CHAPTER 11

A HISTORY OF TOPOLOGICAL KNOT THEORY

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> *The subject is a very much more difficult and intricate one than at first sight one is inclined to think.*
> Peter Guthrie Tait, 1876.

1. Introduction

In this Chapter, *knot theory* will be used as a generic term to signify what mathematicians often distinguish into two separate theories, the one concerned with n-links and the other dealing with braids. To a mathematician a *knot* is a single closed curve that meanders smoothly through Euclidean three-space without intersecting itself. An n-link is composed of n of such components, which may link and intertangle but not intersect each other. This affords a simple and intuitive picture, capturing the most essential aspects of a real-life knotted structure. The mathematical concept of a braid will be treated in a later section.

Any account of the history of mathematical knot theory inevitably will be fragmented over many aspects of the fields across which the subject stretches. Combinatorics, topology and group theory are but a few of these fields. In this exposition I have chosen to give a broad outline which touches upon the main conceptual developments as seen in an historic perspective, in which the more formal theoretical developments feature in the background. Knot theory has now become a subject in its own right, which has grown by leaps and bounds (sometimes in quite unexpected directions) along a multidisciplinary front. This causes it to have a sparkling history, involving a wide diversity of ideas, methods and applications, and linked with the names of many famous mathematicians and scientists.

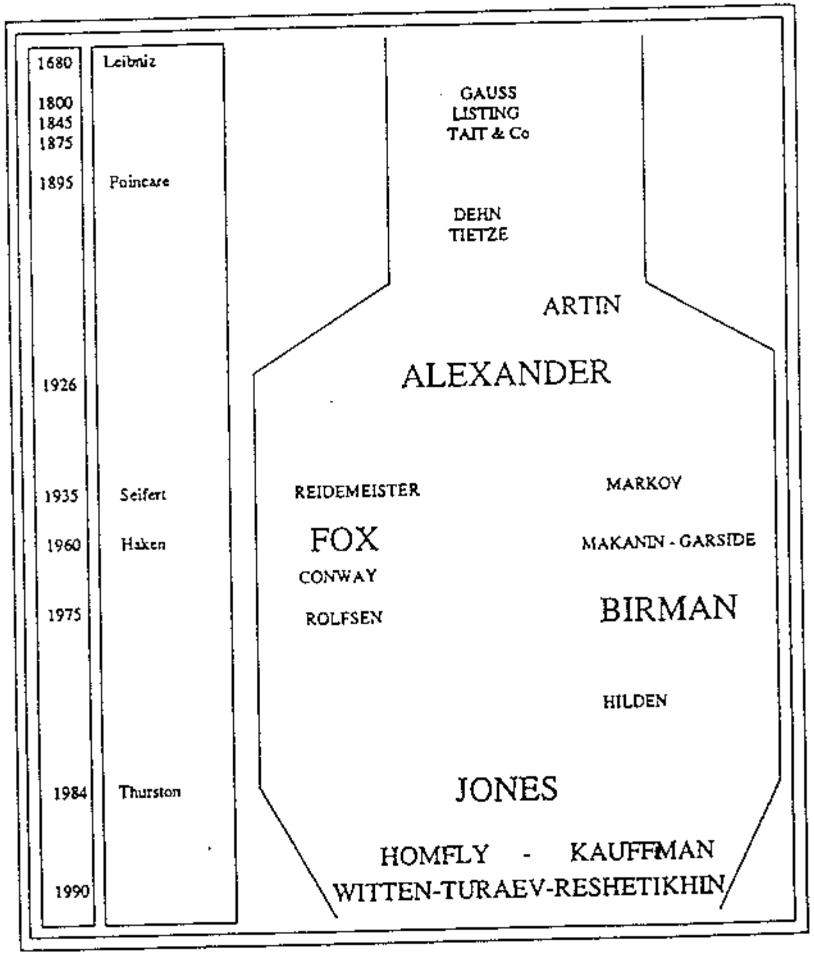


Fig. 1. A Chronology of Topological Knot Theory

From primitive man to present-day scientist, rudimentary knot theory can be traced from the construction of knots and braids for decorative purposes to, what once was held to be, a rather esoteric and deceptively tranquil branch of pure mathematics. However that is a misleading image. From its inception as

a proper mathematical discipline, in the second half of the 19th century, knot theory has been associated with many projects on the frontiers of fundamental scientific research. Although much of knot theory is mathematically abstruse, it has found important application in fields as diverse as atomic modelling, quantum physics, theoretical psychology [82] and molecular biology [95].

The table of (Fig. 1) gives an approximate chronology of the chief discoverers and developers of Knot Theory. The column to the far left indicates the time scale, in years. The second column gives some of the great names in topology. The split column to the right is used to indicate the historic placings of the greatest names in Knot Theory, and also to symbolise its dual nature: the left branch covers mathematical knot theory, whilst the right branch caters for the mathematical theory of braids.

2. The Very Early Days

Throughout history many different interpretations of the phenomenom *knot* have been proposed. In this section we shall consider knots to be those structures which can be realized in a knottable medium, such as a length of rope. In order to show how modelling of the simple act of tying a knot progressively translates into amazingly complex mathematical machinery, we shall first be concerned with questions about *the awareness of mathematical problems associated with knots.* In this context *prehistory* is defined to be the interval of time before the recording of data or information in formalized mathematical ways began to take place.

Although knot theory exists nowadays as a highly specialized and concrete set of mathematical ideas, its origins are not easily traced. Yet somewhere in a far and distant past, inklings of those ideas must have been born. Unfortunately, in a mathematical perspective, the theory's tremendously dispersed history prevents them from being localized with certainty. One may be inclined to believe that there was no theorizing about knots in the very early days of Mankind. Surely there was none in any of the stringent ways we now use to model the phenomenon knot. Nevertheless, by making certain assumptions, plausibility for which is provided by results from anthropological and psychological studies concerning knots [26], [82], one is able to make statements about the roots of the theory. Hence some tracking of the subject's gradual evolution is feasible.

The discovery and use of knots would seem to predate those of fire and the wheel by countless aeons. Knots were used long before the practice of mathematization began to influence the thoughts and actions of Mankind. To primitive Man even the simplest of knots would pose vexing and crucial problems. Yet, it is doubtful whether he would analyze them in any degree other than what was required to employ them as practical tools in his struggle for survival. In order to comprehend their workings, his mind would model knotted structures by translating their relevant spatial, topological and mechanical properties into some kind of logical framework of the mind. We could suitably describe the result, achieved through the cognitive processes of transferring the most obvious properties of knots into a mental model, as *intuitive knot theory*. The most important aspects of intuitive knot theory, being structure and its transformation properties, also constitute the main ingredients of contemporary knot theory. How knots and their workings manifest themselves in the real world was of absolute importance to primitive Man. In many instances his life would literally depend on that kind of vital knowledge. Often certain symmetries determine a knot's ability to operate either in a desired manner or utterly to fail. A point in case is provided by the pair consisting of the Reef Knot and Thief Knot (Fig. 2) which are shown below and whose respective behaviour depends on subtle symmetry properties.

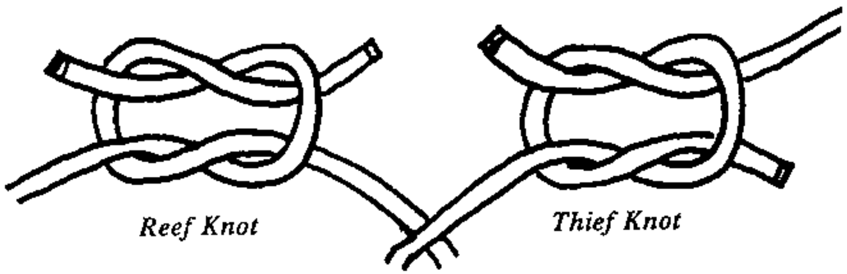


Fig. 2. A Reef Knot and a Thief Knot

To primitive Man, the often incomprehensible and erratic workings of knots were attributable to Divine intervention. The mysterious workings of these topological machines led him to endow them with supernatural powers, placing them into regions of superstition and metaphysics. Such attitudes can still be observed today, with magic knots being found on amulets worn to protect, or bring luck to, the bearer. For similar reasons, knots occasionally were put to decorative uses, in ceremonies for the worshipping of divinities. They were also used for more mundane pastimes, such as in the finger games of cat's cradles, examples of which are to be found in many cultures. The mathematical prehistory terminates at different times in different places. As a rule, this may be said to occur as soon as artefacts, or remnants thereof, for the various cultures come forth.

Knots have been used to represent numbers in different ways. One such numerical application of knots is found in the quipus employed by the Peruvian Incas, a people who used sets of knotted strings for administrative purposes during the better part of a thousand years [11], [53]. The knots themselves functioned only as symbolic and mnemonic devices; but their arrangements

on various lengths of string, connected in ordered, meaningful ways, would encourage deeper mathematical thought. The Incas were aware that their quipu knots (Fig. 3) could not transform themselves (without Divine intervention!); and they were so tied and arranged that cheats could not tamper with them, without considerable difficulty. Thus their bookkeeping was assured of consistency and safety.

In order to employ knots in such a fashion, further demands by their accounting systems would relate to problems of structure recognition. The knot-properties they exploited thus concerned structural stability and mutual structural distinctness. Luckily the favoured Overhand Knots possess quite reliable character in both respects.

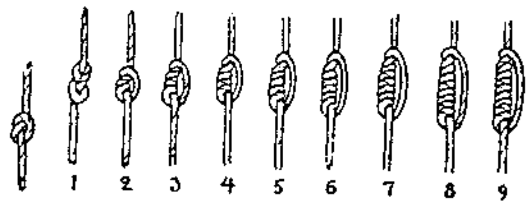


Fig. 3. Quipu Knots

To use knots for decorative purposes, it would be natural to draw pictures of them. This would be an initial step away from intuitive knot theory, as pictures are a first stage of abstraction. Furthermore, drawings entail a process of geometrization, which brings things down to two dimensions. It must be emphasized that this process is not a deliberate attempt to resolve conceptual problems about knots, but arises as a side effect of their application to art. There have been many so-called primitive people that drew fascinatingly complex curves which can be readily recognized as projections of knots. For example, the Bushoong and Tshokwe people in the Zaire-Angola-Zambia region in southwest Africa traced, and their descendants still do trace, complicated and regular figures in the sand. Their unoriented curves, lacking crossings with any distinct parity, are not in any sense knotted and are more akin to graphs. Although their activities have an underlying mathematical base, they seem to be unaware of it [11, p. 34–37].

Evidence of mathematical ideas which tend towards a more Occidental understanding of scientific study concerning knots' planar geometric properties, is discernible in the work of Celtic scribes, work carried out a thousand years ago.

The Celts produced diagrams of knots in which, when following a fixed direction along the curve, the line makes a succession of over-passes and underpasses which alternate regularly throughout the whole curve. We now call the special type of knots which project into such a picture *alternating knots*.

The Celts made extensive use of such pictures (Fig. 4), for decorative and presumably religious purposes. We surmize that their features had to symbolize a number of things. The line would represent time, or possibly life. In the case of a closed curve, the knot's periodicity might relate to the regularity of seasonal changes, with the alternating aspect symbolizing night and day. They seem to have been aware of the non-trivial fact that an alternating knot could be made to correspond with any simple closed planar curve. Their desire to draw such knots posed geometrical problems. This contributed to the process of mathematization of their worldviews, because they had to discover *how* to geometrically create the truly knotted curves and zoömorphics which they employed to adorn surfaces [12], [24].

Fig. 4. Celtic Knotwork

In a sense the foregoing examples of diagrammatic representations of knots and their uses are like an overture. They witness of relatively primitive mathematical thought, and were described in order to illustrate the transition from intuitive knot theory, lacking any apparant formalism, to a vestigial form of the subject. The introduction of a kind of planar geometry was doubtless not directly an attempt to *understand* knots. The geometry of the Celts is of an essentially different kind from Euclidean, but nevertheless it involves elusive properties like transformation and symmetry. The awareness of such problems posed refined demands, requiring the development of new ideas in mathematics.

3. The Birth of Knot Theory

The subject's next steps were related to spirals and closed intertwined curves, and were mainly a German affair. As far back as 1679 Leibniz, in his *Characteristica Geometrica*, tried to formulate basic (geometric) properties of geometrical figures by using special symbols to represent them, and to combine these properties under operations so as to produce other properties. He called

his study *Analysis Situs* or *Geometria Situs*, and it comes closest to what we now would call *Combinatorial Topology* [57], the discipline in which geometrical figures are considered as aggregates of smaller building blocks. Leibniz did not go so far as to study knots; but his endeavours at finding a geometry of this kind, different from the only one known at the time, predated other work in this direction by more than half a century.

Although thinkers like Leibnitz recognized the need for different geometries, it was not until 1771 that the birth of knot theory occurred. In that year Alexandre Theophile Vandermonde (1735–1796) wrote a paper [90] (see also [30]), in which he specifically places knots into the arena of the geometry of position. In the opening paragraphs, Vandermonde includes the lines:

> *Whatever the twists and turns of a system of threads in space, one can always obtain an expression for the calculation of its dimensions, but this expression will be of little use in practice. The craftsman who fashions a braid, a net, or some knots will be concerned, not with questions of measurement, but with those of position: what he sees there is the manner in which the threads are interlaced.*

The possibility for a mathematical study of knots was probably first recognized by the truly great mathematician and physicist, Carl Friedrich Gauss (1777–1855), of Göttingen, Germany. One of the oldest notes found amongst his papers after his death was on a sheet of paper dated 1794, which bore the caption *A Collection of Knots*. It contains thirteen sketches of knots with English names written beside them. It is probably an excerpt he copied from an English book. With it are two additional pieces of paper with a few more sketches of knots. One is dated 1819, the other some eight years later [31]. Notes of Gauss referring to the knotting together of closed curves appear in his collected works [38]. During the period of 1823–1827 he was working on *Geometria Situs* about which he later wrote, on 22 January 1833:

> *Eine Hauptaufgabe aus dem Grenzgebiet der Geometria Situs und der Geometria Magnitudinis wird die sein, die Umschlingungen zweier geschlossener oder unendlicher Linien zu zählen.**

His work on electromagnetism had led him to compute inductance in a system of two linked circular wires; and he introduced the concept of winding numbers (or linking numbers), which are now a basic tool in knot theory and other

*[One of the main tasks in the borderland between Geometria Situs and Geometria Magnitudinus will be to count the 'windings around' of two closed or infinite lines.]

branches of topology. One result, which he gave without proof (after the quotation just given) is the following integral:

$$\iint \frac{(x'-x)(dydz'-dzdy')+(y'-y)(dzdx'-dxdz')+(z'-z)(dxdy'-dydx')}{((x'-x)^2+(y'-y)^2+(z'-z)^2)^{\frac{3}{2}}} = 4m\pi$$

where m is the number of 'windings around' (*Umschlingungen*), and the integral extends over both curves.

Another note of special interest, recorded in December 1844, gave numerous forms which closed curves with four knots can exhibit.

These mere snippets represent Gauss' known researches relating to knots; further mention of his knot work may be found in Stäckel [81]. One can only surmize what further thoughts this genius may have had, and what results gained, on the nature and properties of knots.

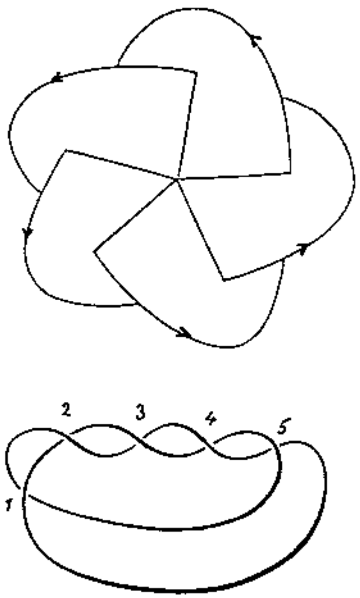


Fig. 5. A Knot by Otto Boeddicker

By contrast with all his other fields of interest, Gauss was not a very active researcher in topology. It has been alleged that Schnürlein, a pupil of Gauss, carried on intensive research with his help, on the application of higher analysis to topology; but no one has been able to verify this. On the other hand, Otto Boeddicker's work from 1876 is with certainty an independent continuation of Gauss' work. In his inaugural dissertation Boeddicker discusses at considerable

length the value of the above-mentioned integral [18]. He later expanded this work to illustrate the connection between knots and Riemann surfaces [19, p. 316]. A diagram from one of Boedicker's papers is shown in Fig. 5.

In any case, Gauss' knotting attempts made him (Gauss) conscious of the semantic difficulties continually to be found in topological studies. This placed him among the first to display, and encourage, deep scientific and mathematical interest in knotted structures. Gauss certainly led some of his students to study the intricacies of topology. Fortunately one of them, Johann Listing, was inspired to pursue vigorously the quest for knot knowledge. He thereby secured for himself a name amongst the founders of the subject. Through his work, which we describe next, the roots of the family tree of modern knot theory are firmly anchored in nineteenth century mathematics.

4. Johann Listing's Complexions-Symbol

Johann Benedict Listing (1806–82) was a student of Gauss in 1834 who later became professor of physics at Göttingen. His topological researches eventually led him to publish some of his work on knots in an essay entitled *Vorstudien zür Topologie*, in 1847 [62]. In this work he discussed what he preferred to call *the geometry of position*, but since this term had been reserved for projective geometry by von Staudt, he used the term *topology* instead. This became the collective name for the mathematical disciplines which study the more general concepts of geometric structures. Listing, even though he carried out quite considerable work on the subject, seems to have published a mere fraction of these researches. *

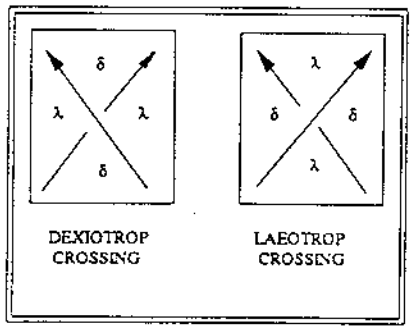


Fig. 6. Handedness of Oriented Crossings

In his 1847 publication he considered the handedness of spirals and discerned between their ability to be either left- or right-handed, which he termed respectively *dexiotropy* and *laeotropy*. Planar projections of these spirals led

*Letter by Listing to the Proc. Roy. Soc. Edinburgh (1877). p. 316.

him to introduce the concept of *handedness for an oriented crossing* (Fig. 6). To each type he assigned a certain symbol distribution consisting of δs and λs. This is illustrated below. The orientation becomes insignificant after the regions have been assigned a type.

From some simple experiments with two- and three-stranded braids and their closures, he came to consider the possibility of listing and classifying all knot projections having fewer than seven crossing points.

Listing was the first to persist in representing *knots* as knotted circles, and obtaining diagrams by projecting these onto a plane. By attaching symbols to the crossings in a diagram, to indicate their types, and considering the resulting symbol distribution in each of the diagram's regions, he was able to propose an 'invariant' for a knot. In general an *invariant* is a mathematical expression (it may be just a number) which carries information about a system, and whose values do not change when the system is transformed in some defined way. An invariant in knot theory is generally an expression derived from a knot diagram which depends solely on the knot or link under consideration, in any of its forms, and not on any particular picture of them. The easiest invariant to visualize, but one which is not very useful for distinguishing between knots, is the number of components in an n-link L. By definition it equals n, and remains so whatever continuous deformations L is subjected to.

Invariants are useful aids in the classification of knots for the following reason. Suppose we compute the value of a particular invariant from two knot diagrams, and obtain two different values. Then we can conclude that the two knot forms from which the diagrams were obtained are different knots. However, the converse is not true; diagrams having the same invariant value may or may not come from the same knot. A perfect knot invariant, which always takes the same value for any particular knot, and a different value for any other knot, has yet to be discovered.

Listing concocted his invariant as follows. He called a region of a knot diagram *monotypical* if all the angles on the region's boundary had been assigned the same type-symbol (that is, were all δ or all λ); in which case, the region was itself given the same type-symbol. If a mixture of δs and λs had occurred, he called the region *amphitypical*. In this way, Listing typified each region, including the unbounded one (which he called the *amplexum*).

He defined a diagram to be *in reduced form* if it had a minimal number of crossings, over all possible diagrams obtainable from the knot. He knew that if one or more regions were amphitypical in a diagram, then the diagram would not necessarily be in reduced form; but he gave no methods for reducing the numbers of crossings in such diagrams.

He proposed an invariant for knots which have a monotypical diagram in reduced form; and he gave it the name *Complexions-Symbol*. Later, we shall give an example which shows that it is not a true invariant.

Briefly, a pair of polynomials are computed from the diagram, one in the 'variable' δ, and the other in the 'variable' λ. The exponents on the terms of these polynomials correspond to the numbers of sides surrounding the regions in the diagram: thus, for example, suppose that there are 5 regions, each having 3 sides and bearing the symbol δ; then the term $5\delta^3$ will appear in the *δ-polynomial* of the Complexions-Symbol.

A full example will clarify the matter. The extraction of the Complexions-Symbol from a diagram of a specific 7-crossing knot is shown below (Fig. 7). Note that all the regions in the diagram are monotypical.

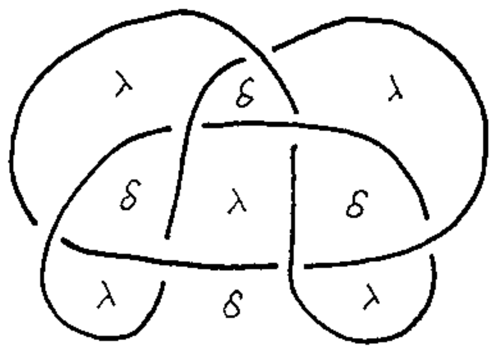


Fig. 7. A 7-crossing knot, with labelled regions

In this case there are four δ-regions, of which three are 3-sided and the unbounded region adjoins five sides. There are five λ-regions, of which two pairs are respectively 2- and 3-sided, while the remaining one is 4-sided. The pair of polynomials for the knot would be shown by Listing thus:

$$\left\{ \begin{array}{c} \delta^5 + 3\delta^3 \\ \lambda^4 + 2\lambda^3 + 2\lambda^2 \end{array} \right\}$$

In general, a Complexions-Symbol has the form

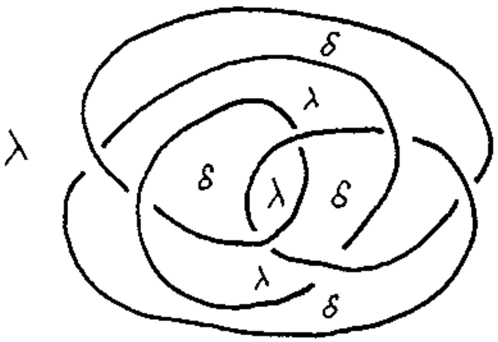
$$\left\{ \begin{array}{c} a_0\delta^n + a_1\delta^{n-1} + \ldots + a_{n-1}\delta^1 \\ b_0\lambda^n + b_1\lambda^{n-1} + \ldots + b_{n-1}\lambda^1 \end{array} \right\}$$

where the coefficients a_i and b_i, with $0 \leq i \leq n-1$, indicate the numbers of the various kinds of region occurring in the diagram.

Listing noted that if a term with exponent unity existed in either of the two polynomials, that term would derive from one or more simple twists in the diagram. These could be removed by 'untwisting'; and so such term(s) could be dropped from the Complexions-Symbol. Note that Euler's polyhedral formula can also be used to check the calculations; the sum of the coefficients in both polynomials is equal to the number of regions in the diagram, which equals $(n+2)$ by Euler's formula.

Listing's Complexions-Symbol has several serious defects. First, it is not defined for the *Unknot*, when projected into a diagram having no crossings. Nor is it defined for non-alternating knots, all of whose diagrams must have at least one amphitypical region; examples of these occur first among knots of eight crossings. The Prussian Heinrich Weith aus Homburg von der Höhe, following up Listing's work, noted this in 1876 [93, pp. 15–16], and gave a diagram of a non-alternating knot to prove the point.

Johann Listing himself noted that occasionally the so-called invariant proved not to be invariant at all! To illustrate how this can happen, we give below an alternative diagram for the 7-crossing knot used above; the Complexions-Symbol derived from *this* diagram is clearly not the same as the one obtained above.



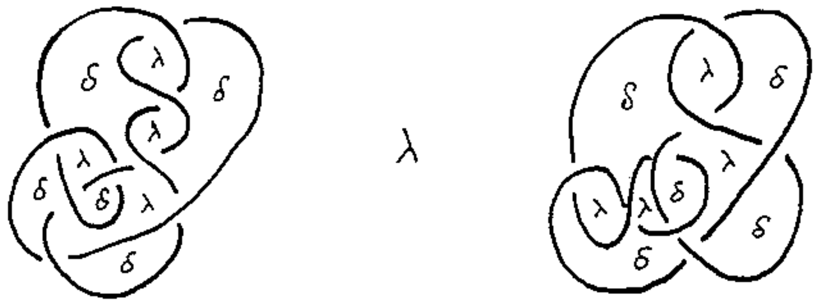
$$\left\{ \begin{matrix} 2\delta^4 + 2\delta^3 \\ \lambda^4 + 2\lambda^3 + 2\lambda^2 \end{matrix} \right\} \neq \left\{ \begin{matrix} \delta^5 + 3\delta^3 \\ \lambda^4 + 2\lambda^3 + 2\lambda^2 \end{matrix} \right\}$$

Thus a single knot can give rise to two different Complexions-Symbols.

Any hopes that Listing might have had that his 'invariant' would be a complete invariant were destroyed by P. G. Tait's finding of two distinct 8-crossing alternating knots which both had the same Complexions-Symbol [84, p. 326]. These two knots, (numbers 8_{11} and 8_{12} from the listings by Reidemeister and Rolfen [74] and [77]), and their Complexions-Symbol, are shown in Fig. 8.

Summarizing the contribution of Listing to Knot Theory, from this brief description of his work, we can see that he established a basis for the mathematical study of knots, working with the natural tool of the diagram of a knot projection. He saw the need for invariants which would help distinguish

between different knots; and he proposed one, to be computed from a knot diagram in terms of his crossing-type symbols.



$$\left\{ \begin{matrix} \delta^5 + \delta^4 + \delta^3 + 2\delta^2 \\ \lambda^5 + \lambda^4 + \lambda^3 + 2\lambda^2 \end{matrix} \right\}$$

Fig. 8. Tait's two different knots with the same complexions-symbol

Although his Complexions-Symbol had too many serious defects for it to be of much use as a knot invariant, it posed a challenge to other workers coming into the field: namely, to find good invariants. As such, it was perhaps Listing's greatest contribution to Knot Theory. The major quest in twentieth century knot research has been for ever stronger invariants.

Before describing this further work on invariants, however, we must mention other work that was done in the final thirty years of the nineteenth century. In particular, we shall describe the monumental achievements of P. G. Tait and his collaborators; and they demand a section of their own. Passing mention will first be made of several other items related to knot research.

H. Weith, in his inaugural dissertation which elaborated on Listing's work, elegantly showed that there is an infinite number of different knot forms [93].

About that time, there was a curious connection made between knot theory and psychic research. The mathematician Felix Klein appears to have observed that no ordinary knot can exist knotted in a space of four dimensions; one can always use the extra dimension in order to untie it, without, of course, cutting the string [56], [83].

This proposition would be experimentally testable if only we had access to a fourth dimension. Several very reputable and distinguished scientists, including J. C. F. Zöllner from Leipzig, began conducting knot experiments which involved psychic mediums [98]. If the mediums were able to untie closed knots, without cutting the string, it would be a reasonable conclusion that they had some kind of access to a fourth dimension. In the account of Zöllner's investigations, a record is made that this kind of experiment was successfully carried out in December 1877, by Mr. Slade, an American medium.

Some time later, Henry Slade's psychic abilities were proved to be fraudulent. It was shown that the phenomena he produced in the experiments were achieved by trickery. Below [Fig. 9] we reproduce an illustration, by Zöllner, which shows the Overhand Knots produced by Slade's conjuring.

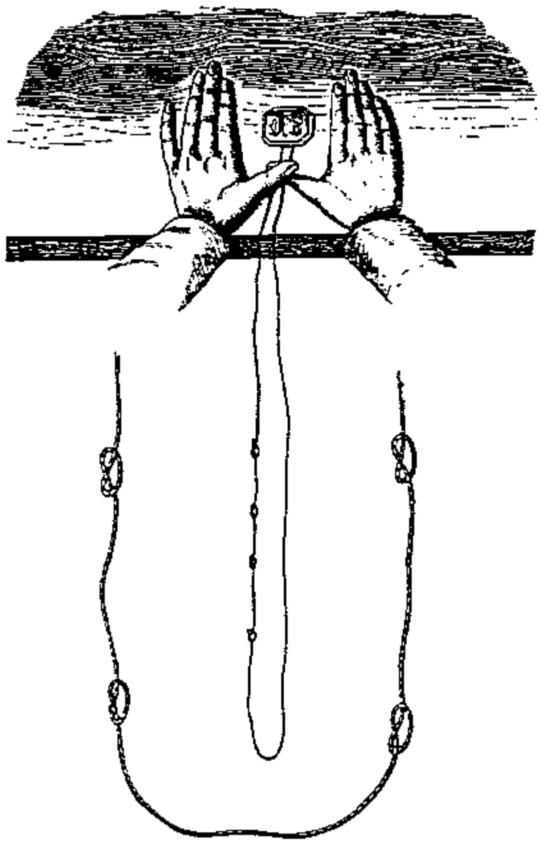


Fig. 9. Overhand knots on Zöllner's sealed cord

There were some publications in which a mathematical explanation was given for 'conjuring' knots from apparently thin air. The first, by Oscar Simony, was based on a prior discovery by Augustin Ferdinand Moebius (1790-1868), that if one makes three or more likewise-handed twists in a flat strip, and pastes the ends together, then on cutting along the centreline of this one-sided surface a knot or knots may occur [80]. In 1890 Friedrich Dingeldey, then

a professor at the Darmstadt technical university, published a more rigorous account, which also gives a detailed overview of the early history of topology [30].

In 1897 Hermann Brunn observed that any knot has a projection with a single multiple-point [20]. This proposition became attributed to James Alexander, some 25 years later.

In this period André Hurwitz did some work on Riemann manifolds, which were first steps leading towards theories of braids.

This completes our summary of German work in the field around that time. We must cross the Channel to England to continue our story of the history of knot theory.

5. The Work of Peter Tait

Further substantial progress in knot theory was not made until Sir William Thompson (1824–1907; Baron Kelvin of Largs) announced his model of the atom, his 'vortex theory'. Thompson began writing about this concept in the mid-1860s. He believed that all material matter was caused by motion in the hypothetical ether medium, which he termed *vortex motion* [99]. How did knots relate to these ideas? Scotsman Peter Tait (Dalkeith, 1831–1901) was a close associate of Thompson. In 1867, having been greatly taken by Helmholtz's papers on vortex motion, Tait devised an apparatus for studying vortex smoke rings in which the rings underwent elastic collisions exhibiting interesting modes of vibration. The experiment gave Thompson the idea of a vortex atom. The imaginative picture painted by the theory he developed subsequently was one of particles as tiny topological twists, or knots, in the fabric of space-time. The stability of matter might be explained by the stability of knots; their topological nature prevents them from untwisting. His aim was to achieve a description of chemistry in terms of knots. More specifically, Thompson wanted to produce a kinetic theory of gases, a theory which could explain multiple lines in the emission spectrum of various elements. A swirling vortex tube would absorb and emit energy at certain fundamental frequencies: linked vortex tubes would explain multiple spectral lines. In short, he believed that the variety of chemical elements could be accounted for by the variety of different knots. The main advantage of Thompson's model was that its indivisible bits would be held together by the 'forces' of topology. This construction would avoid the problems inherent in devising forces to hold together an atom made up of little billiard balls.

The theory was taken seriously for quite some time, and even eminent scientists like James Clark Maxwell stated that it satisfied more of the conditions than any other hitherto considered. In fact, in retrospect one could add *transmutation* to its merit-list. The ability of atoms to change into other atoms at high energies could be interpreted as the cutting and recombining of knots.

Even though Thompson's vortex theory of atoms stood for only about two decades, Tait's fascination for knots had been aroused. Thus the study of vortices stands as the starting point of a highly important pioneering study of the topology of knots [87].

The vortex theory led Tait immediately to the problems at the heart of the subject: with insufficiently developed mathematics to help out, knots and links could not be characterized. As already indicated, the accessible work on knot models was very scattered and fragmentary; and results on knots had usually been arrived at almost simultaneously, and independently, by mathematicians ignorant of each other's work. At that time, it was not even clear (to Tait) whether or not there were finitely many knots. Therefore his first self-appointed task, which gradually became his main occupation, was one of enumeration, in which he tried to *find* and *classify* knotted structures. Tait called this *the census problem*. The main thrust of his work was how to find all possible (distinct) knotted structures which can be represented by plane diagrams of continuous curves having n crossing-points. He studied ways by which such diagrams could be 'reduced', a reduction corresponding to a topological change in a knot which led to fewer crossing-points in the plane diagram. He called the minimal number of crossing-points achievable for a diagram of a given knotted structure, the degree of knottiness of that structure. Tait gave several methods for making the reductions. During his attempts to develop these ideas, he followed (roughly) two lines of combinatorial attack. The first involved the development of a method which he called the *Scheme-method*, one which seems to have been known to Gauss [86, p. 13]. The second attempt he came to develop was partially inspired by information gleaned from Johann Listing's work. He termed it the *Partition-method*.

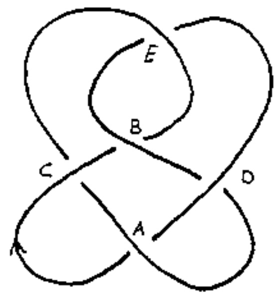
5.1. The Scheme-Method

The central problem, in Tait's case, was to find the combinations of symbol sequences which would encode a connection relation for n points in the plane. He introduced a tool, for which he coined the name *scheme*, which was basically a symbolic shorthand for a connected graph. For ease of demonstration, we shall describe this method in reverse.

Given a diagram D of a 1-link L, with at least one crossing. Choose any crossing from which to start. Call it A. Move from A in either of the two possible directions. Traverse the knot diagram and name the crossings encountered in the odd places respectively B,C,D and so forth until all crossings have been assigned a letter. Traverse the structure over again from any starting-point and jot down the sequence of crossing-points visited. This symbol-string, along

with the respective crossing-parities (i.e. 'over' and 'under' crossing symbols), encodes the structure. These symbol-strings, however, were of no significance to Tait. Very much like Listing before him, at first he only considered knots which were represented by alternating diagrams (i.e. diagrams where 'over' and 'under' crossings alternate throughout, in a traverse around the string). He did so because he erroneously believed that all knots, without changing their minimal number of crossing points, could be made to fit such a diagram [13].

An example will illustrate the procedure. For the knot with 5 crossing-points, as displayed below, we find the following scheme.



$$ACBECADBED|A$$

The $|A$ indicates the starting point of the sequence. Note that the letters A, B, C, D, E occur at the crossing positions 1,3,5,7,9, respectively as met during the knot traverse.

As said, Tait's procedure was based on a reversal of this process. Thus he tried to find, empirically, all sequences which could yield such schemes. It is easy to see that this task would quickly lead, with increasing values of n, to a massive undertaking. An upper bound for the number of schemes, when n crossings are involved, can be obtained by solving the following combinatorial problem:

> *How many arrangements are there of n letters, when A cannot be in the first or last place, B not in the second or third, and so on, and the sequence must have length 2n?*

With the help of Cayley and Muir it was soon found that the number of combinations rises sharply, into thousands, even with a modest number of crossing points [32]. To complicate matters: many combinations do not even represent knots. The hopelessness of manually finding knotted structures by means of the scheme method was realized. It was abandoned in favour of a more effective one, to be described next.

5.2. The Partition-Method

This was also based on graph properties of knot diagrams. An n crossing-point diagram will have $(n+2)$ regions, which can be denoted by R_i, $0 \leq i \leq (n+1)$. Crucial to the method is the observation that for any region R_i the number of corresponding sides (edges of the graph) s_i has to fulfill the inequality: $2 \leq s_i \leq n$. It is known that the total number of sides in the graph equals $2n$. These facts enable one to partition the number n and, via use of polyhedrons, to arrive at graphs which can be assigned a crossing-coding, thereby yielding representations of knotted structures. This method eventually was made to work well. It was also the method which Tait developed further still, after sharing ideas with the Reverend T. P. Kirkman and C. N. Little, an American professor. They had independently pursued the same lines. Their communications led to a happy collaboration, which resulted in the trio collectively listing virtually all alternating knots up to 11-fold knottiness.

5.3. The Final Results

What did Tait and his collaborators eventually achieve? They found 82 types of knots of 9 or less crossings. An especially remarkable achievement was their work in the class of 10-fold knottiness. A mammoth undertaking which, with their tools, took them 6 years to complete; it resulted in some beautiful tables of 10-crossing knot diagrams. Little continued the struggle, and published the results of his attempts on 10-fold knottiness in 1885 [63]. They were finally able to resolve, too, a large number of the alternating 11-crossing knots. Kirkman provided Little with a manuscript of 1581 polyhedral drawings, from which he distinguished 357 different knot-types.

It is impossible to summarise adequately, in a few paragraphs, the extent of Tait's contributions to the birth of Knot Theory. His researches affected all aspects of the subject. He empirically discovered a great number of useful results, while experimenting with many ideas which future researchers would take up. He worked on the so-called *Gordian number*, which is the minimal number of crossing-point changes required in a knot to produce an unknot. He made some pertinent conjectures which were not resolved for well over a century. He had already considered knots such as Moebius braids, and he both toyed and toiled with problems relating to symmetries such as mirroring. He found a nice little theorem on amphicheirality (a knot is said to be amphicheiral if it can be topologically transformed into its mirror image). He was very

much aware of the difficulties which symmetries in mirroring brought along. He introduced a Scottish verb, *to flype*, to denote an operation which can be carried out on certain portions of knots or their diagrams. In fact, one of his foremost contributions was to introduce nomenclature of this kind into knot studies [58].

In order to develop the subject rigorously, however, he needed to discover some form of knot invariant, which would help him to distinguish and identify knot types. He gave no formal proofs that any of his methods actually came to define or to implement one.

The main underlying problem which confronted Tait and his co-workers was deciding when two knotted structures were isotopic, i.e. telling whether either of them could be deformed, by a continuous transformation, into the other. Two knots or links are said to be the *same*, or *isotopy equivalent*, if they can be made to look exactly alike by pushing and pulling, but not cutting, the string(s) in which they are realized. This problem of isotopy became established as the central problem in knot theory, and it became known as the Knot Problem. It was not to be dealt with satisfactorily until the advent of algebraic topology.

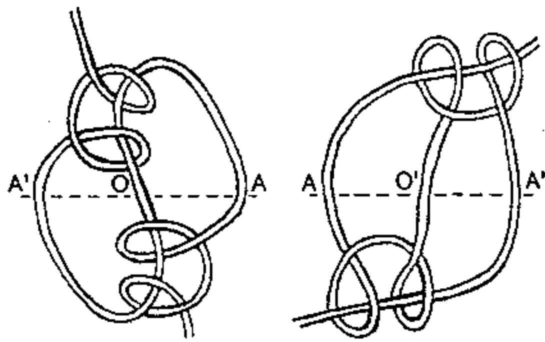


Fig. 10. Knots from Mary Haseman's dissertation

Some thirty years after Tait's endeavours, Mary Gertrude Haseman tackled amphicheiral knots of 12-fold knottiness (Fig. 10). The results of her brave expedition into the then uncharted regions of 12-crossing knot-projections are presented in the charming dissertation [43], which gives a census of amphicheirals in that class of knottiness.

6. The Beginning of the 20th Century

By 1900 there were almost-complete tables available listing knots of up to 11-fold crossings. They represented the fruits of the arduous labor by Tait and his collaborators, and of physicists who had been working in an atmosphere of 'applied mathematics'. Their work had provided sufficient concepts, terminology and knot-diagrams to enable the development of a formalized theory to begin. Tabulating knots had two goals. Completeness of the list was the first. Distinctness of all tabled structures the other. Generally speaking the first goal could be achieved via (cumbersome) combinatorics. The second required methods for dealing with problems involving isotopy; and for that, new mathematical methods were required. Powerful knot invariants had to be discovered.

As the emphasis in the theory of knots turned away from enumeration, under the awareness of the problems due to isotopy, the hunt for good knot invariants began. The ensuing period of transition showed great quantitative and qualitative differences in knot research, as compared with the early and rather empiric work in enumeration. In fact the changes effectively caused the census problem to cease to be the theory's major research topic for the next six decades; although its importance continued to be recognized. Its goals and achievements served as testing grounds for new invariants and other important tools which began to be discovered, in algebraic topology, group theory and other mathematical fields. Much mutual interplay took place between the old and the new approaches.

Knot theory as attempted from the purely topological side became possible only after the development of the required mathematical machinery. This was pioneered by Henri Poincaré (1854-1912) around the turn of the century. Poincaré was a professor at the university of Paris, and a leading mathematician of his day. It has been claimed that he was the last man to possess a universal knowledge of mathematics and its applications. His prime motivation for mathematical research invariably sprang from scientific problems. He was the first person to make a systematic and general attack on the combinatorial theory of a special type of geometric figures called *complexes*. Due to this work he is usually regarded as the founder of combinatorial topology. He decided that a systematic study of the analysis situs of general or n-dimensional figures was not only desirable but also necessary. After some notes which appeared in the *Comptes Rendu* of 1892 and 1893, he published a basic paper in 1895; this was followed by 5 lengthy supplements running in various journals, appearing in the years up to 1904. He did not regard his work on combinatorial topology as a study of topological invariants, but rather a systematic way of studying n-dimensional geometry. However, the influence of his work on subsequent knot theory was to reverse these priorities—the study of topological invariants came to the fore.

Poincaré introduced a number of topological tools, such as the so-called fundamental group of a complex, also known as the Poincaré group in his honour; it was the first in the string of homotopy groups [72]. It came to play

a role of utmost importance in topology.

In his efforts to distinguish complexes, Poincaré came to introduce torsion coefficients, and a method for computing Betti numbers of an n-dimensional complex. These concepts are defined as follows. Given a finitely generated Abelian group A, it can be written as the direct product of a free Abelian group F and a family of cyclic groups A/H_i, where each A/H_i is a finite cyclic group of order h_i and such that $h_1|h_2|\dots|\#A$. The rank of the free Abelian part F and the uniquely defined numbers h_i are invariants of the group A, and completely determine its structure. If A is the homotopy group in dimension, say d, then the rank of F is the d-th Betti number, and the h_i are the torsion coefficients. They are numerical invariants of isomorphism classes of finitely generated Abelian groups. The rank of F is used to calculate the Euler characteristic.

It is interesting to note that Poincaré used only methods of continuous mathematics at the beginning of his series of papers; but by the end he relied heavily on combinatorial techniques. This was not without impact on the newly founded schools that formed to take up and develop his ideas. For the next 30 years researchers concentrated almost exclusively on combinatorial and algebraic methods.

The belief in the power and aptness of combinatorics ran deep. The knot problem's solution demanded a formal definition of a knot, which in true combinatorial spirit became a set of straight arcs making up a closed non self-intersecting polygon in space. Max Dehn and Poul Heegaard in their article [29] in *Encyklopedie der Mathematischen Wissenschaft* in 1907 noted that the knot problem could be formulated entirely in terms of arithmetic, i.e. combinatorics. However this kind of reduction seemed to be of no practical value, nor did it seem to have any theoretical consequences (e.g. for decidability of knot equivalences). There are many natural numerical invariants of knots which may be defined quite easily, such as the already-met number of crossing-points, the Gordian number, the maximal Euler characteristic and so on; but difficulties in computing them by solely combinatorial techniques seem to be inversely proportional to the ease in defining them. There is something general about this matter. There is for instance, to date, still no known algorithm for finding the minimal number of crossing points for an arbitrarily given n-link. In fact there seems to be no hope for finding this number with any tool at all! On the other hand, a recent attack on the Gordian number has yielded good bounds for it (1994). A good account of this work, by William Menasco and Lee Rudolph, can be found in [67].

The first successful algebraic topological invariant attached to a link L was the fundamental group of the 3-manifold, which is constituted by the link's complement in 3-space, namely $(\mathbf{R}^3 - L)$; this invariant is sometimes called *the group of the knot-complement* or, simply, *the knot group*. For an arbitrary link L, and with reference to a basepoint p in L's complement, the knot group is denoted by $\pi_1(\mathbf{R}^3 - L, p)$. This group is one in which the elements are homotopy classes of (unknotted) loops which traverse the complement space of a knotted structure, starting from and terminating at the basepoint p. The binary operation for the group is the composition of two loops, carried out by concatenating them at p. Since this composition is non-commutative, the fundamental group (knot group) is non-Abelian.

The fundamental group expresses in algebraic language some of the topology of the knot-complement, which makes it possible to compare different knots by comparing their algebraic descriptions. A knot's complement, which is three-dimensional, carries a richer topological structure than the knot itself, which is one-dimensional. The topological structure of the complement necessarily contains certain information about the knot. In 1908 Tietze conjectured that it contained *all* such information; an idea that did not become an established fact for 1-links until 1988 [40]. The uncertainties surrounding this conjecture did not prevent this avenue being pursued vigorously; presentations of certain knot groups appeared fairly soon in the literature. General methods for writing down a presentation of the knot group from a knot projection were introduced by Wirtinger, who did not publish them; but he got credit for the idea anyway [65]. Max Dehn, in 1910, also published methods for presenting knot groups.

7. Max Dehn's Work in Knot Theory

It was thought that by considering the knot groups one might be able to classify knotted structures. The initial notions on groups had arisen from 19th-century algebra, analysis and geometry. By the time that Max Dehn began his work on knots, early in the 20th century, group theory had proceeded so far that it was no longer necessary to describe groups by means of their cumbersome Cayley (composition) tables. At the beginning of the 1880's von Dyck had shown how every group is the homomorphic image of a free group, and how one could present such a group by giving so-called *generators* and *defining relations*. Armed with these tools Dehn attacked the knot group.

In his 1910 paper *Über die Topologie des dreidimensionalen Raumes* [27], Dehn discussed a method for extracting a description of the knot group, the so-called *Dehn presentation*. He did so by the following algorithm:

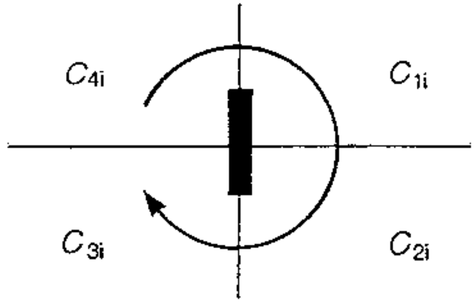
1. List and denote all *bounded* regions of a knot-diagram by $C_1, \ldots, C_n$. These are to be considered the group generators.

2. Every over-crossing yields a relation R_i, $1 \leq i \leq n$ by noting down a relation containing a sequence which is the product of generators as

encountered when traversing clockwise the crossing:

$$R_i = C_{1i}C_{2i}^{-1}C_{3i}C_{4i}^{-1} = 1.$$

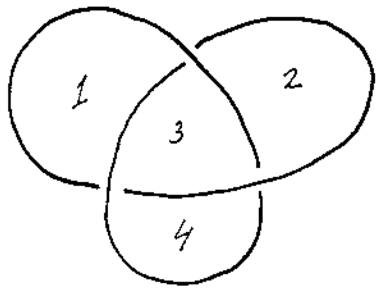
In case the crossing includes the unbounded region, the specific generator's place is taken to be unity. Thus a three-generator relation is obtained.



3. The collection of n generators, and n relations, as written below, is a presentation of the knot group.

$$G_k = \begin{cases} C_1, \ldots, C_n \\ R_1 = R_2 = \ldots = R_n = 1 \end{cases}$$

In order to provide an example, we shall apply the algorithm to the right-handed version of the Trefoil Knot, illustrated below.



The diagram gives rise to the following three relations. Note that we have omitted the identity **1** in each, as we may.

$$\begin{cases} C_1C_2C_3^{-1} = 1 \\ C_1C_3^{-1}C_4 = 1 \\ C_2C_4C_3^{-1} = 1 \end{cases}$$

These yield the following presentation for the fundamental group of the Trefoil Knot:

$$(C_1, C_2, C_3, C_4 : C_1C_2C_3^{-1} = C_1C_3^{-1}C_4 = C_2C_4C_3^{-1} = 1)$$

In fact, this is not the most economical presentation possible: using Tietze operations we can reduce the number of generators to 2, and the number of relations to 1, thus:

$$(C_1, C_2 : C_1C_2C_1 = C_2C_1C_2)$$

Wirtinger's presentation is derived in similar fashion, but with generators being associated with overpass arcs, rather than with regions (see [25] for details). The end-result is, of course, the same.

Further contributions by Dehn are described in the next section on Alexander's work.

8. James Alexander's Influence

Applications of the fundamental group quickly yielded several breakthroughs. Proofs of the existence of non-trivial knots, via knot groups, had already been given by Tietze as early as 1906. However, the first successes from use of the knot group lay in the verification of the correctness of the knot tables. To achieve this, the group was used with other tools which Henri Poincaré and Enrico Betti had introduced. The proof that Betti numbers and torsion coefficients define combinatorial knot-invariants was first given by James Waddel Alexander (1888-1971), a professor of mathematics at Princeton University and later at the Institute for Advanced Studies. Collaborating with G. B. Briggs and using the torsion numbers, he distinguished all tabled knots up to 8 crossings and all, except three pairs, up to 9 crossings [7]. Alexander also showed that two 3-dimensional manifolds may have the same Betti numbers, torsion coefficients and fundamental group and yet *not* be homeomorphic. His example, of course, involved knot complements. Thus he had shown that a knot contains (at least *a priori*) more information than just its group.

With the tools just introduced, Dehn proved that an arbitrary knot K, its mirror image K^*, and its version with reversed orientation $\tilde{K}$, produce three knot-complement groups which are mutually isomorphic. (Later Reidemeister also proved this, more rigorously.) Using π_1 to denote a knot-complement group, this theorem is stated as follows:

$$\pi_1(\mathbf{R}^3 - K, p) \cong \pi_1(\mathbf{R}^3 - K^*, p) \cong \pi_1(\mathbf{R}^3 - \tilde{K}, p)$$

It was thus realized that the complement alone could not provide complete invariants. The situation was repaired by equipping the complement with an

orientation. Let $K \subset \mathbf{S}^3$ be our given smooth knot. By thickening the knot's actual curve to a knotted tube and removing this tube's interior from 3-space we are left with X, the knot's *exterior*. By laying a 'coordinate system' over the tube around the knot, the exterior thus acquires more structure than the complement. The exterior with the coordinate system is called the *peripheral system*. Using additional information from the peripheral system Max Dehn could show by 1914 that neither of the oriented Trefoils (see Figs. 11, 12) is isotopic to its mirror image [28].

Fig. 11. Left-handed Trefoil Knot

Fig. 12. Right-handed Trefoil Knot

He did so by taking one Trefoil, removing it from $\mathbf{S}^3$, reinserting it with opposite orientation, and showing that the result was not homeomorphic to the original knot. This procedure is known as *Dehn surgery*.

The natural question arises as to what extent the peripheral structure is determined by the group alone. It was known at an early date that the Reef Knot and the Granny Knot (see Figs. 13, 14) possess isomorphic groups. Seifert had shown in 1933 that their complements were non-homeomorphic [79]. In 1952, using the peripheral system, R. H. Fox showed that irrespective of the orientations they may have been given, they are two distinct knots [35].

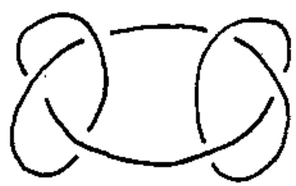


Fig. 13. Reef Knot

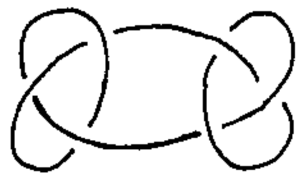


Fig. 14. Granny Knot

The knot group did not immediately fall from grace; but now it was known to be an incomplete knot-invariant. In algebraic topology terminology: the group of a knot determines the knot's complement merely up to homotopy type.

This disturbing example put paid to the generally-held idea that the knot group contained all information about the knot; worse still, it continued to cause trouble over the next few decades. However, despite such examples revealing its shortcomings, the group of a knot was still a powerful invariant. And in the late 1960s the role of the peripheral system was finally clarified; it was shown to be a complete invariant. This demonstration resulted from Waldhausen's work on irreducible, sufficiently large, 3-manifolds, which in turn was based on earlier ideas by Haken [42], [44].

The knot group, even though it was an unwieldy mathematical object, formed the basis for much further research on knots. The new approach via knot groups effectively brought the unknot U into the picture (it was a knot that had been ignored or not taken seriously by early researchers). This knot, a kind of 'limit' in the class of knots, now became an important one in the knot tables, because its group turned out to be a special one, namely the infinite cyclic group with one generator, which is isomorphic to the group of integers under addition. The proof of this was settled in 1956 by the great mathematician Papakyriakopoulos, who also proved that the group of a knot determines the homotopy type of its complement.

Equivalent knots which are projected into distinct diagrams can yield different presentations of their knot group. These presentations must, as theory tells us, relate to isomorphic groups. However, there is no general algorithm which will enable us to decide whether two given representations relate to two isomorphic groups. It is known that no such *general* algorithm is possible. Nevertheless, in working to resolve particular cases, the main efforts in knot research came to concentrate on the problem of finding reduced presentations of knot-groups; in the process, the problem of knot-equivalence was cast into an (algebraic) word-problem mould. The main question became: *When are two presentations of knot groups equivalent?* The complexity of this problem (which is, as already noted, generally unsolvable) led to a quest for simpler invariants, ones more tractable than the knot-group.

This research direction began with a discovery by J. W. Alexander; in 1928 he 'launched' the knot polynomial which was later named after him. It was a totally new idea. He described a method for associating with each knot a polynomial, such that if one form of a knot can be topologically transformed into another form, both will have the same associated polynomial; it quickly proved to be an especially powerful tool in knot theory. For example, the polynomial was able to distinguish 76 knots out of the first 84 in the knot-tables; they were found to have unique Alexander polynomials.

Alexander first obtained his polynomial of a knot K by labelling the regions in the plane bounded by an oriented knot-diagram of K having n crossings. By noting the types of crossing around the knot, in relation to the arc labels, he extracted a certain $n \times n$ matrix (now called the *Alexander matrix*). All of the elements in an Alexander matrix are either 0, or -1, or t, or $1 - t$, where t is a dummy variable or parameter. By removing the last row, and the last column, of the matrix, and taking the determinant of the remaining matrix, a polynomial in t is obtained. This is known as the *Alexander polynomial* of the knot. We may denote it by $\Delta_K(t)$, or simply Δ_K. Alexander was able to show that $\Delta_K(t)$ is an invariant for the knot K (see [6] for full details). In fact, Alexander presented a sequence of polynomials, $\{\Delta_n(t)\}$, with $n = 1, 2, 3, \ldots$, all invariants of the knot K. The first one (case $n = 1$) is the one known as the Alexander polynomial.

Why associate a *polynomial* with a knot diagram? The schemes and partitions which Tait, Kirkman and Little had worked with were unwieldy. Listing's complexions-symbols were not quite what was needed to yield an unambiguous invariant. But why Alexander's polynomial worked the way it did was not clear at the outset. Alexander himself suspected that it was some kind of shorthand for homology groups. A rather reasonable hunch, as later work placed it on a sound homological base.

Alexander's polynomial proved to be a fairly powerful invariant of isotopy in knots. Differently deformed versions of the same knot yield the same polynomial Δ_K. The following comments illustrate a few attractive aspects of the polynomial's behaviour.

Given two prime knots with respective Alexander polynomials, the Alexander polynomial for their knot-composition is given by the multiplication of the two original polynomials. Another aspect almost amounts to a pun: for an alternating knot, Δ_K has coefficients of alternating sign [70]. These, and other more refined pleasant properties, made it knot theory's main tool for almost half a century. The Alexander polynomial's weak points are that it always takes the same value for a knot and its mirror image; and that its power to distinguish between knots terminates for certain example pairs and classes of knots with more than 9 crossings. In 1934, classes of non-trivial knots with trivial Alexander polynomial were discovered [78].

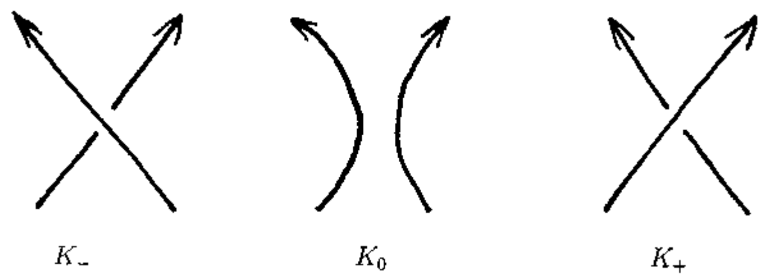


Fig. 15. Set of Alexander crossings

Alexander also discovered a relationship between the polynomials of three oriented knots whose diagrams are identical except within a neighborhood of one fixed crossing where they are as shown in Fig. 15, [6, p. 301]. For further reference the diagrams will be called 'a set of Alexander crossings'.

The relationship is:

$$\Delta_{K_+}(t) - \Delta_{K_-}(t) + (t^{-\frac{1}{2}} - t^{\frac{1}{2}})\Delta_{K_0}(t) = 0, \text{ with } \Delta_U = 1$$

Many years later this relation, and others like it (now called *skein relations*), came to have great significance in the development of recursive methods to produce knot invariants. In spite of its early discovery, the literature has shown a remarkable tendency to remain loyal to the calculation of Δ_K by means of determinants. This is rather strange, as this relation bears within it the possibility of calculating $\Delta_K(t)$ recursively by 'untying—or splitting repeatedly—a knot-diagram'.

The idea of unknotting was not born here, though, as Tait had already considered the Gordian number. The relation given above is in a sense deceptively simple. There is no reason *a priori* why it should define any invariant. It may after all depend on things like projections or properties of the plane. Alexander did not find sufficient conditions to give any recursive process by which to obtain his polynomial. He did however prove the well-definedness of his proposed invariant.

The period of Alexander's work can suitably be called one of change and crossroads. The idea that knots could perhaps be understood by studying braids was one of the most promising ones to be pursued at that time. It caused Emil Artin to introduce the braid group, and Alexander to make some fundamental discoveries which bridged the gap between the two theories of knots and braids. We shall discuss these developments later, when we come to focus on braid theory's contribution to the study of knots.

9. Kurt Reidemeister and His Moves

In 1923 Kurt Werner Friedrich Reidemeister (1893–1971) accepted an associate professorship in Vienna where he did research on the foundations of mathematics. In 1925 he obtained a full professorship in Königsberg where his interests went to the foundations of geometry. It is not surprising that he was the person who disposed of many of the basic problems and early difficulties in the field of knot theory. His thorough work covered fundamental treatments of knot enumeration, projections and isotopy. Tait and his collaborators had found many knots, but they had not catalogued them in any workable manner. Reidemeister ordered and numbered them, using a notation which gave their positions in the list and their minimal numbers of crossing points. His notation, and tables of knot diagrams, stood for many years.

In the field of planar knot-projections, Reidemeister studied small, local changes made to a knot and how they corresponded to changes in the diagram

obtained by projection of the knot into a plane. He discovered that there were three fundamental changes (and their inverses); they are shown in diagram form below (Fig. 16). The left-hand sketch shows how a loop can be untwisted (removing one crossing); the centre one shows the pulling apart of two flaps (removing two crossings); and the right-hand one shows a portion of string passing over a crossing point (leaving the number of crossings unchanged). These three types of change, together with their inverse changes, are known as the *Reidemeister moves*. It should be evident that none of these changes relates to a change in the topological nature of the underlying knot.

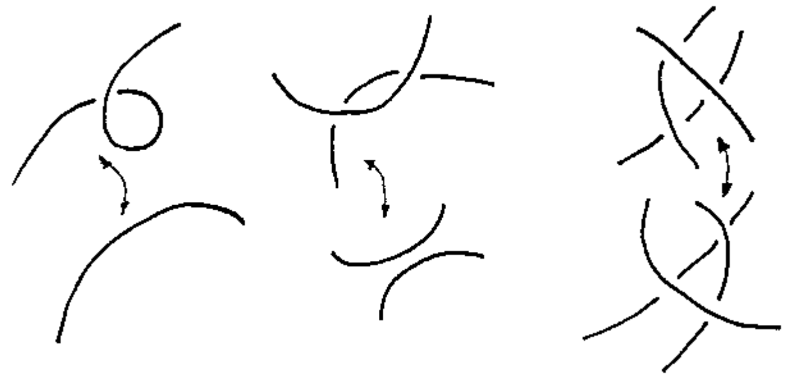


Fig. 16. The Reidemeister moves

The importance of the Reidemeister moves in combinatorial knot theory is embodied in the following key theorem:

If two knots (or links) are topologically equivalent, their diagrams can be transformed one to the other by some (finite) sequence of Reidemeister moves.

It should be noted that in any given case, there are many (indeed an infinite number of) different sequences of the three Reidemeister moves and their inverses which effect a transformation from one diagram to the other.

Reidemeister published the first book*on knot theory, in German, in 1932: an English edition of this book was published in 1983 [74].

Midway during the 20th century the history of knot theory, like much else, was temporarily disrupted by a seemingly global desire to practice politics

*J. B. Listing wrote the first book [62] on topology in 1847; it was dedicated primarily to knot theory. Bernhard Riemann was a student of Listing, and he learned about knots from Listing's book.

with violent means. The influence of combinatorial topology on knot theory declined markedly during this period.

10. The Fifties and Sixties

Ralph Hartzler Fox (1913–1973) was a mathematician who fostered an impressive mathematical environment around his person. Since Alexander's time, Princeton University had been the great name in knot theory's geography; and Fox's extensive publications on the subject made it even greater.

After the interruption in efforts caused by World War II, research came to focus mainly on the knot group, its subgroups and the principal ideals of its group ring. The way to represent a knot group by means of generators and defining relations led Fox to discover a free differential calculus. The ideas behind this calculus caused the Alexander polynomial to emerge as a determinant value of a matrix in which the entries are 'partial derivatives' (in Fox's calculus) of the knot group's relators with respect to its generators. The calculus came to be christened Fox's, and it led to the discovery of links between hitherto unrelated other fields in mathematics. In knot theory itself, it showed that the knot polynomial is determined by the group of the knot, and provided a link between the combinatorial and geometric definitions of the Alexander polynomial. On the practical side, the calculus supplied one more method for calculating Alexander polynomials. Specifically, it became one of the most important tools for studying knot groups defined by generators and relations.

As a person Ralph Fox has left quite an impression. After his death former students dedicated a 350-page book of their research papers to his memory [100]. From his school came people like Joan Birman, whom we shall meet later, and Lee Neuwirth working in knot groups; and Elvira Strasser Rappaport who studied 'knot-like' groups, addressing the question of which groups are knot groups.

Many of the developments in topology during the 1950–1980 period came to affect ideas about knots. Typifying the general development of knot theory and its techniques is that the concept of 'knot', so far treated as a *polygonal non-intersecting curve in 3-space (i.e.* $\mathbf{R}^3$*) upon which certain moves were permitted* became modernized to 'knot' being an *equivalence class of embeddings of the unit circle* $\mathbf{S}^1$ *in* $\mathbf{S}^3$. Topological studies had made it clear that $\mathbf{R}^3$ should be replaced by $\mathbf{S}^3$, in view of compactification properties of the latter. At the end of the fifties this led mathematicians such as André Haefliger and Christopher Zeeman to elaborate upon a theme, traceable back to Emil Artin's work, which considered mappings $\mathbf{S}^n \to \mathbf{S}^m$, for which $m - n = 2$ [41], [97]. These mappings 'tied the n-dimensional unit sphere into a knot' in the m-dimensional unit sphere. The objectives of this higher-dimensional, generalized knot theory included classification of knots with respect to isotopy. A difficulty was that the construction of these knots could not be visualized by simple-minded drawings of knot projections. Classification, and hence finding invariants, had therefore to be coupled to construction methods, showing how the invariants were realizable.

The more formal demands on smoothness of mappings brought in the notions of *tame* and *wild* knots. A knot is *tame* if it is equivalent to some polygonal knot; otherwise it is *wild*. The distinction was of vital importance; the principal invariants of knot type, namely the elementary ideals and the knot polynomials, were not necessarily defined for a wild knot. Knot theory was largely confined to the study of polygonal knots, and it was natural to ask what kinds of knot other than these were tame. An early theorem, and partial answer to this question was: *If a knot parametrized by arc length is continuously differentiable, then it is tame.*

There are infinite classes of wild knots, and their study forms a field of its own within the topological theory of knots.

11. John Conway's Tangling

As we have seen above, the problem of distinguishing knot-types for given numbers n of crossings, and tabulating them, was first tackled by the three men Tait, Kirkman and Little, in the final fifteen years of the 19th century. They succeeded in resolving the problem, by largely empirical methods, for $n = 3, \ldots, 10$ and for most of the alternating knots on 11 crossings.

There the matter rested for some seventy years, until John Horton Conway devised entirely new methods for studying knots, based on a construct called a *tangle*. Essentially, a *tangle* is a portion of a knot-diagram from which a number, usually four, free-end strands emanate; an example (Fig. 17) is given below.

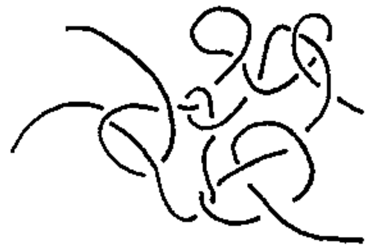


Fig. 17. An example of a Conway tangle

Conway gave a notation for describing knots in terms of their construction from tangles: using this notation, he was able to give rules for determining equivalences between knots. His methods were much simpler than previous ones, and lent themselves to programming for computer analysis. In a 1970 paper [23], Conway presented these ideas, and also listed knot-types in his notation for the following: all the prime alternating and nonalternating knots with crossing numbers $n = 3, \ldots, 11$; the 2-links up to $n = 8$; the 2-, 3-, and 4-links for $n = 9$; and all links for $n = 10$. In addition, for most of the knots in his tables, Conway gave values for several classical, and also new, invariants, obtained by his methods.

Thus, in a very short time*, using his newly devised methods of tangles, Conway had checked and extended the tables of Tait, Kirkman and Little, produced so laboriously about eighty years previously.

Certain manipulations on the Conway tangles gave rise to polynomials. Sample calculations with these were made, and they revealed certain algebraic relations between the polynomials (which in fact obviated the use of a computer for calculating his tables). It was natural to think of tangles as elements in a vector space, in which certain identities became linear relations. There were many natural questions to be asked about these spaces, and study of these led Conway to his discovery of a polynomial knot-invariant. Initially, Conway only wanted to further the enumeration and tabulation of knot-types, which task had been at a standstill for the past six decades when he began his attack upon it. But his contribution turned out to be a major one, in the hunt for knot invariants. Even though his find, in a sense, was Alexander's polynomial disguised in a normalized form, it was obtained by totally new methods. It became known as *the Conway polynomial*, often denoted by ∇_K. It was also a polynomial which could be calculated directly from a diagram by means of a recursive method, not requiring the evaluation of any determinant.

John Conway wanted to call the relationship between three links whose diagrams differ only in a set of Alexander crossings a *potential function*; but instead, this relationship went on to lead its own life in knot research, acquiring the name of *skein relation*. In Conway's original work this potential function had the form:

$$\nabla_{K_+}(t) - \nabla_{K_-}(t) = t\nabla_{K_0}(t)\ , \text{ with } \nabla_{\mathrm{U}} = 1$$

It relates to the Alexander polynomial Δ_K via the equation:

$$\Delta_K(t) = \nabla_K(t^{\frac{1}{2}} - t^{-\frac{1}{2}})$$

The important idea, which set new trends in knot research, was that the skein

*In [23] Conway claims that he could check in a mere afternoon much of the work that Tait and Co. took six years to complete!

relation became the invariant's definition. Its well-definedness could be proved by showing its invariance under the Reidemeister moves.

More so than with Alexander polynomials, which require definition and computations of certain determinants, the preferred way for defining polynomial invariants obtained via skein relations is to proceed from knot-diagrams. A polynomial is computed recursively, by a kind of 'unknotting process' when one systematically obtains diagrams on reducing numbers of crossings, making use of a given skein relation. When diagrams with already known polynomials are arrived at, the process can be retraced, and the polynomial for the original knot is arrived at.

Success with, and increased use of, this procedure caused knot-diagrams to become notational devices at the same level as other symbols in mathematical writings.

Incidentally, as we noted above, Conway did expand the knot-tables; and his work was later continued by Thistlethwaite and Perko [86], [71]. The latter completed the census problem for 10-fold knottiness in 1974, and detected some errors in Little's 1885 table of 11-fold crossing knots. Now we have complete listings of knots with up to 13 crossing-points [86]. And researchers are working to enumerate knots on 14 and 15 crossings [8]. There is an estimate that there exist over 150 000 different prime alternating knots on 15 crossings.

The following table shows the totals of prime alternating knots which have from 3 to 13 crossing-points. The top row gives the numbers of crossing-points, and the bottom row the corresponding frequencies of knots.

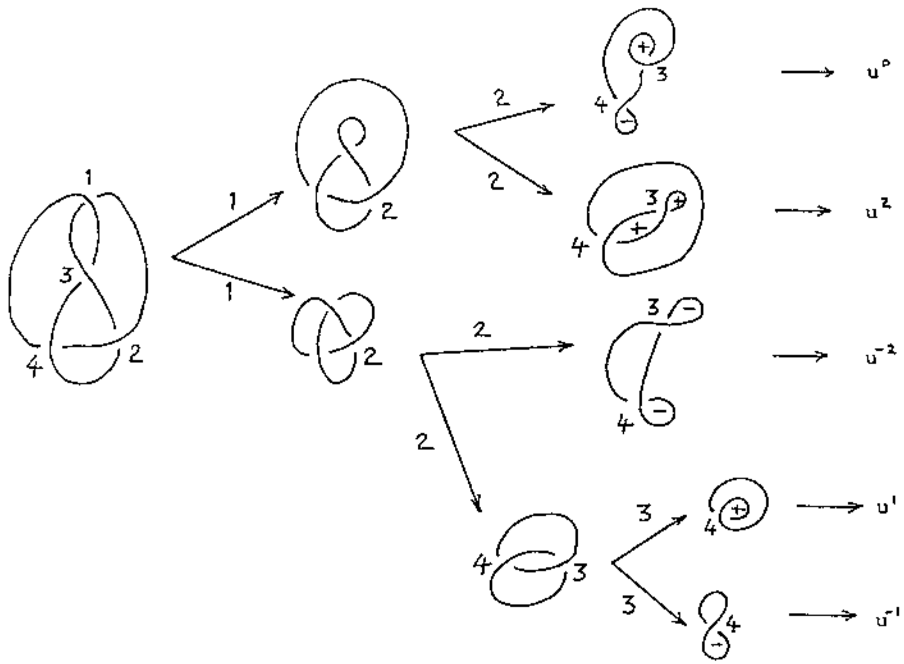
n	3	4	5	6	7	8	9	10	11	12	13
f	1	1	2	3	7	21	49	165	552	2176	9988

In the early 80s, John Turner [89] studied knot-graphs, and experimented with operations similar to Conway's. He obtained various knot invariants, working from both non-oriented and oriented diagrams. One idea he pursued was to take an alternating knot-diagram, and reduce it systematically by a process he called *twinning*. Crossings were 'deleted', one at a time, and two new knots (the 'twins', each with one fewer crossing than the original knot) were formed at each 'deletion' (see the example in Fig. 18). He continued this twinning, producing a binary tree of knots, and stopping the deletion process whenever a twist knot was arrived at.

The ultimate result, from any given starting knot, was a collection of twist knots (situated at the tree leaf-nodes), each of which had well-defined twist-senses, labelled plus or minus. He saw these as being fundamental building units of the original knot, and was able to prove, subject to one of Tait's many conjectures being true, that the end-collection of twists was independent of the order of reduction by 'deletions' of crossings, and that it was a knot invariant.

He assigned the symbol u^n to each n-twist, where n (positive, zero or negative) was the 'sum' of the senses of crossings in the twist (e.g. see Fig. 18).

Collecting all the symbols together, he arrived at a polynomial which he called *the twist spectrum* of the starting knot*. This, then, was a polynomial knot-invariant. In [89], Turner gave tables of knot twist-spectra for all alternating prime knots with $n = 3, \ldots, 9$ crossings, and all alternating 2-links with $n = 2, \ldots, 8$ crossings. The twist spectrum distinguished all these knots. He conjectured that it would distinguish all alternating knots with fewer than 15 crossings. In that sense it clearly outperformed the Alexander polynomial. Moreover, it could be used to provide a simple test for nonamphicheirality in a knot; for if a knot is amphicheiral its twist spectrum is symmetric about the constant term (the converse of this was conjectured, but unproven).



Spectrum: $T(u) = u^{-2} + u^{-1} + u^0 + u^1 + u^2$
Vector of coefficients: $(1, 1, \underline{1}, 1, 1)$

Fig. 18. Computing the Twist Spectrum of Listing's Knot (amphicheiral)

*This was a precursor of Kauffman's bracket polynomial, to be described later. Kauffman used the same deletion process, but continued beyond the twists, until no crossings at all remained. If his process were stopped at n-twists, his polynomial would be the same as the twist spectrum.

Figure 18 demonstrates the above process, producing the twist spectrum for the 4_1 knot (Listing's). The final twists, with their orientations and their corresponding polynomial terms, are shown on the right of the diagram.

An interesting connection between the Alexander polynomial $\Delta(t)$ of a knot, and the twist spectrum $T(u)$, is that the so-called determinant of the knot, given by $|\Delta(-1)|$, is equal to the *torsion coefficient* value $T(1)$. Also like the Alexander polynomial, the twist spectrum of a composition of two knots is equal to the product of the twist spectra of the two knots. Further, this time like the Jones' polynomial, $T_{K^*}(u) = T_K(u^{-1})$ if K^* and K are mirror images. So, for example, the trefoil and its mirror image are distinguished, since their twist spectra coefficients-vectors are $(1, \underline{0}, 1, 1)$ and $(1, 1, \underline{0}, 1)$.

Very soon after the twist spectrum was discovered, Vaughan Jones' great knot polynomial discovery was announced [43]. As we shall see below, this triggered an explosion of discoveries of polynomial invariants, and markedly changed the face of topological knot theory, both pure and applied. Before going on to describe these developments, it is necessary for us to review the history and achievements of braid theory.

12. Researches in Braid Theory

The beginning of the 1920s witnessed an impasse in the theory of knots. With the omnipotence of the knot group fatally punctured, and presentations of knot groups stuck in generally unsolvable word problems, it was not strange that knot theorists should seek new ways for achieving progress. The problems of those days attracted some of the most prominent algebraists and topologists. Minds like Seifert's pursued further research via Riemannian manifolds, while the actions of others appeared more desperate. Reasoning that knots consist of *bits of knotted patterns*, they broke them into smaller pieces which fulfilled certain conditions and called these objects ***braids***. Braids were not a new idea when they entered the scene in the 1920s. Listing and Tait had already studied procedures which generated simple knots after plaiting samples with two and three strands. On the other hand the *breaking up* was something entirely new. Emil Artin (1898-1962), with the help of Otto Schreier, formalized the ideas and provided tools to carry on the halted quest. His landmark paper on them [9], *Theorie der Zöpfe*, appeared in 1925.

Basically, he provided an entirely new algebraic environment for knot studies by introducing the so-called (algebraic) *braid group*, denoted by B_n. This is the set of all braids on n strings satisfying certain conditions, together with a binary operation which consists of the simple process of concatenation of two braids, joining the lower ends of one to the upper ends of the other. Examples of 3- and 4-string braids appear in Fig. 19.

The geometric picture of a braid in $\mathbf{R}^3$ is easy to envisage. Consider n

parallel strings in a plane, all hanging vertically from a line drawn on, for example, the ceiling, and dropping down to a line on the floor. There are thus $2n$ string endpoints, n on the ceiling and n on the floor. In a given braid, the endpoints are all to be regarded as fixed. This first configuration, with all strings parallel, is the *null braid*; it acts as the unit element of the *geometric braid group* $B(n)$. If now the lower endpoints are removed from the floor, the strings interwoven in some manner, and finally the endpoints are refixed to the floor in some order, then a new n-string braid will be achieved; such are the members of $B(n)$. With this geometric picture in mind, it is easy to imagine the concatenation operation, which joins two braids 'one on top of the other'.

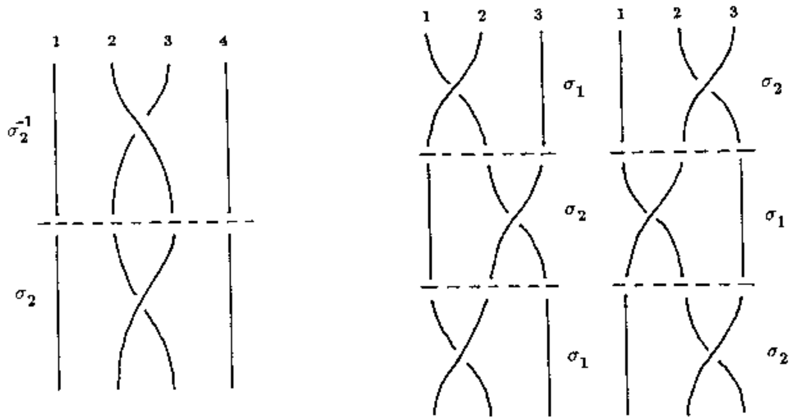


Fig. 19. On the left, a σ_2^{-1} twist in a 4-string braid is shown above a σ_2 twist in another 4-string braid. Concatenated they become a $\sigma_2^{-1}\sigma_2$ 4-string braid; note that the resulting 4-string braid is equivalent to the null braid (one with no twists). The other diagrams show concatenations of 3-string braids; note from these that, isotopically, $\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$

The n-string braid group is generated by the (n-1) *twists* (denoted by σ_i): the twist σ_i indicates a half-twist between the ith and $(i+1)$th strings. Its inverse is a half-twist in the opposite sense. The diagrams above illustrate this, with the 4-string braids.

Artin showed that $B(n) \simeq B_n$, and that they permit a presentation in terms of generators and relators given by

$$(\sigma_1, \ldots, \sigma_{n-1} : r_1, r_2), \qquad \text{in which}$$

$$\begin{cases} r_1 : \sigma_i\sigma_j = \sigma_j\sigma_i\,, \quad |i-j| \geq 2,\ 1 \leq i,j \leq n-1 \\ r_2 : \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}\,, \quad 1 \leq i \leq n-2 \end{cases}$$

Note how the two relators catch the principal features of a braid with three or more strands.

The braid group elements are *open* patches of weaving. They can be closed by linking up the respective ends. The most obvious way this can be done for an arbitrary $\beta \in B_n$ is by pairing the ends at the ceiling one to one with the corresponding ends at the floor. This yields the *closed* braid denoted by $\hat{\beta}$.

The main evidence that braids would be useful in studying knots and links was the existence of an $(n-1)$-dimensional representation of B_n, discovered by Werner Burau in 1936 [21]. This representation is expressed with a parameter t; and from it one can extract the so-called *Burau Matrix*. The following relationship connects Burau's braid group representation with Alexander's knot polynomial:

$$\det(\text{Id - Burau Matrix}(\beta)) = \pm t^n \Delta_{\hat{\beta}}(t) \qquad \text{for some } n$$

Burau showed this by connecting the Burau Matrix of β with a known way of calculating Δ from a presentation of the knot group.

The first firm connection between knot theory on the one hand and braid theory on the other was made by Alexander in 1923 with a theorem which showed that any n-link is equivalent to some closed braid. He also gave a simple algorithm [5] for converting an n-link into a closed braid. The existence of such an algorithm was already noted by Herman Brunn in 1897 [20]. However, troubles with this algorithm are twofold. It may cause one and the same n-link to become a closed braid which, upon cutting can belong to two distinct braid groups B_n and $B_{n'}$, $n \neq n'$. If the algorithm consistently causes the n-link to yield a closure of a braid belonging to just one braid group, then there may be many words representing it.

How does this result affect classification and knot isotopy? Closed braids have two very interesting properties, which are caught by the so-called Markov moves.

1. There is a particularly diabolical way of making a knot from an open braid. If one takes two braids $\alpha, \beta \in B_n$ and constructs the closure: $(\widehat{\alpha\beta\alpha^{-1}})$ then it will equal $\hat{\beta}$. The closing operation causes α to be cannibalized by its inverse. This is worded by saying that conjugate elements in B_n yield equivalent links.

2. Imagine a closed braid on a spar. Adding any number of strands by means of simple twists cast in any of its outer bights over the spar does not change the type of link.

The essence of these two properties was captured by A.A. Markov in his theorem of 1936 which states that closed braids $\hat{\xi}$ and $\hat{\zeta}$ are equivalent as links if and only if they can be connected by a finite sequence of elementary moves, which are precisely those described by the two properties of closed braids above. The braids are then said to be Markov equivalent. This is an equivalence relation on B_∞, the disjoint union of all braid groups.

Markov's theorem is of particular interest because it allows one to restate the knot problem as a purely algebraic problem known as the *Algebraic Link Problem*, which is the classification of Markov classes in B_∞. This comes down to finding a well-behaving class function on the Markov classes. It is worthy of remark that Markov never proved the theorem which got named after him. It had to wait until 1974 when the doyen of Braid Theory, Joan Birman, published a complete proof, achieved by cleverly combining the results of many other workers. It was no mean feat.

Another classical result is due to Artin. His original paper already contained the fundamental isomorphism between braids and automorphisms on B_n. It establishes that these mappings may be used to obtain a presentation for the knot group of any tame n-link [9]. The proof, though, was too intuitive for his liking. In 1947 he published a new paper on braids. This time he gave rigorous definitions and proofs including normal forms of a braid, which may be used to give a complete characterization of knot groups [10]. When Artin first suggested braid theory as an approach to the study of knots and links, he conjectured that the chief obstacle in the approach would be the conjugacy problem in B_n. This is the problem of deciding whether there exists a $\gamma \in B_n$ such that for two braids $\alpha, \beta \in B_n$ the equation $\alpha = (\gamma\beta\gamma^{-1})$ holds. In that case α and β are said to be *conjugated.* There have been many attempts to resolve this conjecture since Artin's 1925 paper. Partial solutions were attained, such as Frölich's in 1936, but it was not until Makanin and Garside around 1968–69 completely solved the problem. Garside [37] invented an ingenious, though rather complicated and hard to prove algorithm, by which he could decide whether or not two braids are conjugate.

Since we have available an algorithmic solution to the conjugacy problem it is natural to ask whether this might lead to a general solution of the knot problem? The answer is no, since there is trouble with the Markov moves. An arbitrary sequence of them applied to the closure of $\alpha \in B_n$ may either increase or decrease the number of strands, but if the final closed braid ultimately returns to B_n then there is no guarantee that one has not replaced $\hat{\alpha}$ with a conjugate of itself. This is not so bad as it may sound. Birman succeeded in finding some relaxed conditions for the knot problem which resulted in simplifying Garside's solution a little. Her work can easily be explained by introducing some nomenclature. A *positive word* denotes an open braid in which all $\sigma_i (1 \leq i \leq (n-1))$ have non-negative exponents. If such a braid is closed then it yields a *positive link.* Birman found that the knot problem on positive links reduces to the conjugacy problem. This implied that such links can be classified.

13. Problems in Paradise

Until this time, the beginning of the 1980s, the overall picture of the knot theoretical arena was one of relative tranquility. Purists insisted that knots and braids were different things. The former lived in the world of topologists, whilst braids belonged to the algebraists. This view catered for a state of peaceful coexistence, the result of an evolution in which both camps more or less went on minding their own business. This changed in a radical manner when a new knot polynomial erupted onto the scene. In the 1980s the New Zealander Vaughan Jones, through work in von Neumann Algebras associated with certain physics problems in statistical mechanics, had found a new way into knot polynomials. In order to study those aspects of theoretical physics he had constructed an algebra J_n given by:

$$(a_1, \ldots, a_{n-1} : r_1, r_2, r_3) \text{ where } \begin{cases} r_1 : a_i^2 = a_i \\ r_2 : a_i a_{i\pm1} a_i = \tau a_i \\ r_3 : a_i a_j = a_j a_i \,,\ |i - j| \geq 2 \end{cases}$$

What happened is perhaps best expressed in his own words [48, p. 53]:

> *In my work I had been astonished to discover expressions that bore strong resemblance to the algebraic expression of certain topological relations among braids.*

He was so struck by the resemblance between the definition of his algebra and that of the braid group B_n, that in May 1984 he journeyed to Columbia University to meet Joan Birman, to discuss his ideas with her. Their initial deliberations were discouraging. But Jones found, very soon after, that a representation of B_n could be transformed into one of J_n; which in turn possesses a trace map. Since trace maps are natural class functions, and Jones' trace also supported the second of the Markov moves, he thus had effectively constructed a link invariant!

General acknowledgement of the importance of this discovery was not immediate; but eventually his ideas came to have immense impact on much mathematical research in topology. In fact, he was later awarded the prestigious Fields' Medal for his contribution to the advancement of mathematics. Not only did Jones' work link knots to statistical mechanics, but also it sparked an interaction between knot-theory and braid-theory, the like of which had not been seen since the times when Artin's and Alexander's ideas became enjoined.

The new Jones knot invariant became known as *the Jones polynomial*; it is denoted by V. Jones himself had published its skein relation in his first article [47] which documents his ideas. The relation has the form:

$$t^{-1} V_{K_+}(t) - t V_{K_-}(t) + (t^{-\frac{1}{2}} - t^{\frac{1}{2}}) V_{K_0}(t) = 0\,, \text{ with } V_{\mathrm{U}} = 1$$

At first it seemed to be yet another polynomial invariant in the long line of such objects which had been proposed previously. Some of them were truly exotic in their definitions, while others were plain monsters in their time complexities; but V did something different. Knot theorists had begun to understand that symmetries, such as handedness and orientation, which are not caught by the Alexander/Conway polynomial, belong to entirely different regions of the mathematical universe. The reason that Alexander's polynomial does not cater for invariance under mirroring is due to the fact that $\Delta_{K^*}(t) = \Delta_K(\pm t^n)$. A key property of V is that (like the twist spectrum):

$$V_{K^*}(t) = V_K(t^{-1})$$

Simple examples show that $V_K(t)$ need not be invariant under $t \to t^{-1}$, so that it can sometimes distinguish knots from their mirror image. For example, in the case of the two Trefoils T_l and T_r we have:

$$\begin{cases} V_{T_r^*}(t) = t + t^3 - t^4 = V_{T_l}(t^{-1}) \\ V_{T_l^*}(t) = t^{-1} + t^{-3} - t^{-4} = V_{T_r}(t^{-1}) \end{cases}$$

V is pretty good at detecting this sort of symmetry, though not infallible. Its merits for this are first dashed with knot 9_{42} from the tables by Reidemeister and Rolfsen [74], [77]. But even so, this property shows that V is not a knot-group invariant like Δ or ∇.

It is a remarkable fact that ∇'s appearance on the knot scene, some 40 years after Δ's, did not trigger any generalizing activity in the mathematical community. Whereas the appearance of Jones' polynomial seemed to present an explicit invitation to do so. It immediately led to an outburst of discoveries of knot polynomials with more than one variable. Moreover, time since then has shown that V was to be generalized in two quite distinct ways.

The first general polynomial to appear was to be known as *Homfly*; it is also called *Homfly-PT*, and also *Thomflyp*. The names are acronyms which are derived from the initials (of 6) of the 8 people who discovered it [36], independently and almost simultaneously. Four of them happened to submit an article on their work to one and the same major mathematical journal on virtually the same day! Although they reached their results via different routes, the main idea was inspired by generalizing the coefficients in the skein relation. They were invited to pool their ideas, and present a single paper, jointly under all their names; this they did. Their two-variable polynomial, denoted by P, has the following skein relation:

$$\ell P_{K_+}(\ell, m) + \ell^{-1} P_{K_-}(\ell, m) + m P_{K_0}(\ell, m) = 0\,, \text{ with } P_{\mathrm{U}} = 1$$

For the sake of comparison all four skein relations are listed below:

$$\begin{cases} \Delta_{K_+}(t) - \Delta_{K_-}(t) + (t^{-\frac{1}{2}} - t^{\frac{1}{2}})\Delta_{K_0}(t) = 0\,, \text{ with } \Delta_{\mathrm{U}} = 1 \\ \nabla_{K_+}(t) - \nabla_{K_-}(t) - t\nabla_{K_0}(t) = 0\,, \text{ with } \nabla_{\mathrm{U}} = 1 \\ t^{-1}V_{K_+} - tV_{K_-} + (t^{-\frac{1}{2}} - t^{\frac{1}{2}})V_{K_0}(t) = 0\,, \text{ with } V_{\mathrm{U}} = 1 \\ \ell P_{K_+}(\ell, m) + \ell^{-1} P_{K_-}(\ell, m) + m P_{K_0}(\ell, m) = 0\,, \text{ with } P_{\mathrm{U}} = 1 \end{cases}$$

However, very soon after the public announcement of P's discovery, examples of indistinguishable pairs of mirror image knots emerged. Of course, one counter-example is sufficient to torpedo a conjecture, but in 1986 Taizo Kanenobu produced infinitely many classes of, in turn, infinitely many *distinct* knots with the same P-polynomial. Using the second elementary ideal of the Alexander module, he showed [52] that for the knots $K_{p,q}$ (see Fig. 20):

$$P(K_{p,q}) = P(K_{p',q'}) \quad \text{if and only if} \quad p + q = p' + q'.$$

P contains the information of Δ, ∇, V and more; but there the similarity ends. And to-date all attempts to interpret V in the same topological framework as Δ have failed.

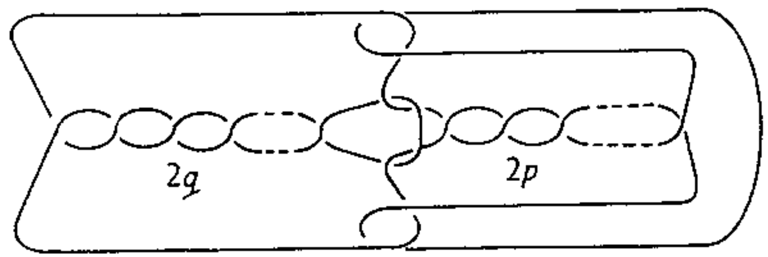


Fig. 20. The knot $K_{p,q}$ used by Taizo Kanenobu

Louis Kauffman's F-polynomial was the second distinct generalization of V to appear. He obtained his polynomial by cutting up a knot in a special way [54]. The skein relations we have seen so far are recursive definitions on knot diagrams which differ in the set of crossings which was shown in Fig. 18. Kauffman's polynomial is based on the very imaginative idea of a *state model*, which sees an unoriented knot diagram as a *state* σ, and which provides information carried by the diagram. Given a diagram (of a knot K, say), he proposed splitting a crossing in two ways, thereby obtaining two new states (the same operation as used to obtain Turner's twist spectrum). These were assigned a 'weight' A or A^{-1} according to their type of split (if a new state included a circle, a more complex weight, a function of A and A^{-1}, had to be assigned). He continued this process until no crossings were left; and then took the weighted sum over all the resulting states, obtaining what is now called the *Bracket polynomial* of K. This is denoted by:

$$< K > = \sum_{\sigma} < \sigma | K > (-A^2 - A^{-2})^{||\sigma|| - 1}$$

The Bracket can be generalized to yield a two-variable polynomial invariant, named the *L-polynomial*. By using L, divided by a factor involving one of the variables raised to the sum of the knot's crossing parities (the so-called *writhe* $\omega(K)$), one can construct yet another polynomial invariant, which is customarily denoted by F, and is known as the *Kauffman polynomial*. Thus:

$$F_K(\alpha, z) = \alpha^{-\omega(K)} L_K(\alpha, z)$$

In [60] Lickorish shows that the Jones and Kauffman polynomials, V and F, are related by the following equation:

$$V_K(t) = F_K(-t^{-\frac{3}{4}}, t^{-\frac{1}{4}} + t^{\frac{1}{4}})$$

The F-polynomial is quite good at detecting chirality. It is probably a little better for this purpose than P, because it originates from four instead of three terms; but it has its shortcomings too. However the Bracket polynomial is a truly amazing construct. For instance, it was used to confirm the old conjecture (made by Tait) that *the number of crossing points in a connected, reduced, alternating projection of a link is a topological invariant*. Several other long-standing problems were also dealt with quickly by means of this and the other new knot polynomials [69], [85].

In spite of its more powerful generalizations, $V_K(t)$ has retained its interest and value in knot research and applications. Ironically, one reason for this is, simply, that it has only one variable; which makes it easier to work with!

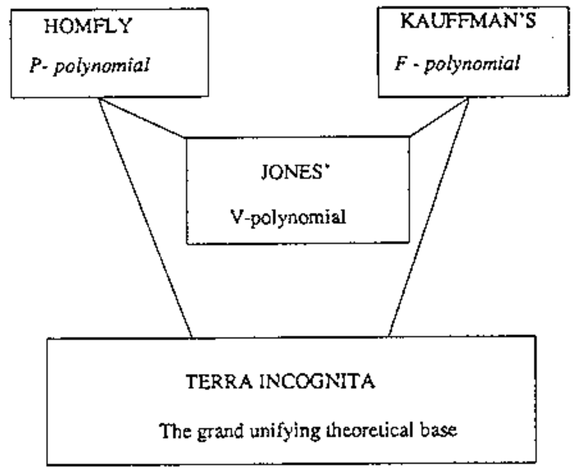
Amongst all the questions asked after the discovery of the new knot polynomials, the dominating one was: *Granting its existence, how may these polynomials be placed in a sound mathematical setting?* Many researchers have tried to shed light on this question, in the past decade, working from a variety of viewpoints, moving in uncharted territories of topology, algebra and statistical mechanics. The schematic diagram on the next page indicates the need to underpin satisfactorily the array of new knot invariants with a unifying theoretical base. And the next section summarizes some of the work that has been done in this 'terra incognita' since the invariants appeared.

14. Charting Terra Incognita

Jones' discovery implied that statistical mechanics must hold clues for an understanding of his polynomial. Therefore the discovery of V, F and

P triggered off even more ambitious research, which came from roughly two (interacting) directions. One of these was from physicists churning out link invariants. The other was from the mathematical camp of knot-theorists, trying hard to understand them.

It became obvious that via Markov's and Alexander's Theorem, one should be able to relate algebraic interpretations for link invariants to the braid groups. Algebraically, the link problem translates into domesticating a class function on the Markov classes. However, a head-on attack on Markov equivalence in B_∞ is hopelessly difficult. Luckily, representation theory's richness provides plenty of room for finding invariants. As the endpoints of braids define a permutation in a natural way, the symmetric group S_n thus exists as a quotient in B_n. It is natural, then, to study how representations of S_n and B_n are related. It turns out that *every* irreducible representation of S_n transforms to a parametrised family of irreducible representations of B_n. In fact, S_n transforms to an algebra $H_n(q)$, the so-called Hecke algebra, when $q \to 1$.



Adrian Ocneanu discovered Homfly as a trace function on the algebras H_∞, which supported a Markov trace as a weighted sum of matrix traces on their irreducible summands. The quadratic defining relation of a Hecke algebra afforded an explanation for its skein relation. The Japanese research team of Akutsu and Wadati found new link invariants by interpreting further statistical mechanical concepts [1], [2], [3], [4]. Their invariants turned out to be P again, but now in terms of 'cubic' Hecke relations, and supporting skein relations for triplets of links which are equivalent except in one crossing point, where they may have left- or right-handed spiralling segments instead of a single crossing. This was an idea that Conway had already explored when he discovered ∇ [23]. It is called *cabling*.

Unlike V, and to a certain extent P, the F-polynomial was discovered by purely combinatorial techniques, and it seemed at first glance to be completely unrelated to braids. However, so-called BWM algebras were constructed by Joan Birman, Hans Wenzl and also, independently, by Jun Murakami [17], [68]. Geometrically speaking, they extended the braid groups with U-turns, making them into (braid) monoids. The BWM algebras are quotients of the complex group algebra CB_n, and they support a 2-parameter family of Markov traces whose associated link invariant is the Kauffman polynomial. Each of these algebras contains H_n as a direct summand, and the Markov trace that associates to Homfly is the restriction to H_n of the Markov trace that defines F.

Each of the algebras just described supported a Markov trace, and so determined a link-type invariant. In this way a uniform picture of the old and new link invariants gradually emerged, with the representation theory of B_n being an important central part of the picture.

However, the various generalizations of link polynomials have been subsumed under an even more general and unifying procedure, via the so-called *Yang-Baxter Equation* (YBE). A Yang-Baxter operator on a vector space V is a linear isomorphism $R : V \otimes V \to V \otimes V$ such that the following hexagon commutes:

$$\begin{array}{ccccc} & V\otimes V\otimes V & \overset{R\otimes 1}{\rightarrow} & V\otimes V\otimes V & \\ {}^{1\otimes R}\nearrow & & & & \searrow^{1\otimes R} \\ V\otimes V\otimes V & & & & V\otimes V\otimes V \\ {}_{R\otimes 1}\searrow & & & & \nearrow_{R\otimes 1} \\ & V\otimes V\otimes V & \overset{1\otimes R}{\rightarrow} & V\otimes V\otimes V & \end{array}$$

This is equivalent to requiring that:

$$(R \otimes Id_V) \circ (Id_V \otimes R) \circ (R \otimes Id_V) = (Id_V \otimes R) \circ (R \otimes Id_V) \circ (Id_V \otimes R)$$

This equation is the Yang-Baxter Equation (YBE), introduced by C. N. Yang in 1967 in the context of the 1-dimensional quantum n-body problem as a factorization condition on the S-matrix. Later it was used by Rodney Baxter to obtain explicit formulae for the partition function of the 8-vertex model by the transfer matrix method. The YBE plays a fundamental role in two physical theories: namely the theory of exactly solvable models in statistical mechanics, and the theory of completely integrable systems.

Driven by an idea of Vaughan Jones [49], and returning to the mathematics behind statistical mechanics, Vladimir Turaev [88] showed that R-matrices (i.e. YBE solutions) yield a mechanism to produce new representations for B_n, each of which lead to new polynomial invariants. Turaev imposed conditions which ensured that the representations so obtained supported a Markov trace. Thus was born a machine, ready to produce further link invariants provided it was fed with R-matrices. Since then the history of the YBE has intertwined and interacted with developments in knot theory. For example, Hans Wenzl discovered how, by cabling braids one can find new representations for B_n, which in turn yield new solutions to the YBE [94].

The YBE can be written in several forms. There are general methods which allow one to construct R-matrices for the various versions [59], [75]. Originally, it was not written in the form given above, but as an equation in a Lie algebraical setting and called the classical YBE (the CYBE). This is easily explained. Let $\mathcal{L}$ be a Lie algebra, and let $r : \mathcal{L}^{\otimes 2} \to \mathcal{L}^{\otimes 2}$ be an isomorphism. Let r_{12} be $r \otimes 1 : \mathcal{L}^{\otimes 3} \to \mathcal{L}^{\otimes 3}$ and r_{ij} the image of r_{12} under the automorphism of $\mathcal{L}^{\otimes 3}$ induced by the permutation $(1,i)(2,j)$. The isomorphism r is said to be a solution for the CYBE if:

$$[r_{12}, r_{23}] + [r_{13}, r_{23}] + [r_{12}, r_{13}] = 0$$

The CYBE has been well studied. To any finite-dimensional representation of a simple complex Lie algebra on a vector space V endowed with an automorphism of the Dynkin diagram, there is a matrix $R \in \mathrm{End}(V \otimes V)$ which satisfies the CYBE [50], [59]. It was found that CYBE solutions satisfied Turaev's extra conditions. Michio Jimbo gives the R-matrices corresponding to the fundamental vectorial representations of the non-exceptional Lie algebras of the series $A_n^1, B_n^1, C_n^1, D_n^1, A_n^2$ and D_n^2, where the upper index denotes the order of the automorphism of the Dynkin diagram [45]. Jimbo showed that the R-matrices of the series A_n^1 result in operator invariants of ordered tangles and the Homfly polynomial of a link. The R-matrices of the series B_n^1, C_n^1, D_n^1 and A_n^2 result in operator invariants of framed tangles and the Kauffman polynomial of a link. These are 'constant' YBE-solutions. A daunting idea is to parametrise them and the YBE, much like a Hecke algebra is a parametrised version of S_n. A quantised universal enveloping algebra is such a transformation of a classical Lie algebra that depends on a transformation parameter q and recovers the classical algebra in the limit as $q \to 1$. Furthermore, it is endowed with a comultiplication, as well as an antipode and a co-unit, which gives it the structure of a Hopf algebra. These objects are better known as quantum groups, due to their intimate relationship with the quantum inverse scattering method, and in that connection first studied by the St. Petersburg school of L. D. Faddeev. Quantum groups arose in 1982 as algebraic structures describing symmetry properties encountered in solvable statistical models. However, they have not much to do with quantum mechanics, and are not groups either! Drinfel'd, who introduced the term *quantum group*, defined the structure as a Hopf algebra, essentially a bialgebra with an antipode. Quantum groups constitute an exciting generalization of the concept of symmetry. In this context, the parametrised YBE becomes the Quantum YBE (i.e. QYBE).

The invariants from the quantum group setting are polynomials. They are gathered under the heading quantum invariants, also called generalized Jones invariants. Since transfer matrices satisfy the YBE if and only if they are a representation for B_n, every quantum invariant is obtained from a trace function on an R-matrix representation for B_n.

The renaissance in the interactions between between physics and knot theory (recall Gauss's use of properties of knots in the solution of an electromagnetic problem, and Thompson's plans to describe chemistry in terms of knots) was due to statistical mechanics studies. But nowadays there are at least three other, different, ways in which physics and knot theory are related. Not only is there topological quantum field theory, but the theory of quantum invariants has also proved to be closely related to conformal field theories. In this connection one should mention Edward Witten's papers [96], where it is shown that Jones' polynomial and its generalizations are related to the topological Chern Simons actions.

So, order is emerging from chaos, and new results are being achieved continuously. The order appears to be part of an even larger order, which involves the physics of conformal field theory, and leads to further invariants, now in arbitrary 3-manifolds. As we have seen, the theory of R-matrices gives a systematic description for the quantum invariants. It has been known for a long time that any compact orientable 3-manifold arises (up to PL-homeomorphism or diffeomorphism, depending on the choice of category) as the boundary of a 4-manifold M, where M is obtained by attaching 2-handles to the 4-ball, along some framed link in S^3, i.e. by surgery along framed links. Lickorish and Wallace proved this at the beginning of the 1960s [61], [92]. One can think of framing as the thickening of the knot into a ribbon-like object. Any closed oriented 3-manifold may thus be obtained by performing surgery along *different* framed links in the 3-sphere. This yields an equivalence relation on framed links. Robion Kirby proposed a set of moves which generated this equivalence relation [55]. Roger Fenn and Colin Rourke simplified them. Their moves may be described by means of tangle generators [33]. By using Kirby, Fenn and Rourke calculus, Nicolai Reshetikhin and Vladimir Turaev in 1991 defined 3-manifold invariants using the theory of quantum groups. They produced new 3-manifold invariants, which can be defined from any simple Lie algebra, provided the associated quantum groups have the structure of a finite dimensional

modular Hopf algebra [76].

The theory of quantum invariants had led to the discovery of tangle categories. By looking at the most generalized form of 'knot', which is a graph with 'knotted ribboned parts', one can construct so-called ribbon categories. The finite dimensional representations for quantum groups comprise such a ribbon category. This connection could thus be used to find the new invariants for 3-manifolds. Other results made Reidemeister's theorem a covariant functor between the categories of links and diagrams, while quantum invariants became covariant functors from tangle categories to categories of modules. On the whole, category theory has been able to cut a lot of cake, as the language was quite effective to formulate and extend very general ideas about the central link polynomials.

So far, the story has been one of 'simple' hierarchical generalizations. An exciting change of perspective comes from V. A. Vassiliev [91]. He considers the so-called *knot space* $\mathcal{M}$, which is the space of all embeddings $\gamma : \mathbf{S}^1 \hookrightarrow \mathbf{S}^3$. This allows one to study more than just a single knot and the ways in which distinct knots relate to each other. The object of utmost interest is the natural stratification of $\mathcal{M}$. Deforming knots to the level where we permit self-intersections of the cord in which they are realised leads to the notion of *chambers* in $\mathcal{M}$. The discriminant $\mathcal{C}$ of $\mathcal{M}$ is defined to be the set of mappings which are not embeddings. This is a singular hypersurface in $\mathcal{M}$. The components of $\mathcal{M}-\mathcal{C}$ are clearly in one-to-one correspondence with the knot types. Thinking of a numerical knot invariant as a function on the components of $\mathcal{M}-\mathcal{C}$ one is led to study the cohomology of $\mathcal{M}$. Vassiliev introduced a system of subgroups of $\tilde{H}^0(\mathcal{M}-\mathcal{C})$:

$$0 = G_1 \subset G_2 \subset G_3 \subset \ldots \subset \tilde{H}^0(\mathcal{M}-\mathcal{C})$$

where $\tilde{H}^0$ is reduced cohomology with integer coefficients and G_i is free abelian of finite rank. The evaluation of an element in G_i/G_{i-1} on the component of $\mathcal{M}-\mathcal{C}$ corresponding to an oriented knot type K yields a rational number $u_i(K)$ associated to K. This is a Vassiliev invariant of order i. Like the Jones invariants, one computes Vassiliev invariants from a diagram by changing crossings. However, the combinatorics of the computation are very much more difficult.

Vassiliev's invariants are rational numbers, while generalized Jones invariants are Laurent polynomials over $\mathbf{Z}[q, q^{-1}]$. Joan Birman and Xiao-Song Lin showed that there is a relation between them [16]. Let a *knot* K have Jones polynomial $V_K(t)$. Set $U_x(K) = V_K(e^x)$ and express it as a power series in x:

$$U_x(K) = \sum_{i=0}^{\infty} u_i(K) x^i$$

then $u_0(K) = 1$ and each $u_i(K)$, for $i \geq 1$, is a Vassiliev invariant of order i. Although this scheme so far works only for knots, the Vassiliev invariants seem to offer at least part of the topological framework we seek for the quantum invariants. Hence there are speculations abounding. For instance, it is well-known that quantum groups do not detect invertibility of knots, but Vassiliev's invariants just might. This would make them stronger than quantum invariants. The research with ribbon categories and Vassiliev invariants have caused the singular braid monoid to place itself permanently on the scene. Moreover it threatens to become just as fundamental a mathematical tool as the braid groups. However, amongst all of these recent developments, mainly involving the braid groups, the knot complement group is far from forgotten. David Joyce, Colin Rourke and Roger Fenn have concocted an algebraic structure, which dates back to John Conway and van Brieskorn in the 1960s, and is now called a *rack*. It generalizes the knot group, but also captures aspects of the peripheral system. This completely classifying invariant seems to be a promising and exciting new part of the overall picture [34], [51].

15. The Future?

In the foregoing sections we have seen how vague, intuitive notions about knotted structures, beginning with the work of Listing, Gauss, Kirkman, Tait and Little of last century, were gradually developed until they reached, by the last decade of this century, extremely high levels of abstraction and complexity. Concerning the future of this process, one can only speculate on how far, and in which directions, the current attempts to solve a variety of outstanding problems will take us.

The prime knots with up to 13 crossings have been distinguished and tabled; and these knots are relatively simple objects. Attacks on the classification of 14- and 15-fold crossing knots are in progress; there are very many more of such knots, and no doubt it will require combinations of several of the available invariants to distinguish them all. It will be difficult, and perhaps not sensible, to produce diagrams for these vast numbers of knots; most will be 'known' only by their corresponding invariant values, arranged in classes and stored in some digitised form. The baffling problem of finding a single, complete, knot invariant (if one exists) still remains.

Future research will certainly be affected by the amazing developments stemming from Jones' discovery. They continue unrelentingly; yet many simple questions remain unanswered. It is still not known whether a non-trivial link can have the same V-, P- or F-polynomial as has the trivial link of the same number of components; we know this can happen for Δ and ∇. Resolution of this question would lead to significant conceptual progress. No generalization of it to knots and links of higher dimension has yet been achieved.

From the point of view of contemporary topological knot theory, the chief problem is to find an interpretation of the new invariants in terms of classical algebraic topology (homology theory, homotopy theory and such) or differential geometry (differential forms, connections, etc.). Here considerable speculation has produced little of note. Perhaps there can be no such interpretation, and states-model theories from statistical mechanics must be incorporated into topology. Perhaps we shall witness the emergence of other, exciting and entirely new theories, such as quantum mathematics, enriching mathematics. Who can tell? The recent interactions between knot theory and the rest of mathematics are really *quite* bewildering. They indicate that there is much still to be done.

Knot theory as it stands today represents a significant stream of ideas, flowing from the challenging difficulties of describing and understanding the phenomenon *knot* and its observable properties. The abstract heights it has reached, and the applications it has so surprisingly found in the wake of Jones' discoveries, give eloquent support to the often-mentioned notion of: *'the seemingly inevitable utility of mathematics conceived symbolically without reference to the real world.'*

It has been said that knots are more numerous than the stars, and are equally mysterious and beautiful. Like the stars seen at night, knots pervade our senses and challenge us to understand them. This happens now, not only in our everyday working world but also, as we learn from the quantum physicists, in our deeper philosophical efforts to explain the mysteries of fundamental physical and biological phenomena. The needs to understand these mysteries will continue to give impetus to the currently widening spread of research into theories and applications of knot theory.

Bibliographic Notes

Knot theory has a substantial literature, albeit very scattered; literature on the history of the subject is also scattered, fragmentary and sporadic. The earliest works, before the turn of this century, tend to mention many interesting sources; but as a rule authors on knot theory after 1900 are rather sparing with their historical information. Luckily there are a few exceptions such as Dehn/Heegaard [29]. From a mathematician's point of view, undoubtedly the most impressive accounts of knot theory's history may be found in Gordon [39] and Thistlethwaite [86]. The encyclopaedic work by Burde/Zieschang [22] evaluates and records the state of the field immediately before the spectacular discovery of the Jones polynomial. Their book supplies fragmentary historical data; but their bibliographic listing has over 1000 entries to compensate. Wilhelm Magnus has written about the early history of braid theory in [64]. Józef Przytycki has described parts of the modern history of knot theory in

[73]; and Joan Birman has written a speculative and exciting overview article [15] of the very latest developments.

After the discovery of the Alexander polynomial, knot literature came in a steadily increasing flow. Nowadays one may speak of an explosive growth of papers in the field. Yet comprehensive books, both monographs and textbooks, are still few and far between. Kurt Reidemeister's pioneering work *Knoten Theorie* of 1932 appeared in English translation [74] in 1983. Its approach, of course, follows the combinatorial spirit of its times, and so only supplies a historical introduction to the subject. Another book still of much value is *Introduction to Knot Theory*, by Richard Crowell and Ralph Fox (1963) [25]. This gives a beautiful introduction to the subject from the classical algebraic topological point of view, and is a fine tribute to the developments which emerged in the post World War II period. *Knots and Links* by Dale Rolfsen (1976) is remarkable for a number of reasons [77]. It is a giant leap into (geometric) topology, and introduces all developments up to the mid-70s. An excellent introduction to the theory of braids is *Braids, Links and Mapping Class Groups* by Joan Birman [14].

Following the explosion of activity in applied knot studies in the late 80s, a stream of books on the topic has been published. For example, Louis Kauffman's *Knots and Physics* [54], hard on the heels of books such as *Braid Group, Knot Theory and Statistical Mechanics* (edrs. C. N. Yang and M. L. Ge, 1989) and *New Developments in the Theory of Knots* (edr. Toshitake Kohno, 1990); these last two are volumes 9 and 11 in World Scientific's *Advanced Series in Mathematical Physics*. This present book is volume 11 in World Scientific's *Series on Knots and Everything*. And in January 1992 the first edition of *Journal of Knot Theory and its Ramifications* appeared, also published by World Scientific; the subject has, at last, its own Journal.

It is inevitable that the new ideas and theories about knots will gradually be introduced into syllabuses for graduate and undergraduate mathematicians and physicists. Textbooks for teaching the subject will come forth. An excellent recent example is *Knot Theory*, by Charles Livingston (Mathematical Association of America, 1993); he covers much of the classical theory, and continues through to high-dimensional knots and the combinatorial techniques of various of the new polynomial invariants. He includes many exercises suitable for undergraduates, to whet their appetites and help them come to grips with this exciting but demanding subject.

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CHAPTER 12

ON THEORIES OF KNOTS

John Turner

> *"'And everybody praised the Duke,*
> *Who this great fight did win.'*
> *'But what good came of it at last?'*
> *Quoth little Peterkin.*
> *'Why, that I cannot tell,' said he,*
> *'But 'twas a famous victory!'"*
>
> [On G. T. Fechner turning psychology into an exact science; quoted in *The World of Mathematics*, J. R. Newman, p. 1165.]

1. Is Knot Theory Topology?

The earliest scientific paper we know in which a mathematician discusses the problem of constructing a mathematical theory of knots, contains the following paragraph:

> *Whatever the twists and turns of a system of threads in space, one can always obtain an expression for the calculation of its dimensions, but this expression will be of little use in practice. The craftsman who fashions a braid, a net, or some knots will be concerned, not with questions of measurement, but with those of position: what he sees there is the manner in which the threads are interlaced.*
>
> Alexandre Theophile Vandermonde (1735–1796)