ISOTOPIC CLASSIFICATION OF ODD-DIMENSIONAL
SIMPLE LINKS OF CODIMENSION TWO

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ABSTRACT. This paper gives a PL-isotopy classification of odd-dimensional simple links of
dimension \( \geq 5 \) in terms of their Seifert matrices.

Bibliography: 11 titles.

This paper gives an isotopy classification of odd-dimensional (of dimension \( \geq 5 \)) simple
links in terms of their Seifert matrices. A similar ambient-isotopy classification of simple
links was obtained by Liang [1], following the ambient-isotopy classification of simple
knots carried out by Levine [5]. In contrast to ambient isotopy, isotopy of knots is an
uninteresting equivalence relation: any knot is isotopic to the trivial knot.

The formulation in this paper is given for two-component links, but everything
generalizes easily to the case of a larger number of components.

The main results of the paper are formulated in the Introduction (§1.6); §§2 and 3 are
devoted to the proofs, and in a supplement (§4) some isotopy invariants of links connected
with cobordism invariants of matrices are considered.

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§1. Introduction

1.1. Links. By an \( n \)-dimensional link is meant a piecewise-linear locally flat oriented
submanifold \( L \) of the sphere \( S^{n+2} \), homeomorphic to the disjoint sum \( S^n \sqcup S^n \) of two
\( n \)-dimensional spheres. Denote by \( K_1 \) and \( K_2 \) the connected components of \( L \). It is well
known that each submanifold \( K_i \) bounds a piecewise-linear locally flat compact oriented
submanifold \( V_i \) of \( S^{n+2} \), called a Seifert surface of the knot \( K_i \). If there exist nonintersecting
Seifert surfaces \( V_1 \) and \( V_2 \) of the components \( K_1 \) and \( K_2 \), then \( L \) is called a boundary
link, and the manifold \( V = V_1 \cup V_2 \) is called a Seifert surface of the link \( L \). A link of
dimension \( 2q - 1 \) is called simple if it has a Seifert surface consisting of \( (q - 1) \)-connected
components.

1.2. Seifert matrices of links. Let \( L \) be a boundary link of dimension \( 2q - 1 \), and let \( V \)
be a Seifert surface of \( L \). We denote by \( TV \) a regular neighborhood of the surface \( V \) in
\( S^{2q+1} \). The manifold \( TV \) is homeomorphic to the product \( V \times J \), where \( J = [-1, 1] \), and
where the surface \( V \), lying in \( TV \), is carried to \( V \times \{0\} \) by this homeomorphism. Putting
\[ V_+ = V \times \{1\}, \text{let } i: V \to V_+ \text{ be the homeomorphism defined by translation } V \times \{0\} \to V \times \{1\} \text{ in the product } V \times J, \text{ and let } j: V_+ \to S^{2q+1} \setminus V \text{ be the inclusion. We define a form } \]

\[ l: H_q(V; \mathbb{Z})/\text{Tors} \times H_q(V; \mathbb{Z})/\text{Tors} \to \mathbb{Z} \]

as follows: if \( x, y \in H_q(V; \mathbb{Z})/\text{Tors} \), then \( l(x, y) = \text{lk}((j \circ i)_*x, y) \), where

\[ \text{lk}: H_q(S^{2q+1} \setminus V; \mathbb{Z})/\text{Tors} \times H_q(V; \mathbb{Z})/\text{Tors} \to \mathbb{Z} \]

is the linking coefficient in the sphere \( S^{2q+1} \).

If \( V_1 \) and \( V_2 \) are Seifert surfaces of the components of the link \( L \), then in the basis of the group \( H_q(V; \mathbb{Z})/\text{Tors} \), consisting of bases of \( H_q(V_1; \mathbb{Z})/\text{Tors} \) and \( H_q(V_2; \mathbb{Z})/\text{Tors} \), the matrix of the form \( l \) can be written as

\[
M = \begin{pmatrix}
M_1 & P \\
-\epsilon P' & M_2
\end{pmatrix},
\]

where \( M_1 \) is the matrix of the restriction of \( l \) to \( H_q(V_1; \mathbb{Z})/\text{Tors} \), called a Seifert matrix of the knot \( K_i, \epsilon = (-1)^q \), and the prime denotes transposition (see [1]). The matrix \( M \) of the form \( l \), equipped with such a decomposition, will be called a Seifert matrix of the link \( L \). Since \( M_1 \) is a Seifert matrix of the component \( K_i \), the matrix \( M_1 + \epsilon M' \) is unimodular (see [7]). This is obviously equivalent to unimodularity of the matrix \( M + \epsilon M' \).

1.3. **L-matrices.** Let \( \epsilon = \pm 1 \). By an L-matrix is meant a square matrix \( M \), equipped with a decomposition of the form

\[
M = \begin{pmatrix}
M_1 & P \\
-\epsilon P' & M_2
\end{pmatrix},
\]

where \( M_1 \) and \( M_2 \) are square matrices such that \( M + \epsilon M' \) is unimodular. We denote the number \( \epsilon \) in the definition of \( M \) by \( \epsilon(M) \).

**Theorem 1.3.1 (Liang [1]).** For any L-matrix \( M \) and any integer \( n \geq 1 \), there exists a simple link of dimension \( 4n + 2 + \epsilon(M) \), having \( M \) as its Seifert matrix.

1.4. **\( \epsilon \)-equivalence of L-matrices.** Let

\[
M = \begin{pmatrix}
M_1 & P \\
-\epsilon P' & M_2
\end{pmatrix} \quad \text{and} \quad N = \begin{pmatrix}
N_1 & Q \\
-\epsilon Q' & N_2
\end{pmatrix}
\]

be L-matrices, where the matrices \( M_1 \) and \( N_1 \) have size \( m_1 \times m_1 \), and \( M_2 \) and \( N_2 \) have size \( m_2 \times m_2 \). The L-matrices \( M \) and \( N \) are called \( \epsilon \)-congruent if there exist unimodular \( B_1 \) of size \( m_1 \times m_1 \) and \( B_2 \) of size \( m_2 \times m_2 \) such that

\[
N = \begin{pmatrix}
B_1 & 0 \\
0 & B_2
\end{pmatrix} M \begin{pmatrix}
B_1' & 0 \\
0 & B_2'
\end{pmatrix}.
\]

We shall say that the L-matrix \( M' \) is an \( \epsilon \)-enlargement of the L matrix

\[
M = \begin{pmatrix}
M_1 & P \\
-\epsilon P' & M_2
\end{pmatrix}
\]
if

\[
M^1 = \begin{pmatrix}
0 & X & 0 & 0 & 0 \\
Y & Z & Q_3 & Q_1 & Q_2 \\
0 & -\epsilon Q'_3 & M_3 & 0 & 0 \\
0 & -\epsilon Q'_1 & 0 & M_1 & P \\
0 & -\epsilon Q'_2 & 0 & -\epsilon P' & M_2
\end{pmatrix}
\]

or

\[
M^1 = \begin{pmatrix}
M_1 & P & 0 & 0 & Q_1 \\
-\epsilon P' & M_2 & 0 & 0 & Q_2 \\
0 & 0 & M_3 & 0 & Q_3 \\
0 & 0 & 0 & 0 & X \\
-\epsilon Q'_3 & -\epsilon Q'_2 & -\epsilon Q'_1 & Y & Z
\end{pmatrix}
\]

where \(X, Y, Z\) and \(M_3\) are square matrices. It is clear that the matrices \(X + \epsilon Y'\) and \(M_3 + \epsilon M'_3\) are unimodular. The matrix \((0\ Y_2)\) will be called the nontrivial part of the \(I\)-enlargement. The \(L\)-matrix \(M\) will, in its turn, be called an \(I\)-reduction of the \(L\)-matrix \(M^1\). We shall say that \(L\)-matrices \(M^0_0\) and \(M^1\) are \(I\)-equivalent if they can be connected by a chain of \(I\)-enlargements, \(I\)-reductions and \(I\)-congruences.

If in the definition of \(I\)-equivalence we restrict ourselves to the special form of \(I\)-enlargements and \(I\)-reductions in which the matrix \(M_3\) has size zero and the nontrivial part is equal to \((0\ 0)\) or \((0\ 0)\), we obtain the definition of \(I\)-equivalence of \(L\)-matrices (see [1]).

**Remark.** It is easy to show that any \(I\)-enlargement of an \(L\)-matrix \(M\) is \(I\)-congruent to an \(I\)-enlargement of the \(L\)-matrix \(M\) for which \(Z + \epsilon Z' = 0\) and the matrix \(X + \epsilon Y'\) is the identity.

### 1.5. Isotopy of links.

Links \(L_0\) and \(L_1\) of dimension \(n\) are called isotopic if there exists a piecewise-linear embedding

\[ F: (S^n_1 \sqcap S^n_2) \times I \to S^{n+2}_n \times I \]

(not necessarily locally flat) such that

1. \(F((S^n_1 \sqcap S^n_2) \times \{\alpha\}) \subset S^{n+2}_n \times \{\alpha\}\) for all \(\alpha \in I\), and
2. \(F((S^n_1 \sqcap S^n_2) \times \{i\}) = L_i\), where \(i = 0, 1\).

### 1.6. Formulation of the main results.

**Theorem 1.6.1.** If odd-dimensional boundary links are isotopic, then their Seifert matrices are \(I\)-equivalent.

**Theorem 1.6.2.** If the Seifert matrices of simple odd-dimensional links of dimension \(\geq 5\) are \(I\)-equivalent, then these links are isotopic.

These assertions, taken together with Theorem 1.3.1, yield an isotopy classification of simple odd-dimensional links of dimension \(\geq 5\) in terms of their Seifert matrices.

### §2. Proof of Theorem 1.6.1

#### 2.1. Local isotopy and Rolfsen's theorem.

Let \(L_0\) and \(L_1\) be links of dimension \(n\), and suppose that there exists in the ambient sphere \(S^{n+2}_n\) a piecewise-linear ball \(D\) of dimension \(n + 2\) such that

1. \(L_i\) and \(\partial D\) intersect transversally,
(2) the paris \((D, D \cap L_i)\) are proper ball pairs, and
(3) outside \(D\) the links \(L_0\) and \(L_1\) coincide.

We shall then say that the links \(L_0\) and \(L_1\) are \emph{locally isotopic} (see [3]), or that \emph{they can be connected by a local isotopy in the ball} \(D\). It is clear that locally isotopic links are isotopic.

Links \(L_0\) and \(L_1\) of dimension \(n\) are called \emph{ambient-isotopic} if there exists a piecewise-linear homeomorphism \(F: S^{n+2} \times I \to S^{n+2} \times I\) such that

1. \(F(S^{n+2} \times \{\alpha\}) \subset S^{n+2} \times \{\alpha\}\) for all \(\alpha \in I\).
2. \(F|_{S^{n+2} \times \{0\}} = \text{id. and}
3. \(F(L_0 \times \{1\}) = L_1\).

Obviously, ambient-isotopic links are isotopic. All these concepts are connected by the following theorem.

**THEOREM 2.1 (ROLFSEN [2]).** If \(L_0\) and \(L_1\) are isotopic links, then \(L_1\) can be obtained from \(L_0\) by a finite sequence of local and ambient isotopies; moreover, two local isotopies, one for each component, are sufficient.

In view of this theorem and the \(i\)-equivalence of Seifert matrices of ambient-isotopic links (see [1]), to prove Theorem 1.6.1 it suffices to prove the following lemma.

**2.2. LEMMA.** Seifert matrices of locally isotopic links are \(i\)-equivalent.

Let \(L_0\) and \(L_1\) be locally isotopic links of dimension \(2q - 1\), and let \(D\) be the ball of this local isotopy. It is easy to show that there exist Seifert surfaces \(V_0\) and \(V_1\) of the links \(L_0\) and \(L_1\) satisfying the following conditions:

1. \(V_0\) and \(V_1\) coincide outside \(D\),
2. \(V_i\) and \(\partial D\) intersect transversally, and
3. \(V_i \cap \partial D\) is connected.

It is clear that in such a case the surface \(W = V_0 \cap \partial D = V_1 \cap \partial D\) is a connected Seifert surface of the knot \(K = L_0 \cap \partial D = L_1 \cap \partial D\) in the sphere \(\partial D\). Let \(U\) be the connected component of \(V_0\) intersecting \(D\), and put \(W_1 = U \cap D\) and \(W_2 = U \setminus W_1\), so that \(U = W_1 \cup W_2\).

We first prove Lemma 2.2 for local isotopies such that the group \(H_q(W; \mathbb{Z})\) has no torsion. We denote \(H_i(X; \mathbb{Z})/\text{Tors} \rightarrow \overline{H}_i(X)\).

**2.3. A special case:** \(\text{Tors} H_q(W; \mathbb{Z}) = 0\). Let \(G\) be a finitely generated abelian group and \(H\) a subgroup. We denote by \(S(H, G)\) the smallest pure subgroup of \(G\) containing \(H\).

It is clear that rank \(S(H, G) = \text{rank } H\).

Let \(\overline{\in_i}: \overline{H}_q(W) \to \overline{H}_q(W_i)\), for \(i = 1, 2\), be the homomorphisms induced by the inclusion homomorphisms \(\in_i: H_q(W; \mathbb{Z}) \to H_q(W_i; \mathbb{Z})\). We put \(G_i = S(\text{Im } \overline{\in_i}, \overline{H}_q(W_i))\). It is clear that the group \(B_i = \overline{H}_q(W_i)/G_i\) has no torsion, so that \(\overline{H}_q(W_i) = B_i \oplus G_i\). We consider a segment of the exact sequence of the triad \((U, W_1, W_2)\):

\[
\cdots \to H_q(W; \mathbb{Z}) \to H_q(W_1; \mathbb{Z}) \oplus H_q(W_2; \mathbb{Z}) \stackrel{\partial}{\to} H_q(U; \mathbb{Z}) \stackrel{\partial}{\to} H_{q-1}(W; \mathbb{Z}) \to \cdots
\]

and a segment of the induced sequence:

\[
\cdots \to \overline{H}_q(W) \to \overline{H}_q(W_1) \oplus \overline{H}_q(W_2) \stackrel{\overline{\partial}}{\to} \overline{H}_q(U) \stackrel{\overline{\partial}}{\to} \overline{H}_{q-1}(W) \to \cdots.
\]

Since \(\text{Tors} H_{q-1}(W; \mathbb{Z}) = 0\), the induced sequence is exact at the term \(\overline{H}_q(U)\), and therefore \(\overline{H}_q(U) = \text{Im } \overline{\rho} \oplus \overline{G}\), where \(\overline{G} \cong \text{Im } \overline{\delta}\).
Put $\bar{B}_i = S(\tilde{\rho}(B_i), \tilde{H}_q(U))$. Let $\bar{H}_q(W) \rightarrow \bar{H}_q(U)$ be the homomorphism induced by the inclusion homomorphism in: $H_q(W; \mathbb{Z}) \rightarrow H_q(U; \mathbb{Z})$, and put $T = S(\text{Im} \tilde{\rho}, \bar{H}_q(U))$. It is easy to see that $\text{Im} \tilde{\rho} = B_1 \oplus B_2 \oplus T$, and so $\bar{H}_q(U) = B_1 \oplus B_2 \oplus T \oplus G$.

Let $\tau: \bar{H}_q(U) \times \bar{H}_q(U) \rightarrow \mathbb{Z}$ be the intersection index in the manifold $U$; in a basis of $\bar{H}_q(U)$, consisting of bases for the subgroups $\bar{B}_1$, $\bar{B}_2$, $T$ and $G$, the matrix of the form $\tau$ obviously can be written as

\[
A = \begin{pmatrix}
X_1 & 0 & 0 & P_1 \\
0 & X_2 & 0 & P_2 \\
0 & 0 & 0 & S_1 \\
eP_1' & eP_2' & eS_1' & S_2
\end{pmatrix},
\]

where $\epsilon = (-1)^q$ and the matrix $S_1$ is unimodular. Adding to the generators of the subgroups $\bar{B}_1$ and $\bar{B}_2$ elements of the subgroup $T$, we make the matrices $P_1$ and $P_2$ vanish. These elements, obtained as a result of such a modification of the generators of $\bar{B}_1$ and $\bar{B}_2$, generate subgroups $C_1$ and $C_2$ of $\bar{H}_q(U)$, which is represented in the form of a direct sum $\bar{H}_q(U) = C_1 \oplus C_2 \oplus T \oplus G$.

We turn now to the Seifert from constructed on the manifold $U$:

\[
l_U: \bar{H}_q(U) \times \bar{H}_q(U) \rightarrow \mathbb{Z}.
\]

Let $N$ be the matrix of the form $l_U$, in a basis of $\bar{H}_q(U)$ consisting of bases of the subgroups $C_1$, $C_2$, $T$ and $G$. As is easily seen, the subgroup $T$ is orthogonal to the subgroup $C_1 \oplus C_2 \oplus T$ with respect to $l_U$, and the subgroups $C_1$ and $C_2$ are also orthogonal. Taking into account that

\[
N + \epsilon N' = \begin{pmatrix}
X_1 & 0 & 0 & 0 \\
0 & X_2 & 0 & 0 \\
0 & 0 & 0 & S_1 \\
0 & 0 & \epsilon S_1' & S_2
\end{pmatrix},
\]

we obtain the following form for the matrix $N$:

\[
N = \begin{pmatrix}
M_1 & 0 & 0 & Q_1 \\
0 & M_2 & 0 & Q_2 \\
0 & 0 & 0 & X \\
-\epsilon Q_1' & -\epsilon Q_2' & Y & Z
\end{pmatrix},
\]

thus the Seifert matrix $N_0$ of the link $L_0$ has the form

\[
N_0 = \begin{pmatrix}
M_1 & 0 & 0 & Q_1 & 0 \\
0 & M_2 & 0 & Q_2 & P \\
0 & 0 & 0 & X & 0 \\
-\epsilon Q_1' & -\epsilon Q_2' & Y & Z & Q_3 \\
0 & -\epsilon P' & 0 & -\epsilon Q_3' & M_3
\end{pmatrix},
\]

i.e., it is an $I$-enlargement of the $L$-matrix

\[
M = \begin{pmatrix}
M_2 \\
-\epsilon P' \\
M_3
\end{pmatrix}.
\]
Analogous reasoning, carried out for the Seifert surface $V_x$ of the link $L$, shows that its Seifert matrix $N_x$ is also an $I$-enlargement of $M$. Lemma 2.2 is proved for the case $\text{Tor}_q(W;\mathbb{Z}) = 0$.

Now let $H_{q-1}(W;\mathbb{Z})$ be arbitrary; we restrict consideration to the case $q \geq 4$. The general proof for all $q$ is essentially analogous to that carried out above, only more cumbersome; in the case $q \geq 4$ we shall reduce the proof of Lemma 2.2 to the special case examined above, utilizing the concept of a cross-section of a Seifert surface of a knot (see [4]).

2.4. Cross-sections of Seifert surfaces of knots. Let $K$ be a knot of dimension $2q - 2$, $W$ its Seifert surface, and suppose that in the ambient sphere $S^{2q}$ there exists a piecewise-linear $2q$-dimensional ball $B$ such that

1. $\partial B$ intersects $K$ and $W$ transversally,
2. the pair $(B, B \cap K)$ is a proper ball pair, and
3. the pairs $(W \cap B, W \cap \partial B)$ and $(W \cap (S^{2q} \setminus B), W \cap \partial B)$ are $(q - 1)$-connected.

Then the surface $W \cap \partial B$ is called a cross-section of the Seifert surface $W$ of the knot $K$.

**Lemma 2.4** (Kearton [4]). A Seifert surface of a $(2q - 2)$-dimensional knot for $q \geq 4$ has a cross-section.

Let

$$i_1: H_{q-1}(W \cap \partial B;\mathbb{Z}) \to H_{q-1}(W \cap B;\mathbb{Z})$$

and

$$i_2: H_{q-1}(W \cap \partial B;\mathbb{Z}) \to H_{q-1}(W \cap (S^{2q} \setminus B);\mathbb{Z})$$

be inclusion homomorphisms; we shall say that the surface $W \cap \partial B$ is a regular cross-section of the surface $W$ if $\text{Ker} i_1 = \text{Ker} i_2$. It is easy to show that if $W$ has a regular cross-section, then $\text{Tor}_q(W;\mathbb{Z}) = 0$.

2.5. Conclusion of the proof of Lemma 2.2: reduction to the case $\text{Tor}_q(W;\mathbb{Z}) = 0$. We shall represent the given local isotopy in the form of a composition of three local isotopies satisfying the conditions of §2.3. We adopt the notation of §2.2.

Let $Z_0$ and $Z_1$ be the principal components of the links $L_0$ and $L_1$, such that $Z_0 \cap D = L_0 \cap D$, and let $W' = W \cap \partial B$ be a cross-section of the surface $W$. Put $U_1 = W \cap B$ and $U_2 = W \setminus U_1$, and let $S = K \cap \partial B$ and $C = K \cap B$.

Thicken the ball $B$ to a cylinder $B \times J$, where $J = [-1, 1]$, in the ambient sphere $S^{2q+1}$, so that $\partial D \cap (B \times J) = B \times \{0\}$. The links $L_0$ and $L_1$ coincide near $\partial D$, so that $L_0 \cap (B \times J) = L_1 \cap (B \times J)$. We can suppose that $L_i \cap (B \times J) = Z_i \cap (B \times J)$. The link $L_0$ intersects the cylinder $B \times J$ in the disk $C \times J$, and intersects $\partial (B \times J)$ in the sphere $Q$, where

$$Q = (C \times \{-1\}) \cup (S \times (-1)) (S \times J) \cup (S \times (1)) (C \times \{1\}),$$

and the Seifert surface $F$ of the knot $Q$ in the sphere $\partial (B \times J)$:

$$F = (U_1 \times \{-1\}) \cup (W \times (-1)) (W' \times J) \cup (W' \times (1)) (U_1 \times \{1\})$$

has the surface $W'$ as a regular cross-section. We subject the link $L_0$ to a local isotopy, transforming (within the ball $B \times J$) the disk $C \times J$ to a disk which is symmetric, with respect to $\partial (B \times J)$, to the disk $Z_0 \setminus (C \times J)$; the resulting link is denoted by $L_0'$. Since $F$ has a regular cross-section, $\text{Tor}_q(F;\mathbb{Z}) = 0$, and for the above local isotopy Lemma 2.2 is valid.
We subject the link \( L \), to an analogous local isotopy (which transforms the disk \( C \times J \) to a disk symmetric, with respect to \( \partial (B \times J) \), to the disk \( Z^C \times / \)); the resulting link is denoted by \( L' \). The links \( L' \) and \( L'' \) are connected by a local isotopy in \( D \), and the Seifert surface \( F' = U_2 \cup \mu' U_2 \) of the knot \( K' = L'_0 \cap \partial D = L'_1 \cap \partial D \) in \( \partial D \) has a regular cross-section \( W' \), so that for this local isotopy, and consequently also for the given local isotopy, the assertion of Lemma 2.2 is valid, which completes the proof of Theorem 1.6.1.

§3. Proof of Theorem 1.6.2.

3.1. Scheme of the proof. Let the \( L \)-matrix \( N_i \) be an \( I \)-enlargement of the \( L \)-matrix \( N_0 \). For each \( q \geq 1 \) we construct simple \((4q + 2 + e(N_0))\)-dimensional links \( L_0 \) and \( L_1 \), connected by a local isotopy in a ball \( D \), whose Seifert matrices are the \( L \)-matrices \( N_0 \) and \( N_i \). From this, in view of the results of [1], Theorem 1.6.2 will be proved.

First, using the nontrivial part of the given \( I \)-enlargement, we construct a Seifert surface \( W \) of the knot \( L' \cap \partial D \) in \( 3D \); and then we will construct Seifert surfaces of the desired links \( L_0 \) and \( L_1 \), intersecting \( 3D \) in \( W \).

3.2. Construction of the surface \( W \). It obviously suffices to restrict attention to one of the two forms of an \( I \)-enlargement, since one form is obtained from the other by renumbering the components of links. Consider the \( L \)-matrix

\[
N_0 = \begin{pmatrix}
   M_1 & P \\
   -eP' & M_2
\end{pmatrix}
\]

and its \( I \)-enlargement

\[
N_i = \begin{pmatrix}
   0 & X & 0 & 0 & 0 \\
   Y & Z & Q_3 & Q_1 & Q_2 \\
   0 & -eQ'_3 & M_3 & 0 & 0 \\
   0 & -eQ'_1 & 0 & M_1 & P \\
   -eQ'_2 & 0 & -eP' & M_2
\end{pmatrix}
\]

Let \( X, Y \) and \( Z \) be \( m \times m \) matrices. As we remarked in §1.4, we can assume that \( Z + eZ' = 0 \) and that the matrix \( X + eY' \) is the identity. Let \( 4q + 2 + e(N_0) = 2n - 1 \) (such an \( n \) can always be found, since \( e(N_0) = \pm 1 \)).

In \( R^{2n+2} \) with coordinates \( x_1, \ldots, x_{2n+2} \) we consider the unit sphere \( S^{2n+1} \). Denote by \( D \) the hemisphere in \( S^{2n+1} \) given by the inequality \( x_1 \geq 0 \). The hemisphere \( D \) will be the ball of our local isotopy. Put \( S_0 = \partial D \), let \( S_1 \) be the sphere defined in \( S^{2n+1} \) by \( x_1 = x_2 = 0 \), and let \( D_1 \) be the ball defined in \( S^{2n+1} \) by \( x_1 = x_2 = 0 \) and \( x_3 \geq 0 \). Put \( S_2 = \partial D_1 \), and let \( \Omega_1, \ldots, \Omega_m \) be disjoint smooth \((2n - 2)\)-dimensional balls in \( S_2 \). To \( D_1 \) in \( S_1 \) we attach \( m \) handles \( h^n_1, \ldots, h^n_m \) of index \( n \), so that the attaching sphere \( \xi_i \) of the handle \( h^n_i \) lies in \( \Omega_i \). For the resulting manifold \( D'_1 \), generators \([a_1], \ldots, [a_m]\) of the group \( H_n(D'_1; Z) \) are realized by disjoint smoothly embedded spheres \( a_1, \ldots, a_m \) of dimension \( n \). Let \( \gamma_1, \ldots, \gamma_m \) be a family of \( n \)-dimensional smooth balls in \( S_2 \) such that \( \gamma_i \in \Omega_i \) and \( \partial \gamma_i = \xi_i \), and let \( \tau_1, \ldots, \tau_m \) be a family of smooth disjoint \((n - 2)\)-dimensional spheres in \( S_2 \) such that \( \tau_i \) intersects only \( \gamma_i \), in exactly one of its interior points. To \( D'_1 \) in \( S_0 \) we attach \( m \) handles \( h^{n-1}_1, \ldots, h^{n-1}_m \) of index \( n - 1 \), using \( \tau_1, \ldots, \tau_m \) as attaching spheres, obtaining a manifold we denote by \( W' \). The handles \( h^{n-1}_1, \ldots, h^{n-1}_m \) yield generators \([\beta_1], \ldots, [\beta_m]\) of \( H_{n-1}(W'; Z) \), realized by disjoint smoothly embedded spheres \( \beta_1, \ldots, \beta_m \) of dimension \( n - 1 \), and the incidence number of the classes \([a_i]\) and \([\beta_j]\) in \( W' \) is equal to \( \delta_{ij} \). It is easy to see that the boundary of \( W' \) is a \((2n - 2)\)-dimensional sphere.
Finally, modifying, if necessary, the embeddings of the handles \( h_1^{n-1}, \ldots, h_m^{n-1} \) without changing their attaching spheres, we obtain a submanifold \( W \) of \( S_0 \) such that
\[
l_w : H_n(W; \mathbb{Z}) \times H_{n-1}(W; \mathbb{Z}) \to \mathbb{Z}
\]
is the Seifert pairing constructed for the submanifold \( W \) of \( S_0 \), and \( x_i \) is an element of the matrix \( X \).

### 3.3. Construction of the link \( L_1 \)
Let \( J = [-\epsilon_0, \epsilon_0] \), where \( \epsilon_0 \) is a sufficiently small positive number. Thicken the sphere \( S_0 \) in \( S_2^{n+1} \) to a strip \( R(-\epsilon_0, \epsilon_0) \), defined by \(-\epsilon_0 < x < \epsilon_0\), correspondingly thicken the manifold \( W \) to a cylinder \( H^*_n \times J \) in \( R(-\epsilon_0, \epsilon_0) \), and let \( W_1 \) and \( W_2 \) be the upper and lower bases of \( W \times J \). Generators of \( H_n(W_i; \mathbb{Z}) \) and \( H_n(W; \mathbb{Z}) \) are realized by smoothly embedded spheres, which we denote by the corresponding spheres in \( W \) supplied with a superscript, for example \( \alpha_1^1, \beta_1^2 \) etc.

Attach to \( W \times J \) in \( R(\epsilon_0, 2\epsilon_0) \), defined in \( S_2^{n+1} \), the inequalities \( \epsilon_0 < x_1 < 2\epsilon_0 \), \( m \) handles \( H_1^{n1}, \ldots, H_{m1}^{n1} \) of index \( n \), using \( \beta_1^1, \ldots, \beta_{m1}^1 \) as attaching spheres. Attach also to \( W \times J \) in \( R(-2\epsilon_0, -\epsilon_0) \), defined in \( S_2^{n+1} \), the inequalities \( -2\epsilon_0 < x_1 < -\epsilon_0 \), \( m \) handles \( H_1^{n2}, \ldots, H_{m2}^{n2} \) of index \( n \), using \( \beta_1^2, \ldots, \beta_{m2}^2 \) as attaching spheres. It is clear that such attachings are realizable. Denote the resulting manifold by \( T = (W \times J) \cup H_1^{n1} \cup \cdots \cup H_{m1}^{n1} \cup H_1^{n2} \cup \cdots \cup H_{m2}^{n2} \).

A simple argument shows that the boundary of \( T \) is a \((2n - 1)\)-dimensional sphere. The manifold \( T \) is \((n - 1)\)-connected, and \( H_n(T; \mathbb{Z}) = \mathbb{Z} \oplus \cdots \oplus \mathbb{Z} \) (2\( m \) summands). As a basis for \( H_n(T; \mathbb{Z}) \) we can take the classes \([\alpha_1], \ldots, [\alpha_m]\) and \([\delta_1], \ldots, [\delta_m]\), where the notation \([\alpha_i]\) is preserved for the image of the classes \([\alpha_i]\) under the inclusion homomorphism in: \( H_n(W; \mathbb{Z}) \to H_n(T; \mathbb{Z}) \), and a representative \( \delta_i \) of the class \([\delta_i]\) is obtained, by gluing, from the middle disks of the handles \( H_{i1}^n \) and \( H_{i2}^n \) and the collar \( \beta_i \times J \). For such a basis the Seifert matrix of the knot \( \partial T \) will obviously take the form \( (X) \), where \( F \) is an \( m \times m \) matrix. We may suppose that \( F + \epsilon F' = 0 \) (this can be attained by modifying the generators \([\delta_i]\)). Finally, we may suppose that \( F = I \) (this equality can be attained by modifying the embeddings of the handles \( H_{i1}^n \) and \( H_{i2}^n \) with changing their attaching spheres). Thus the Seifert matrix of the knot \( \partial T \) is equal to \( (X) \).

We construct in the ball \( D(-3\epsilon_0) \), defined in \( S_2^{n+1} \) by \( x_1 < -3\epsilon_0 \), a simple \((2n - 1)\)-dimensional link \( L \) with Seifert matrix
\[
\begin{pmatrix}
M_1 & P \\
-\epsilon P' & M_2
\end{pmatrix},
\]
which can be done in view of Theorem 1.3.1. Denote by \( B_1 \) and \( B_2 \) the components of \( L \), and let \( A_1 \) and \( A_2 \) be \((n - 1)\)-connected disjoint Seifert surfaces of the components. The manifold \( A_1 \) can be represented as a \( 2n \)-dimensional disk to which are attached \( m \) handles \( g_{i1}, \ldots, g_{im} \) of index \( n \). Analogously [7], we transform \( A_1 \) into a submanifold \( A'_1 \) of \( D(-\epsilon_0) \), by modifying the embeddings of the handles \( g_{i1}, \ldots, g_{im} \), without changing their attaching spheres, in order that the link consisting of the components of \( \partial T \) and \( \partial A'_1 \) have Seifert matrix \( N'_2 \), where
\[
N'_2 = \begin{pmatrix}
0 & X & 0 \\
Y & Z & Q_1 \\
0 & -\epsilon Q'_1 & M_1
\end{pmatrix}.
\]
Using $Q_2$, we transform $A_2$ into a submanifold $A'_2$ of $D(-e_0)$ in order that the link consisting of the components of $\partial T$ and $\partial A'_2$ have Seifert matrix $N_3$, where

$$N_3 = \begin{pmatrix} 0 & X & 0 \\ Y & Z & Q_2 \\ 0 & -eQ'_2 & M_2 \end{pmatrix}.$$

We construct in the ball $D(3e_0)$, defined in $S^{2n+1}$ by $x_i \geq 3e_0$, a simple knot $B_3$ with Seifert matrix $M_3$, and let $A_3$ be an $(n-1)$-connected Seifert surface of $B_3$. Using the matrix $Q_3$, we transform $A_3$ into a submanifold $A'_3$ of $D(e_0)$, in order that the link consisting of the components of $\partial T$ and $\partial A'_3$ have Seifert matrix $N_4$, where

$$N_4 = \begin{pmatrix} 0 & X & 0 \\ Y & Z & Q_3 \\ 0 & -eQ'_3 & M_3 \end{pmatrix}.$$

Denote by $L_1$ the link consisting of the components of $\partial (T \# A'_3 \# A'_1)$ and $\partial A'_2$, where the symbol $\#$ denotes connected sum along the boundary. It is clear that the link $L_1$ is simple, and that its Seifert matrix is equal to $N_1$.

### 3.4. Construction of the link $L_0$

Put

$$L_0 \cap \left(S^{2n+1} \setminus D\right) = L_1 \cap \left(S^{2n+1} \setminus D\right).$$

For the construction of the part of $L_0$ lying in $D$, to the cylinder $W \times J$ in the strip $R(e_0, 2e_0)$ we glue $m$ handles $q_i^{n+1}, \ldots, q_m^{n+1}$ of index $n + 1$, using $a_i, \ldots, a_m$ as attaching spheres. From the construction of $W$ it is clear that such an attaching is possible. Let $S = (W \times J) \cup q_i^{n+1} \cup \cdots \cup q_m^{n+1}$, and then put $L_0 \cap D = \partial S \cap D$. It is easy to see that the submanifold $L_0$ so defined is a simple link with the Seifert matrix

$$N_0 = \begin{pmatrix} M_1 & P \\ -eP' & M_2 \end{pmatrix}.$$ 

Moreover, by construction, the links $L_0$ and $L_1$ are connected by a local isotopy in the ball $D$. Theorem 1.6.2 is proved.

### §4. Supplement: some isotopy invariants of links

#### 4.1. Cobordism of matrices

Matrices $M_1$ and $M_2$ are called cobordant if their block difference

$$\begin{pmatrix} M_1 & 0 \\ 0 & -M_2 \end{pmatrix}$$

is congruent to a matrix of the form $(0 \ A \ B)$, where $A$ and $B$ are square matrices of the same size. Let $\epsilon = \pm 1$; we shall say that a matrix $M$ satisfies property $\epsilon$ if the matrix $M + \epsilon M'$ is unimodular.

As Levine showed, cobordism classes of matrices satisfying property $\epsilon$ form an abelian group with respect to block addition, and the cobordism class of a matrix $M$ is completely determined by invariants $\epsilon_0(M), \sigma^0(M)$ and $\mu^0(M)$ (see [6] and [8]).

#### 4.2. Enlargements of Seifert matrices of links

Let a boundary link $L$ have Seifert matrix

$$\begin{pmatrix} M_1 & P \\ -eP' & M_2 \end{pmatrix}.$$
We call the matrix $\overline{M}$ an enlargement of the Seifert matrix of the link, where

$$\overline{M} = \begin{pmatrix} M_1 & P & 0 & 0 \\ -\varepsilon P' & M_2 & 0 & 0 \\ 0 & 0 & -M_1 & 0 \\ 0 & 0 & 0 & -M_2 \end{pmatrix}.$$ 

From Theorem 1.6.1 easily follows

**Assertion 4.2.1.** Enlargements of Seifert matrices of isotopic links are cobordant.

This assertion can also be obtained differently, making use of the following theorem.

**Theorem 4.2.2** ([Rolfsen [2]]). Isotopic links are cobordant if and only if their corresponding components are cobordant.

From Assertion 4.2.1 there plainly follows the isotopy invariance of the family $\{\epsilon_\lambda(\overline{M}), \sigma_\lambda(\overline{M}), \mu_\lambda(\overline{M})\}$, where $\overline{M}$ is an enlargement of the Seifert matrix of a link.

**4.3. Examples.** A link not cobordant to a split link; nonisotopic links not distinguished by Rolfsen’s invariants. As O. Ya. Viro has told me, the signature of a symmetrized enlargement of the Seifert matrix of a link provides an elementary way to disprove the erroneous theorem of Gutierrez [9], in which it was asserted that every link of codimension 2 is cobordant to a split link (i.e. to a link whose components can be separated by disjoint embedded balls). In fact, it is obviously a cobordism invariant, and is equal to zero for split links. On the other hand, for a $(4q - 3)$-dimensional link $L$ with Seifert matrix $M$, where

$$M = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

it is equal to 2 by an easy computation.

Other counterexamples to the formulation of Gutierrez were found by Cappell and Shaneson [10] and Kawauchi [11].

Moreover, the link $L$ just mentioned and the link $-L$ provide an example of nonisotopic links that are not distinguished by localized Alexander invariants (see [3]) but (obviously) distinguished by the invariant $\sigma(\overline{M} + \overline{M}')$.

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**Bibliography**


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