to put some extra Whitehead torsion information into the obstruction sitting in a group similar to a Wall group denoted as $L_m^{s,\tau}(\pi_1(B), w_1(\xi))$. This is related to the ordinary Wall group $L_m^s(\pi_1(B), w_1(\xi))$ by an exact sequence (see § 4):

$$0 \longrightarrow L_m^s(\pi,w) \longrightarrow L_m^{s,\tau}(\pi,w) \longrightarrow Wh(\pi).$$

If $m = 2n$ is even and $k = n$ the obstruction sits even in a quotient group of $L_m^{s,\tau}(\pi,w)$. Let $S = S(M)$ be the subgroup of $\Lambda = \mathbb{Z}[\pi_1(M)]$ representing the elements $\mu(S^n)$, where S^n is an immersed sphere in $M \times I$ representing an element of Ker $\pi_n(M) \longrightarrow \pi_n(B)$ and μ the self-intersection number in $Q_{(-1)^n}$ (as defined in [41], § 5). Then we define $L_m^{s,\tau}(\pi,w,S)$ as the group of stable isomorphism classes of $(-1)^n$-quadratic forms (λ,μ) where μ has values in $\Lambda/_S$. In the case k = n the obstruction sits in $L_m^{s,\tau}(\pi_1(B), w_1(\xi), S(M))$. A similar group was introduced by Bak [1] for purely algebraic reasons. Our results show that such groups are also useful for the geometry.

If $k = [(m-1)/_2] -1$ the obstruction behaves as expected and is formulated in terms of a very complicated abelian monoid denoted as $l_m^{s,\tau}(\pi,w)$ or $l_m^{s,\tau}(\pi,w,S)$ if m is even and $S \subset \Lambda$ similar as above.

It is surprising that for m = 2n+1 odd this obstruction disappears if we allow stabilization of M and N by connected sum with $(S^n \times S^n)$'s. If $M \# r(S^n \times S^n)$ is diffeomorphic to $N \# r'(S^n \times S^n)$ we call M and N <u>stably diffeomorphic.</u> So, the result is in this case that M and N are stably diffeomorphic if and only if they are (n-1)-normally bordant (Theorem 2.1) and the question of a stable classification is reduced to homotopy theory, especially stable homotopy theory. In the moment, this is the case with the most applications.

As it is comparatively easy to decide whether M and N are stably diffeomorphic one would like to solve the cancellation problem for connected sum with $(S^n \times S^n)$'s. If $f: M \# r(S^n \times S^n) \longrightarrow N \# r(S^n \times S^n)$ is a diffeomorphism we denote the composition

$$\Lambda^{2r} = \pi_n(r(\widetilde{S^n \times S^n}) \longrightarrow \pi_n(M\#r(S^n \times S^n)) \longrightarrow \pi_n(N\#r(S^n \times S^n)) \rightarrow \pi_n(r(\widetilde{S^n \times S^n})) = \Lambda^{2r}$$

by $\vartheta(f)$. If $\vartheta(f)$ is an isometry with respect to the hamiltonian form (λ, μ) on Λ^{2r}, $\vartheta(f)$ represents an element in $L^{s,\tau}_{2n+1}(\pi_1(M), w_1(M))$. If this vanishes then M and N are diffeomorphic (s-cobordant if n=2) (Theorem 3.1, see the reformulation at the end of § 4). It is perhaps interesting to note that the obstruction $\vartheta(f)$ occurs naturally as an isometry whereas the obstructions for W in $L^{s,\tau}_{2n+1}(\pi_1(B), w_1(\xi))$ considered above is better understood as a formation. In general the cancellation problem of $S^n \times S^n$ seems to be very difficult even if $L^{s,\tau}_{2n+1}(\pi_1(M), w_1(M))$ vanishes (compare [21]).

It is natural to ask for a given (B, ξ) whether there exists a normal k-smoothing in (B, ξ). If B is a n-dimensional CW-complex ($n \geq 5$) and $k = n$ this is the question of the existence of a closed n-dimensional manifold homotopy equivalent to B. In this case B has to be a finite Poincaré-complex. If we only look for a normal k-smoothing we only require <u>k-partial Poincaré duality</u>: There exists $\alpha \in H_n(B; \mathbb{Z}^t)$ called a <u>k-partial fundamental class</u> such that $\cap\alpha : H^{n-r}(B; K) \longrightarrow H_r(B; K)$ is an isomorphism for $r < k+1$ and $n-r < k+1$, surjective for $r=k+1$ and injective for $r=n-k-1$ (this definition shouldn't be mixed with partial Poincaré complexes in the sense of [13] where duality between the low and high dimensions is required). Furthermore we require that B has finite $([n/2]+1)$-skeleton.

Then if $(M,f,\bar{f})$ is a map into (B, ξ) such that $f_*[M] = \alpha$ is a k-partial fundamental class (this replaces the degree 1 condition and $(M,f,\bar{f})$ is then called a <u>k-admissible map</u>) there is an obstruction in an abelian group $L^{h,p}_n(\pi_1(B), w_1(\xi))$ for $k > [n/2]$ and in a monoid $l^{h,p}_n(\pi_1(B), w_1(\xi))$ for $k = [n/2]$ which vanishes if and only if $(M,f,\bar{f})$ is bordant to a normal k-smoothing. If $k = [n/2] - 1$ there is no obstruction. If B is a finite

Poincaré complex and k=n we have the original problem of Browder, Novikov and Wall and obtain an invariant in $L_n^h(\pi_1(B), w_1(\xi))$. In general, for $k < n$, as we only assume that B has finite $([n/2]+1)$-skeleton some finiteness obstruction is contained in the invariant in $L_n^{h,p}$ or $l_n^{h,p}$.

If one applies the approach described above to the question whether M and N are diffeomorphic the problem occurs to find a (B, ξ) to compare M and N in. This should be universal in the sense that if M and N have a handle decomposition with same k-skeleton then both manifolds should admit a normal k-smoothing in (B, ξ). The right answer to this problem is to consider the k-th step of a Postnikov decomposition of the normal Gauß map ν_M. This is a commutative diagram

$$\begin{tikzcd} & B \arrow[d, "p"] \\ M \arrow[ur, "\bar{\nu}_M"] \arrow[r, "\nu_M"'] & BO \end{tikzcd}$$

where $p: B \longrightarrow BO$ is a (k+1)-co-connected fibration (homotopy groups of the fibre vanish in dimension $\geq k+1$) and $\bar{\nu}_M$ is a (k+1)-equivalence. This fibration $B \longrightarrow BO$ is uniquely determined by M and k and denoted by $B_k(M)$, <u>the normal k-type of M.</u> If we restrict B to $BO(N)$, $N \gg 2n$, we can take for ξ the pullback of the universal vector bundle over $BO(N)$. But instead of working with the bundle ξ and pairs $(f, \bar{f})$ one can equivalently work with homotopy lifts $\bar{\nu}_M$ of ν_M (see the discussion in §1). We will work with this notation which seems to be very natural in our context. If one prefers one can everywhere replace $\bar{\nu}_M$ by a pair $(f, \bar{f})$ or a pair (f, α), α a framing of $\nu_M - f^*\xi$. In the language of lifts a <u>normal k-smoothing</u> in B, $B \longrightarrow BO$ a fibration, is a homotopy lift $\bar{\nu}_M$ of the normal Gauß map into B which is a (k+1)-equivalence. We denote the set of normal k-smoothings in B by $NS_{n,k}^B$. With this notation the classification approach described above can be formulated more systematically (see Theorem 7.3). We also have relative versions for manifolds with boundary.

Given a fibration $B \longrightarrow BO$, we denote the set of diffeomorphism classes (s-cobordism classes, if the dimension is 4) of n-dimensional manifolds M admitting a normal k-smoothing in B by $\mathfrak{M}^B_{n,k}$. The difference between $NS^B_{n,k}$ and $\mathfrak{M}^B_{n,k}$ is measured by the group of fibre homotopy self equivalences of B, $\mathrm{Aut}(B,p)$. This acts on $NS^B_{n,k}$ by composition and the orbit space is isomorphic to $\mathfrak{M}^B_{n,k}$ (Proposition 7.4), if $B \to BO$ is $(k+1)$-co-connected.

With these notations the principle of classifying manifolds with our approach can be formulated as follows. Given M and N decide whether they have the same normal k-type B (for some $k \geq [n/2] - 1$), choose normal k-smoothings $\bar{\nu}_M$ and $\bar{\nu}_N$ in B and decide whether $(M, \bar{\nu}_M)$ and $(N, \bar{\nu}_N)$ are contained in the same orbit under the action of $\mathrm{Aut}(B,p)$.

This is perhaps a good place to explain the basic philosophy of the approach. In a way we classify manifolds with prescribed k-skeleton (given by a normal k-type B). If $k \geq [n/2] - 1$ (what we always assume) duality phenomena should control a lot of informations about the manifolds which have this k-skeleton. For instance Poincaré duality is the explanation why the obstruction for replacing a bordism W between M and N by a s-cobordism is contained in the same group if only $k \geq [n/2]$.

Our results can be considered as a generalization of ([41] ,§ 5 and § 6). All basic ideas follow from Wall as well as from Browder [5] and Novikov [29] and, of course, from Kervaire and Milnor [15]. The experts will notice some differences, for instance in the proof that the surgery obstructions are bordism invariants which in the even dimensional case follows easily from our Theorem 2.1. I have tried to give a rather self-contained presentation meaning that it should be readable if one knows the basic ideas of surgery below the middle dimension as described very well in Browder's book ([5] , § IV, 1). Another information one has to take from the literature is the definition of the intersection - and self intersection forms contained in ([41] , § 5 or [34] , § 5). I have collected all necessary informations about the obstruction groups (monoids) in § 4 but for some proofs I have referred to [33] . The basic definitions and properties of Whitehead groups should be taken from [28] .

If one generalizes a general theory this needs a justification: Some problem must be solved with it or it must give more insight into the topic. I hope that a little bit both is true. Some applications of this approach had been obtained before I had formulated it in this generality. For instance my computation of the bordism group of diffeomorphisms can be considered as an application of this approach. I have collected this and some other applications in § 8.

The paper is organized as follows. After introducing the basic notations and definitions in § 1, we prove the stable classification Theorem 2.1 (up to connected sum with $S^n \times S^n$'s) in § 2. This is the shortest way to a result which allows already some interesting applications. On the other hand the proof is a good exercise for the methods used to produce s-cobordisms including the control of Whitehead torsion. § 3 deals with the question of cancelling $(S^n \times S^n)$'s and brings in the odd Wall obstruction groups in a very natural way. §§ 5 und 6 contain the main technical results separately for even- and odd-dimensional manifolds. Finally in § 7 the results are collected in form of exact sequences and the relation between normal k-smoothings and manifolds of type B is pro ved.

I began to think about this approach during a stay in Aarhus, summer 1981. I finished it more or less when I visited the Max-Planck-Institut / SFB in Bonn from September 1981 to September 1982. I would like to thank both places for the invitation as well as the mathematicians there for several fruitful discussions, especially with Ib Madsen. In the winter term 1982/83 and in the summer term 1983 I gave a course about it in Mainz. I would like to thank my students and assistents, especially Stephan Stolz, for their interest and stimulating contributions.

§ 1 Normal k-smoothings

We will formulate our results in the smooth category. Everything works with appropriate modifications over PL or TOP (replace the differentiable normal bundle by the corresponding PL- or TOP-bundles).

Let $p: B \longrightarrow BO$ be a fibration s.t. B is connected and has the homotopy of a CW-complex. Consider a n-dimensional manifold $(M, \partial M)$ embedded into $(\mathbb{R}_+ \times \mathbb{R}^N, \{0\} \times \mathbb{R}^N)$, $N \gg 2n$ where ∂M meets $\{0\} \times \mathbb{R}^N$ transversally. Denote the corresponding normal Gauß map by ν_M. We are going to study homotopy lifts $\bar{\nu}_M$ of ν_M:

$$\begin{tikzcd} & B \arrow[d, "p"] \\ M \arrow[ur, "\bar{\nu}_M"] \arrow[r, "\nu_M"'] & BO \end{tikzcd}$$

If $(M, \partial M)$ is in a different way embedded into $(\mathbb{R}_+ \times \mathbb{R}^N, \{0\} \times \mathbb{R}^N)$ we can join the two embeddings by an isotopy. The isotopy leads to a homotopy between ν_M and ν'_M, the Gauß map corresponding to the second embedding. If one lifts this homotopy extending a given lift $\bar{\nu}_M$ and restricts to $M \times \{1\}$ one gets a homotopy lift of ν'_M. As two isotopies between two embeddings are themselves isotopic one obtains a map from the homotopy lifts of ν_M to the homotopy lifts of ν'_M which is bijective.

A <u>B-structure on M</u> is an equivalence class of such homotopy lifts where two lifts of ν_M and ν'_M are equivalent if they correspond to each other under the map defined above. We will always consider a B-structure as given by a representative $\bar{\nu}_M$. A pair $(M, \bar{\nu}_M)$ is denoted as a <u>n-dimensional B-manifold</u>.

The restriction of a B-structure $\bar{\nu}_M$ to the boundary defines a B-manifold $\partial(M, \bar{\nu}_M) := (\partial M, \bar{\nu}_M|_{\partial M})$. A B-structure $\bar{\nu}_M$ on $M \times \{0\} \subset M \times I$ (straighten the angle if M has a boundary ([9] , p.9)) can be extended to $M \times I$. Its restriction to $M \times \{1\}$ is denoted as the <u>inverse</u> of $\bar{\nu}_M$: $-\bar{\nu}_M$. For $(M, -\bar{\nu}_M)$ we write $-(M, \bar{\nu}_M)$.

If $(M, \bar{\nu}_M)$ is a B-manifold and $f: N \longrightarrow M$ a diffeomorphism then f induces a B-structure on N : $f^* \bar{\nu}_M$. Two B-structures $(M, \bar{\nu}_M)$ and $(N, \bar{\nu}_N)$ are <u>diffeomorphic</u> if there exists a diffeomorphism $f : N \longrightarrow M$ with $f^* \bar{\nu}_M = \bar{\nu}_N$. If $f : \partial(M, \bar{\nu}_M) \longrightarrow \partial(N, -\bar{\nu}_N)$ is a diffeomorphism then there is a B-structure on $M \cup_f N$ extending the given B-structure on M and N. We write for this B-manifold: $(M, \bar{\nu}_M) \cup_f (N, \bar{\nu}_N)$.

Two closed B-manifolds $(M, \bar{\nu}_M)$ and $(N, \bar{\nu}_N)$ are called <u>B-bordant</u> if there exists a compact B-manifold $(W, \bar{\nu}_W)$ with $\partial W = M + N$ and $\bar{\nu}_W|_M = \bar{\nu}_M$ and $\bar{\nu}_W|_N = -\bar{\nu}_N$. The set of these bordism classes forms a group denoted by Ω_n^B (For more details compare [37] , [23]).

If there exists a B-bordism $(W, \bar{\nu}_W)$ between $(M, \bar{\nu}_M)$ and $(N, \bar{\nu}_N)$ which is a s-cobordism (i.e. the inclusions $M \longrightarrow W$ and $N \longrightarrow W$ are simple homotopy equivalences [28]) then $(M, \bar{\nu}_M)$ and $(N, \bar{\nu}_N)$ are called <u>s-cobordant.</u>

A n-dimensional B-manifold $(M, \bar{\nu}_M)$ is called a <u>normal k-smoothing in B</u> if $\bar{\nu}_M$ is a (k+1)-equivalence. The set of s-cobordism classes of n-dimensional normal k-smoothings in B is denoted by $NS_{n,k}^B$. There is an obvious map

$$NS_{n,k}^B \longrightarrow \Omega_n^B .$$

One can also introduce relative versions. Let $(V, \overline{\nu}_V)$ be a B-manifold. A <u>relative B-manifold</u> is a triple $(M, \overline{\nu}_M, f)$ where $f: (\partial M, \overline{\nu}_{\partial M}) \longrightarrow (V, \overline{\nu}_V)$ is a diffeomorphism. Two compact relative B-manifolds $(M, \overline{\nu}_M, f)$ and $(N, \overline{\nu}_N, g)$ are called B-bordant <u>rel. boundary</u> if there exists a B-manifold $(W, \overline{\nu}_W)$ with $\partial W = M \cup_{g^{-1}f} N$, $\overline{\nu}_W|_M = \overline{\nu}_M$ and $\overline{\nu}_W|_N = -\overline{\nu}_N$. If in addition W is a s-cobordism we call $(M, \overline{\nu}_M)$ and $(N, \overline{\nu}_N)$ <u>s-cobordant rel. boundary</u>.

Given a closed (n-1)-dimensional B-manifold $(V, \overline{\nu}_V)$ we denote the set of B-bordism classes rel. boundary of compact relative B-manifolds by $\Omega_n^{(B,V)}$ and the corresponding set of s-cobordism classes rel. boundary of compact relative normal k-smoothings by $NS_{n,k}^{(B,V)}$. Again we have a map

$$NS_{n,k}^{(B,V)} \longrightarrow \Omega_n^{(B,V)}$$

If $k \geq 0$ and $(M, \overline{\nu}_M)$ a normal k-smoothing in B then M is orientable if and only if $p^* w_1 = 0$ in $H^1(B;\mathbb{Z}_2)$, where $w_1 \in H^1(BO;\mathbb{Z}_2)$ is the first Stiefel-Whitney class. Or equivalently if $p^*(\gamma|_{BO(N)})$ is orientable, the stable universal vector bundle. We will always assume that B has a base point $*$ and that $p^*(\gamma|_{BO(N)})$ is <u>oriented in $*$</u>, called a <u>local orientation of B</u>. Furthermore we will equip all manifolds with a base point (normally without special mention) and require <u>all maps into B to be base point preserving</u>. Thus a B-structure induces <u>a local orientation on M</u>. One further convention: <u>We will always assume that B is connected.</u> For our purpose, to classify manifolds, this is no real restriction as one can study the components separately.

<u>Remark 1.1</u> : A Postnikov decomposition of a map $f : X \longrightarrow Y$ between connected CW-complexes is a diagram

$$\begin{tikzcd}
 & \vdots \arrow[d] \\
 & B_k \arrow[d] \\
 & B_{k-1} \arrow[d] \\
 & \vdots \arrow[d] \\
 & B_1 \arrow[d] \\
X \arrow[ruuuu] \arrow[ruuu] \arrow[ru] \arrow[r, "f"] & Y
\end{tikzcd}$$

where $B_k \longrightarrow B_{k-1}$ is a fibration with fibre a $(K\pi_k,k)$ and $X \longrightarrow B_k$ a $(k+1)$-equivalence (compare [3], Theorem 5.3.1).

Given another Postnikov decomposition

$$\begin{tikzcd}
 & \vdots \arrow[d] \\
 & B'_k \arrow[d] \\
 & B'_{k-1} \arrow[d] \\
 & \vdots \arrow[d] \\
X \arrow[ruuu] \arrow[ruu] \arrow[r, "f"] & Y
\end{tikzcd}$$

then the fibrations $B_k \longrightarrow Y$ and $B'_k \longrightarrow Y$ are fibre homotopy equivalent (this follows from [3], Corollary 5.3.8).

Thus we obtain for connected M.

Lemma + Definition 1.2 : Let $B_k \longrightarrow BO$ be a (k+1)-co-connected fibration, i.e. $\pi_r(F) = 0$ for $r > k$, and we assume that B_k has the homotopy type of a CW-complex.

i) A diagram
$$\begin{tikzcd} & B_k \arrow[d] \\ M \arrow[ur, "\bar{\nu}_M"] \arrow[r, "\nu_M"'] & BO \end{tikzcd}$$
is a normal k-smoothing if and only if it is the k-th step of a Postnikov decomposition of $M \xrightarrow[\nu_M]{} BO$.

ii) The fibre homotopy equivalence class of the fibration $B_k \longrightarrow BO$ is an invariant of M. We call it the normal k-type of M. Notation: $B_k(M)$.

The case of k-co-connected fibrations is most interesting as it is the universal situation.

It is sometimes useful to work with the following equivalent description of B-manifolds (normal k-smoothings)

For a fibration $B \longrightarrow BO$ one can consider the sequence of fibrations $B|_{BO(N)} \longrightarrow BO(N)$. A B-manifold (normal k-smoothing) can be described as a compatible sequence of pairs (f_N, α_N), $N \gg \dim M$, $f_N\colon M \longrightarrow B|_{BO(N)}$ a map and α_N a trivialisation of $\nu_N(M) - f_N^* \gamma(B|_{BO(N)})$, where $\nu_N(M)$ is the N-dimensional normal bundle of M and $\gamma(B|_{BO(N)})$ is the pullback of the N-dimensional universal bundle. As the sequence (f_N, α_N) is compatible with the diagram
$$\begin{tikzcd} B|_{BO(N)} \arrow[r] \arrow[d] & B|_{BO(N+1)} \arrow[d] \\ BO(N) \arrow[r] & BO(N+1) \end{tikzcd}$$
the maps f_N fit together to a map $f\colon M \longrightarrow B = \lim_{N \to \infty} B|_{BO(N+1)}$. For a k-normal smoothing we require that f is a (k+1)-equivalence.

The trivialisation α_N of $\nu_N(M) - f_N^* \gamma(B|_{BO(N)})$ is equivalent to an isomorphism $\nu_N(M) \longrightarrow f_N^* \gamma(B|_{BO(N)})$ which itself is equivalent to an explicit homotopy between the maps ν_M and $p \circ f$. If one lifts this homotopy and restricts to the map covering $p \circ f$ one gets for each $N \gg n$ a lift $\bar{\nu}_M : M \longrightarrow B|_{BO(N)}$. These lifts fit together to a lift $\bar{\nu}_M : M \longrightarrow B$ which is a B-structure on M (a normal k-smoothing). We normally will work with the definition of B-manifolds and normal k-smoothings in terms of homotopy lifts of the normal Gauß map but sometimes it is more convenient to think of it as a sequence (f_N, α_N).

Example : Let X be a n-dimensional Poinvaré complex and ξ a stable vector bundle over X or equivalently a map $\xi : X \longrightarrow BO$. Consider the n-th step of a Postnikov decomposition of ξ.

$$\begin{tikzcd} & B = B_n(X,\xi) \arrow[d] \\ X \arrow[ur, "\bar{\xi}"] \arrow[r, "\xi"'] & BO \end{tikzcd}$$

As $\bar{\xi}$ is a (n+1)-equivalence we consider X as the n-skeleton of a CW-decomposition of B and $\bar{\xi}$ as an inclusion.

Let $(M, \bar{\nu}_M)$ be a n-dimensional normal n-smoothing in B. Then we can modify $\bar{\nu}_M$ by a homotopy into a map g, so that im $g \subset X$. Then $g : M \longrightarrow X$ is a homotopy equivalence or a smoothing of M in X.

Thus we obtain a map $NS^B_{n,n} \longrightarrow S^h(X)$, where $S^h(X)$ is the set of smoothings. (h stands for homotopy equivalence. If one considers simple homotopy equivalences, the set of smoothings is denoted as $S^s(X)$.)

A bundle isomorphism $\alpha : \xi \longrightarrow \xi'$ induces a bijection $NS^{B_n(X,\xi)}_{n,n} \longrightarrow NS^{B_n(X,\xi')}_{n,n}$ so that the diagram

$$\begin{tikzcd} NS^{B_n(X,\xi)}_{n,n} \arrow[rr] \arrow[dr] & & NS^{B_n(X,\xi')}_{n,n} \arrow[dl] \\ & S^h(X) & \end{tikzcd}$$

commutes. Especially the group of bundle automorphisms of ξ acts on $NS^B_{n,n}$. This group is isomorphic to $[X,O]$ (ξ is a stable vector bundle), so $[X,O]$ acts on $NS^B_{n,n}$ and we have a map

$$NS^B_{n,n}\ /[X,O] \longrightarrow S^h(X)$$

defined for the isomorphism class of ξ.

Given a smoothing $g : M \longrightarrow X$ one obtains a normal n-smoothing in $B_n(X,(g^{-1})^*\nu(M))$. Thus the map

$$\bigcup_{\xi\in \widetilde{KO}(X)} NS^{B_n(X,\xi)}_{n,n}\ /[X,O] \longrightarrow S^h(X)$$

is surjective. It is not difficult to show that it is also injective.

<u>Lemma 1.3</u> : The construction above gives a bijection

$$\bigcup_{\xi\in \widetilde{KO}(X)} NS^{B_n(X,\xi)}_{n,n}\ /[X,O] \longrightarrow S^h(X)\ .$$

Thus our program can be considered as an extension of the results of Browder, Novikov and Wall about smoothings of manifolds.

§ 2 Stable classification of normal (n-1)-smoothings on 2n-manifolds

Two 2n-dimensional manifolds M an M' are called <u>stably diffeomorphic</u> if there exist $r, s \in \mathbb{N}$ s.t. $M \# r\cdot(S^n \times S^n) \cong M' \# s\cdot(S^n \times S^n)$. This connected sum is well defined even if M an M' are non-oriented as $S^n \times S^n$ admits an orientation reversing diffeomorphism. The difference r-s is measured by the Euler characteristic. If a diffeomorphism $f : \partial M \longrightarrow \partial M'$ is given we say f can be <u>stably extended</u> if there exists a diffeomorphism $M \# r(S^n \times S^n) \longrightarrow M' \# s(S^n \times S^n)$ extending f.

We want to introduce this stabilization process for normal (n-1)-smoothings. Let $B \longrightarrow BO$ be a fibration of pointed spaces. A B-strucutre on S^n, $\bar{\nu}_{S^n}$, is called elementary if $(\bar{\nu}_{S^n})_*[S^n] = 0$ in $\pi_n(B)$. An elementary B-structure will be denoted as (S^n, α). (Warning: α is not uniquely determined by this property. The differnt α's correspond to framings on S^n.) The composition $S^n \times D^{m+1} \longrightarrow S^n \xrightarrow{\alpha} B$ is a B-structure on $S^n \times D^{m+1}$. We denote its restriction to the boundary by $(S^n \times S^m, \alpha)$.

Given two B-manifolds $(M, \bar{\nu}_M)$ and $(N, \bar{\nu}_N)$ one can form their connected sum denoted by $(M, \bar{\nu}_M) \# (N, \bar{\nu}_N)$. This is well defined in terms of the local orientations on M an N (see §1).

Given a 2n-dimensional normal smoothing $(M, \bar{\nu}_M)$ in B and a B-manifold $(S^n \times S^n, \alpha)$ as above, the connected sum $(M, \bar{\nu}_M) \# (S^n \times S^n, \alpha)$ is again a normal (n-1)-smoothing. We say $(M, \bar{\nu}_M)$ and $(M', \bar{\nu}_{M'})$ are <u>stably s-cobordant</u> if there exist elementary B-structures $\alpha_1, \dots, \alpha_r$ and $\beta_1, \dots, \beta_s$ on S^n s.t. $(M, \bar{\nu}_M) \# (S^n \times S^n, \alpha_1) \# \dots \# (S^n \times S^n, \alpha_r)$ and $(M', \bar{\nu}_{M'}) \# (S^n \times S^n, \beta_1) \# \dots \# (S^n \times S^n, \beta_s)$ are s-sobordant.

By the s-cobordism Theorem [14] (the stable s-cobordism Theorem, if n=2 [32]) stably s-cobordant manifolds are stably diffeomorphic. Given a closed (2n-1)-dimensional B-manifold $(V, \bar{\nu}_V)$ we denote the set of stable s-cobordism classes of relative 2n-dimensional normal (n-1)-smoothings in B by

$$NSt_{2n}^{(B,V)}$$

As $(S^n \times S^n, \alpha)$ bounds $(S^n \times D^{n+1}, \alpha)$ it is zero bordant. Thus we have a map

$$NSt_{2n}^{(B,V)} \longrightarrow \Omega_{2n}^{(B,V)}$$

There is an obvious condition for the existence of a normal k-smoothing of B namely B must have a CW-decomposition with finite (k+1)-skeleton.

Theorem 2.1: If $B \longrightarrow BO$ is a fibration of pointed spaces and B has up to homotopy equivalence a CW-decomposition with finite n-skeleton

$$NSt_{2n}^{(B,V)} \longrightarrow \Omega_{2n}^{(B,V)}$$

is a bijection for $n \geq 2$.

Remark 2.2: If one has two 2n-dimensional simple smoothings in $S^s(X)$, X a Poincaré complex: $f: M \longrightarrow X$ and $g: N \longrightarrow X$, which are normally cobordant then one can deduce from results of Wall [41] that they are stably diffeomorphic, if $n \geq 3$. The argument is the following. In ([41], Theorem 6.5) Wall proves that the bordism is rel. boundary bordant to a bordism W of the following form. $W_1 = W_1 \cup W_2$ with $W_1 = M \times I$ with r trivial n-handles attached and $W_2 = N \times I$ with r trivial n-handles attached. Thus

$$M \# r\,(S^n \times S^n) \cong W_1 \cap W_2 \cong N \# r\,(S^n \times S^n).$$

Theorem 2.1 shows that even under far weaker conditions and also for n=2, namely if one has two 2n-dimensional normal (n-1)-smoothings in (B,V) which are bordant in Ω_{2n} (B,V), then M and N are stably s-cobordant and thus stably diffeomorphic. (For n=2 one has to apply the stable s-cobordism theorem [32]). It is perhaps a little bit surprising that no L-group like obstructions occur at the stable classification of manifolds.

Proof of Theorem 2.1: A) Surjectivity.

As we will use this and the corresponding odd-dimensional statement later again we will formulate it as a seperate Lemma.

Lemma 2.3 : Suppose that B has finite $[n/2]$-skeleton. A B-manifold $(M^n, \bar{\nu}_M)$ is bordant rel. boundary to $(M', \bar{\nu}_{M'})$ s.t. $\bar{\nu}_{M'}$ is a $[n/2]$-equivalence.

Proof :

The idea is simple. Given a relative n-dimensional B-manifold $(M, \bar{\nu}_M)$ we first replace it in its bordism class by a manifold $(M', \bar{\nu}_M{}')$ such that $\bar{\nu}_M{}'$ is surjective in homotopy groups up to dimension $n/2$. Then we kill the kernel by surgery.

For the first step it is enough to find a closed zero-bordant B-manifold which is surjective in homotopy up to dimension $n/2$. Take the connected sum with it to obtain $(M', \bar{\nu}_M{}')$. Let X be a finite $[\frac{n}{2}]$-skeleton of B. Thicken X up to a compact n+1-dimensional manifold NX so that

$$\begin{array}{ccc} X & \longrightarrow & B \\ \downarrow & & \downarrow \\ NX & \xrightarrow[\nu_{Nx}]{} & BO \end{array}$$

is homotopy commutative. This can for instance be obtained by replacing cells of X by handles and choosing the framings so that the diagram above becomes homotopy commutative (or use [44]). As $X \longrightarrow NX$ is a homotopy equivalence, NX admits a B-structure, in fact a normal $([\frac{n}{2}]-1)$-smoothing as $X \longrightarrow B$ is a $[\frac{n}{2}]$-equivalence. Then ∂NX is the desired zero-bordant B-manifold. For ∂NX is a normal $([\frac{n}{2}]-1)$-smoothing as $\partial NX \longrightarrow NX$ is a $[\frac{n}{2}]$-equivalence.

For the second step we assume now $\bar{\nu}_M$ induces surjections $\pi_r(M) \longrightarrow \pi_r(B)$ for $r \leq [n/2]$. For $r < [\frac{n}{2}]$ we can kill the kernel by a finite sequence of surgeries. More precisely we assume inductively that in addition $\pi_r(M) \rightarrow \pi_r(B)$ is an isomorphism for $r < m$. Then if $m < [\frac{n}{2}]$ we have a short exact sequence

$$0 \longrightarrow \pi_{m+1}(B,M) \longrightarrow \pi_m(M) \longrightarrow \pi_m(B) \longrightarrow 0.$$

By the Hurewicz isomorphism one knows that

$$\pi_{m+1}(B,M) \cong H_{m+1}(B,M;\Lambda)$$

as Λ-modules where $\Lambda = \mathbb{Z}[\pi_1(B)]$ is the group ring and $H_{m+1}(B,M;\Lambda)$ is the ordinary homology of the universal cover with (left) Λ-module structure given by the covering transformations. As B has finite $[\frac{n}{2}]$-skeleton the Λ-modul $H_{m+1}(B,M;\Lambda)$ is finitely generated. Surgery on a set of generators kills the kernel of $\pi_m(M) \longrightarrow \pi_m(B)$ (See [5], §II.1). This leads to a B-manifold $(M', \bar{\nu}_M')$ which is B-bordant to $(M, \bar{\nu}_M)$ such that $\pi_r(M') \longrightarrow \pi_r(B)$ is an isomorphism for $r \leq m$. As $m < [\frac{n}{2}]$ a general position argument shows that the maps $\pi_r(M') \longrightarrow \pi_r(B)$ for $r \leq [\frac{n}{2}]$ are again surjective.

B) Injectivity

Let $(W^{2n+1}, \bar{\nu}_W)$ be a zero bordism of $M \cup_{g^{-1}f} N$. Then $\pi_r(W) \longrightarrow \pi_r(B)$ is surjective for $r \leq n$. By Lemma 2.3 we can kill the kernel for $r < n$ by a sequence of surgeries in W. Thus we can assume that $\bar{\nu}_W : W \longrightarrow B$ is a n-equivelence.

To obtain a s-cobordism we have to kill $\pi_n(W,M)$ and $\pi_n(W,N)$, so that the resulting homotopy equivalences $M \longrightarrow W$ and $N \longrightarrow W$ are simple. For this we use the following modification of subtraction of trivial handles: Consider an embedding $S^n \times D^{n+1} \hookrightarrow W$. Join $\partial(S^n \times D^{n+1})$ and M (or N) by an embedded thickened are $I \times D^{2n}$ meeting $\partial(S^n \times D^{n+1})$ and M (or N) transversally in $\{0\} \times D^{2n}$ and $\{1\} \times D^{2n}$. Remove $S^n \times D^{n+1}$ and $I \times D^{2n}$ from W and straighten the resulting angles (compare [9], p. 9). The resulting manifold W' has boundary $\partial(W) \# S^n \times S^n$, where the connected sum takes place in M (in N). If $S^n \times \{0\} \longrightarrow W \longrightarrow B$ is null-homotopic then the B-structure on $S^n \times D^{n+1}$ is given by some elementary B structure on S^n as described at the beginning of this §. Thus in this case W' is a bordism between

$$(M, \bar{\nu}_M) \# (S^n \times S^n, \alpha) \quad \text{and} \quad (N, \bar{\nu}_N)$$

(or with M and N interchanged).

Thus we are finished if we can find a finite sequence of such disjoint embeddings $S^n \times D^{n+1} \hookrightarrow W$ with $S^n \longrightarrow W \longrightarrow B$ null-homotopic and s.t. if we remove these trivial handles joining them either with M or N the resulting manifold is a relative s-cobordism.

Again as in part A) the Λ-modules $\pi_n(W,M)$ and $\pi_n(W,N)$ are finitely generated. Choose a set of generators of $\pi_n(W,M)$. From the exact sequences

$$\begin{tikzcd}
& \pi_{n+1}(B,W) \arrow[d] \arrow[dr] & & \\
\pi_n(M) \arrow[r] \arrow[dr] & \pi_n(W) \arrow[r] \arrow[d] & \pi_n(W,M) \arrow[r] & 0 \\
& \pi_n(B) \arrow[d] & & \\
& 0 & &
\end{tikzcd}$$

and the fact that $\pi_n(M) \longrightarrow \pi_n(B)$ is surjective it follows that $\pi_{n+1}(B,W) \longrightarrow \pi_n(W,M)$ is surjective. Thus we can find counterimages of the generators of $\pi_n(W,M)$ in $\pi_n(W)$ mapping to zero in $\pi_n(B)$. Thus they can be represented by disjoint embeddings $(S^n \times D^{n+1})_i \subset W$. Join them with M and remove them to obtain W'.

It is obvious that $\bar{\nu}_{W'} = \bar{\nu}_W|_{W'} : W' \longrightarrow B$ is again a n-equivalence. By Hurewicz isomorphism and excision we have isomorphisms $\pi_n(W', M\#r(S^n \times S^n))$ $\cong H_n(W', M\#r(S^n \times S^n); \Lambda) \cong H_n(W, M \cup ((S^n \times D^{n+1})_i \cup I \times D^{2n}); \Lambda) = \{0\}$, as the $(S^n \times D^{n+1})_i$ form a set of generators of $H_n(W,M;\Lambda)$ (Consider the long exact homology sequence of the triple).

Poincaré duality (compare [41], Theorem 2.1) implies that with arbitrary coefficients K $H_r(W', N; K)$ $\cong H^{2n+1-r}(W', M\#r(S^n \times S^n); K) = \{0\}$ for $r > n$. On the other hand it vanishes for $r < n$ by assumption. Thus $H_n(W', N; K)$ is the only possibly non-vanishing homology group and consequently is a stably free Λ-module ([41], Lemma 2.3), if $K = \Lambda$.

It is easy to see that if we add to our system of generators of $\pi_n(W,M)$ used before to kill this group a set of spheres $S^n \times D^{n+1}$ sitting unlinked in a ball $D^{2n+1} \subset W$ then for each $S^n \times D^{n+1}$ we enlarge $H_n(W',N;\Lambda)$ by direct sum with Λ. Thus we can assume that $H_n(W',N;\Lambda)$ is a free Λ-module.

If one distinguishs a basis of $H_n(W',N;\Lambda)$, the Whitehead torsion of (W',N) is defined ([28], p. 378). Obviously, stabley one can always choose a basis so that the torsion vanishes. Following Wall such a basis is called <u>preferred</u> ([41], §2).

As before we can represent such a preferred basis by s disjoint embeddings $(S^n \times D^{n+1})_i \subset W'$ mapping to zero in $\pi_n(B)$. Join them with N and remove the union of them denoted by U to obtain W''.

We claim that W'' is a s-cobordism between $M\#r(S^n \times S^n)$ and $N\#s(S^n \times S^n)$. This is equivalent to: $\pi_1(M\#r(S^n \times S^n)) \longrightarrow \pi_1(W'')$ and $\pi_1(N\#s(S^n \times S^n)) \rightarrow \pi_1(W'')$ are isomorphisms, $H_*(W'',N\#s(S^n \times S^n);\Lambda) = \{0\}$ and the Whitehead torsion $\tau(W'',N\#s(S^n \times S^n))$ vanishes (compare [35], 1.2.1).

The first condition is obviously fullfilled. For the second and third we first remark that excision induces an isomorphism $H_*(W'',N\#s(S^n \times S^n);\Lambda) \longrightarrow H_*(W',N \cup U;\Lambda)$ and that the Whitehead torsion of both pairs is the same if they are acyclic. We sonsider the homology sequence:

$$\cdots\cdots \longrightarrow H_{n+1}(W',N \cup U;\Lambda) \longrightarrow H_n(N \cup U,N;\Lambda) \longrightarrow$$

$$H_n(W',N;\Lambda) \longrightarrow H_n(W',N \cup U;\Lambda) \longrightarrow \cdots\cdot$$

By assumption the only non-vanishing homology groups of $(N \cup U,N)$ and (W',N) are in dimension n, when both modules are Λ^s. A basis of

$H_n(N \cup U, N; \Lambda)$ is given by the $(S^n \times \{0\})_i$ which is a preferred basis (the Whitehead torsion vanishes). By assumption the image of these basis elements forms a preferred basis for $H_n(W', N; \Lambda)$.

All this implies that $H_*(W', N \cup U; \Lambda) = \{0\}$ and that the Whitehead torsion of the acyclic complex given by the exact homology sequence is 0. By the additivity formula for the Whitehead torsion ([28], Theorem 3.2) this implies:

$$\tau(W'', N\#s(S^n \times S^n)) = \tau(W', N \cup U) = \tau(W', N) - \tau(N \cup U, N) = 0.$$

q.e.d.

By Theorem 2.1 there is raised the following natural question: Can one introduce a geometrically defined group structure on NSt_{2n}^B such that the map into Ω_{2n}^B is a group isomorphism?

Let X be a fixed finite (n-1)-skeleton of B or equivalently consider a fixed (n-1)-equivalence of a finite CW-complex X of dimension n-1 to B. If $\bar\nu_M : M^{2n} \longrightarrow B$ is a normal (n-1)-smoothing then by obstruction theory there is a lift $X \longrightarrow M$ s.t. the diagram

$$\begin{tikzcd} & X \arrow[dl] \arrow[dr] & \\ M \arrow[rr, "\bar\nu_M"'] & & B \end{tikzcd}$$

commutes.

One can choose $X \longrightarrow M$ as an embedding. We denote a regular neighbourhood of such an embedding by $U_M(X)$. Consider the same information for another normal (n-1)-smoothing $\bar\nu_N : N \longrightarrow B$. Then $U_M(X)$ and $U_N(X)$ are diffeomorphic as B-manifolds. Thus the union along the boundary $M - U_M(X) \cup N - U_N(X)$ is again a B-manifold. It is easy to check that it is a normal (n-1)-smoothing. Further by construction it is bordant to $(M, \bar{\nu}_M) + (N, \bar{\nu}_N)$. Thus by Theorem 2.1 the stable s-cobordism class of $M - U_M(X) \cup N - U_N(X)$ is independent of all choices and denoted by $[M, \bar{\nu}_M] \#_X [N, \bar{\nu}_N]$. This is the desired geometric addition on NSt^B_{2n}. As $NSt^B_{2n} \longrightarrow \Omega^B_{2n}$ is a homomorphism and this map is bijective, $(NSt^B_{2n}, \#_X)$ is an abelian group.

§ 3 Cancellation of $S^n \times S^n$

In the last chapter we have seen that under comparatively weak conditions M and N are stably diffeomorphic. On the other hand, under the stronger condition that M and N are normally cobordant there exists an obstruction in the L-group $L^s_{2n+1}(\pi_1(M), w_1(M))$ for M and N to be s-cobordant ([41]). We will describe under certain conditions for a diffeomorphism $f: M \# r(S^n \times S^n) \longrightarrow N \# r(S^n \times S^n)$ invariants in the Whitehead group $Wh(\pi_1(M))$ and in $L^s_{2n+1}(\pi_1(M), w_1(M))$. If they vanish, M and N are s-cobordant (diffeomorphic for dim $M > 4$).

We have canonical splittings $H_n(M \# r(S^n \times S^n);\Lambda) = H_n(M;\Lambda) \oplus \Lambda^{2r}$ and $\pi_n(M \# r(S^n \times S^n)) = \pi_n(M) \oplus \Lambda^{2r}$ which are compatible with the Hurewicz map. Wall ([41], §5) had defined a $(-1)^n$-quadratic form (λ, μ) for immersed spheres $S^n \looparrowright M$. (For the algebraic notation of a quadratic form compare §4). Geometrically λ is given by intersections measured in $\pi_1(M)$, thus $\lambda(\alpha, \beta) \in \Lambda = \mathbb{Z}[\pi_1(M)]$. μ is given by self-intersections which are only well defined up to indeterminacy, thus $\mu(\alpha) \in \Lambda/\{a-(-1)^n\bar{a}\} = Q_{(-1)^n}$. If $n \neq 1,3,7$ every element in $\pi_n(M)$ with stably trivial normal bundle can be represented by a unique immersion with (unstable) trivial normal bundle (compare [41], p.44). Thus if $K\pi_n(M)$ is the kernel of the normal Gauß map $\pi_n(M) \longrightarrow \pi_n(BO)$, we get for $n \neq 1,3,7$ a quadratic form (λ, μ) on $K\pi_n(M)$. The basic property of μ is that for $\alpha \in K\pi_n(M)$ if $\mu(\alpha) = 0$ then the immersion can be represented by an embedding with trivial normal bundle. If $n = 1,3,7$, λ is also defined but μ is only well defined if we impose on elements of $K\pi_n(M)$ an additional structure, a stable framing. All we do in this chapter for $n \neq 3,7$ can be extended to case $n = 3,7$ with appropriate modifications. For simplicity we do this only for 1-connected manifolds. In the situation above the restriction of (λ, μ) to $r(S^n \times S^n)$ in $\pi_n(M \# r(S^n \times S^n))$ is the hyperbolic form or hamiltonian $H^r_{(-1)^n}$ (for the notation compare § 4).

If $f : M \# r(S^n \times S^n) \longrightarrow N \# r(S^n \times S^n)$ is a diffeomorphism we identify $\pi_1(M)$ and $\pi_1(N)$ by f_*. Then our condition is that the composition

$$H^r_{(-1)^n} \to \pi_n(M) \oplus H^r_{(-1)^n} \xrightarrow{f_*} \pi_n(N) \oplus H^r_{(-1)^n} \longrightarrow H^r_{(-1)^n}$$

or equivalently the corresponding sequence with homology groups is an isometry (with respect to (λ, μ)). We denote this composition by $\vartheta(f)$. If $\vartheta(f)$ is invertible the Whitehead torsion in $Wh(\pi_1(M))$ is defined and called the Whitehead torsion of f on $r(S^n \times S^n)$: $\tau(f)$. If this vanishes then $\vartheta(f)$ is a simple isometry of H^r_ε representing an element of $L^s_{2n+1}(\pi_1(M), w_1(M)) = SU^{(-1)^n}(\Lambda)/RSU^{(-1)^n}(\Lambda)$ (for the definition compare ([41], § 6) or § 4). We denote this element in $L^s_{2n+1}(\pi_1(M), w_1(M))$ by $[\vartheta(f)]$.

Theorem 3.1: Let $f : M \# r(S^n \times S^n) \longrightarrow N \# r(S^n \times S^n)$ be a diffeomorphism and $n \neq 1, 3$ or 7. If $\vartheta(f) : H^r_{(-1)^n} \to H^r_{(-1)^n}$ is a simple ($\tau(f) = 0$) isometry and $[\vartheta(f)] \in L^s_{2n+1}(\pi_1(M), w_1(M))$ vanishes then M and N are diffeomorphic under a diffeomorphism extending $f|_{\partial M}: \partial M \longrightarrow \partial N$ (s-cobordant rel. boundary, if n = 2). If n=3 or 7 and M is 1-connected we obtain the same conclusion if $\vartheta(f)$ is an isometry of the intersection form.

Remark: In § 4 we will combine the two invariants $\tau(f)$ and $[\vartheta(f)]$ to a single invariant and give a reformulation of this theorem.

Proof: If $[\vartheta(f)] = 0$ in $L^s_{2n+1}(\pi_1(M), w_1(M))$ then after eventually further stabilization we can assume that $\vartheta(f) = \mathrm{Id}$. For $[\vartheta(f)] = 0 \Leftrightarrow \vartheta(f) \in RSU^\varepsilon(\Lambda)$. For every matrix $\alpha \in RSU^\varepsilon(\Lambda)$ one can find for an appropriate s a selfdiffeomorphism $g : M \# (r+s)(S^n \times S^n)$ preserving the basepoint, inducing the identity on $\pi_1(M)$ so that $\vartheta(g) = \alpha \oplus \mathrm{Id}$. This is proved in the more difficult case $n = 2$ in ([8], Theorem A.2) and the same proof extends to $n > 2$. Then $f' = (f \# \mathrm{Id}) \circ g^{-1} : M \# (r+s)(S^n \times S^n) \longrightarrow N \# (r+s)(S^n \times S^n)$ fulfils: $\vartheta(f') = \mathrm{Id}$. If $n = 3, 7$ and $\pi_1(M)$ is 1-connected a similar argument shows that if $\vartheta(f)$ is an isometry we can replace it by f' with $\vartheta(f') = \mathrm{Id}$.

Now, we consider the bordism rel. boundary

$$M \times I \natural_r (S^n \times D^{n+1}) \cup_f N \times I \natural_r (D^{n+1} \times S^n)$$

between M and N. We claim that this is a relative s-cobordism if $\vartheta(f) = \mathrm{Id}$.

For this we write the bordism as $X \cup_f Y$. Obviously $\pi_1(M) \longrightarrow \pi_1(X \cup_f Y)$ and $\pi_1(N) \longrightarrow \pi_1(X \cup_f Y)$ are isomorphisms. To show that $H_*(X \cup_f Y, N; \Lambda)$ vanishes and that $(X \cup_f Y, N)$ has trivial Whitehead torsion we consider the exact homology sequence:

$$\ldots \to H_{n+1}(X \cup_f Y, N; \Lambda) \to H_{n+1}(X \cup_f Y, Y; \Lambda) \to H_n(Y, N; \Lambda) \to H_n(X \cup_f Y, N; \Lambda) \to \ldots$$

By excision $H_*(X \cup_f Y, Y; \Lambda) = H_*(X, M \# r(S^n \times S^n); \Lambda) = \begin{cases} 0 & * \neq n+1 \\ \Lambda^r & * = n+1 \end{cases}$

and the disks $\{*\} \times D^{n+1}$ represent a preferred basis in dimension $n+1$.

By construction $H_*(Y, N; \Lambda) = \begin{cases} 0 & * \neq n \\ \Lambda^r & * = n \end{cases}$ and the spheres $\{*\} \times S^n$ form a preferred basis in dimension n. By construction the boundary map is $\vartheta(f)$ restricted to the half rank subspace $(\{0\} \times \Lambda)^r$. If $\vartheta(f) = \mathrm{Id}$ it is a simple isomorphism and by the same argument as at the end of the proof of Proposition 2.3 it follows that $H_*(X \cup_f Y, N; \Lambda) = \{0\}$ and $(X \cup_f Y, N)$ has trivial Whitehead torsion.

q.e.d.

One might think that this result is completely theoretical as one almost never has an explicit diffeomorphism f to check the conditions, especially that $\vartheta(f)$ is an isometry. But there are some cases where this is automatically the case.

Proposition 3.2: $n \neq 1,3,7$. If (λ,μ) vanishes on $K\pi_n(M)$ and $K\pi_n(N)$ ($K\pi_n(M) = \mathrm{Ker}(\nu_M)_* : \pi_n(M) \longrightarrow \pi_n(BO)$) then for each diffeomorphism $f\colon M\#r(S^n \times S^N) \longrightarrow N\#r(S^n \times S^n)$, $\vartheta(f)$ is an isometry.
If $n = 3,7$ and M is 1-connected, $\vartheta(f)$ is an isometry of the intersevtion form.

Proof: The radical of (λ,μ) on $K\pi_n(M\#r(S^n \times S^n))$ is by assumption $K\pi_n(M)$. Thus if we divide out the radical, the isometry $f_*\colon K\pi_n(M\#r(S^n \times S^n)) \longrightarrow K\pi_n(N\#r(S^n \times S^n))$ induces an isometry $H_\varepsilon^r \longrightarrow H_\varepsilon^r$ which is $\vartheta(f)$.

q.e.d.

Besides trivial cases like $K\pi_n(M)$ and $K\pi_n(N) = \{0\}$ the condition of this Proposition is for instance fulfilled if $K\pi_n(\partial M) \longrightarrow K\pi_n(M)$ and $K\pi_n(\partial N) \longrightarrow K\pi_n(N)$ are surjective and μ vanishes on $K\pi_n(M)$ and $K\pi_n(N)$. For, the image of $K\pi_n(\partial M) \longrightarrow K\pi_n(M)$ is always contained in the radical of λ. Sometimes one can apply this by the following considerations also to closed manifolds.

Let $M^{2n} = X \cup_f Y$ and $N^{2n} = X' \cup_g Y$ and $K\pi_n(\partial X) \longrightarrow K\pi_n(X)$ and $K\pi_n(\partial X') \longrightarrow K\pi_n(X')$ be surjective and μ the trivial map as above. Then M is diffeomorphic to N if the diffeomorphism $g^{-1} \circ f : \partial X \longrightarrow \partial X'$ can be extended to a diffeomorphism $X \longrightarrow X'$. Suppose that X and X' have the same normal (n-1) type B, (see Lemma-Definition 1.2). If $X \underset{g^{-1}\circ f}{\cup} X'$ bounds a B-manifold s.t. the restrictions of the B-structures to X and X' are normal (n-1)-smoothings then $g^{-1} \circ f$ extends stably to X. By Proposition 3.2 there are obstructions in $Wh(\pi_1(x))$ and $L^s_{2n+1}(\pi_1(x), w_1(x))$ and if they vanish (for instance if X is 1-connected) $g^{-1} \circ f$ extends to X.

§ 4 Definition of the surgery obstruction groups (monoids).

In § 1 we have introduced the map

$$NS_{n,k}^{(B,V)} \longrightarrow \Omega_n^{(B,V)}$$

If B is a Poincaré complex and $k = n$ the image of this map is contained in the subset of $\Omega_n^{(B,V)}$ represented by degree ± 1 maps. In §5 we will generalize the concept of degree ± 1 maps to so called k-admissible maps. We denote the subset of $\Omega_n^{(B,V)}$ represented by k-admissible maps by $A\Omega_{n,k}^{(B,V)}$. Every normal k-smoothing is k-admissible, thus the map above ends in $A\Omega_{n,k}^{(B,V)}$.

The typical scheme of a surgery classification is to introduce obstruction groups (monoids) L and L', a map $A\Omega_{n,k}^{(B,V)} \longrightarrow L'$ such that the fibre over $\{0\}$ is the image of the map above and to define an action of L on $NS_{n,k}^{(B,V)}$ such that the orbit space injects into $A\Omega_{n,k}^{(B,V)}$. This is often written in form of an "exact" sequence

$$L \longrightarrow NS_{n,k}^{(B,V)} \longrightarrow A\Omega_{n,k}^{(B,V)} \longrightarrow L'$$

If $k=n$ and B is given by a finite Poincaré complex with boundary V together with a stable vector bundle over it then as discussed in § 1 we are in the situation of Wall's book [41] and thus L and L' are then ordinary obstruction groups. More precisely as $NS^{(B,V)}_{n,n}$ corresponds then to smoothings (and not to simple smoothings) L' is then $L^h_n(\pi,w)$ whereas L is $L^C_{n+1}(\pi,w)$ in the notation of ([41], § 17 D) which we will denote by $L^{s,\tau}_{n+1}(\pi,w)$.

One would expect that L and L' are more complicated if $k < n$. Surprisingly this is not the case for L as long as $k \geq [n/2]$. In fact the groups L for $k \geq [n/2]$ are all the same group $L^{s,\tau}_{n+1}(\pi,w)$ except for n odd and $k = [n/2]$ where it is surprisingly even easier namely a quotient of this group denoted by $L^{s,\tau}_{n+1}(\pi,w,S)$.

L' is also unexpectedly easy if $k > [n/2]$ namely equal to a group $L^{h,p}_n(\pi,w)$. If $k = [n/2]$ L' is more complicated, it is then an abelian monoid $l^{h,p}_n(\pi,w)$.

The upper indices s,τ and h,p shall indicate that these groups sit in the first case between the Wall group L^s and the Whitehead group Wh and the other case between the Wall group L^h and the projective class group $\tilde{K}_0$. Thus besides the ordinary Wall group obstructions our obstruction contains some information about Whitehead torsion (a finiteness obstruction).

If one goes over to $k = [n/2]-1$ the picture changes further. Then the map $NS^{(B,V)}_{n,[n/2]-1} \longrightarrow \Omega^{(B,V)}_n$ is surjective by Lemma 2.3. The obstruction monoids $l^{s,\tau}_{2n+1}(\pi,w)$ and $l^{s,\tau}_{2n}(\pi,w,S)$ in which elements of a fibre of $NS^{(B,V)}_{n,\,[n/2]} \longrightarrow \Omega^{(B,V)}_{n}$ can be classified up to an indeterminacy are far more complicated than $L^{s,\tau}_{n+1}(\pi,w)$. As usual all groups and monoids are periodic of order 4 in n.

Let $\Lambda = \mathbb{Z}[\pi]$ be a group ring and $w: \pi \longrightarrow \mathbb{Z}_2$ a homomorphism. Let $- : \mathbb{Z}[\pi] \longrightarrow \mathbb{Z}[\pi]$ be the involution sending $g \in \pi$ to $w(g)\cdot g^{-1}$. (The following definitions can be extended in a natural way to rings A with involution). If $A = (a_{ij})$ is a matrix, $A^* := (\bar{a}_{ji})$. If M is a left Λ-module we denote the dual left Λ-module by M^* (compare [33], p.102).

The basic ingrediants of our definition are quadratic forms and formations (compare [41], [33], [1], [2]). For a fixed $\varepsilon = \pm 1$ consider a subgroup $S \subset \Lambda$ such that i) $a \in S \Rightarrow a + \varepsilon\bar{a} = 0$ ii) $a \in S \Rightarrow b \cdot a \cdot \bar{b} \in S$ for all $b \in \Lambda$ and iii) S contains the group $\{a - \varepsilon\bar{a} \mid a \in \Lambda\}$. The properties guarant that for $[a] \in \Lambda/S$, $a + \varepsilon\bar{a}$ is a well defined element of Λ and that for $[a] \in \Lambda/_S$ and $b \in \Lambda$, $b\cdot[a]\cdot\bar{b}$ is well defined. If we don't specify S we mean $\{a - \varepsilon\bar{a}\}$ in which case $\Lambda/_S$ is denoted by Q_ε in ([41], § 5). A subgroup S as above is called a <u>form parameter</u> in [1].

A ε<u>-quadratic form</u> over (Λ, S) consists of a triple (M, λ, μ) where M is a left Λ-module, $\lambda : M \times M \longrightarrow \Lambda$ a ε-hermitian form and $\mu : M \longrightarrow \Lambda/_S$ a quadratic refinement of λ. That means λ and μ have to fulfill the following properties:

i) For $y \in M$ fixed the map $M \mapsto \Lambda$, $x \longmapsto \lambda(x,y)$, is a Λ- homomorphism.

ii) $\lambda(x,y) = (-1)^k\ \lambda(y,x)^-$.

iii) $\lambda(x,x) = \mu(x) + (-1)^k \mu(x)^- \in \Lambda$.

iv) $\mu(x+y) = \mu(x) + \mu(y) + [\lambda(x,y)]$

v) $\mu(a\,x) = a \cdot \mu(x) \cdot \bar{a}$ for $x \in M$ and $a \in \Lambda$.

For $S = \{a - \varepsilon\,\bar{a}\}$ such forms were introduced in ([41], § 5), for general S they are studied in [1] and [2] where they are called <u>S-quadratic modules</u>.

An important special case is the <u>ε -hyperbolic form</u> (= <u>hamiltonian</u>) $H_\varepsilon^r = H \oplus \ldots \oplus H$ (r summands), $H_\varepsilon = (\Lambda \oplus \Lambda\ ,\ \lambda = \begin{pmatrix} 0 & 1 \\ \varepsilon & 0 \end{pmatrix})$, $\mu(e) = \mu(f) = 0 \in \Lambda/_S)$ where (e,f) is the canonical basis of $\Lambda \times \Lambda$.

A form isomorphic to H_ε^r is called <u>trivial</u>. We call (M,λ,μ) <u>projective</u> or <u>free</u> if M has this property. (M,λ,μ) is called <u>weakly based</u> if an equivalence class of bases is distinguished where two bases are equivalent if the transformation matrix has vanishing Whitehead torsion in $Wh(\pi)$. We say that (M,λ,μ) is <u>non-singular</u> if the adjoint of λ is an isomorphism. If (M,λ,μ) is weakly based, non-singular and the adjoint of λ is a simple isomorphism, then (M,λ,μ) is <u>based</u>. We always assume that H_ε^r is based by the e_i, f_i.

(M,λ,μ) and (M',λ',μ') are <u>stably equivalent</u> if for some r and s, $(M,\lambda,\mu) \oplus H_\varepsilon^r$ is isomorphic to $(M',\lambda',\mu') \oplus H_\varepsilon^s$. If M and M' are weakly based we require that the isomorphism is simple (trivial Whitehead torsion in $Wh(\pi)$).

We define $L_{2n}^{s,\tau}(\pi,w)$ as the set of stable equivalence classes of weakly based non-singular $(-1)^n$-quadratic forms over Λ. Let S be as above. Then $L_{2n}^{s,\tau}(\pi,w,S)$ is the set of stable equivalence classes of weakly based non-singular $(-1)^n$-quadratic forms over (Λ, S). If $S = \{a-(-1)^n \bar{a} \mid a \in \Lambda\}$ then $L_{2n}^{s,\tau}(\pi,w,S)$ is of course equal to $L_{2n}^{s,\tau}(\pi,w)$. In general we have a map $L_{2n}^{s,\tau}(\pi,w) \longrightarrow L_{2n}^{s,\tau}(\pi,w,S)$. One can easily show that this map is surjective.

Orthogonal sum defines an abelian monoid-structure on these sets. They are even groups. This follows easily if one compares them with the Wall groups $L_{2n}^{s}(\pi,w)$. First we note that as $L_{2n}^{s,\tau}(\pi,w) \longrightarrow L_{2n}^{s,\tau}(\pi,w,S)$ is a surjective homomorphism, it is enough to show that $L_{2n}^{s,\tau}(\pi,w)$ is a group. Recall that the <u>Wall group</u> $L_{2n}^{s}(\pi,w)$ is the set of stable equivalence classes of based non-singular $(-1)^n$-quadratic forms over Λ with group structure given by orthogonal sum. (If one forgets the bases one obtains $L_{2n}^{h}(\pi,w)$). We have an obvious injective homomorphism $L_{2n}^{s}(\pi,w) \longrightarrow L_{2n}^{s,\tau}(\pi,w)$ and the image is the kernel of the homomorphism $L_{2n}^{s,\tau}(\pi,w) \longrightarrow Wh(\pi)$ mapping (M,λ,μ) to the Whitehead torsion of the adjoint of λ : $\tau(\lambda^*)$. Thus $L_{2n}^{s,\tau}(\pi,w)$ is a group if the image in $Wh(\pi)$ is a subgroup. The image is represented by all non-singular matrices A over Λ such that $A = (-1)^n A^*$ (meaning that A defines a $(-1)^n$-hermitian form) and A of the form $B + (-1)^n B^*$ form some matrix B (meaning that λ has a quadratic refinement or that λ is even). It is easy to show that these matrices form a group.

We summarize these considerations in the following Lemma which plays the role of the Rothenberg exact sequence in our context.

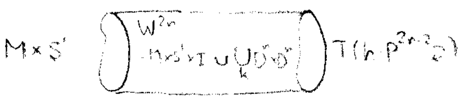
<u>Lemma 4.1:</u> We have exact sequences of abelian groups

$$
\begin{tikzcd}
0 \arrow[r] & L_{2n}^{s}(\pi,w) \arrow[r] & L_{2n}^{s,\tau}(\pi,w) \arrow[r] \arrow[d] & Wh(\pi) \\
& & L_{2n}^{s,\tau}(\pi,w,S) \arrow[d] & \\
& & 0 &
\end{tikzcd}
$$

Bak [1] has computed the kernel of the horizontal map. We will describe this at the end of this chapter.

To define $l_{2n}^{s,\tau}(\pi,w,S)$, S as above, we consider triples $(V \xleftarrow{f} M \xrightarrow{g} W, \lambda, \mu)$, M a finitely generated Λ-module, $f \oplus g : M \to V \oplus W$ injective, V and W based modules, $\lambda : V \longrightarrow W^*$ a simple isomorphism such that the induced form on M is $(-1)^n$-symmetric and $\mu : M \longrightarrow \Lambda/_S$ a quadratic refinement of this form. If (M, λ, μ) is a weakly based non-singular ε-quadratic form over (Λ, S) we can consider it as such a triple : $(M \xleftarrow{Id} M \xrightarrow{Id} M, \lambda, \mu)$ where the left M is equipped with the given basis and the right M is based such that $\lambda : V \longrightarrow W^*$ is simple. Especially $H^k_{(-1)^n}$ is contained in the set of those triples and we can define the stable equivalence relation as above. We define $l_{2n}^{s,\tau}(\pi,w,S)$ as the abelian monoid under orthogonal sum of stable equivalence classes of such triples $(V \longleftarrow M \longrightarrow W, \lambda, \mu)$. The construction above defines an injective homomorphism $L_{2n}^{s,\tau}(\pi,w,S) \longrightarrow l_{2n}^{s,\tau}(\pi,w,S)$.

$(K, \lambda) \in W_{2n}(\mathbb{Z}[\pi]) = L_{2n}(\Omega^{-1}\mathbb{Z}[\pi][z,z^{-1}]) \ni (\Omega^{-1}K[z,z^{-1}], (z-1)\lambda)$ $\quad \pi = \pi_1(M^{2n-2})$



$K_n(W) = K \xleftarrow{1} K_n(T(h)) = K$

$K_{n-1}(T(h)) = K^* \xleftarrow{(\lambda\ 1)} K_{n-1}(P) = K \oplus K^*$

$$
\begin{tikzcd}
K_n(W) \arrow[r] \arrow[d, equal] & K_n(W,P) \arrow[r, "\sim"] \arrow[d, equal] & K^n(W,P) \arrow[r] \arrow[d, equal] & K_n(W) \arrow[d, equal] \\
K \arrow[r, "\begin{pmatrix}1\\0\\0\end{pmatrix}"] \arrow[rrr, bend right, "\lambda"'] & K \oplus K \oplus K^* \arrow[r, "\lambda \oplus \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}"] & K^* \oplus K \oplus \ldots \arrow[r, "(1\ 0\ 0)"] & K^*
\end{tikzcd}
$$

But $f^{-1*}g \neq (-)^n g^* f^*$

$W_{2n}(\mathbb{Z}[\pi]) \longrightarrow l_{2n}(\pi)$; $(K, \lambda) \longrightarrow (V = K \xleftarrow{f=1} M = K \xrightarrow{g=1} W = K,\ \lambda : V = K \xrightarrow{\sim} W^* = K^*)$

Finally we <u>define $L_{2n}^{h,p}(\pi,w)$</u> as the group of stable equivalence classes of finitely generated projective non-singular $(-1)^n$-quadratic forms over Λ. If we replace projective modules by lattices (i.e. free as abelian groups) we obtain <u>the monoid $l_{2n}^{h,p}(\pi,w)$</u>. The fact that $L_{2n}^{h,p}(\pi,w)$ is a group follows similarly as for $L_{2n}^{s,\tau}(\pi,w)$ from:

<u>Lemma 4.2:</u> There is an exact sequence of groups, where $\tilde{K}_0(\pi)$ is the projective class group:

$$0 \longrightarrow L_{2n}^{h}(\pi,w) \longrightarrow L_{2n}^{h,p}(\pi,w) \longrightarrow \tilde{K}_0(\pi).$$

$$[M,\lambda,\mu] \longmapsto [M]$$

<u>Proof:</u> All we have to show is that every element in im $L_{2n}^{h,p}(\pi,w) \longrightarrow \tilde{K}_0(\pi)$ has an inverse in this image, so that the image is a subgroup. Let M be in the image and N an inverse in $\tilde{K}_0(\pi)$: $M \oplus N$ is free. We have to show that $N \oplus \Lambda^s$ for some s admits a quadratic form. But $N \oplus \Lambda^s = N \oplus N^* \oplus M^*$ where we have a quadratic form given by the orthogonal sum of the hyperbolic form on $N \oplus N^*$ and the dual of the given form on M.

q.e.d.

To define the odd L-groups (monoids) we recall and slightly modify some definition from [33]. In this part we <u>assume that all quadratic forms</u>

are free. A lagrangian of a non-singular ε-quadratic form (M,λ,μ) is a direct summand $L \subset M$ s.t. $L = L^{\perp}$ and $\mu|_L = 0$. A non-singular ε-quadratic form is trivial if and only if it has a lagragian. ([33], Theorem 1.1). If L is a lagrangian of (M,λ,μ) then a lagrangian L' is called a hamiltonian complement if $M = L \oplus L'$. The adjoint of λ defines then an isomorphism $L \longrightarrow (L')^{*}$. If L is based we consider the (dual) basis on L' induced by this isomorphism. The sum of these bases on L and L' defines a preferred basis on M depending only on the given basis on L. If L and M are based in this way we call $((M,\lambda,\mu,),L)$ a based lagranian.

A sublagrangian of (M,λ,μ) is a submodule $L \subset M$ s.t. $L \subseteq L^{\perp}$ and $\mu|_L = 0$ (in contrast to [33] we don't require L to be a direct summand). A ε-formation is a triple $((M,\lambda\,\mu),F,G)$, (M,λ,μ) a non-singular ε-quadratic form, F a lagrangian and G a sublagrangian. A formation is non-singular if also G is a lagrangian. The formation $(H_{\varepsilon}^{r},(\Lambda \times \{0\})^{r},(\{0\} \times \Lambda)^{r})$ is called hamiltonian, every formation isomorphic to a hamiltonian is called trivial. A ε-formation is weakly based if $((M,\lambda,\mu,),F)$ is based and G is free and based. If in addition $((M,\lambda,\mu),G)$ is based then the formation is called based. The hamiltonian formation is based by the standard bases. A ε-formation is elementary if it is isomorphic to $(H^{r},(\Lambda \times \{0\})^{r}$, $\{x + (C-\varepsilon C^{*})x \mid x \in (\Lambda \times \{0\})^{r}\})$ where C is a homomorphism $(\Lambda \times \{0\})^{r} \longrightarrow (\{0\} \times \Lambda)^{r}$. The elementary formations are based by the standard bases. Two ε-formations $((M,\lambda,\mu),F,G)$ and $((M',\lambda',\mu'),F',G')$ are weakly isomorphic if (M,F,G) is isomorphic to (M',F'',G') and F' and F'' have a common hamiltonian complement. If the formations are (weakly) based we require that the isomorphism preserves the bases and the induced bases on the common hamiltonian agree. (The notion weakly isomorphic is not needed in [33]. The reason is that if (M',F',G') and (M',F'',G')

are non-singular, L a common hamiltonian complement of F' and F'' then if L' is a hamiltonian complement of G', $(M',F',G') \oplus (-M,L,L')$ and $(M',F'',G') \oplus (-M,L,L')$ are both elementary or equivalently the formations have the same imverse in the L-group.)

Two ε- formations X and X' are stably elementary equivalent if there exist trivial formations A and A' and elementary formations B and B' such that $X \oplus A \oplus B$ is weakly isomorphic to $X' \oplus A' \oplus B'$. As discussed above we can replace weakly isomorphic by isomorphic if X and X' are non-singular.

To demonstrate the effect of the relation "weakly isomorphic" we prove the following Lemmas.

Lemma 4.3 : Let $((M,\lambda,\mu),F,G)$ be a non-singular ε-formation and $((M,\lambda,\mu),G,H)$ be a ε-formation. Then $((M,\lambda,\mu),F,G) \oplus ((M,\lambda,\mu),G,H)$ is stably elementary equivalent to $((M,\lambda,\mu),F,H)$.

Proof : Verbally the same as the proof of ([33], Lemma 3.3) where the special case of this proof follows from our weak isomorphism relation.

q.e.d.

Lemma 4.4 : Let A be an isometry of H^r_ε and $(H^r_\varepsilon,(\Lambda \times \{0\})^r,G)$ an ε-formation. Then $(H^r_\varepsilon,(\Lambda \times \{0\})^r,G) \oplus (H^r_\varepsilon,(\Lambda \times \{0\})^r,A(\Lambda \times \{0\})^r)$ and $(H^r_\varepsilon,(\Lambda \times \{0\})^r,A(G))$ are stably elementary equivalent.

Proof : By 4.3 : $(H^r_\varepsilon,(\Lambda \times \{0\})^r,A(G))$ and $(H^r_\varepsilon,(\Lambda \times \{0\})^r,A(\Lambda \times \{0\})^r) \oplus (H^r_\varepsilon,A(\Lambda \times \{0\})^r,A(\Lambda \times \{0\})^r)$ are stably equivalent.

q.e.d.

Now we can define the odd L - groups (monoids). $L^{s,\tau}_{2n+1}(\pi,w)$ is the set of stable elementary equivalence classes of non-singular weakly based $(-1)^n$-formations. Orthogonal sum defines the structure of an abelian momoid.

As in the case of even L - groups one can show that $L^{s,\tau}_{2n+1}(\pi,w)$ is a group. The Wall group $L^{s}_{2n+1}(\pi,w)$ is defined as $L^{s,\tau}_{2n+1}$ above replacing weakly based formations by based formations. There is an obvious injective homomorphism $L^{s}_{2n+1}(\pi,w) \longrightarrow L^{s,\tau}_{2n+1}(\pi,w)$ and the image is the kernel of the homomorphism $L^{s,\tau}_{2n+1}(\pi,w) \longrightarrow Wh(\pi)$ mapping $((M,\lambda,\mu),F,G)$ to the Whitehead torsion of the transformation matrix between the given basis on M and the basis induced on M from the basis of G as described above. This transformation matrix is a unitary transformation of H^r_{ε}. We denote the stable group of unitary transformations by $U^{\varepsilon}(\Lambda)$.

Thus we have a short exact sequence:

$$0 \longrightarrow L^{s}_{2n+1}(\pi,w) \longrightarrow L^{s,\tau}_{2n+1}(\pi,w) \longrightarrow Wh(\pi)$$

and $L^{s,\tau}_{2n+1}(\pi,w)$ is an abelian group. The image of the map on the right side is the subgroup represented by $U^{\varepsilon}(\Lambda)$.

Originally Wall defined $L^{s}_{2n+1}(\pi,w)$ as the quotient $SU^{(-1)^n}(\Lambda)/RSU^{(-1)^n}(\Lambda)$ where $SU^{\varepsilon}(\Lambda)$ is the subgroup of $U^{\varepsilon}(\Lambda)$ with trivial Whitehead torsion and $RSU^{\varepsilon}(\Lambda)$ ($RU(\Lambda)$ in the notation of([41],§6)) generated by simple isometries of H^r_{ε} preserving $(\Lambda \times \{0\})^r$ by a simple isomorphism and by the isometry $\sigma = \begin{pmatrix} 0 & (-1)^n \\ 1 & 0 \end{pmatrix}$ of H_{ε}. The same arguments as in ([33],Theorem 5.6) give an isomorphism $U^{(-1)^n}(\Lambda)/RSU^{(-1)^n}(\Lambda) \longrightarrow L^{s,\tau}_{2n+1}(\pi,w)$ mapping $A \in U^{\varepsilon}_r(\Lambda)$ to $(H^r_{\varepsilon},(\Lambda \times \{0\})^r, A(\Lambda \times \{0\})^r)$, where H^r_{ε} and $(\Lambda \times \{0\})^r$ are equipped with the standard bases and $A(\Lambda \times \{0\})^r$ with the image of the standard basis under A. (That the map is a homom orphism follows from Lemma 4.4.)

We summarize these considerations as follows.

Lemma 4.5 : There is a commutative diagram of exact sequences of abelian groups: ($\varepsilon = (-1)^n$):

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & \\
 & RSU^{\varepsilon}(\Lambda) \arrow[r, equal] \arrow[d] & RSU^{\varepsilon}(\Lambda) \arrow[d] & \\
0 \arrow[r] & SU^{\varepsilon}(\Lambda) \arrow[r] \arrow[d] & U^{\varepsilon}(\Lambda) \arrow[r] \arrow[d] & Wh(\pi) \arrow[d, equal] \\
0 \arrow[r] & L^{s}_{2n+1}(\pi, w) \arrow[r] \arrow[d] & L^{s,\tau}_{2n+1}(\pi, w) \arrow[r] \arrow[d] & Wh(\pi) \\
 & 0 & 0 &
\end{tikzcd}$$

To define $l^{s,\tau}_{2n+1}(\pi, w)$ we consider $\underline{\varepsilon\text{-quasi-formations}}$: triples $((M,\lambda,\mu),F,G)$, F a based lagrangian and G a half rank based direct summand in M (no lagrangian!). In a similar way as above the relation of stable elementary equivalence is defined. We define $l^{s,\tau}_{2n+1}(\pi, w)$ as the abelian monoid under orthogonal sum of stable elementary equivalence classes of $(-1)^n$-quasi-formations over Λ. We have an injective homomorphism $L^{s,\tau}_{2n+1}(\pi, w) \hookrightarrow l^{s,\tau}_{2n+1}(\pi, w)$.

As for $L^{s,\tau}_{2n+1}(\pi, w)$ we have an interpretation of $l^{s,\tau}_{2n+1}(\pi, w)$ in terms of matrices. Let $GL_r(\Lambda)$ be the group of invertible 2rx2r-matrices over Λ. This should be considered as automorphisms of H^r_{ε} (not respecting λ and μ). $GL(\Lambda) := \lim_{r\to\infty} GL_r(\Lambda)$. We have a map $GL(\Lambda) \longrightarrow l^{s,\tau}_{2n+1}(\pi, w)$, $A \in GL_r(\Lambda)$ is mapped to $(H^r_{\varepsilon}, (\Lambda \times \{0\})^r, A(\Lambda \times \{0\})^r)$. A given $((M,\lambda,\mu),F,G)$ is isomorphic to $(H^r_{\varepsilon}, (\Lambda \times \{0\})^r, G')$ which is in the image of this map: Extend the given basis of G' to a basis of H^r_{ε} and take the transformation matrix to the standard basis. Thus the map is surjective. A and A' are mapped to the same quasi formation if and only if there exists a $T \in GL_r(\Lambda)$ such that $A' = A\,T$ and $T \big|_{(\Lambda \times \{0\})^r}$ is a simple automorphism of this subspace. We denote the subgroup of $GL(\Lambda)$ of such matrices by $TsL(\Lambda)$ (the little s points to the fact that $T \big|_{(\Lambda \times \{0\})^r}$ is a simple automorphism but T itself is not simple).

Thus we have an induced map $GL(\Lambda)/_{TsL(\Lambda)}$ onto $l^{s,\tau}_{2n+1}(\pi, w)$. $RSU^{(-1)^n}(\Lambda)$ acts from the left and we also get a map $_{RSU^{\varepsilon}(\Lambda)}\backslash GL(\Lambda)/_{TsL(\Lambda)} \longrightarrow l^{s,\tau}_{2n+1}(\pi, w)$. For, by 4.4, $B \cdot A$ with $B \in RSU^{\varepsilon}(\Lambda)$ and $A \in GL(\Lambda)$ is mapped to $(H^r_\varepsilon, (\Lambda \times \{0\})^r, A(\Lambda \times \{0\})^r) \oplus (H^r_\varepsilon, (\Lambda \times \{0\})^r, B(\Lambda \times \{0\})^r)$ and the last formation is equivalent to zero in $l^{s,\tau}_{2n+1}(\pi, w)$.

<u>Lemma 4.6</u> : The map of double cosets

$$_{RSU^{\varepsilon}(\Lambda)}\backslash GL(\Lambda)/_{TsL(\Lambda)} \longrightarrow l^{s,\tau}_{2n+1}(\pi, w)$$

is an isomorphism of abelian monoids under orthogonal sum.

<u>Proof</u> : If A and $A' \in GL(\Lambda)$ are mapped to the same element in $l^{s,\tau}_{2n+1}(\pi, w)$ there exist $B \in SU^{\varepsilon}_1(\Lambda)$ preserving $(\Lambda \times \{0\})^1$ (thus $B \in RSU^{\varepsilon}(\Lambda)$), $r, r' \in \mathbb{N}$ and matrices $X = \begin{pmatrix} 1 & 0 \\ C-\varepsilon C^* & 1 \end{pmatrix}$ and $X' = \begin{pmatrix} 0 & 0 \\ C'-\varepsilon C'^* & 1 \end{pmatrix}$ such that $B(A \oplus r\sigma \oplus X)$ and $(A' \oplus r'\sigma \oplus X')$ are mapped to the same quasi formation. Thus there exists a $T \in TsL(\Lambda)$ such that $B \cdot (Id \oplus r\sigma \oplus X) \cdot A \cdot T = (Id \oplus r'\sigma \oplus X') \cdot A'$. The injectivity follows as $r\sigma \oplus X \in RSU^{\varepsilon}(\Lambda)$.

q.e.d.

Finally we define $L^{h,p}_{2n+1}(\pi,w)$ as the group of stable elementary equivalence classes of non-singular $(-1)^n$-formations $((M,\lambda,\mu),F,G)$ over Λ where we require F is free. If G is also free we obtain Wall's group $L^h_{2n+1}(\pi,w)$. If we only require that G is a sublagrangian we obtain the monoid $l^{h,p}_{2n+1}(\pi,w)$. Again, that $L^{h,p}_{2n+1}(\pi,w)$ is a group follows from

Lemma 4.7: There is an exact sequence of groups

$$0 \longrightarrow L^h_{2n+1}(\pi,w) \longrightarrow L^{h,p}_{2n+1}(\pi,w) \longrightarrow \tilde{K}_0(\pi)$$

$$[(M,\lambda,\mu),F,G] \longmapsto [G]$$

Proof: As for Lemma 4.1. a) we have to show that $G \in \operatorname{im} L^h_{2n+1}(\pi,w) \to \tilde{K}_0(\pi)$ has an inverse in the image. Let H be an inverse of G in $\tilde{K}_0(\pi)$. Then $H \oplus G^*$ is contained in the image, as it is a hamiltonian in the formation $((H \oplus G \oplus H^* \oplus G^*$, hyperbolic form), $H \oplus G \oplus \{0\} \oplus \{0\}$, $H \oplus \{0\} \oplus \{0\} \oplus G^*)$. Thus $H \oplus G^* \oplus G$ is contained in the image. As $G^* \oplus G \cong M$ is free we are finished.

q.e.d.

Obviously all elements in the different odd L-groups (monoids) can be represented by formations $(H^r_\varepsilon, (\Lambda \times \{0\})^r, G)$. On the set of these formations we have a left action of $RSU^\varepsilon(\Lambda)$: $(B,(H^r_\varepsilon,(\Lambda \times \{0\})^r, G)) \longmapsto (H^r_\varepsilon,(\Lambda \times \{0\})^r, B(G))$. Similar considerations as for the proof of Lemma 4.6 show that the orbit space is isomorphic to the corresponding odd L-groups (monoids) if we restrict G appropriately.

<u>Lemma 4.8:</u> $\varepsilon = (-1)^n$.

$$RSU^{\varepsilon}(\Lambda)\backslash\{(H^{\tau}_{\varepsilon},(\Lambda\times\{0\})^r,G)\} \cong \begin{cases} L^s_{2n+1}, & \text{if } (H^r_{\varepsilon},G) \text{ is a based lagrangian} \\ L^{s,\tau}_{2n+1}, & \text{if } G \text{ is a lagrangian and based} \\ l^{s,\tau}_{2n+1}, & \text{if } G \text{ is a half rank direct summand} \\ L^h_{2n+1}, & \text{if } G \text{ is a lagrangian and free} \\ L^{h,p}_{2n+1}, & \text{if } G \text{ is a lagrangian} \\ l^{h,p}_{2n+1}, & \text{if } G \text{ is a sublagrangian} \end{cases}$$

Most of the obstruction groups defined in this chapter are contained already in the literature. I mentioned already those groups which were defined by Wall in [41] . In several cases there were no geometric applications known. For instance Bak [1] introduced similar groups like our $L^{S,\tau}_{2n}(\pi,w,S)$ for purely algebraic reasons. In this as well as in the other cases we will give geometric applications in the next chapters. Bak introduced the group of stably equivalence classes of based non-singular $(-1)^n$-quadratic forms over (Λ,S) denoted by $WQ_0^{(-1)^n}(\Lambda,S)_{\text{based}} - [\pm\pi]$ which I will denote by $L^s_{2n}(\pi,w,S)$. If $S \subseteq S'$ are form parameters Bak computes the kernel of the surjective map $L^s_{2n}(\pi,w,S) \longrightarrow L^s_{2n}(\pi,w,S')$ in terms of an exact sequence ([1], §11). If Λ is a so called trace noetherian ring (which is for instance true if π is finite) ([1] ,p.202) the answer is very explicit: The kernel is then isomorphic to $(S'/S \otimes_{\Lambda} S'/S) / \{a\otimes b - b\otimes a,\ a\otimes b - a\otimes ba\bar{b}\}$ where Λ operates on $S'/_S$ from the left by $x \longmapsto \lambda x \bar{\lambda}$ and from the right by $x \longmapsto \bar{\lambda} x \lambda$ ([1] , Theorem 1.2). Obviously the maps $L^S_{2n}(\pi,w,S) \longrightarrow L^S_{2n}(\pi,w,S')$ and $L^{S,\tau}_{2n}(\pi,w,S) \longrightarrow L^{S,\tau}_{2n}(\pi,w,S')$ have same kernel. Thus we obtain

Theorem 4.9 (Bak [1] , Theorem 1.2): If $S \subseteq S'$ are form parameters and Λ is trace neotherian (for instance π finite) there is a split exact sequence

$$0 \longrightarrow (S'/_S \otimes_\Lambda S'/_S)/_{\{a\otimes b - b\otimes a,\ a\otimes b - a\otimes b\, a\,\bar{b}\}} \longrightarrow L^{S,\tau}_{2n}(\pi,w,S) \longrightarrow L^{S,\tau}_{2n}(\pi,w,S') \longrightarrow 0$$

The map on the left side is given by ($\varepsilon = (-1)^n$)

$$[a\otimes b] \longmapsto [\Lambda \oplus \Lambda, \begin{pmatrix} 0 & \varepsilon \\ 1 & 0 \end{pmatrix}, \mu(e_1) = a \text{ and } \mu(e_2) = b]$$

Example 4.10: $\pi = \{1\}$. If n is even the only form parameter group is $\{0\}$. If n is odd there are two form parameter groups: $S = 2\mathbb{Z} \subset \mathbb{Z} = \Lambda$ and $S' = \mathbb{Z}$. $L^S_{2n}(1\ , S=2\mathbb{Z})$ is the ordinary surgery obstruction group in the 1-connected case and isomorphic to $\mathbb{Z}_2$ [41], [5] . It is easy to check that $L^S_{2n}(1\ , S' = \mathbb{Z}) = \{0\}$.

Bak defined corresponding odd-dimensional surgery obstruction groups with respect to a form parameter S. In our situation we only need the additional case $S = \{a+\bar{a}\} + \mathbb{Z}$ and only the corresponding monoid denoted by $l^{S,\tau}_3(\pi,w,\mathbb{Z})$ The definition is obvious: equivalence classes of quasi formations $((M,\lambda,\bar{\mu}),F,G)$ where $\bar{\mu}$ is a quadratic refinement with values in $\Lambda/_{\{a+\bar{a}\}} + \mathbb{Z}$.

The following diagram describes the obvious maps between the different obstruction groups (monoids). The vertical sequences in the diagram are exact.

$$
\begin{tikzcd}
 & 0 \arrow[d] & & 0 \arrow[d] & \\
 & L_n^s(\pi,w) \arrow[d] & & L_n^h(\pi,w) \arrow[d] & \\
l_n^{s,\tau}(\pi,w) & L_n^{s,\tau}(\pi,w) \arrow[l, hook'] \arrow[d] \arrow[urr] & & L_n^{h,p}(\pi,w) \arrow[r, hook] \arrow[d] & l_n^{h,p}(\pi,w) \\
 & Wh(\pi) & & \widetilde{K}_0(\pi) &
\end{tikzcd}
$$

and we have maps:

$$L_{2n}^{s,\tau}(\pi,w) \twoheadrightarrow L_{2n}^{s,\tau}(\pi,w,S) \hookrightarrow l_{2n}^{s,\tau}(\pi,w,S)$$

where the kernel of the map on the left side is described by Bak ([1], compare Theorem 4.9), and we have a map $I_{2n+1}^{s,\tau}(\pi,w) \longrightarrow I_{2n+1}^{s,\tau}(\pi,w,\mathbb{Z})$.

We finish this chapter with a reformulation of Theorem 3.1. If $f : M \# r(S^n \times S^n) \longrightarrow N \# r(S^n \times S^n)$ is a diffeomorphism and $\vartheta(f)$: $H_\varepsilon^r \longrightarrow H_\varepsilon^r$ is an isometry then $\vartheta(f)$ defines an element in $L_{2n+1}^{s,\tau}(\pi_1(M),w_1(M))$. By the exact sequence of Lemma 4.5, $\vartheta(f) = 0$ if and only if $\tau(f) = 0$ and $\vartheta(f)$ vanishes in $L_{2n+1}^{s}(\pi_1(M),w_1(M))$. Thus we obtain:

<u>Reformulation of Theorem 3.1:</u> Let $f : M \# r(S^n \times S^n) \longrightarrow N \# r(S^n \times S^n)$ be a diffeomorphism, $n \neq 1,3,7$. Suppose that $\vartheta(f)$ is an isometry. If $\vartheta(f)$ vanishes in $L_{2n+1}^{s,\tau}(\pi_1(M),w_1(M))$ then $f|_{\partial M} : \partial M \longrightarrow \partial N$ extends to a diffeomorphism (relative s-cobordism, if $n = 2$) between M and N.

§ 5 Classification of normal k-smoothings I

Let $p : B \longrightarrow BO$ be a fibration and $(V, \bar{\nu}_V)$ a fixed closed (n-1)-dimensional B-manifold. We are going to study the following problem: Given a relative n-dimensional compact B-manifold $(M, \bar{\nu}_M, f)$, under which conditions is it B-bordant rel. boundary to a relative normal k-smoothing (i.e. $\bar{\nu}_M$ is a (k+1)-equivalence)? Classify the relative normal k-smoothings within a given bordism class.

There is an obvious necessary condition on (B,V). Suppose $(M, \bar{\nu}_M, f)$ is a normal k-smoothing, $k \geq [n/2]$. The local orientation of B induces a local orientation on M. This specifies a generator $[M, \partial M]$ of $H_n(M, \partial M; \mathbb{Z}^t) \cong \mathbb{Z}$, the homology group with twisted integer coefficients (compare [43]). Denote the image of $[M, \partial M]$ under $\bar{\nu}_M$ by $\alpha \in H_n(B,V;\mathbb{Z}^t)$. Consider the commutative diagram, where K is some coefficient module over Λ

$$\begin{tikzcd}
H_r(M, \partial M; K) \arrow[r] & H_r(B,V;K) \\
H^{n-r}(M;K) \arrow[u, "\cap [M,\partial M]"] & H^{n-r}(B;K) \arrow[l] \arrow[u, "\cap \alpha"']
\end{tikzcd}$$

where we identify $\pi_1(M)$ and $\pi_1(B)$ by $\bar{\nu}_M$ and $\Lambda = \mathbb{Z}[\pi_1(B)]$. If $r < k+1$ and $n-r < k+1$ then all maps except $\cap\alpha$ and thus $\cap\alpha$ are isomorphisms. If $r = k+1$, $\cap\alpha$ is surjective, if $r = n-k-1$ it is injective.

Given a map $V^{n-1} \longrightarrow B$, V a closed manifold, $w : \pi_1(B) \longrightarrow \mathbb{Z}_2$ a homomorphismus such that the composition $\pi_1(V) \longrightarrow \pi_1(B) \longrightarrow \mathbb{Z}_2$ is the first Stiefel-Whitney class of V and $\alpha \in H_n(B,V;\mathbb{Z}^t)$ we call $(B;V;w,\alpha)$ a n-dimensional <u>relative k-partial Poincaré complex</u> (assume $k \geq n/2$) if the map $\cap\alpha$ fulfils the properties above and $d(\alpha) \in H_{n-1}(V;\mathbb{Z}^t)$ is a fundamental class of V. We call α then a k-partial fundamental class. If $B \longrightarrow BO$ is a fibration we always assume that w is the homomorphism $w_1(B) : \pi_1(B) \longrightarrow \pi_1(BO) = \mathbb{Z}_2$ without mentioning it further. If (B,V,α) is a k-partial Poincaré complex for all k then it is a Poincaré complex in the ordinary sense. For instance as used above all compact manifolds are Poincaré complexes ([41], Theorem 2.1).

If $B \longrightarrow BO$ is a fibration, $(V,\bar{\nu}_V)$ a closed B-manifold, a relative B-manifold $(M,\bar{\nu}_M,f)$ is k-admissible if $(\bar{\nu}_M)_*[M,\partial M] = \alpha$ is a k-partial fundamental class. This generalizes the degree one condition for normal maps into a Poincaré complex. The homolgy class $(\bar{\nu}_M)_*[M,\partial M] \in H_n(B,V;\mathbb{Z}^t)$ is a bordism invariant. We denote the set of bordism classes represented by k-admissible maps by $A\Omega_{n,k}^{(B,V)}$. If $k=[n/2]-1$, every $(M,\bar{\nu}_M,f)$ is k-admissible and thus $A\Omega_{n,k}^{(B,V)} = \Omega_n^{(B,V)}$.
To formulate our results we have to introduce some invariants. Let $(M,\bar{\nu}_M,f)$ be a n-dimensional k-admissible relative (B,V)-manifold or $(M,\bar{\nu}_M)$ a n-dimensional B-bordism rel. boundary between two relative normal (k-1)-smoothings $(M_0,\bar{\nu}_{M_0},f_0)$ and $(M_1,\bar{\nu}_{M_1},f_1)$ in (B,V). We assume B to have finite $[n/2]+1$-skeleton. By Lemma 2.3 we can assume up to bordism rel. boundary that $\bar{\nu}_M$ is a $[n/2]$-equivalence.

In this chapter we study the case n even, $n=2m$. We consider the homotopy sequence

$$\pi_{m+1}(B,V) \longrightarrow \pi_m(M) \longrightarrow \pi_m(B) \longrightarrow 0$$

As mentioned in §3 there is a $(-1)^m$ - quadratic form (λ,μ) defined on immersions $S^m \looparrowright M$. Every element of $\pi_{m+1}(B,M)$ determines an immersion $S^m \times D^m \looparrowright M$ ([41], p.10). Thus (λ,μ) is defined on $\pi_{m+1}(B,M)$. In most cases $(\pi_{m+1}(B,M),\lambda,\mu)$ will represent our even-dimensional surgery obstruction.

In some cases we have to replace $\pi_{m+1}(B,M)$ by a quotient of $K\pi_m(M)$, meaning here the kernel of $\pi_m(M) \xrightarrow{\bar{\nu}_m} \pi_m(B)$. If $m \neq 1,3,7$ (λ,μ) is also defined on $K\pi_m(M)$ as discussed in §3. If $m = 3,7$ it is not difficult to show that (λ,μ) is defined on $K\pi_m(M) = \pi_{m+1}(B,M)/\pi_{m+1}(B)$ if for all $\alpha \in \pi_{m+1}(B)$, $w_{m+1}(\alpha^* p^* \gamma) = 0$. The reason for this is that for $m = 3,7$ the value of μ on $\pi_{m+1}(B)$ is determined by this Stiefel Whitney number. If this Stiefel Whitney number is non-trivial for some α then we get a form $\bar{\mu} : K\pi_m(M) \longrightarrow \Lambda/\langle a+\bar{a}\ ,\ \mathbb{Z}\rangle = \mathbb{Q}_-/\mathbb{Z}_2$. In this case the quadratic form $\bar{\mu}$ works as good to control surgery as μ. For then $\bar{\mu}(x) = 0$ implies x can be represented by an embedding with framed normal bundle such that we can do surgery on it compatible with the B-structure.

To decide in which obstruction group or monoid the invariant sits we have to study $\pi_{m+1}(B,M)$ and λ. As (B,M) is m-connected, $\pi_{m+1}(B,M) \cong H_{m+1}(B,M;\Lambda)$ by the Hurewicz isomorphism and is finitely generated as B has finite (m+1)-skeleton. <u>We first investigate the case where $(M,\bar{\nu}_M,f)$ is a k-admissible map and $k>m$.</u> We will show that then $H_{m+1}(B,M;\Lambda)$ is <u>projective</u>.

This is true if $H^{m+2}(B,M)$ vanishes with all coefficients in a Λ-module ([31] , Lemma 2.3). This follows by partial Poincaré duality from the following diagram with arbitrary coefficient module:

$$\begin{tikzcd}
H^{m+1}(B) \arrow[r, hook] \arrow[d, "\cong"] & H^{m+1}(M) \arrow[r] \arrow[d, "\cong"] & H^{m+2}(B,V) \arrow[r] & H^{m+2}(B) \arrow[r] \arrow[d] & H^{m+2}(M) \arrow[d, "\cong"] \\
H_{m-1}(B,V) & H_{m-1}(M,\partial M) \arrow[l, "\cong"'] & & H_{m-2}(B,V) & H_{m-2}(M,\partial M) \arrow[l, "\cong"']
\end{tikzcd}$$

If $k \geq m$ partial Poincaré duality implies that the sequence

$$0 \longrightarrow H_{m+1}(B,M;\Lambda) \longrightarrow H_m(M;\Lambda) \longrightarrow H_m(B;\Lambda) \longrightarrow 0$$

is short exact and splits. This implies that the form λ on $H_{m+1}(B,M;\Lambda)$ is non-singular (compare [41] , Lemma 2.2). Furthermore it implies that $H_{m+1}(B,M;\Lambda)$ is free as abelian group, a lattice.

If B is a finite Poincaré complex we are in the situation of Wall's book. Then a similar argument as above shows that $H_{m+1}(B,M;\Lambda)$ is the only non-vanishing homology group and thus is stably free ([41] , Lemma 2.3).

We can stabilize $H_{m+1}(B,M;\Lambda)$ if we make connected sum of M with $r(S^m \times S^m)$ as described in §2. Thus we can assume in this case $H_{m+1}(B,M;\Lambda)$ to be free.

Thus the ε-quadratic form $(\pi_{m+1}(B,M), \lambda, \mu)$ represents an element in $L^{h,p}_{2m}(\pi_1(B), w_1(B))$ if $k = m$, in $L^{hp}_{2m}(\pi_1(B), w_1(B))$ if $k > m$ and in $L^{h}_{2m}(\pi_1(B), w_1(B))$ if B is a finite Poincaré complex. We will show that this is a bordism invariant and denote it by $\Theta(M, \bar{\nu}_M, f)$. If $k = m-1$ we define it as zero.

Now we study the case where $(M, \bar{\nu}_M)$ is a B-bordism between two (2m-1)-dimensional relative normal (k-1)-smoothings $(M_0, \bar{\nu}_{M_0}, f_0)$ and $(M_1, \bar{\nu}_{M_1}, f_1)$, $k \geq m-1$. As (B,M) is assumed to be m-connected and (B,M_i) is at least (m-1)-connected, (M,M_i) is (m-1)-connected. Poincaré duality implies that $H_m(M,M_i;K)$ is then the only non-vanishing homology group (In this case we have Poincaré duality: $H_*(M,M_i;K) \xleftarrow{\cong} H^{2m-*}(M,M_i;K)$ where i is counted mod 2). Thus as above $H_m(M,M_i;\Lambda)$ is stably free and we assume that it is free. We can choose a preferred class of bases on $H_m(M,M_i;\Lambda)$ (Whitehead torsion vanishes). Finally as (M,M_i) is (m-1)-connected, Poincaré duality implies the intersection form

$$\lambda : H_m(M,M_0;\Lambda) \times H_m(M,M_1;\Lambda) \longrightarrow \Lambda$$

is non-singular and simple ([41] ,Theorem 2.1). This intersection form is related to the intersection form on $\pi_m(M)$ by: $\lambda(x,y) = \lambda(i_0 x, i_1 y)$

where $x,y \in \pi_m(M)$ and i_0, i_1 are the maps $\pi_m(M) \longrightarrow \pi_m(M,M_i)$.

If $k > m$ our surgery obstruction will again be represented by $(\pi_{m+1}(B,M), \lambda,\mu)$. We have to show that $\pi_{m+1}(B,M)$ is free with a preferred basis and λ is non-singular to obtain an element in $L_{2m}^{s,\tau}(\pi_1(B),w_1(B))$. We will see that the boundary homomorphism $H_{m+1}(B,M;\Lambda) \longrightarrow H_m(M,M_i;\Lambda)$ is an isomorphism. This implies by the non-singularity of λ: $H_m(M,M_0;\Lambda) \times H_m(M,M_1;\Lambda) \longrightarrow \Lambda$ that λ is non-singular on $H_{m+1}(B,M;\Lambda) = \pi_{m+1}(B,M)$. The preferred basis on $H_{m+1}(M,M_0;\Lambda)$ is obtained from the preferred basis on $H_m(M,M_0;\Lambda)$ under this isomrphism. We denote the corresponding element in $L_{2m}^{s,\tau}(\pi_1(B),w_1(B))$ by $\Theta(M,\bar{\nu}_M)$. The isomorphism $H_{m+1}(B,M;\Lambda)$ $H_m(M,M_i;\Lambda)$ follows from the diagram of exact suquences with Λ-coefficients:

$$\begin{tikzcd}
 & 0 \arrow[d] & & \\
 & H_{m+1}(B,M) \arrow[d] \arrow[dr] & & \\
H_m(M_i) \arrow[r] \arrow[dr, "\cong"'] & H_m(M) \arrow[r] \arrow[d] & H_m(M,M_i) \arrow[r, dash] & 0 \\
 & H_m(B) \arrow[d] & & \\
 & 0 & &
\end{tikzcd}$$

Remark 5.1 : The definition is not symmetric in M_0 and M_1 . The only difference is the preferred basis. The Whitehead torsion of the base change is the torsion of the composite isomorphism

$$H_m(M,M_0;\Lambda) \xleftarrow{\cong} H_{m+1}(B,M;\Lambda) \xrightarrow{\cong} H_m(M,M_1;\Lambda) .$$

If B is a simple Poincaré complex and $\bar{\nu}_{M_0}$ and $\bar{\nu}_{M_1}$ are simple isomorphisms then this Whitehead torsion vanishes and we obtain the ordinary Wall surgery obstruction in $L^s_{2m}(\pi_1(B),w_1(B))$. A similar asymmetry will occur for the odd-dimensional surgery obstructions in §6. The reason for introducing $L^{s,\tau}$ is that the Whitehead torsion of a normal k-smoothing is in general not defined so that we don't have a definition of simple normal k-smoothings.

If $k = m$ or $m-1$ our surgery obstruction sits in $L^{s,\tau}_{2m}(\pi_1(B),w_1(B);S)$ or $l^{s,\tau}_{2m}(\pi_1(B),w_1(B),S)$. We have to define S. On the image of $K\pi_m(\partial M) \to K\pi_m(M)$ the intersection form λ vanishes. If $m \neq 3,7$ or $m = 3,7$ and $w_{m+1}(\alpha^* p^* \gamma) = 0$ for all $\alpha \in \pi_{m+1}(B)$ we have the quadratic refinement μ of λ defined on $K\pi_m(M)$. μ is, restricted to the image $\operatorname{im} K\pi_m(\partial M) \longrightarrow K\pi_m(M)$, a homomorphism. Especially it is a homomorphism on $\operatorname{im} K\pi_m(M_0) \cap \operatorname{im} K\pi_m(M_1)$. We denote S(M) as the subgroup of Λ which projects to the image of μ on this intersection submodule. As μ is there a homomorphism S(M) fulfils our properties for S. If $m = 3$ or 7 and $w_{m+1}(\alpha^* p^* \gamma) \neq 0$ for some $\alpha \in \pi_{m+1}(B)$ we have to work with $\bar{\mu}$ instead of μ. We replace S(M) then by its sum with $\mathbb{Z} \subset \Lambda$. In the following we will not give seperate arguments for this case as everything works with appropriate modifications.
If $k = m$ we will show $\operatorname{im} K\pi_m(M_0) = \operatorname{im} K\pi_m(M_1)$ in $K\pi_m(M)$. S(M) is then equal to $S(M_i \times I)$. The map $\pi_m(M) \longrightarrow \pi_m(M,M_i)$ induces an isomorphism $K\pi_m(M)/K\pi_m(M_0) \xrightarrow{\cong} \pi_m(M,M_i)$. This follows from the diagram

$$\begin{array}{ccccccc}
 & & \pi_{m+1}(B,M) & & & & \\
 & & \downarrow & & & & \\
\pi_m(M_i) & \longrightarrow & \pi_m(M) & \longrightarrow & \pi_m(M,M_i) & \longrightarrow & 0 \\
 & \searrow & \downarrow & & & & \\
 & & \pi_m(B) & & & & \\
 & & \downarrow & & & & \\
 & & 0 & & & &
\end{array}$$

The form $(\lambda,\mu)^{\vee}$ (resp. $\lambda,\bar{\mu}$) on $K\pi_m(M)$ induces a form over $(\Lambda, S(M_i \times I)$ on $K\pi_m(M)/K\pi_m(M_0)$ representing an element $\Theta(M,\bar{\nu}_M) \in L^{s,\tau}_{2m}(\pi_1(B),w_1(B),S(M_0 \times I) = S(M_1 \times I))$.

To show that $\operatorname{im} K\pi_m(M_0) = \operatorname{im} K\pi_m(M_1)$ we assume there exists $x \in \operatorname{im} K\pi_m(M_0)$ but $x \notin \operatorname{im} K\pi_m(M_1)$ (or if we interchange the role of M_0 and M_1). Then x represents a non-trivial element in $\pi_m(M,M_1)$. As λ is non-singular and $\pi_m(M) \longrightarrow \pi_m(M,M_0)$ is surjective there exists a $y \in \pi_m(M)$ s.t. $\lambda(x,y) \neq 0$. This is a contradiction as x vanishes in $\pi_m(M,M_0)$ and thus $\lambda(x,y) = 0$.

Finally if $k = m-1$, $\mu^{\vee}$ (resp. $\bar{\mu}$) induces a map $K\pi_m(M)/K\pi_m(M_0) \cap K\pi_m(M_1) \longrightarrow \Lambda/S(M)$ again denoted as $\mu^{\vee}$ or $\bar{\mu}$. $K\pi_m(M)/K\pi_m(M_0) \cap K\pi_m(M_1)$ is a finitely generated Λ - module as the finitely generated Λ - module $H_{m+1}(B,M;\Lambda)$ maps surjectively onto $K\pi_m(M)$. Our surgery obstruction is in this case $\Theta(M,\bar{\nu}_M)$, represented by $(\pi_m(M,M_0) \longleftarrow K\pi_m(M)/K\pi_m(M_0) \cap K\pi_m(M_1) \longrightarrow \pi_m(M,M_1)$, $\lambda, \mu^{\vee}$ or $\bar{\mu}$) in $l^{s,\tau}_{2m}(\pi_1(B),w_1(B);S(M))$, where $\lambda : \pi_m(M,M_0) \times \pi_m(M,M_1) \longrightarrow \Lambda$ is the intersection form.

Before we formulate the main result for even-dimensional manifolds we introduce the following notation: An element of $l^{s,\tau}_{2m}(\pi,w,S)$ is zero bordant if there is a representative $(V \leftarrow M \rightarrow W, \lambda, \mu)$ which has a based submodule $U \subset M$ s.t. i) $U \subset U^{\perp}$, ii) the image of U in V and W denoted as U_V and U_W is a direct summand and λ induces a simple isomorphism $U_V \longrightarrow (W/U_W)^*$ and iii) $\mu|_U = 0$ or $\bar{\mu}|_U = 0$.

Remark : If the element is in the image of $L^{s,\tau}_{2m}(\pi,w,S)$ then it is zero bordant if and only if it is trivial.

Theorem 5.2: $m \geq 3$.

a) Let $(M^{2m}, \bar{\nu}_M, f)$ be a 2m-dimensional k-admissible ($k \geq m-1$) relative (B,V)-manifold, where B has finite m+1-skeleton. Then the invariant

$$\theta(M, \bar{\nu}_M, f) \quad \in \begin{cases} \{0\}, & k = m-1 \\ l_{2m}^{h,p}(\pi_1(B), w_1(B)), & k = m \\ L_{2m}^{h,p}(\pi_1(B), w_1(B)), & k > m \\ L_{2m}^{h}(\pi_1(B), w_1(B)), & k = 2m \text{ and } B \text{ a finite Poincaré complex} \end{cases}$$

is a bordism invariant rel. boundary.

$\theta(M, \bar{\nu}_M, f) = 0 \Leftrightarrow (M, \bar{\nu}_M, f)$ is bordant rel. boundary to a normal k-smoothing.

b) Let $(M, \bar{\nu}_M)$ be a 2m-dimensional B-bordism rel. boundary between two relative normal (k-1)-smoothings $(M_0, \bar{\nu}_{M_0}, f_0)$ and $(M_1, \bar{\nu}_{M_1}, f_1)$ in (B,V), where again B has finite (m+1)-skeleton.

Then for k = m,m-1 the subgroup $S(M) \subset \Lambda$ (which as we have shown above for k = m is equal to $S(M_0 \times I) = S(M_1 \times I)$) and the invariant

$$\theta(M, \bar{\nu}_m) \quad \in \begin{cases} l_{2m}^{s,\tau}(\pi_1(B), w_1(B), S(M)), & k = m-1, m \neq 3, 7 \\ L_{2m}^{s,\tau}(\pi_1(B), w_1(B), S(M)), & k = m, m \neq 3, 7 \\ L_{2m}^{s,\tau}(\pi_1(B), w_1(B)), & k > m \\ L_{2m}^{s}(\pi_1(B), w_1(B)), & k=2m \text{ and } B \text{ a finite simple Poincaré complex} \end{cases}$$

is a bordism invariant rel. boundary. If $k=m-1$ or m and $m=3,7$ the same result holds if $\pi_{m+1}(B)\to\mathbb{Z}_2,\ \kappa\mapsto\langle w_{m+1}(p^*\gamma),\alpha\rangle$, is trivial. If not we have to replace the obstruction monoid (group) by $\mathcal{L}^{s,\tau}_{2m}(\pi_1,w_1,S(M)+\mathbb{Z})$ or $L^{s,\tau}_{2m}(\pi_1,w_1\,S(M)+\mathbb{Z})$.
$(M,\bar{\nu}_M)$ is bordant rel. boundary to a s-cobordism $\Longleftrightarrow$

$$\begin{cases} \theta(M,\bar{\nu}_M)=0 & \text{if } k\geq m \\ \theta(M,\bar{\nu}_M) \text{ zero bordant} & \text{if } k=m-1. \end{cases}$$

<u>Proof:</u> Suppose $(M,\bar{\nu}_M,f)$ or $(M,\bar{\nu}_M)$ bordant rel. boundary to $(N,\bar{\nu}_N,f)$ or $(N,\bar{\nu}_N)$ and assume that $\bar{\nu}_N$ is also a m-equivalence. That means that $(M,\bar{\nu}_M)$ and $(N,\bar{\nu}_N)$ are normal (m-1)-smoothings which are bordant rel. boundary. Theorem 2.1 implies that they are stably s-cobordant. This implies that $S(M)=S(N)$ and $\theta(M,\bar{\nu}_M,f)=\theta(N,\bar{\nu}_N,g)$ and $\theta(M,\bar{\nu}_M)=\theta(N,\bar{\nu}_N)$.

Now, suppose $\theta(M,\bar{\nu}_M,f)=0$. We will show that we can make (B,M) then (m+1)-connected by a sequence of surgeries. Partial Poincaré duality implies then that we have obtained a normal k-smoothing. Similarly if $\theta(M,\bar{\nu}_M)=0$ (or $\theta(M,\bar{\nu}_M)$ zero bordant, if k=m-1) we will kill $\pi_*(M;M_1)$ by a sequence of surgeries. Then Poncaré duality implies that M is a h-cobordism. Working carefully with preferred bases M will even be a s-cobordism.

We will carry the proof through for the case $\theta\,(M, \bar{\nu}_M)$ zero bordant and $k = m-1$. All other cases can with some obvious modifications be proved in the same way.

After possibly stabilizing M by a connected sum with $(S^n \times S^n)$'s we can assume that for $(\pi_m(M,M_0) \longleftarrow K\,\pi_m(M)/K\pi_m(M_0) \cap K\pi_m(M_1) \longrightarrow \pi_m(M,M_1), \lambda, \mu)$ there exists a based submodule $U \subset K\,\pi_m(M)/K\pi_m(M_0) \cap K\pi_m(M_1)$ with the properties i) - iii) in the definition of "zero bordant" above. The definition of $S(M)$ implies that we can find $x_1,\dots,x_k \in K\,\pi_m(M)$ representing a basis of U and $\mu(x_i) = 0$ in $Q_{(-1)^n}$. As also $\lambda(x_i,x_j) = 0$ one can find disjoint embeddings $(S^m \times D^m)_i$ representing x_i([41], Theorem 5.2 or [34], Proposition 5.2; this is proved there only for a single element but the same Whitney trick argument shows that one can choose the $(S^m \times D^m)_i$ disjointly). Furthermore we can assume the embeddings compatible with the B-structure ([5] , § IV.1) so that we can make surgery with them. We claim that the resulting B-manifold $(M', \bar{\nu}_{M'})$ is a s-cobordism.

It is clear that $\pi_1(M_i) \longrightarrow \pi_1(M')$ is an isomorphism. To compute $H_*(M',M_1;\Lambda)$ we consider the following exact sequences with Λ-coefficients. Write $X = \cup\,(S^m \times D^m)_i$

$$\begin{tikzcd}
 & H_{m+1}(M',M-\mathring{X}) \arrow[d, "d"] & & & & \\
0 \arrow[r] & H_m(M-\mathring{X},M_1) \arrow[r] \arrow[d] & H_m(M,M_1) \arrow[r, "j"] & H_m(M,M-\mathring{X}) \arrow[r] & H_{m-1}(M-\mathring{X},M_1) \arrow[r] \arrow[d, "\cong"] & 0 \\
 & H_m(M',M_1) \arrow[d] & & & H_{m-1}(M',M_1) & \\
 & 0 & & & &
\end{tikzcd}$$

By excision $H_* (M,M-\mathring{X}) \cong H_* (\tilde{X}, \partial \tilde{X})$ where $\tilde{X}$ means the counterimage of X $\overbrace{\text{in } \tilde{M}}^{\text{the universal covering}}$. Thus $H_* (M,M-\mathring{X})$ is trivial except for $* = m$ where it is Λ^k with basis $(\{*\} \times D^m)_i$ or $* = 2m$ where it is again Λ^k with basis $[(S^m \times D^m)_i, \partial]$. This basis represents a preferred basis of $H_* (M,M-\mathring{X})$.

Similarly $H_* (M',M-\mathring{X})$ has a preferred basis represented by $(D^{m+1} \times \{*\})_i$ in dimension $m+1$ and by $[(D^{m+1} \times S^{m-1})_i, \partial]$ in dimension $2m$.

With respect to this basis the homomorphism $H_m(M,M_1) \xrightarrow{j} H_m(M,M-\mathring{X})$ is given by $x \longmapsto (\lambda(x,x_1), \dots, \lambda(x,x_k))$. If we denote the image of U in $H_m(M,M_i)$ by U_i, condition ii) of the definition of zero bordant in $l_{2m}^{s,\tau}$ implies that $H_m(M,M_1)$ splits as $U_1 \oplus H_m(M,M_1)/U_1$ and that j is a simple isomorphism from $H_m(M,M_1)/U_1$ to $H_m(M,M-\mathring{X})$. Thus $H_{m-1}(M-\mathring{X},M_1)$ and with it $H_{m-1}(M',M_1)$ vanishes. Furthermore $H_m(M-\mathring{X},M_1)$ is isomorphic to U_1 and if we equip $H_m(M-\mathring{X},M_1)$ with the preferred basis of U_1 then d: $H_{m+1}(M',M-\mathring{X})$ $H_m(M-\mathring{X},M_1)$ is a simple isomorphism.

As $H_*(M,M_1)$ vanishes for $* \neq m$, $H_{2m}(M,M-\mathring{X}) \xrightarrow{\cong} H_{2m-1}(M-\mathring{X},M_1)$. If we equip $H_{2m-1}(M-\mathring{X},M_1)$ with the preferred basis of $H_{2m}(M,M-\mathring{X})$ then d: $H_{2m}(M',M-\mathring{X}) \longrightarrow H_{2m-1}(M-\mathring{X},M_1)$ is a simple isomorphism.

These considerations imply that $H_*(M',M_1) = \{0\}$ and that the Whitehead torsion of the acyclic complex given by the based horizontal and vertical homology sequences vanishes. As (M,M_1) and $(M,M-\mathring{X})$ have trivial Whitehead torsion the additivity formula implies that $(M-\mathring{X},M_1)$ has trivial torsion. The same argument applied to the vertical sequence implies that (M',M_1) has trivial torsion. Thus M' is a s-cobordism.

If $(M, \bar{\nu}_M, f)$ is bordant rel. boundary to a normal k-smoothing then obviously $\theta\,(M, \bar{\nu}_M, f) = 0$ and if $(M, \bar{\nu}_M)$ is bordant rel. boundary to a s-cobordism then it is clear that $\theta(M, \bar{\nu}_M) = 0$ except in the case $k=m-1$. The difficulty there comes from the fact that $\theta\,(M, \bar{\nu}_M)$ is only defined after transforming $\bar{\nu}_M$ into a m-equivalence. Thus if $M = M_0 \times I$ and $\bar{\nu}_{M_0}$ is a (m-1)-equivalence $\theta\,(M_0 \times I, \bar{\nu}_{M_0} \cdot P)$ is only defined if we first transform it into a m-equivalence.

Given $(M, \bar{\nu}_M)$, $\bar{\nu}_M$ a m-equivalence, which is bordant to a s-cobordism and suppose that the bordism is obtained by a sequence of surgeries on disjoint embeddings $(S^m \times D^m)_i$ as above. Then the considerations above show in turn that $\theta\,(M, \bar{\nu}_M)$ is zero bordant. But there is always after possibly stabilizing $(M, \bar{\nu}_M)$ by connected sum with $(S^m \times S^m)$'s a bordism of this type between $(M, \bar{\nu}_M)$ and a s-cobordism $(N, \bar{\nu}_N)$ if $(M, \bar{\nu}_M)$ is bordant to $(N, \bar{\nu}_N)$. Namely by similar considerations as in Lemma 2.3 we can transform $(N, \bar{\nu}_N)$ by surgeries on disjoint embeddings $(S^{m-1} \times D^{m+1})_i$ into a normal (m-1)-smoothing $(N', \bar{\nu}_{N'})$.† by surgeries on $(D^m \times S^m)_i$. On the other hand, as $(N', \bar{\nu}_{N'})$ and $(M, \bar{\nu}_M)$ are bordant normal (m-1)-smoothings they are stably diffeomorphic by Theorem 2.1. This ends the proof of Theorem 5.2.

† In turn $(N, \bar{\nu}_N)$ is obtained from $(N', \bar{\nu}_{N'})$

§ 6 Classification of normal k-smoothings II

In this chapter we want to extend our results to the odd-dimensional case. Suppose that $(M, \bar{\nu}_M, f)$ is a (2m+1)-dimensional k-admissible relative (B,V)-manifold or that $(M, \bar{\nu}_M)$ is a B-bordism between two 2m-dimensional relative normal (k-1)-smoothings in (B,V) and that B has finite (m+1)-skeleton. By Lemma 2.3 we can assume up to bordism rel. boundary that $\bar{\nu}_M$ is a m-equivalence.

We begin by describing the surgery obstruction for a normal k-map $(M, \bar{\nu}_M, f)$, $k \geq m$. By assumption we have an exact sequence

$$\pi_{m+1}(B,M) \xrightarrow{d} \pi_m(M) \to \pi_m(B) \to 0$$

As $\pi_{m+1}(B,M) \cong H_{m+1}(B,M;\Lambda)$, this Λ-module is finitely generated. Choose disjoint embeddings $(S^m \times D^{m+1})_i \subset M$, $1 \leq i \leq k$, re presenting a system of generators of im d and compatible with the B-structure ([5], § IV.1).

We denote $\bigcup_i (S^m \times D^{m+1})_i = U$. On $\pi_m(\partial \tilde{\mathcal{U}})$ we have for $m \neq 3,7$ the $(-1)^m$-quadratic form (λ, μ). This vanishes on the kernel of the inclusion from $\partial\mathcal{U}$ into a bounding manifold ([41], p. 52). If we consider the inclusion $\partial\tilde{\mathcal{U}} \longrightarrow \tilde{\mathcal{U}}$ we obtain as kernel a lagrangian denoted by F. If we consider the inclusion $\pi_m(\partial\tilde{\mathcal{U}}) \longrightarrow \pi_m(M-U)$ we obtain a sublagrangian G. If k=m our surgery obstruction will be respresented by $((\pi_m(\partial\tilde{\mathcal{U}}), \lambda, \mu), F, G)$ in $l^{h,p}_{2m+1}(\pi_1(B), w_1(B))$ denoted as $\Theta(M, \bar{\nu}_M, f)$.

If m = 3,7 the quadratic refinement μ is defined on $\pi_{m+1}(B, \partial\tilde{U})$. As discussed in § 5 this induces a form μ on $\pi_m(\partial\tilde{U}) = \pi_{m+1}(B, \partial\tilde{U})/\pi_{m+1}(B)$ if the homomorphism $\pi_{m+1}(B) \to \mathbb{Z}_2$, $\alpha \mapsto \langle w_{m+1}(p^*\gamma), \alpha \rangle$ is trivial. As for every manifold M of dimension $\leq$ 2m+1 the map $\pi_{m+1}(M) \longrightarrow \mathbb{Z}_2$, $\alpha \mapsto \langle w_{m+1}(M), \alpha \rangle$ is trivial, the vanishing of $\pi_{m+1}(B) \longrightarrow \mathbb{Z}_2$ is a necessary condition for the existence of a manifold of dimension $\leq$ 2m+1 of normal k-type B if $k \geq m$. It is easy to see that if $k \geq m$ and there exists any k-admissible B-manifold B has this property.

If $k > m$ we will show that G is a direct summand , $G = G^{\perp}$ and thus $\theta(M, \bar{\nu}_M, f)$ is contained in the obstruction group $L^{h,p}_{2m+1}(\pi_1(B), w_1(B))$.

To show that $G = G^{\perp}$ and G is a direct summand we first note that $\mathrm{Ker}\ \pi_m(\partial \tilde{U}) \longrightarrow \pi_m(M-\mathring{U})$ is equal to $\mathrm{Ker}\ H_m(\partial \tilde{U}) \longrightarrow H_m(M-\mathring{U}; \Lambda)$. This follows by diagram chasing from the commutative diagram:

$$\begin{tikzcd}
 & & 0 \arrow[d] \\
 & \pi_{m+1}(B,M-\mathring{U}) \arrow[r, "\cong"] \arrow[d] & H_{m+1}(B,M-\mathring{U};\Lambda) \arrow[d] \\
H_m(\partial \tilde{U}) = \pi_m(\partial \tilde{U}) \arrow[r] \arrow[dr, "0"'] & \pi_m(M-\mathring{U}) \arrow[r] \arrow[d] & H_m(M-\mathring{U};\Lambda) \arrow[d] \\
 & \pi_m(B) \arrow[r] & H_m(B;\Lambda)
\end{tikzcd}$$

The isomorphism in the diagram follows as by assumption $(B,M-\mathring{U})$ is m-connected. The injectivity of $H_{m+1}(B,M-\mathring{U};\Lambda) \longrightarrow H_m(M-\mathring{U};\Lambda)$follows as k-partial Poincaré duality implies that

$$H_{m+2}(M-\mathring{U};\Lambda) \longrightarrow H_{m+2}(B;\Lambda) \text{ is surjective.}$$

Next we will show that $\mathrm{Im}\ H_m(\partial \tilde{U}) \longrightarrow H_m(M-\mathring{U};\Lambda)$ is projective which implies that G is a direct summand. For this we will show that this image is equal to $H_{m+1}(B,M-\mathring{U};\Lambda)$. If $k > m$, k-partial Poincaré duality implies that $H^{m+2}(B,M-\mathring{U})$ vanishes with all coefficients. Thus $H_{m+1}(B,M-\mathring{U};\Lambda)$ is projective ([41], Lemma 2.3).

To determine $\operatorname{Im} H_m(\partial\tilde{\mathfrak{U}}) \longrightarrow H_m(M-\mathring{U};\Lambda)$ as $H_{m+1}(B,M-\mathring{U};\Lambda)$ we observe that the sequence

$$H_m(\partial\tilde{\mathfrak{U}}) \longrightarrow H_m(M-\mathring{U}) \longrightarrow H_m(B) \longrightarrow 0$$

is exact. This follows from the diagram:

$$\begin{tikzcd}
H_{m+1}(\tilde{\mathfrak{U}},\partial\tilde{\mathfrak{U}}) \arrow[r, phantom, "="] \arrow[d] & H_{m+1}(\tilde{\mathfrak{U}},\partial\tilde{\mathfrak{U}}) \arrow[d] & & \\
H_m(\partial\tilde{\mathfrak{U}}) \arrow[r] \arrow[d] & H_m(M-\mathring{\mathfrak{U}}) \arrow[r] \arrow[d] & H_m(B) \arrow[r] \arrow[d, phantom, "\parallel"] & 0 \\
H_m(\tilde{\mathfrak{U}}) \arrow[r] & H_m(M) \arrow[r] & H_m(B) \arrow[r] & 0
\end{tikzcd}$$

in which both vertical sequences and the lower horizontal sequence are exact, the latter follows from the fact that $H_m(\tilde{\mathfrak{U}})$ generates Ker $H_m(M) \longrightarrow H_m(B)$.

Finally we have to show that $G = G^{\perp}$. The diagram

$$\begin{tikzcd}
G \arrow[r, phantom, "\subset"] \arrow[d] & G^{\perp} \arrow[r, phantom, "\subset"] \arrow[d, phantom, "\cong"] & H_m(\partial\tilde{\mathfrak{U}}) \arrow[d, phantom, "\cong"] \\
(H_m(\partial\tilde{\mathfrak{U}})/_G)^* \arrow[r, phantom, "="] & H_m(\partial\tilde{\mathfrak{U}}) / {}_G)^* \arrow[r] & H_m(\partial\tilde{\mathfrak{U}})^*
\end{tikzcd}$$

in which the vertical maps are given by the pairing λ shows that

$G = G^{\perp} \Leftrightarrow G \longrightarrow (H_m(\partial\tilde{\mathfrak{U}})/_G)^*$ is surjective. We consider the following diagram with Λ-coefficients:

$$\begin{array}{ccccccccc}
H_{m+1}(M-\mathring{U}) & \to & H_{m+1}(M-\mathring{U},\partial U) & \longrightarrow & H_m(\partial\tilde{U}) & \to & H_m(M-\mathring{U},\partial M) & \xrightarrow{j_*} & H_m(M-\mathring{U},\partial\mathfrak{U}\cup M) \\
\uparrow{\scriptstyle S\wr} & & \uparrow{\scriptstyle S\wr} & & & & & & \\
H^m(M-\mathring{U},\partial\mathfrak{U}\cup\partial M) & \xrightarrow[j^*]{} & H^m(M-\mathring{U},\partial M) & & & & & &
\end{array}$$

The sequence is exact as $H_m(\partial\tilde{U}) \longrightarrow H_m(M-\mathring{U},\partial M)$ factorizes through $H_{m+1}(B,M-U)$:

$H_m(\partial\tilde{\mathfrak{U}}) \longrightarrow H_{m+1}(B,M-U) \rightarrowtail H_m(M-U,V)$ and thus Ker $H_m(\partial\tilde{\mathfrak{U}}) \longrightarrow H_m(M-\mathring{U},\partial M)$ = Ker $H_m(\partial\tilde{\mathfrak{U}}) \longrightarrow H_m(M-\mathring{U})$.

We have to show that $G \longrightarrow (H_m(\partial\tilde{\mathfrak{U}})/_G)^*$ is surjective. By the diagram G = Koker j^* and $H_m(\partial\tilde{\mathfrak{U}})/_G$ = Ker j_* and the map corresponds to the Kronnecker pairing. Thus we are finished if the Kronnecker pairing induces a surjection

$$\text{Koker } j^* \longrightarrow (\text{Ker } j_*)^*.$$

We note that Ker j_* = Ker $H_{m+1}(B,M-U) \longrightarrow H_{m+1}(B,M-U/_{\partial U})$ which follows from the obvious diagram and the information proved above. We have the similar statement for Koker j^*. Furthermore the same argument used above to show that $H_{m+1}(B,M-U)$ is projective shows that $H_{m+1}(B,M-U\ /_{\partial U})$ is projective. As $H_{m+1}(B,M-U) \longrightarrow H_{m+1}(B,M-U\ /_{\partial U})$ is surjective this implies that the map splits. Thus we have the following diagram:

$$\begin{array}{ccccccc}
H^{m+1}(B,M-U\ /_{\partial U}) & \longrightarrow & H^{m+1}(B,M-U) & \longrightarrow & \text{Koker } j^* & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
H_{m+1}(B,M-U\ /_{\partial U})^* & \longrightarrow & H_{m+1}(B,M-U)^* & \longrightarrow & (\text{Ker } j^*)^* & \longrightarrow & 0
\end{array}$$

As $(B,M-U)$ is m-connected the map $H^{m+1}(B,M-U) \longrightarrow H_{m+1}(B,M-U)^*$ is surjective implying Koker $j^* \longrightarrow (\text{Ker } j^*)^*$ surjective.

If B is a finite Poincaré complex then the same type of arguments shows that G is a lagrangian (in general G will only be stably free but one can stabilize it by adding new unlinked 0-homotopic embeddings $S^m \times D^{m+1} \hookrightarrow M$ to the given system of embeddings as described in the proof of Theorem 2.1) and we obtain an element $\theta(M,\bar{\nu}_M,f) \in L^h_{2m+1}(\pi_1(B),\ w_1(B))$ as in Wall's book.

Now, we describe the surgery obstruction of a B-bordism $(M,\ \bar{\nu}_M)$ between two 2m-dimensional relative normal (k-1)-smoothings in a (k-1)-partial Poincaré complex (B,V,α) with finite (m+1)-skeleton, $k \geq m$. Again we assume that (B,M) is m-connected. As (B,M_i) is at least m-connected, (M,M_i) is (m-1)-connected and $\pi_m(M) \longrightarrow \pi_m(M,M_i)$ is surjective.

We have the following diagram of exact sequences:

$$\begin{tikzcd}
 & \pi_{m+1}(B,M) \arrow[d, "d"'] \arrow[dr] & & \\
\pi_m(M_i) \arrow[r] \arrow[dr, two heads] & \pi_m(M) \arrow[r] \arrow[d] & \pi_m(M,M_i) \arrow[r] & 0 \\
 & \pi_m(B) \arrow[d] & & \\
 & 0 & &
\end{tikzcd}$$

It implies $\pi_{m+1}(B,M) \longrightarrow \pi_m(M,M_i)$ is surjective. As (B,M) is m-connected and B has finite $(m+1)$-skeleton $\pi_{m+1}(B,M) \cong H_{m+1}(B,M;\Lambda)$ is finitely generated. We choose disjoint embeddings $(S^m \times D^{m+1})_i \subset M$ as above representing generators of im d and denote $\bigcup (S^m \times D^{m+1})_i$ by U.

As discussed at the beginning of this chapter the definition of μ on $\pi_m(\partial \tilde{U})$ needs some explanation if $m = 3,7$. We have explained this if the homomorphism $\pi_{m+1}(B) \to \mathbb{Z}_2$, $\alpha \mapsto \langle w_{m+1}(p^* \gamma), \alpha \rangle$ is trivial which holds automatically if $k \geq m$. If this is non-trivial we replace μ by $\bar{\mu}: \pi_m(\partial \tilde{U}) \to \Lambda/_{\{a+\bar{a}\}} + \mathbb{Z}$

as discussed in § 5. In this case the obstruction will be contained in $l^{s,\tau}_{2m+1}(\pi_1(B)), w_1(B), \mathbb{Z})$. To simplify arguments we will not mention this special case in the following arguments separately. The necessary modifications are obvious.

We consider the exact sequence with Λ-coefficients:

$$H_{*+1}(M-\mathring{U}, \partial U \cup M_0) \longrightarrow H_*(\partial \tilde{U}) \longrightarrow H_*(M-\mathring{U}, M_0).$$

We will show i) that this sequence vanishes except for $* = 0,2m$ where the left or right maps are obviously isomorphisms and the corresponding modules are free with a canonical geometric basis and for $* = m$. In this case we will show that all terms are stably free and after stabilization we equip these moduls with a preferred basis such that the Whitehead torsion of all three pairs is 0.

ii) $G = H_{m+1}(M-\mathring{U}, \partial U \cup M_0)$ is a half rank submodule in $H_m(\partial \tilde{U})$ (if $k=m-1$, this in only true if the Euler numbers $e(M_0)$ and $e(M_1)$ agree) and iii) (λ, μ) vanishes on G, if $k \geq m$, thus it is a based lagrangian in this case.

$\theta(M, \bar{\nu}_M)$ is then the element represented by $(\pi_m(\partial \tilde{U}), F, G)$ in $L^{s,\tau}_{2m+1}(\pi_1(B), w_1(B))$ if $k \geq m$ and in $l^{s,\tau}_{2m+1}(\pi_1(B), w_1(B))$ if $k = m-1$.

i) It is enough to show that $H_m(M-\mathring{U},M_0)$ is stably free. This module is unchanged if we join all components of ∂U with M_0 by a thickened arc as described in the proof of Theorem 2.1. But then it is a simple consequence of Poincaré duality and the fact that (M,M_i) is (m-1)-connected to show that the m-th homology groups are the only non vanishing modules with arbitrary coefficients. Then by ([41] , Lemma 2.3) the module is stably free.

ii) From now on we assume that $H_{m+1}(M-\mathring{U}, \partial U \cup M_0)$ and $H_m(M-\mathring{U},M_0)$ are free with a preferred basis. We have to show that $\operatorname{rank}_\Lambda H_{m+1}(M-\mathring{U}, \partial U \cup M_0;\Lambda)$ $= \operatorname{rank}_\Lambda H_m(M-\mathring{U},M_0;\Lambda)$ or by Poincaré-duality that $\operatorname{rank}_\Lambda H_m(M-\mathring{U},M_1;\Lambda)$ $= \operatorname{rank}_\Lambda H_m(M-\mathring{U},M_0;\Lambda)$. As the modules are free this is equivalent to : $\operatorname{rank}_{\mathbb{Z}} H_m(M-\mathring{U},M_1;\mathbb{Z}) = \operatorname{rank}_{\mathbb{Z}} H_m(M-\mathring{U},M_0;\mathbb{Z})$. As M_0 and M_1 have same Betti-numbers up to dim m-1 this is equivalent to: $e(M_0) = e(M_1)$. This is fulfilled if $k \geq m$ and we require it for $k = m-1$.

iii) We have to show for $k \geq m$ that (λ,μ) vanishes on $\operatorname{Ker} H_m(\partial\tilde{U}) \longrightarrow H_m(M-\mathring{U},M_0;\Lambda)$ or as the spaces are (m-1)-connected on $\operatorname{Ker} \pi_m(\partial\tilde{U}) \longrightarrow \pi_m(M-\mathring{U},M_0)$. On the other hand (λ,μ) vanishes on $\operatorname{Ker} \pi_m(\partial\tilde{U}) \rightarrow \pi_m(M-\mathring{U})$. ([41] , p.52). Thus we are finished if $\operatorname{Ker} \pi_m(\partial\tilde{U}) \longrightarrow \pi_m(M-\mathring{U},M_0)$ is contained in $\operatorname{Ker} \pi_m(\partial\tilde{U}) \longrightarrow \pi_m(M-\mathring{U})$.
But this follows from the commutative diagram:

$$\begin{tikzcd}
\pi_m(\partial\tilde{U}) \arrow[r] \arrow[dd, "0"'] \arrow[dr] & \pi_m(M-\mathring{U},M_0) \\
 & \pi_m(M-\mathring{U}) \arrow[u] \arrow[dl] \\
\pi_m(B) & \pi_m(M_0) \arrow[l, "\cong"] \arrow[u]
\end{tikzcd}$$

In analogy to the even-dimensional case we introduce the following notation:

(or $l^{s,\tau}_{2m+1}(\pi, w, \mathbb{Z})$)

An element of $l^{s,\tau}_{2m+1}$ (π, w) is <u>zero bordant</u> if it has a representative of the form $(H^r_{(-1)^m}, (\Lambda \times \{0\})^r, \{x+C(x) \mid x \in (\{0\} \times \Lambda)^r\})$, $C\colon (\{0\} \times \Lambda)^r \longrightarrow (\Lambda \times \{0\})^r$ a homomorphism. This looks similar to an elementary formation but it is only equivalent (after interchanging $\Lambda \times \{0\}$ with $\{0\} \times \Lambda$) if C is of the form $A \pm A^*$. Again elements in the image of $L^{s,\tau}_{2m+1}(\pi, w) \longrightarrow l^{s,\tau}_{2m+1}(\pi, w)$ are zero bordant if and only if they are trivial.

Now we are ready to prove the main result in the odd dimensional case.

<u>Theorem 6.1:</u> $m \geq 2$.

a) Let $(M^{2m+1}, \bar{\nu}_M, f)$ be a $(2m+1)$-dimensional k-admissible $(k \geq m-1)$ (B,V)-manifold, where B has finite $(m+1)$-skeleton. Then the invariant

$$\theta(M, \bar{\nu}_M, f) \in \begin{cases} \{0\} & k = m-1 \\ l^{h,p}_{2m+1}(\pi_1(B), w_1(B)) & k = m \\ L^{h,p}_{2m+1}(\pi_1(B), w_1(B)) & k > m \\ L^{h}_{2m+1}(\pi_1(B), w_1(B)) & k = 2m+1 \text{ and } B \text{ a finite Poincaré complex} \end{cases}$$

is a bordism invariant rel boundary.

$\theta(M, \bar{\nu}_M, f) = 0 \Leftrightarrow (M, \bar{\nu}_M, f)$ is bordant rel. boundary to a normal k-smoothing.

b) Let $(M, \bar{\nu}_M)$ be a $(2m+1)$-dimensional B-bordism rel. boundary between two relative normal $(k-1)$-smoothings $(M_0, \bar{\nu}_{M_0}, f_0)$ and $(M_1, \bar{\nu}_{M_1}, f_1)$ in (B,V), where again B has finite $(m+1)$-skeleton. If $k = m-1$ we require $e(M_0) = e(M_1)$. Then the invariant

$$\theta(M, \bar{\nu}_M) \in \begin{cases} l^{s,\tau}_{2m+1}(\pi_1(B), w_1(B)) & k = m-1\ , m \neq 3,7 \\ L^{s,\tau}_{2m+1}(\pi_1(B), w_1(B)) & k \geq m \\ L^{s}_{2m+1}(\pi_1(B), w_1(B)) & k=2m+1 \text{ and } B \text{ a finite simple Poincaré complex} \end{cases}$$

is a bordism invariant rel. boundary. If $k = m-1$ and $m = 3,7$ the same holds if $\pi_{m+1}(B) \to \mathbb{Z}_2$, $\kappa \mapsto \langle w_{m+1}(p^* \tau), \kappa \rangle$, is trivial. If not we have to replace the monoid by $l^{s,\tau}_{2m+1}(\pi_1, w_1, \mathbb{Z})$.

$(M, \bar{\nu}_M)$ is bordant rel. boundary to a s-cobordism $\iff$

$$\begin{cases} \theta(M, \bar{\nu}_M) = 0 & \text{if } k \geq m \\ \theta(M, \bar{\nu}_M) \text{ zero bordant} & \text{if } k = m-1 \end{cases}$$

<u>Proof:</u> To show that $\theta(M, \bar{\nu}_M, f)$ and $\theta(M, \bar{\nu}_M)$ are bordism invariants we first study the change of the system of embeddings $U = \bigcup_i (S^m \times D^{m+1})_i$. Let $V = \bigcup_j (S^m \times D^{m+1})_j$ be an other set of embeddings s.t. $(S^m \times D^{m+1})_j$ generates $\operatorname{im} \pi_{m+1}(B,M) \longrightarrow \pi_m(M)$. We can assume that $U \cap V = \emptyset$ und thus consider $U \cup V$. The invariant is independent of the embeddings if it agrees for

U and $U \cup V$. It is even enough to show that it agrees for U and U together with an additional disjoint embedding $S^m \times D^{m+1} \hookrightarrow M$ s.t. $S^m \times \{*\}$ represents an element of im $\pi_{m+1}(B,M) \longrightarrow \pi_m(M)$.

Let $U' = U \cup S^m \times D^{m+1}$. We have to study the kernel of $\pi_m(\partial \tilde{U}') \to \pi_m(M-\mathring{U}')$ (or to $\pi_m(M-\mathring{U}', M_0)$ where the arguments are similar). We denote the kernel of $\pi_m(\partial \tilde{U}) \longrightarrow \pi_m(M-\mathring{U})$ by G and the standard basis of $S^m \times S^m$ by $e = S^m \times \{*\}$ and $f = \{*\} \times S^m$. As U generates im $\pi_{m+1}(B,M) \longrightarrow \pi_m(M)$ there exist $x \in \pi_m(\partial \tilde{U})$, $a \in \Lambda$ s.t. $e + a \cdot f + x \in G' = \operatorname{Ker} \pi_m(\partial \tilde{U}') \longrightarrow \pi_m(M-\mathring{U}')$. And for every $y \in G$ there exists $b_y \in \Lambda$ s.t. $y + b_y \cdot f \in G'$. b_y can be computed in terms of the x above and of y: $b_y = -\lambda(x,y)$. For, $x + e + a \cdot f \in \operatorname{Ker} \pi_m(\partial \tilde{U}') \longrightarrow \pi_m(M-\mathring{U}')$ and $y + b_y f \in \operatorname{Ker} \pi_m(\partial \tilde{U}') \longrightarrow \pi_m(M-\mathring{U}')$ (or $\longrightarrow \pi_m(M-\mathring{U}', M_0)$) $\Rightarrow 0 = \lambda(x+e+a\cdot f, y + b_y \cdot f) = \lambda(x,y) + b_y$. It is easy to check that the element $e + a\cdot f + x$ and the elements $\{y - \lambda(x,y)\cdot f \mid y \in G\}$ generate G'.

We will show that $((\pi_m(\partial \tilde{U}'), \lambda, \mu), F, G')$ is isomorphic to $((\pi_m(\partial \tilde{U}), \lambda, \mu), F, G) \oplus (H_{(-1)^m}, \Lambda \times \{0\}, \{0\} \times \Lambda)$ under a simple isometry preserving F by a simple isomorphism. This isometry is given by

$$\pi_m(\partial \tilde{U}) \oplus H_{(-1)^m} \longrightarrow \pi_m(\partial \tilde{U}') = \pi_m(\partial \tilde{U}) \oplus H_{(-1)^m},$$

$$y \in \pi_m(\partial \tilde{U}) \longmapsto y - \lambda(x,y)\cdot f, \quad e \longmapsto x + e + a\cdot f, \quad f \longmapsto f.$$

Now, the bordism invariance follows by the same argument as in ([41] , §6). We first note that surgery on $(S^m \times D^{m+1})_i$ replaces $\theta(M, \bar{\nu}_M, f)$ or $\theta(M, \bar{\nu}_M)$ by the product with $\begin{pmatrix} 1 & & 0 \\ & \sigma & \\ 0 & & 1 \end{pmatrix}$. Thus by Lemma 4.6 the element in the surgery obstruction group (monoid) is unchanged. But in ([41] , p. 61) it is prooved that if M and M' are bordant rel.

under a highly connected bordism

such

boundary then one can pass form M to M' by a sequence of such surgeries, compatible with the B-structure. As we are free in the choice of our system of generators of $\operatorname{im} \pi_{m+1}(B,M) \longrightarrow \pi_m(M)$ we can assume that the surgeries are all performed on $(S^m \times D^{m+1})'$s contained in U.

Now, suppose $\theta(M, \overline{\nu}_M, f) = 0$, $\theta(M, \overline{\nu}_M) = 0$ if $k \geq m$ or $\theta(M, \overline{\nu}_M)$ zero bordant if $k = m-1$. As in the even-dimensional case the proof is very similar in all these situations and we carry only the case $\theta(M, \overline{\nu}_M)$ zero bordant and $k = m-1$ through.

If $\theta(M, \overline{\nu}_M)$ is zero bordant it is after stabilizing with hamiltonian and elementary formations within the bordism class isomorphic to $(H^r_{(-1)^n}, (\Lambda \times \{0\})^r, \{x + C(x) \mid x \in (\{0\} \times \Lambda)^r\})$ for some homomorphism. $C: (\{0\} \times \Lambda)^r \longrightarrow (\Lambda \times \{0\})^r$. (Weakly isomorphic can here be replaced by isomorphic as the first lagrangian $(\Lambda \times \{0\})^r$ is the same.) Let's assume for a moment that we can realize the stabilization process geometrically so that $\theta(M, \overline{\nu}_M)$ is itself isomorphic to the quasi-formation above.

The claim is that under these conditions M is already an s-cobordism. For this we consider the exact sequences with Λ-coefficients:

$$\begin{tikzcd}
 & & & 0 \arrow[d] & & \\
 & & & H_{m+1}(M-\mathring{U}, M_0 \cup \partial U) \arrow[d] & & \\
 & & H_{m+1}(\tilde{U}, \partial \tilde{U}) \arrow[r] \arrow[d, "\cong"] & H_m(\partial \tilde{U}) \arrow[d] & & \\
0 \arrow[r] & H_{m+1}(M, M_0) \arrow[r] & H_{m+1}(M, M-\mathring{U}) \arrow[r] & H_m(M-\mathring{U}, M_0) \arrow[r] \arrow[d] & H_m(M, M_0) \arrow[r] & 0 \\
 & & & 0 & &
\end{tikzcd}$$

Similar considerations concerning preferred bases as in the proof of Theorem 5.2 show that (M,M_0) is contractible with trivial Whitehead torsion if and only if the map $H_{m+1}(\tilde{U}, \partial \tilde{U}) \longrightarrow H_m(M-\mathring{U},M_0)$ is a simple isomorphism where $H_m(M-\mathring{U},M_0)$ is in such a way based that the vertical sequence has trivial Whitehead torsion. But this is obviously the case if $\theta\ (M, \overline{\nu}_M)$ is of the form assumed above.

We have shown already in the proof of Theorem 2.1 how one can stabilize with hamiltonian formations. Instead of adding an elementary formation $((H^r_{(-1)^n}, (\Lambda \times \{0\})^r, \{x + (C - (-1)^n C^*)\cdot x \mid x \in (\Lambda \times \{0\})^r\})$ we can add with its image unter the action of $r\cdot\sigma$ (replacing $x \in (\Lambda \times \{0\})^r$ by $x \in (\{0\} \times \Lambda)^r$) as we can realize the action of $r \cdot \sigma$ geometrically by surgeries. To realize the addition with this formation we first stabilize with a r-dimensional hamiltonian formation by adding to U r times a null-momotopic unlinked $(S^m \times D^{m+1})$. This corresponds to the case $C = 0$. The general case can be obtained if one changes the framings of these embeddings appropriately and introduces linkings between $(S^m \times D^{m+1})_i$ and $(S^m \times D^{m+1})_j$, replacing $(S^m \times D^{m+1})_i$ by the connected sum of $(S^m \times D^{m+1})_i$ and $c_{i,j}$ times a thickening of the embedding $(\{*\} \times S^m)_j$ (compare [41], p. 58).

Finally we have for $k = m-1$ to show that if $(M, \overline{\nu}_M)$ is bordant rel. boundary to a s-cobordism then $\theta\ (M, \overline{\nu}_M)$ is zero bordant (all other cases are trivial). This case is easier than the even-dimensional case as $\theta\ (M, \overline{\nu}_M)$ is defined without changing the manifold within its bordism class for an s-cobordism. It is easy to see that for an s-cobordism $\theta\ (M, \overline{\nu}_M)$ is zero bordant.

q.e.d.

§ 7 Summary and reformulation of the results

In this chapter we are going to study the question which elements in the obstruction groups are surgery obstructions, describe an action of

$L^{s,\tau}_{n+1}(\pi_1(B), w_1(B))$ on $NS^{(B,V)}_{n,k}$ leading to an exact sequence relating

$NS^{(B,V)}_{n,k}$ and $A\Omega^{(B,V)}_{n,k}$ and finally we study the difference between

$NS^{(B,V)}_{n,k}$ and the set of s-cobordism classes of manifolds which admit a normal-k-smoothing in (B,V). We always assume that B has finite $[(n+3)/2]$ -skeleton.

We begin with the additivity formula for $\theta(M, \bar{\nu}_M)$. Let $(M, \bar{\nu}_M)$ be a B-bordism rel. boundary between n-dimensional relative normal k-smoothing $(M_0, \bar{\nu}_{M_0}, f_0)$ and $(M_1, \bar{\nu}_{M_1}, f_1)$ and $(N, \bar{\nu}_N)$ such a bordism between $(M_1, \bar{\nu}_{M_1}, f_1)$ and $(M_2, \bar{\nu}_{M_2}, f_2)$. Then $(M \underset{M_0}{\cup} N, \bar{\nu}_M \cup \bar{\nu}_N)$ is a B-bordism rel. boundary between $(M_0, \bar{\nu}_{M_0}, f_0)$ and $(M_2, \bar{\nu}_{M_2}, f_2)$.

<u>Proposition 7.1:</u> If $k \geq [n/2]$ then $\theta(M \underset{M_0}{\cup} N, \bar{\nu}_M \cup \bar{\nu}_N)$ (sitting in $L^{s,\tau}_{n+1}(\pi_1(B), w_1(B)$ or $L^{s,\tau}_{n+1}(\pi_1(B), w_1(B), S(M_0 \times I))$ if n is odd and $k = [n/2]$) is equal to $\theta(M, \bar{\nu}_M) + \theta(N, \bar{\nu}_N)$.

<u>Proof:</u> If $n = 2m-1$ is odd the obstruction is represented by a ε-quadratic form on $H_{m+1}(B,M;\Lambda)$ or on $K\pi_m(M)/K\pi_m(M_0)$. As (B,M_i) is m+1-connected $H_{m+1}(B,M\cup N;\Lambda) \cong H_{m+1}(B,M;\Lambda) \oplus H_{m+1}(B,N;\Lambda)$ and $K\pi_m(M\cup N)/K\pi_m(M_0)$ $\cong K\pi_m(M)/K\pi_m(M_0) \oplus K\pi_m(N)/K\pi_m(M_1)$ and the forms split as orthogonal sum.

If $n = 2m$ is even, $\theta(M, \bar{\nu}_M)$ is represented by the ε-formations on $\pi_m(\partial\tilde{\mathcal{U}})$ represented by the kernel of the map into $\pi_m(M-\mathring{U})$ where U represents a system of generators of the image of $\pi_{m+1}(B,M) \cong H_{m+1}(B,M;\Lambda)$ in $\pi_m(M)$. As (B,M_i) is (m+1)-connected, $H_{m+1}(B,M \cup N;\Lambda) \cong H_{m+1}(B,M;\Lambda) \oplus H_{m+1}(B,N;\Lambda)$ so, if V generates im $\pi_{m+1}(B,N) \longrightarrow \pi_m(M)$, $U \cup V$ generates im $\pi_{m+1}(B,M \cup N) \longrightarrow \pi_m(M \cup N)$ and the kernels slit as the direct sum.

q.e.d.

If $k = [n/2] - 1$, the additivity problem seems to be more difficult.

Proposition 7.2: Let $(M_0, \bar{\nu}_{M_0}, f)$ be a n-dimensional relative normal k-smoothing $k \geq [n/2]$, $n \geq 5$. For every $\theta \in L^{s,\tau}_{n+1}(\pi_1(B), w_1(B))$ ($L^{s,\tau}_{n+1}(\pi_1(B), w_1(B), S(M_0 \times I))$ if n is odd and $k = [n/2]$) there exists a relative normal k-smoothing. $(M_1, \bar{\nu}_{M_1}, f)$ with $\partial M_0 = \partial M_1$ and a B-bordism $(M, \bar{\nu}_M)$ rel. boundary between M_0 and M_1, such that $\theta(M, \bar{\nu}_M) = \theta$. Up to bordism rel.boundary $(M, \bar{\nu}_M)$ is uniquely determined by θ. Especially $(M_1, \bar{\nu}_{M_1}, f)$ is up to s-cobordism completely determined by θ.

Proof: This is proved in ([41] , Theorem 5.8 and Theorem 6.5) for the case of simple normal n-smoothings (= simple normal homotopy equivalences) in a finite Poincaré complex B. The proof extends verbally to our situation.

q.e.d.

Thus we get an action of $L^{s,\tau}_{n+1}(\pi_1(B), w_1(B))$ on $NS^{(B,V)}_{n,k}$ for $k \geq [n/2]$ ($k > [n/2]$ if n is odd), $(\theta, [M_0, \bar{\nu}_0, f]) \longmapsto [M_1, \bar{\nu}_1, f]$, where $(M_1, \bar{\nu}_1, f)$

is as in Proposition 7.2. If $k = [n/2]$, n odd and $k \neq 2,6$ the obstructions are contained in $L^{s,\tau}_{n+1}(\pi_1(B), w_1(B), S(M_0 \times I))$. Then $S(M_1 \times I) = S(M_0 \times I)$, as both are equal to $S(M)$.

Given $S \subset \Lambda$ we denote the subset of $NS^{(B,V)}_{n,k}$ represented by (M_0, ν_{M_0}, f) with $S(M_0 \times I) = S$ by $NS^{(B,V,S)}_{n,k}$ (similarly $A\Omega^{(B,V,S)}_{n,k}$). Thus, if $k = [n/2]$ and n odd (k≠2,6) we obtain an action of $L^{s,\tau}_{n+1}(\pi_1(B), w_1(B), S)$ on $NS^{(B,V,S)}_{n,k}$.

With these preparations we can reformulate our Theorems 5.2 and 6.1 as follows

Theorem 7.3: Let $B \longrightarrow BO$ be a fibration s.t. B has finite $([n/2]+1)$-skeleton and $(V, \overline{\nu}_V)$ a fixed (n-1)-dimensional B-manifold.

$S \subset \Lambda = \mathbb{Z}[\pi_1(B)]$ a subgroup as described in the beginning of § 4 (a form parameter). $n \geq 5$.

The following sequences are exact in the sense that the map on the left side corresponds to the action defined above, exact meaning that the orbit space injects into the bordism group.

A) B a finite Poincaré complex (k = n)

$$L^{s,\tau}_{n+1}(\pi_1(B), w_1(B)) \longrightarrow NS^{(B,V)}_{n,n} \longrightarrow A\Omega^{(B,V)}_{n,n} \xrightarrow{\theta} L^h_n(\pi_1(B), w_1(B))$$

B) $k > [n/2]$

$$L^{s,\tau}_{n+1}(\pi_1(B), w_1(B)) \longrightarrow NS^{(B,V)}_{n,k} \longrightarrow A\Omega^{(B,V)}_{n,k} \xrightarrow{\theta} L^{h,p}_n(\pi_1(B), w_1(B))$$

C) $k = [n/2]$ and $n = 2m$.

$$L^{s,\tau}_{2m+1}(\pi_1(B), w_1(B)) \longrightarrow NS^{(B,V)}_{2m,m} \longrightarrow A\Omega^{(B,V)}_{2m,m} \xrightarrow{\theta} l^{h,p}_{2m}(\pi_1(B), w_1(B))$$

D) $k = [n/2]$ and $n = 2m-1$ and $m \neq 3,7$

$$L_{2m}^{s,\tau}(\pi_1(B),w_1(B),S) \longrightarrow NS_{2m-1,m-1}^{(B,V,S)} \longrightarrow A\Omega_{2m-1,m-1}^{(B,V,S)} \xrightarrow{\Theta} l_{2m-1}^{h,p}(\pi_1(B),w_1(B))$$

If $m = 3,7$ the corresponding statement is contained in Theorems 5.2 and 6.1.

E) For $k = [n/2] - 1$ and $n \geq 4$, $n \neq 5,6,13,14$

$$NS_{n,[n/2]-1}^{(B,V)} \longrightarrow \Omega_n^{(B,V)}$$

is surjective and elements in a fibre are up to an indeterminacy classified in $l_{n+1}^{s,\tau}(\pi_1(B),w_1(B))$ (or $l_{n+1}^{s,\tau}(\pi_1(B),w_1(B),S)$ if n is odd) which means: If $(M,\bar{\nu}_M)$ is a B-bordism rel. boundary between $(M_0,\bar{\nu}_{M_0},f_0)$ and $(M_1,\bar{\nu}_{M_1},f_1)$ (with $e(M_0) = e(M_1)$, if n is even) then $(M,\bar{\nu}_M)$ is bordant rel. boundary to an s-cobordism if and only if $\Theta(M,\bar{\nu}_M)$ vanishes in $l_{n+1}^{s,\tau}(\pi_1(B),w_1(B))$ (in $l_{n+1}^{s,\tau}(\pi_1(B),w_1(B),S(M))$ if n is odd).

If $n = 5,6,13,14$ the corresponding statements are contained in Theorems 5.2 and 6.1.

F) Similary if $n = 4$, in the cases A) - C) elements in the fibres of the map $NS_{4,k} \longrightarrow \Omega_4$ are up to an indeterminacy classified in $L_5^{s,\tau}(\pi_1(B),w_1(B))$ (or $L_5^{s,\tau}(\pi_1(B),w_1(B),S)$.

Remark: If we replace smooth by topological manifolds then A) - C) of Theorem 7.3 is true for $n = 4$ if $\pi_1(B)$ is poly finite-cyclic. This follows from the extension of topological surgery to dimension 4 by Freedman and Quinn [12].

Remark: In case B) the sequence has an extension to the left. This and applications to block-diffeomorphism is discussed in [46].

Theorem 7.3 provides a method for classifying normal k-smoothings in (B,V) $B \longrightarrow BO$ a fibration. It is perhaps more natural to classify all manifolds M which admit a normal k-smoothing in (B,V). We call such a manifold of <u>normal k-type (B,V).</u> The set of diffeomorphism classes (s-cobordism classes, if n = 4) of manifolds of normal k-type (B,V) is denoted by $\mathfrak{M}_{n,k}^{(B,V)}$. There is a projection map

$$NS_{n,k}^{(B,V)} \twoheadrightarrow \mathfrak{M}_{n,k}^{(B,V)}.$$

Let Aut((B,p) rel $\bar{\nu}_V$) be the group of homotopy classes of fibre homotopy self equivalences h of $B \longrightarrow BO$ such that $h \circ \bar{\nu}_V = \bar{\nu}_V$. This group operates on $NS_{n,k}^{(B,V)}$ as well as on $A\Omega_{n,k}^{(B,V)}$ by composition,

$(h, (M, \bar{\nu}_M, f)) \longmapsto (M, h \circ \bar{\nu}_M, f)$.

<u>Proposition 7.4:</u> Let $(M, \bar{\nu}_M)$ be a fixed relative normal k-smoothing and $B \longrightarrow BO$ a (k+1)-coconnected fibration ($\Leftrightarrow$ $\bar{\nu}_M$ a Postnikov-factorization). The map $h \longmapsto h \circ \bar{\nu}_M$ defines a bijection between Aut((B, p) rel $\bar{\nu}_V$) and the different normal k-smoothing of M.

Thus $\quad NS_{n,k}^{(B,V)} / \text{Aut}((B,p) \text{ rel } \bar{\nu}_V) \xrightarrow{\cong} \mathfrak{M}_{n,k}^{(B,V)}$.

<u>Proof:</u> If $\bar{\nu}'_M$ is another normal k-smoothing, the uniqueness of a Postnikov decomposition ([3] , Corollary 5.3.8) implies that there exists a unique fibre homotopy equivalence $h: B \longrightarrow B$ such that $h \circ \bar{\nu}_M$ and $\bar{\nu}'_M$ are homotopic over BO. Thus the map $h \mapsto h \circ \bar{\nu}_M$ is bijective.

q.e.d.

Remark: If one considers the set of (simple) smoothings $S^h(X)$ $(S^s(X))$ of a Poincaré complex X then the set of diffeomorphism classes (s-cobordism classes in dimension 4) of manifolds (simply) homotopy equivalent to X is the orbit space under the action of Aut (X) (Aut_s (X) = group of simple self equivalence).

If we define the set of 2n-dimensional stable diffeomorphism classes of manifolds admitting a relative normal (n-1)-smoothing in (B,V) by $\mathfrak{M}St_{2n}^{(B,V)}$, n = 2, we obtain for a n-co-connected fibration $B \longrightarrow BO$ an identification:

$NSt_{2n}^{(B,V)}/\mathrm{Aut}((B,p)\mathrm{rel}\,\bar{\nu}_V) \xrightarrow{\cong} \mathfrak{M}St_{2n}^{(B,V)}$ or by Theorem 2.1 :

Theorem 7.5 : $\mathfrak{M}St_{2n}^{(B,V)} \cong \Omega_{2n}^{(B,V)}/\mathrm{Aut}((B,p)\mathrm{rel}\,\bar{\nu}_V)$

It should be remarked that Aut $((B,p)\mathrm{rel}\,\bar{\nu}_V)$ operates linearly on $\Omega_{2n}^{(B,V)}$.

With these results one can try to decide whether two compact manifolds M and N of dimension ≥ 5 with diffeomorphic boundary are diffeomorphic:

Check if M and N have same normal k-type for some $k \geq \left[\frac{n}{2}\right] - 1$: $B_k(M) = B_k(N) = B \longrightarrow BO$. Choose normal k-smoothings $\bar{\nu}_M$ and $\bar{\nu}_N$ to obtain elements in $NS_{n,k}^{(B,V)}$.

Compute this set by Theorem 7.3 and check whether $(M, \bar{\nu}_M)$ and $(N, \bar{\nu}_N)$ are in the same orbit under $\mathrm{Aut}((B,p)\mathrm{rel}\,\bar{\nu}_V)$.

If n = 2m is even one can use instead Theorem 7.5 to decide the stable diffeomorphism problem and can try then to cancel the $(S^n \times S^n)$'S with the methods of § 3.

One main problem of this approach is of course to decide whether $B_k(M) = B_k(N)$. This is more or less equivalent to deciding if M and N have homotopy equivalent (k+1)-skeleton and same normal bundle over the (k+1)-skeleton. Similary $Aut(B_k(M),p)$ is more or less the group of homotopy self equivalence classes of a (k+1)-skeleton together with an isomorphism of the normal bundle.

More precisely, given a map $p: X \longrightarrow Y$ we denote by $Aut_k(X,p)$ the semigroup of homotopy classes of pairs (f,h), where $f: X \longrightarrow X$ is a (k+1)-equivalence and h a homotopy between p and $p \circ f$. If p is the constant map we denote the semigroup by $Aut_k(X)$. If $p: B \longrightarrow BO$ is (k+1)-coconnected and X a (k+1)-skeleton of B we denote $p|_X$ again by p.

We obtain a map $Aut_k(X,p) \longrightarrow Aut\, B$ as follows. Consider the composition $i \circ f: X \longrightarrow B$. h gives a homotopy between $p \circ i \circ f$ and p. The restriction of a lift of this homotopy to $X \times \{1\}$ gives a map $g: X \longrightarrow B$ commuting with p. As $p: B \longrightarrow BO$ is (k+1)-connected this map extends to a unique automorphism of B. Obviously this map $Aut_k(X,p) \longrightarrow Aut\, B$ is surjective. It is easy to see that $Aut_k(X,p)$ fits into two exact sequences so that we obtain

<u>Proposition 7.6:</u> Let $B \longrightarrow BO$ be (k+1)-coconnected and X a (k+1)-skeleton of B. There is a surjective map $Aut_k(X,p) \twoheadrightarrow Aut\, B$ and we have exact sequences:

$$[X, O] \longrightarrow Aut_k(X, p) \longrightarrow Aut_k(X) \longrightarrow \widetilde{KO}(X)$$

and

$$0 \longrightarrow Aut(X) \longrightarrow Aut_k(X) \longrightarrow End\, \pi_{k+1}(X) / Aut\, \pi_{k+1}(X).$$

Thus the unstable homotopy theory needed for this approach is the classification of k-complexes and their automorphisms for some $k \geq [n/2]$. The rest is stable homotopy theory computing $\Omega_n^B \cong \pi_n(M\gamma B)$ (by Pontrjagin Thom construction, see [37]) and algebra to determine the L-group (monoids) and of course the determination of the maps in the exact sequences which is often a very difficult problem.

The difference to the information needed for the approach of Browder and Novikov is rather clear if one considers stably parallelizable manifolds. Then $B_k(M) = P_k(M) \times BO\langle k+2\rangle \xrightarrow{P_2} BO\langle k+2\rangle \longrightarrow BO$ where $P_k(M)$ is the k'th stage of a Postnikov tower of M ($M \longrightarrow P_k(M)$ a (k+1)-equivalence and $\pi_r(P_k(M)) = \{0\}$ for $r \geq k+1$) and $BO\langle k+2\rangle$ is the (k+1)-connected cover of BO. The smaller k is the smaller is the information of unstable homotopy theory one needs and the latter seems to be the most difficult part.

The following two exact sequences might be useful in computing $\mathrm{Aut}((B,p)\mathrm{rel}\,\bar{\nu}_V)$.

Proposition 7.7: Let $p: B \longrightarrow BO$ be a fibration with B homotopy equivalent to a CW-complex. $\bar{\nu}_V : V \longrightarrow B$ a B-structure on some manifold V. Then we have exact sequences

$$\cdots \to \pi_1(B^B) \to \pi_1(BO^B \times' B^V) \to \mathrm{Aut}((B,p)\mathrm{rel}\,\bar{\nu}_V) \to \mathrm{Aut}\,B \longrightarrow \pi_0(BO^B \times' B^V)$$

and

$$\cdots \to \pi_2(BO^B) \times \pi_2(B^V) \to \pi_2(BO^V) \to \pi_1(BO^B \times' B^V) \to \pi_1(BO^B) \times \pi_1(B^V) \to \pi_1(BO^V)$$
$$\to \pi_0(BO^B \times' B^V) \to \pi_0(BO^B) \times \pi_0(B^V) \to \pi_0(BO^V)$$

where $BO^B \times' B^V$ is given by the fibre square

$$\begin{array}{ccc} BO^B \times' B^V & \longrightarrow & B^V \\ \downarrow & & \downarrow \\ BO^B & \longrightarrow & BO^V \end{array}$$

The base points in the sets of maps are given by Id in B^B, $\bar{\nu}_V$ in B^V, p in BO^B and $p \circ \bar{\nu}_V$ in BO^V.

Proof: The result follows from the fact $B^B \longrightarrow BO^B \times' B^A$ is a *Serre* fibration (compare Spanier: Algebraic Topology, p.416, Corollary 10).

q.e.d.

§ 8 Some applications

In this chapter we want to demonstrate how the results of this paper can be used for solving certain problems. Some applications were obtained before I developped the machinery in this generality, for instance the computation of bordism groups of diffeomorphisms. To a certain extent these results were the motivation for the research which led to this paper.

Our aim in this chapter is not to give a complete survey about known applications but to demonstrate typical cases at some rather simple examples. In some cases we only sketch the proof and refer to the original article for details. Another good example to demonstrate the difference between the ordinary surgery approach and ours is the application to the classification of 1-connected manifolds up to finite ambiguity extending to corresponding results of Sullivan. This is carried through in [22].

I) Bordism of diffeomorphisms. Let Δ_n^0 (Δ_n) be the bordism group of (orientation preserving) self diffeomorphisms (M,f) on (oriented) closed n-manifolds modulo those bounding a self diffeomorphism on a compact manifold Y [17].

There are two obvious invariants, the bordism class of M and of the mapping torus $M_f := M \times I/_{(x,0)\,\sim\,(f(x),1)}$. It was first proved by myself in the oriented category [16] and with different methods by Frank Quinn ([31], [17]) in both categories that for n odd these invariants determine the bordism class.

Theorem 8.1: The homomorphisms

$$\Delta^0_{2n-1} \longrightarrow \mathfrak{N}_{2n-1} \oplus \mathfrak{N}_{2n}, \quad [M,f] \longmapsto ([M], [M_f]) \text{ and}$$

$$\Delta_{2n-1} \longrightarrow \Omega_{2n-1} \oplus \Omega_{2n}, \quad [M,f] \longmapsto ([M], [M_f])$$

are injective.

Proof: The case $n = 1$ is trivial thus we assume $n > 1$. If $[M] = 0$ then $M = \partial Y$. By Lemma 2.3 we can assume $\nu_y: Y \longrightarrow BO$ (or $Y \longrightarrow BSO$ in the oriented case) is a n-equivalence. Here we consider the trivial fibration $B = BO \longrightarrow BO$ or the orientation covering $B = BSO \longrightarrow BO$ in the oriented case.

If $[M_f] = 0$ then $Y \cup_f Y$ bounds a B-manifold. For $Y \cup_f Y$ and M_f are bordant, a bordism is given by identifying in $(Y \cup_f Y) \times I$ two copies of Y embedded into $(Y \cup_f Y) \times \{1\}$ such that the image of the embedding is the complement of a bicollar of ∂Y.

By Theorem 2.1 f extends to a diffeomorphism $F: Y \# r(S^n \times S^n) \longrightarrow Y \# r(S^n \times S^n)$.

q.e.d.

Remark: If (M^{2n},f) is an even-dimensional diffeomorphism ($n > 1$) and $[M] = 0$, $[M_f] = 0$ the same argument as above shows that M bounds a $B = BO$ (or BSO)-manifold Y s.t. the normal bundle map is a n-equivalence and $Y \cup_f Y$ bounds a B-manifold $(W, \bar{\nu}_W)$. Now we are in the situation of Theorem 5.2. There is an obstruction $\theta\,(W, \bar{\nu}_W) \in l^{s,\tau}_{2n+2}(\pi_1(B), w_1(B), S(W))$.

In general this obstruction is non-trivial. For there is a third invariant, the isometric structure of a diffeomorphism represented in the corresponding Witt group by the operation of the diffeomorphism on $H_n(M; \mathbb{Z})$ as an isometry of the intersection form. The isometric structure is a surjective map onto this Witt group which is not finitely generated. This implies that the monoids $l_{2n+2}^{s,\tau}(\pi,w,S)$ are <u>not finitely generated</u> even if π is the trivial group. But one can show that if the isometric structure of (M,f) vanishes $\Theta(W,\bar{\nu}_W)$ is zero bordant if W is chosen appropriately. In the oriented case this is carried through in [17], again in both categories F. Quinn gave a different computation in ([31], [17]). In all cases the image of the invariants can be computed by similar methods.

Other computations of stable diffeomorphism classes and applications are contained in [19], [20], [30].

<u>II) Complete intersections.</u> This is a report about the results of my student Claudia Traving [39]. For $\underline{d} = (d_1,\dots,d_r) \in \mathbb{N}^r$ we denote the corresponding smooth complete intersection given as the set of zeros of r-homogenous polynomials of degree d_i by $X_n(\underline{d}) \subset \mathbb{C}P^{n+r}$; $\dim_{\mathbb{C}} X_n(\underline{d}) = n$. The diffeomorphism class of $X_n(\underline{d})$ is independent of these polynomials depending only of $\underline{d}$.

C. Traving has given a classification up to diffeomorphism of a certain class of complete intersections. She applies the methods of § 2 and 3. For this one has to determine the normal (n-1)-type of $X_n(\underline{d})$. According to Lefschetz the map $i: X_n(\underline{d}) \longrightarrow \mathbb{C}P^\infty$ representing a generator of $H^2(X_n(\underline{d}); \mathbb{Z}) \cong \mathbb{Z}$ is a n-equivalence and the stable normal bundle of $X_n(\underline{d}) = i^* \xi(n,\underline{d})$, where $\xi(n,\underline{d}) = H^{d_1} \oplus \dots \oplus H^{d_r} - (n+r+1)\ H$, H the Hopf bundle. It is not difficult to show this implies that the normal $(n-1)$-type of $X_n(\underline{d})$ is given by the fibration $B = \mathbb{C}P^\infty \times BO\langle n+1\rangle \xrightarrow{\xi(n,\underline{d})\oplus p} BO$, where p: $BO\langle n+1\rangle \longrightarrow BO$ is the n-connected cover of BO.

To classify complete intersections up to stable diffeomorphism one has to decide whether two complete intersections of same normal $(n-1)$-type B are B-bordant. For this she determines the filtration in the classical Adams spectral sequence of the element in the B-bordism group represented by a complete intersection. If the filtration of a torsion element of the B-bordism group is higher than a certain vanishing line then the element is trivial. Thus if two complete intersections are bordant in the rational B-bordism group (which can be controlled by characteristic numbers) and have such a high Adams filtration they are B-bordant and thus by Theorem 2.1 stably diffeomorphic.

It turns out that under these conditions the complete intersections are even diffeomorphic. For this she applies Proposition 3.2. If n is even Libgober and Wood ([24], Theorem B) have proved that $X_n(\underline{d}) = W_1 \cup_\varphi W_2$ where W_1 is a disk bundle over $\mathbb{C}P_{n/2}$. If the Adams filtration is high enough the manifold W_2 is completely determined by the stable diffeomorphism type. If $X_n(\underline{d'})$ is another complete intersection which is stably diffeomorphic to $X_n(\underline{d})$ then $X_n(\underline{d'}) = W_1' \cup_\psi W_2$. By assumption $\psi^{-1} \circ \varphi$: $W_1 \longrightarrow W_1'$ extends to a stable diffeomorphism. In this situation the conditions of Proposition 3.2 are fulfilled and as the odd L-groups of the trivial group are zero one can cancel the $S^{n/2} \times S^{n/2}$'s. A similar argument works for n odd.

This was the idea of C. Travings main result:

Theorem 8.2 ([39]): Write the total degree $d = d_1 \cdot \ldots \cdot d_r = \prod_{p \text{ prime}} p^{\nu(p)}$. If for all $p \leq \sqrt{n + \frac{5}{4}} + \frac{1}{2}$, $\nu(p) \geq (2n+1)/2(p-1) + 1$ and the same holds for $\underline{d}'$ then two complete intersections $X_n(\underline{d})$ and $X_n(\underline{d}')$ with $n > 2$ are diffeomorphic if and only if the total degrees are the same : $d = d'$, the Pontrjagin classes $p_i(\xi(n,\underline{d})) = p_i(\xi(n,\underline{d}'))$ for $i \leq [n/2]$ and the Euler characteristics are the same : $e(X_n(\underline{d})) = e(X_n(\underline{d}'))$.

III) Simply connected 4-manifolds. The classification of the h-cobordism classes of 1-connected smooth 4-manifolds by the intersection form is well known [40], [41] . We want to give another computation using the normal 2-type as a demonstration of the methods summarized in § 7. Another reason is to discuss the difference to the computation of [41] which shows some typical information.

The normal 2-type of a 1-connected smooth 4-manifold M can easily be described. We denote the fibration $p: K(\pi_2(M), 2) \times B\,\mathrm{Spin} \longrightarrow BO$ given by the sum of a complex line bundle ξ over $K(\pi_2(M),2)$ with second Stiefel-Whitney class equal to $w_2(M)$ and the universal Spin-bundle by $B(\pi_2(M), w_2(M))$. This is the normal 2-type of M. For, if $k: M \longrightarrow K(\pi_2(M),2)$ is the characteristic map then the difference of the normal bundle of M and the pull back of ξ under k has vanishing second Stiefel-Whitney class. Thus the classifying map of this difference bundle factorizes over $B\,\mathrm{Spin}$ by a map $f: M \longrightarrow B\,\mathrm{Spin}$. Obviously the map $k \times f$ is a normal 2-smoothing of M in $B(\pi_2(M), w_2(M))$.

We abbreviate $B(\pi_2(M), w_2(M))$ by B. By Theorem 7.3 we have an exact sequence

$$L_5^{s,\tau}(\{e\}) \longrightarrow NS^B_{4,2} \longrightarrow \Omega^B_4$$

As $\quad L_5^{s,\tau}(\{e\}) = L_5\{e\} = 0 \quad$ we have an injective map $NS^B_{4,2} \longrightarrow \Omega^B_4$. By Pontrjagin-Thom construction $\Omega^B_4 = \pi_4(M\xi \wedge M\,\mathrm{Spin})$.

If we consider the Atiyah-Hirzebruch spectral sequence (with coefficients Ω_4^{Spin}) we obtain exact sequences:

$$0 \longrightarrow F_4^B \longrightarrow \Omega_4^B \longrightarrow H_4(K;\mathbb{Z})$$

$$[M, \bar{\nu}_M] \longmapsto (\bar{\nu}_M)_*[M]$$

and

$$0 \longrightarrow \Omega_4^{\mathrm{Spin}} \longrightarrow F_4^B \longrightarrow \mathrm{Kok}\,\alpha(w_2) \longrightarrow 0$$

where F_4^B is the kernel of the map $\Omega_4^B \longrightarrow H_4(K;\mathbb{Z})$, $K = K(\pi_2(M), 2)$ and $\alpha(w_2)$ is the dual of the map $H^2(K;\mathbb{Z}_2) \longrightarrow H^4(K;\mathbb{Z}_2)$, $w_2 = w_2(M)$.

$$\begin{matrix} \psi & & \psi \\ x & \longmapsto & x^2 + x w_2 \end{matrix}$$

Ω_4^{Spin} is isomorphic to $\mathbb{Z}$, the isomorphism is given by the signature divided by 16. If $w_2(M) = 0$, $\mathrm{Kok}\,\alpha(w_2) = 0$. This follows as $K = \prod \mathbb{C}P^\infty$ and thus its cohomology is a polynomial algebra. If $w_2(M) \neq 0$, $\mathrm{Kok}\,\alpha(w_2) = \mathbb{Z}_2$. In this case the sequence is non-split and thus in both cases $F_4^B \cong \mathbb{Z}$ under the signature divided by 16 and divided by 8 in the second case. That the sequence is non split is equivalent to the existence of a B-manifold in F_4^B with signature 8. It is easy to construct two B-structures $\bar{\nu}$ and $\bar{\nu}'$ on $\mathbb{C}P^2$ such that $g(\bar{\nu})_*[\mathbb{C}P^2] = (\nu')_*[\mathbb{C}P^2]$ in $H_4(K;\mathbb{Z})$ (Hint: consider the case $K = \mathbb{C}P^\infty$ and $w_2 \neq 0$). Thus $g[\mathbb{C}P^2,\bar{\nu}] - [\mathbb{C}P^2,\bar{\nu}']$ is contained in F_4^B and it has signature 8.

If $(N,\bar{\nu}_N)$ is a normal 2-smoothing in B the class $(\bar{\nu}_N)_*[N] \in H_4(K;\mathbb{Z})$ is equivalent to the intersection form on N: If $x, y \in H^2(N;\mathbb{Z})$, $\langle x \cup y, [N]\rangle = \langle \bar{\nu}_N^* x' \cup \bar{\nu}_N^* y', (\bar{\nu}_N)_* [N]\rangle$ as $(\bar{\nu}_N)^*: H^2(K;\mathbb{Z}) \longrightarrow H^2(N;\mathbb{Z})$ is an isomorphism. On the other hand given the intersection form the formula above determines $(\bar{\nu}_N)_* [N]$ as $H^*(K;\mathbb{Z}) \cong H^*(\prod \mathbb{C}P^\infty;\mathbb{Z})$ is a polynomial ring. Thus the B-bordism class of a normal 2-smoothing in B is completely determined by the intersection form. By the injection $NS_{4,2}^B \hookrightarrow \Omega_4^B$ the isomorphism class of a normal 2-smoothing in B is determined by the intersection form.

The normal 2-type of a smooth 1-connected 4-manifold M is determined by $\pi_2(M)$ and $w_2(M)$. As $w_2(M)$ is determined by the intersection form we finally obtain another proof of

<u>Theorem 8.3 ([40], [41]):</u> Two 1-connected smooth 4-manifolds are h-cobordant if and only if they have isomorphic intersection forms.

If one applies the results of Browder and Novikov to prove this result one has first to decide whether M and N are homotopy equivalent. This is equivalent to showing that M and N have isomorphic intersection forms as was proved by Milnor [25] using the methods of Whitehead. Even if the proof is not difficult it is more complicated than determining the normal 2-type. The next step with this program is to determine the set of smoothings S(M). This is done by Wall ([41] , Theorem 16.5): $S(M) = \mathrm{Ker}(H_2(M;\mathbb{Z}_2) \longrightarrow \mathbb{Z}_2)$.

$$x \longmapsto \langle w_2(M), x \rangle$$

Thus a smoothing in M is not determined by the intersection form. But Wall shows that S(M) consists only of self equivalences of M and thus one obtains the result above. Again this is not difficult in this case but in our proof above no operation of Aut (B,p) has to be considered.

<u>Remark:</u> The diffeomorphism classification of smooth 1-connected 4-manifolds is still open as well as a complete answer to the question which forms can be realized by intersection forms of smooth 1-connected 4-manifolds. In the last years S. Donaldson [10] has made a big breakthrough concerning this problem showing that various forms cannot be realized. The homeomorphism classification of 1-connected 4-manifolds was given by Freedmann [11].

<u>IV) Inertia group.</u> We have introduced 4 types of obstruction groups (monoids) for the classification of manifolds: $L_n^{s,\tau}(\pi,w)$, $l_n^{s,\tau}(\pi,w)$, $L_{2n}^{s,\tau}(\pi,w,S)$, $l_{2n}^{s,\tau}(\pi,w,S)$. We want to finish our examples of applications with a result using the group $L_{2n}^{s,\tau}(\pi,w,S)$ to demonstrate that the concept of form parameters is useful for geometric problems.

Suppose n odd, $n \neq 1,3,7$ and M^{2n-1} is 1-connected and that there is an embedding S^n into $M \times I$ with normal bundle the tangent bundle of S^n.

This means $\mu(S^n) = 1 \in \mathbb{Z}_2 \cong Q_{-1}$ [5]. Thus $S(M \times I) = \mathbb{Z}$ and as mentioned at the end of § 4 we know $L_{2n}^{s,\tau}(\{e\}, S(M\times I)) = \{0\}$.

Now suppose that N^{2n-1} is another manifold with the same property which has a normal (n-1)-smoothing in $B_{n-1}(M)$ which is bordant to some normal (n-1)-smoothing of M in $B_{n-1}(M)$. Then by Theorem 7.3 the obstruction to transforming a bordism into a h-cobordism is contained in $L_{2n}^{s,\tau}(\{e\}, S(M\times I)) = \{0\}$ and thus M and N are diffeomorphic $(n \geq 3)$.

For instance let Σ^{2n-1} be the Kervaire sphere generating bP_{2n} [15] and M as above. Then $M \# \Sigma$ and M admit bordant normal (n-1)-smoothings as described above. A bordism is given by $M \times I \natural X$, X a framed manifold with $\partial X = \Sigma$. Thus $M \# \Sigma \cong M$ or Σ is contained in the inertia group $I(M)$: In fact we obtain a slightly stronger result.

<u>Theorem 8.4:</u> Let M^{4k+1} be a closed manifold, $k \neq 1,3$ with the following property: There exists $\alpha \in \pi_{2k+1}(M) = \pi_{2k+1}(M \times \mathbb{R})$ with $\mu(\alpha) = [1] \in \Lambda/\{a+\bar{a}\}$. Then $M \# \Sigma$ is diffeomorphic to M where Σ is the Kervaire sphere.

<u>Remark:</u> This property is fulfilled if there exists an embedding $S^{2k+1} \hookrightarrow M \times \mathbb{R}$ with non-trivial stably trivial normal bundle. If M is highly connected the same result was proved in ([36], Satz 15.4). A special case namely if M is the Stiefel manifold given by the sphere bundle of the tangent bundle over S^{2k+1} was proved in [7].

Proof: By construction ([41] , § 5) $M \# \Sigma$ is the image in the set $\mathfrak{M}(M)$ of manifolds normally homotopy equivalent to M under the action of the non-trivial element in $L^{s,\tau}_{4k+2}\{e\} = \mathbb{Z}_2$ on M. The existence of an α with $\mu(\alpha) = [1]$ implies $\mathbb{Z} \subset S(M)$. Thus we have a homomorphism $L^{s,}_{4k+2}(\{e\}, \mathbb{Z}) \longrightarrow L^{s,\tau}_{4k+2}(\pi_1(M), w_1(M), S(M))$. As $L^{s,\tau}_{4k+2}(\{e\}, \mathbb{Z}) = 0$ the statement follows from the commutative diagram

$$\begin{tikzcd}
L^{s,\tau}_{4k+2}\{e\} \arrow[r] \arrow[d] & \mathfrak{M}(M) \arrow[dd, hook] \\
L^{s,\tau}_{4k+2}(\{e\}, \mathbb{Z}) \arrow[d] & \\
L^{s,\tau}_{4k+2}(\pi_1(M), w_1(M), S(M)) \arrow[r] & \mathfrak{M}(B_{2k}(M))
\end{tikzcd}$$

q.e.d.

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